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ON LARGE VALUES OF L^2 HOLOMORPHIC FUNCTIONS

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1. Introduction

Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded domain of holomorphy and $p \in \Omega$. Let $\mathcal{O}(\Omega)$ denote the holomorphic functions on Ω, $L^2(\Omega)$ the square-integrable functions on Ω and consider the following extreme-value problem:

$$M_\Omega(p) = \sup\{|f(p)|^2 : f \in \mathcal{O}(\Omega) \cap L^2(\Omega), ||f||_{L^2} \leq 1\}. \tag{1.1}$$

The solution to (1.1) gives the value of the Bergman kernel function associated to Ω (the kernel of the operator projecting $L^2(\Omega)$ orthogonally onto $\mathcal{O}(\Omega)$) at (p,p). If $n = 1$, it is a classical fact that $M_\Omega(p)$ is bounded, from above and below, by a constant factor times $\text{dist}(p, b\Omega)^{-2}$. In higher dimensions, the geometry of $b\Omega$ influences the size of $M_\Omega(p)$ in non-trivial ways. A general lower bound for (1.1) was proved by Ohsawa-Takegoshi, [O-T] (see also [P] for a slightly weaker, prior estimate): $M_\Omega(p) \geq C\text{dist }(p, b\Omega)^{-2}$ for a constant C independent of p. It was also shown by Ohsawa, [O], that each positive eigenvalue of the Levi form (of $b\Omega$ near p) adds -1 to the exponent on the right side of this inequality. Trivial upper bounds on (1.1) related to the dimension are easy to obtain from the maximum principle, but only in some special cases have upper and lower bounds of the same order of magnitude in the boundary distance been obtained, see [C], [F], [H], [Mc1-2]. In all these cases, the first assumption is that the Levi form associated to $b\Omega$ has finite degeneracy at points in $b\Omega$ near p.

In this paper, we show that (1.1) has a lower bound, which sharpens that given in [O-T], for a class of domains whose Levi forms do not, necessarily, degenerate to finite order. For notational convenience, we state the result for an infralevel domain of Ω, instead of Ω itself.

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Theorem 1.1. *Let $\{z : r(z) < 0\} = \Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded, pseudoconvex domain. Suppose that $z_0 \in b\Omega$ and that $b\Omega$ admits a (local, weak) holomorphic support surface, S, at z_0. Let ν_{z_0} denote the inward unit normal for $b\Omega$ at z_0.*

Then there exists a constant $c > 0$ so that if $p = z_0 + \delta\nu_{z_0}$, then

$$c\delta^{-2} M_{S_p\cap\Omega}(p) \leq M_{\Omega_\delta}(p), \tag{1.2}$$

where $S_p = \{z : z - \delta\nu_{z_0} \in S\}$ and $\Omega_\delta = \{z : r(z) < -\frac{\delta}{2}\}$.

The hypothesis of this theorem is restrictive, as evidenced by the Kohn-Nirenberg type domains [K-N]. Note that $S_p \cap \Omega$ is (essentially) a domain in $\mathbb{C}^{n-1}$ and so the result should be interpreted as having sliced away a complex dimension from the extreme-value problem (at a cost of the factor δ^{-2}). The exponent -2 reflects the fact that the $\bar{\partial}$-Neumann problem is elliptic in the component transverse to $b\Omega$ on any smoothly bounded pseudoconvex domain; finding the correct exponent for general slices of the domain would be a difficult problem, essentially equivalent to understanding exact, non-isotropic regularity of the $\bar{\partial}$-Neumann problem.

This theorem and previous results do not give an induction on dimension result for the full asymptotics of $M_\Omega(p)$. Consider $\Omega \subset\subset \mathbb{C}^3$ which satisfies the hypothesis of Theorem 1.1 and is, additionally, of finite type (see [D'A] for the definition of this concept). Although Theorem 1.1 relates $M_\Omega(p)$ to an extreme-value problem on a (finite type) domain in $\mathbb{C}^2$, it does not combine with the above mentioned result of Catlin to give a lower bound on $M_\Omega(p)$ solely as a function of δ. This is because the point p, though absolutely close to $bS_p \cap \Omega$, is far from $bS_p \cap \Omega$ relative to the diameter of $S_p \cap \Omega$. Understanding $M_{S_p\cap\Omega}(p)$ is, therefore, not so much a question about the boundary behavior of the Bergman kernel but, rather, related to the asymptotics of the volume (and various moments) of small domains, as their diameters go to zero. We will discuss this problem in a future paper.

The method of proof for Theorem 1.1 revolves around solving a certain system of partial differential equations, which are pertubations of the ordinary $\bar{\partial}$-equations, with estimates in weighted L^2 spaces. The presence of the pertubation factor gives a useful new term in the standard L^2 estimates for the $\bar{\partial}$-equations and manipulation of this factor and the weight factor in a somewhat independent manner is the key step in our proof. The idea of perturbing the $\bar{\partial}$-equations in the manner that we do (section 2) is taken directly from the work of Ohsawa-Takegoshi, [O-T], in which they obtain very similar inequalities on complex mainfolds with a complete Kähler metric. Indeed, the only new point to our manipulations in section 2 is to show that the hypothesis of a complete metric is unnecessary for the type of estimates that [O-T] consider. It is, of course, useful to work with non-complete metrics (e.g., the euclidean metric) as this allows the study of various questions about boundary behavior.

In addition to [O-T], we mention several works which overlap to some extent with this paper. In particular, we point out the recently received papers of Berndtsson [B] and Siu [S] which contain closely related results. Berndtsson, [B], in work on sup-norm estimates for $\bar{\partial}$, obtains a differential equality which is essentially equivalent to a non-integrated version of our identity (2.14). Siu, [S], in work concerning the construction of certain singular metrics, has independently obtained a result equivalent to our Theorem 2.1. We also mention the works of Donnelly-Fefferman [D-F] and Witten [W], where pertubation schemes (of the exterior derivative) similar to that discussed in section 2 appear (explicitly in [W] and implicitly in [D-F]). It is worthwhile to point out that the asymmetric twisting of either the d-complex or $\bar{\partial}$-complex, used in [O-T] and [D-F], is crucial for the useable, new term to appear in the weighted L^2 estimates. Conjugation of the operators by a smooth function, as in [W], leads only to the standard estimates of Hörmander, [H], for a shifted weight function.

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2. L^2 estimates for a twisted $\bar{\partial}$ complex

The manipulations in this section are direct applications of material found in [F-K]. The reader is referred there for further elaboration of the (un-weighted) $\bar{\partial}$-Neumann problem.

Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded, pseudoconvex domain defined by a real-valued function r, normalized so that $|dr| \equiv 1$ on $b\Omega$. We choose the sign of r so that $r < 0$ in Ω.

We use the symbols $\Lambda^{p,q}(\overline{\Omega})$ and $\Lambda^{p,q}(\Omega)$ to denote the forms of type (p, q) which are smooth on $\overline{\Omega}$ and Ω, respectively. If $\phi, \psi \in \Lambda^{p,q}(\Omega)$, we denote the pointwise euclidean inner product of ϕ and ψ at z by $< \phi, \psi >_z$ and drop the subscript z if no confusion is likely to arise. If λ is a function defined on Ω, we define a global inner product of $\phi, \psi \in \Lambda^{p.q}(\Omega)$ by setting

$$(\phi, \psi)_\lambda = \int_\Omega < \phi, \psi >_z \ e^{-\lambda(z)} \, dV(z), \tag{2.1}$$

where $dV(z)$ is the (euclidean) volume element at z. We denote the norm determined by (2.1) by $|| \cdot ||_\lambda$.

Let $\Pi^{0,1}$ denote the projection of the complexified cotangent bundle of $\mathbb{C}^n$ onto its (0,1) subspace and $\Pi^{p,q}$ the naturally induced projection of forms of order $p+q$ onto $\Lambda^{p,q}$. The operator $\bar\partial : \Lambda^{p,q}(\Omega) \longrightarrow \Lambda^{p,q+1}(\Omega)$ is defined as $\bar\partial\phi = \Pi^{p,q+1} d\phi$, where d is the exterior derivative operator. The formal adjoint of $\bar\partial$ with respect to the inner product (2.1), ϑ_λ, maps $\Lambda^{p,q}(\Omega)$ to $\Lambda^{p,q-1}(\Omega)$ and is defined by the condition

$$(\vartheta_\lambda\phi, \psi)_\lambda = (\phi, \bar\partial\psi)_\lambda,$$

for all $\psi \in \Lambda^{p,q-1}(\Omega)$ with compact support.

If ξ is a 1-form, the symbol of $\bar\partial$ is the linear map $\sigma(\bar\partial, \xi) : \Lambda^{p,q}(\Omega) \longrightarrow \Lambda^{p,q+1}(\Omega)$ defined as

$$\sigma(\bar\partial,\xi)\phi = \Pi^{0,1}(\xi) \wedge \phi, \qquad \phi \in \Lambda^{p,q}(\Omega).$$

The symbol of ϑ_λ, $\sigma(\vartheta_\lambda, \xi)$, is the adjoint of $\sigma(\bar\partial,\xi)$ in the euclidean inner product. Integration by parts then reads as:

$$(\phi, \bar\partial\psi)_\lambda = (\vartheta_\lambda\phi, \psi)_\lambda - \int_{b\Omega} < \sigma(\vartheta_\lambda, dr)\phi, \psi > e^{-\lambda}\, dS, \tag{2.2}$$

where dS is the volume element of $b\Omega$ and $\phi \in \Lambda^{p,q}(\overline{\Omega})$ and $\psi \in \Lambda^{p,q-1}(\overline{\Omega})$.

An operator of greater interest in the following is the Hilbert space adjoint of $\bar\partial$, $\bar\partial^*_\lambda$. Define

$$\mathcal{D}^{p,q} = \{\phi \in \Lambda^{p,q}(\overline{\Omega}) : \sigma(\vartheta_\lambda, dr)\phi = 0 \text{ on } b\Omega\}. \tag{2.3}$$

The operator $\bar\partial^*_\lambda$ (defined as an operator on L^2) is defined on Dom $(\bar\partial^*_\lambda)$, a set which contains $\mathcal{D}^{p,q}$. It happens that the operators $\bar\partial^*_\lambda$ and ϑ_λ agree on $\mathcal{D}^{p,q}$, see [F-K].

In the sequel, we will work with forms in $\Lambda^{0,q}(\overline{\Omega})$, $q \le 2$, and it will be convenient to express the relevant quantities in terms of coordinates. If $(z_1, \dots, z_n)$ are holomorphic coordinates on $\mathbb{C}^n$ and $\phi = \sum_{i=1}^n \phi_i\, d\bar z_i \in \Lambda^{0,1}(\overline{\Omega})$ then

$$\bar\partial\phi = \sum_{l<k} \Big(\frac{\partial\phi_l}{\partial\bar z_k} - \frac{\partial\phi_k}{\partial\bar z_l}\Big) d\bar z_l \wedge d\bar z_k. \tag{2.4}$$

The condition that $\phi \in \mathcal{D}^{0,1}$ is equivalent to $\sum_{l=1}^n \phi_l \frac{\partial r}{\partial z_l} = 0$ on $b\Omega$. Also, if $\phi \in \mathcal{D}^{0,1}$,

$$\begin{aligned} \bar\partial^*_\lambda\phi &= -2\sum_{j=1}^n e^\lambda \frac{\partial}{\partial z_j}(e^{-\lambda}\phi_j) \\ &= -2\sum_{j=1}^n \delta_j(\phi_j), \end{aligned} \tag{2.5}$$

where the last equality defines the operators δ_j, $j = 1, \dots, n$.

Theorem 2.1. *Let $\Omega \subset\subset \mathbb{C}^n$ be smoothly bounded and pseudoconvex. Let λ, g, h be smooth functions defined on Ω, $g, h \geq 0$, and suppose, for all $\phi \in \mathcal{D}^{0,1}$,*

(2.6)
$$\int_\Omega g \sum_{j,k=1}^n \frac{\partial^2 \lambda}{\partial z_j \partial \bar{z}_k} \phi_j \bar{\phi}_k \, e^{-\lambda} - \int_\Omega \sum_{j,k=1}^n \frac{\partial^2 g}{\partial z_j \partial \bar{z}_k} \phi_j \bar{\phi}_k \, e^{-\lambda} \\ - \int_\Omega h^{-1} | \sum_{j=1}^n \frac{\partial g}{\partial z_j} \phi_j |^2 e^{-\lambda} \geq (P\phi, \phi)_\lambda,$$

for some positive definite matrix of functions P. Then, for any $\alpha \in \Lambda^{0,1}(\Omega)$ with $\bar{\partial}\alpha = 0$, there exists a solution u to $\bar{\partial}(\sqrt{g+h} \cdot u) = \alpha$ with the estimate

$$||u||_\lambda^2 \leq C(P^{-1}\alpha, \alpha)_\lambda, \tag{2.7}$$

for an independent constant C.

Proof. If $\phi \in \mathcal{D}^{0,1}$ and $g \geq 0$, then (2.4) implies

$$(g\bar{\partial}\phi, \bar{\partial}\phi)_\lambda = 4 \sum_{l<k} ||\sqrt{g}(\frac{\partial \phi_l}{\partial \bar{z}_k} - \frac{\partial \phi_k}{\partial \bar{z}_l})||_\lambda^2 \\ = 4 \sum_{l,k=1}^n ||\sqrt{g} \frac{\partial \phi_l}{\partial \bar{z}_k}||_\lambda^2 - 4 \sum_{l,k=1}^n (g \frac{\partial \phi_l}{\partial \bar{z}_k}, \frac{\partial \phi_k}{\partial \bar{z}_l})_\lambda.$$

It follows from (2.5) that

$$(g\bar{\partial}_\lambda^* \phi, \bar{\partial}_\lambda^* \phi)_\lambda = 4 \sum_{l,k=1}^n (g\delta_l \phi_l, \delta_k \phi_k)_\lambda.$$

Thus,

(2.8)
$$(g\bar{\partial}\phi, \bar{\partial}\phi)_\lambda + (g\bar{\partial}_\lambda^* \phi, \bar{\partial}_\lambda^* \phi)_\lambda = 4 \sum_{l,k=1}^n ||\sqrt{g} \frac{\partial \phi_l}{\partial \bar{z}_k}||_\lambda^2 + \\ 4 \sum_{l,k=1}^n \int_\Omega \{ g\delta_l \phi_l \overline{\delta_k \phi_k} - g \frac{\partial \phi_l}{\partial \bar{z}_k} \overline{\frac{\partial \phi_k}{\partial \bar{z}_l}} \} e^{-\lambda} \\ = 4 \sum_{l,k=1}^n ||\sqrt{g} \frac{\partial \phi_l}{\partial \bar{z}_k}||_\lambda^2 + M.$$

We record three facts:

$$[\delta_l, \frac{\partial}{\partial \bar z_k}] = \frac{\partial^2}{\partial z_l \partial \bar z_k}. \tag{2.9}$$

If u and v are smooth functions, then

$$(\frac{\partial u}{\partial \bar z_k}, v)_\lambda = -(u, \delta_k v)_\lambda - \int_{b\Omega} \frac{\partial r}{\partial \bar z_k} u \bar v \, e^{-\lambda}. \tag{2.10}$$

$$(\delta_l u, v)_\lambda = -(u, \frac{\partial v}{\partial \bar z_l})_\lambda - \int_{b\Omega} \frac{\partial r}{\partial z_l} u \bar v \, e^{-\lambda}. \tag{2.11}$$

The equality (2.9) follows immediately from the definitions and (2.10) and (2.11) follow by integration by parts.

Moving the derivatives to the left in (2.8) gives

$$\begin{aligned} M = 4 \sum_{l,k=1}^n \int_\Omega \{\delta_l (g \frac{\partial \phi_l}{\partial \bar z_k}) - \frac{\partial}{\partial \bar z_k}(g \delta_l \phi_l)\} \overline{\phi_k} \, e^{-\lambda} \, - \\ 4 \sum_{l,k=1}^n \int_{b\Omega} \frac{\partial r}{\partial z_l} \cdot g \frac{\partial \phi_l}{\partial \bar z_k} \bar\phi_k \, e^{-\lambda} + 4 \sum_{l,k=1}^n \int_{b\Omega} \frac{\partial r}{\partial \bar z_k} \cdot g \delta_l \phi_l \bar\phi_k \, e^{-\lambda}. \end{aligned} \tag{2.12}$$

Note that the second boundary integral vanishes since $\phi \in \mathcal{D}^{0,1}$, while the first boundary integral can be re-written in the standard way. Namely: since $\sum_{k=1}^n g \bar\phi_k \frac{\partial}{\partial \bar z_k}$ is a tangential derivative and $\sum_{l=1}^n \phi_l \frac{\partial r}{\partial z_l} = 0$ on $b\Omega$, then on $b\Omega$

$$\begin{aligned} 0 &= \sum_{k=1}^n g \phi_k \frac{\partial}{\partial \bar z_k} (\sum_{l=1}^n \phi_l \frac{\partial r}{\partial z_l}) \\ &= \sum_{l,k=1}^n g \frac{\partial^2 r}{\partial z_l \partial \bar z_k} \phi_l \bar\phi_k + \sum_{l,k=1}^n g \bar\phi_k \frac{\partial \phi_l}{\partial \bar z_k} \frac{\partial r}{\partial z_l}. \end{aligned}$$

Substituting into (2.12) gives

(2.13)

$$
\begin{aligned}
M = {} & 4\sum_{l,k=1}^{n}\int_{\Omega}\{\delta_l(g\frac{\partial\phi_l}{\partial\bar{z}_k}) - \frac{\partial}{\partial\bar{z}_k}(g\delta_l\phi_l)\}\overline{\phi_k}\, e^{-\lambda} + \\
& 4\sum_{l,k=1}^{n}\int_{b\Omega} g\frac{\partial^2 r}{\partial z_l\partial\bar{z}_k}\phi_l\bar{\phi}_k\, e^{-\lambda} \\
= {} & 4\sum_{l,k=1}^{n}\int_{\Omega} g[\delta_l, \frac{\partial}{\partial\bar{z}_k}]\phi_l\bar{\phi}_k\, e^{-\lambda} + 4\sum_{l,k=1}^{n}\int_{b\Omega} g\frac{\partial^2 r}{\partial z_l\partial\bar{z}_k}\phi_l\bar{\phi}_k\, e^{-\lambda} + \\
& 4\sum_{l,k=1}^{n}\int_{\Omega}\{\frac{\partial g}{\partial z_l}\frac{\partial\phi_l}{\partial\bar{z}_k} - \frac{\partial g}{\partial\bar{z}_k}(\delta_l\phi_l)\}\bar{\phi}_k\, e^{-\lambda}.
\end{aligned}
$$

The last equality holds since $\delta_l(uv) = u\delta_l v + \frac{\partial u}{\partial z_l}v$.

In the first piece of the last integral above, move $\frac{\partial}{\partial\bar{z}_k}$ to the left and use that $\phi \in \mathcal{D}^{0,1}$ to obtain

$$
\begin{aligned}
& 4\sum_{l,k=1}^{n}\int_{\Omega}\frac{\partial g}{\partial z_l}\frac{\partial\phi_l}{\partial\bar{z}_k}\bar{\phi}_k\, e^{-\lambda} \\
& \quad = 4\sum_{l,k=1}^{n}\int_{b\Omega}\frac{\partial r}{\partial\bar{z}_k}\frac{\partial g}{\partial z_l}\phi_l\bar{\phi}_k\, e^{-\lambda} - 4\sum_{l,k=1}^{n}\int_{\Omega}\frac{\partial}{\partial\bar{z}_k}(\frac{\partial g}{\partial z_l}\bar{\phi}_k e^{-\lambda})\phi_l \\
& \quad = -4\sum_{l,k=1}^{n}\int_{\Omega}\frac{\partial g}{\partial z_l}\cdot\frac{\partial}{\partial\bar{z}_k}(\bar{\phi}_k e^{-\lambda})\phi_l - 4\sum_{l,k=1}^{n}\int_{\Omega}\frac{\partial^2 g}{\partial z_l\partial\bar{z}_k}\phi_l\bar{\phi}_k\, e^{-\lambda} \\
& \quad = -4\sum_{l,k=1}^{n}\int_{\Omega}\frac{\partial g}{\partial z_l}\phi_l\overline{\delta_k\phi_k}\, e^{-\lambda} - 4\sum_{l,k=1}^{n}\int_{\Omega}\frac{\partial^2 g}{\partial z_l\partial\bar{z}_k}\phi_l\bar{\phi}_k\, e^{-\lambda}.
\end{aligned}
$$

Thus (2.13) becomes

$$
\begin{aligned}
M = {} & 4\sum_{l,k=1}^{n}\int_{\Omega} g[\delta_l, \frac{\partial}{\partial\bar{z}_k}]\phi_l\bar{\phi}_k\, e^{-\lambda} + 4\sum_{l,k=1}^{n}\int_{b\Omega} g\frac{\partial^2 r}{\partial z_l\partial\bar{z}_k}\phi_l\bar{\phi}_k\, e^{-\lambda} - \\
& 4\sum_{l,k=1}^{n}\int_{\Omega}\frac{\partial^2 g}{\partial z_l\partial\bar{z}_k}\phi_l\bar{\phi}_k\, e^{-\lambda} - 4[\sum_{l,k=1}^{n}\int_{\Omega}(\frac{\partial g}{\partial z_l}\phi_l\overline{\delta_k\phi_k} + \frac{\partial g}{\partial\bar{z}_k}(\delta_l\phi_l)\bar{\phi}_k)\, e^{-\lambda}].
\end{aligned}
$$

Consequently, (2.8) yields the following indentity:

(2.14)

$$(g\bar\partial\phi,\bar\partial\phi)_\lambda+(g\bar\partial^*_\lambda\phi,\bar\partial^*_\lambda\phi)_\lambda=4\sum_{l,k=1}^n||\sqrt{g}\frac{\partial\phi_l}{\partial\bar z_k}||^2_\lambda+$$
$$4\int_\Omega g\sum_{l,k=1}^n\frac{\partial^2\lambda}{\partial z_l\partial\bar z_k}\phi_l\bar\phi_k\,e^{-\lambda}-4\sum_{l,k=1}^n\int_\Omega\frac{\partial^2 g}{\partial z_l\partial\bar z_k}\phi_l\bar\phi_k\,e^{-\lambda}+$$
$$4\sum_{l,k=1}^n\int_{b\Omega}g\frac{\partial^2 r}{\partial z_l\partial\bar z_k}\phi_l\bar\phi_k\,e^{-\lambda}-8\mathrm{Re}\,\{\int_\Omega\sum_{l,k=1}^n\frac{\partial g}{\partial z_l}\phi_l\overline{\delta_k\phi_k}\,e^{-\lambda}\}.$$

Suppose that λ,g,h satisfy (2.6). The Cauchy-Schwarz inequality applied to (2.14) gives, if Ω is pseudoconvex,

$$(g\bar\partial\phi,\bar\partial\phi)_\lambda+((g+h)\bar\partial^*_\lambda\phi,\bar\partial^*_\lambda\phi)_\lambda\geq(P\phi,\phi)_\lambda,\qquad\phi\in\mathcal{D}^{0,1}.$$

Let T denote the operator $\bar\partial\circ\sqrt{g+h}$ and let S denote the operator $\sqrt{g+h}\circ\bar\partial$. Using mollifiers, it is not hard to show that $\mathcal{D}^{0,1}$ is dense in both Dom (T^*) and Dom (S) in the graph norm $||T^*\phi||_\lambda+||S\phi||_\lambda$. Thus, we obtain

$$||T^*\phi||^2_\lambda+||S\phi||^2_\lambda\geq(P\phi,\phi)_\lambda,\qquad\phi\in\text{Dom }(T^*)\cap\text{Dom }(S).\tag{2.15}$$

The equation $Tu=\alpha$, for $S\alpha=0$, is equivalent to

$$(u,T^*\psi)_\lambda=(\alpha,\psi)_\lambda,\qquad\psi\in\text{Dom }(T^*).$$

If $\psi\in$ Dom $(T^*)\cap$Dom (S), then the Cauchy-Schwarz inequality and (2.15) imply

$$\begin{aligned}|(\alpha,\psi)_\lambda|^2&\leq|(P^{-1}\alpha,\alpha)_\lambda||(P\psi,\psi)_\lambda|\\&\leq|(P^{-1}\alpha,\alpha)_\lambda|\{||T^*\psi||^2_\lambda+||S\psi||^2_\lambda\}.\end{aligned}\tag{2.16}$$

However, this implies

$$|(\alpha,\psi)_\lambda|^2\leq|(P^{-1}\alpha,\alpha)_\lambda|||T^*\psi||^2_\lambda,\qquad\psi\in\text{Dom }(T^*).\tag{2.17}$$

This follows immediately from (2.16) if $\psi\in$ Ker (S), and, if ψ is orthogonal to Ker (S), then both sides of (2.17) vanish since $S\circ T=0$.

The map $T^*\phi\longrightarrow(\phi,\alpha)_\lambda$ is, therefore, a bounded linear functional on Dom T^* and the Riesz representation theorem then gives a solution u to $Tu=\alpha$ with the estimate claimed in (2.7). This completes the proof. $\square$

3. The auxilary functions and proof of (1.2)

As before, $\Omega \subset\subset \mathbb{C}^n$ is pseudoconvex with smooth $b\Omega$ and, for $\delta > 0$ given, $p = z_0 - \delta\nu_{z_0}$.

Definition. If $D \subset \mathbb{C}^n$ and $q \in bD$, a weak, local holomorphic support surface for D at q is a complex analytic manifold S, $\dim_{\mathbb{C}} S = n-1$, defined in a neighborhood U of q such that $q \in S$ and $S \cap U = \mathbb{C}^n \setminus D$.

Note that S may intersect bD at points other than q.

Suppose f is a holomorphic function on U and $S = \{z \in U : f(z) = 0\}$. If, in some holomorphic coordinate system $(w_1, \ldots, w_n)$, the defining function of $b\Omega$ locally takes the form

$$r(w) = 2\mathrm{Re}\ w_n + R(w', \mathrm{Im}\ w_n), \tag{3.1}$$

where $z_0 = 0$ and R vanishes to order ≥ 2 at 0, then the fact that S is supporting implies that $\frac{\partial f}{\partial w_n}(0) \neq 0$. Choose new coordinates $(z_1, \ldots, z_n)$ such that $p = 0$ and $S = \{z \in U : z_n + \delta = 0\}$. The defining function r does not, necessarily, have the form (3.1) in the coordinates $(z_1, \ldots, z_n)$, but the vector field $\frac{\partial}{\partial z_n}$ is transverse to $b\Omega$ in U.

Since Ω is bounded, we may choose $N > 0$ so that for $z \in \Omega$, $|z|^2 + \delta^2 < Ne^{-2}$ for all $\delta < 1$. Note that $|z_n + \delta|^2 > b\delta^2$ on $\{z \in U : r(z) < -\frac{\delta}{2}\} = \Omega_\delta \cap U$ for an independent constant $b > 0$, since $\{z_n + \delta = 0\}$ is supporting. Thus

$$\frac{b}{2+b} \leq \frac{|z_n + \delta|^2}{|z_n|^2 + \delta^2} \leq 1 \qquad \text{on} \qquad \Omega_\delta \cap U.$$

Choose $a > 0$ small so that

$$-\log(|z_n|^2 + \delta^2) + \log(a|z_n + \delta|^2) > 2, \qquad (z', z_n) \in \Omega_\delta \cap U.$$

Define the twist factor

$$\begin{aligned} g(z) &= -\log(|z_n|^2 + \delta^2) + \log(a|z_n + \delta|^2) \\ &\quad + \log(-\log(|z_n|^2 + \delta^2) + \log(a|z_n + \delta|^2)) \\ &= \kappa(z_n) + \log\kappa(z_n). \end{aligned} \tag{3.2}$$

Direct computation yields

$$-\frac{\partial^2 g}{\partial z_n \partial \bar{z}_n} \geq \frac{\delta^2}{(|z_n|^2 + \delta^2)^2} + \frac{1}{[\kappa(z_n)]^2} \cdot |\frac{\partial \kappa}{\partial z_n}|^2$$

and
$$|\frac{\partial g}{\partial z_n}|^2 \leq (1+\frac{1}{[\kappa(z_n)]^2})|\frac{\partial \kappa}{\partial z_n}|^2.$$
Using the facts $g(z) \geq 2$ and $g(z) \geq \kappa(z_n)$, it follows that if $h = g^3$, then
$$-\frac{\partial^2 g}{\partial z_n \partial \bar{z}_n} - 1/h|\frac{\partial g}{\partial z_n}|^2 \geq \frac{1}{2}\frac{\delta^2}{(|z_n|^2+\delta^2)^2}. \tag{3.3}$$

Define the weight function
$$\lambda_\epsilon(z) = \log(|z_n|^2+\epsilon) - \log(|z_n+\delta|^2) + |z|^2, \tag{3.4}$$
for $\epsilon > 0$ and $\epsilon \leq \delta^2$. Note that λ_ϵ is plurisubharmonic of Ω and that there exists a constant $K > 0$, independent of δ and ϵ, such that
$$|e^{\lambda_\epsilon}|, |g|, |h| \leq K \tag{3.5}$$
on $\Omega_\delta \cap U$.

Proof of Theorem 1.1. First, we note that the localization theorem in [O] implies that
$$\tilde{K}_1 M_{\Omega_\delta \cap U}(p) \leq M_{\Omega_\delta}(p) \leq \tilde{K}_2 M_{\Omega_\delta \cap U}(p),$$
for independent constants $\tilde{K}_1, \tilde{K}_2$ and U the neighborhood of z_0 discussed above.

Let $\mathcal{S} = \{z_n = 0\} \cap \Omega$. Choose $f \in \mathcal{O}(\mathcal{S})$ such that $|f(p)|^2 = M_{\mathcal{S}}(p)$ and $||f||_{L^2(\mathcal{S})} \leq 1$. Let $\chi \in C_0^\infty(\triangle(0;1))$ be a one variable cut-off function with $\chi(\zeta) \equiv 1$ if $|\zeta| < 1/2$. Then
$$\begin{aligned} F(z', z_n) &= \frac{1}{\delta} \cdot \chi(\frac{|z_n|^2}{c\delta^2}) f(z', 0) \\ &= \frac{1}{\delta}\chi_\delta(z_n) f(z', 0) \end{aligned}$$
is a C^∞ function on $\Omega_\delta \cap U$. Let $\alpha = \bar{\partial} F = 1/\delta \frac{\partial \chi_\delta}{\partial \bar{z}_n} f$ and note that $|\frac{\partial}{\partial \bar{z}_n}\chi_\delta| \leq K\frac{1}{\delta}$ for an independent constant K.

Applying Theorem 2.1, solve $\bar{\partial}(\sqrt{g+h} \cdot u_\epsilon) = \alpha$ with the estimate
$$\int_{\Omega_\delta} |u_\epsilon|^2 e^{-\lambda_\epsilon} \leq \int_{\Omega_\delta} < P^{-1}\alpha, \alpha > e^{-\lambda_\epsilon}. \tag{3.6}$$
It follows from (3.3-3.5) that
$$\begin{aligned} \int_{\Omega_\delta} < P^{-1}\alpha, \alpha > e^{-\lambda_\epsilon} &\leq A \int_{\mathrm{supp}\bar{\partial}\chi_\delta} \frac{1}{\delta^2}\frac{(|z_n|^2+\delta^2)^2}{\delta^2}\frac{1}{\delta^2}|f|^2 e^{-\lambda_\epsilon} \\ &\leq \tilde{A}, \end{aligned}$$

A and $\tilde{A}$ independent of δ. Thus, it follows from (3.5)

$$\int_{\Omega_\delta} |\sqrt{g+h}u_\epsilon|^2 \le \sup |(g+g^3)e^{\lambda_\epsilon}| \int_{\Omega_\delta} |u_\epsilon|^2 e^{-\lambda_\epsilon} \le K\tilde{A}. \tag{3.7}$$

Additionally,

$$\begin{aligned} \int_{\Omega_\delta} |u_\epsilon|^2 e^{-\lambda_\epsilon} &\ge B \int (\int_{|z_n|<c\delta} \frac{|z_n+\delta|^2}{|z_n|^2+\epsilon} |u_\epsilon(z',z_n)|^2 \, dV(z_n)) \, dV(z') \\ &\ge \tilde{B}\delta^2 \int \int_{|z_n|<c\delta} \frac{|u_\epsilon(z',z_n)|^2}{|z_n|^2+\epsilon}. \end{aligned} \tag{3.8}$$

Take a weak limit of the solutions u_ϵ as $\epsilon \to 0$; call the limit u. Then u solves the same equation as u_ϵ and (3.8) implies that $u(z',0) \equiv 0$. Therefore the function

$$H(z) = \frac{1}{2}(F(z) - \frac{1}{K\tilde{A}}\sqrt{g(z)+h(z)}u(z))$$

is a candidate for $M_{\Omega_\delta \cap U}(p)$ and has the claimed value at p. This implies (1.2) and completes the proof. $\square$

Remark. A trivial upper bound on $M_{\Omega_\delta}(p)$, related to the lower bound of Theorem 1.1, may be obtained by noting

$$\{z \in U : |z_n| < c\delta, z' \in \{z_n = 0\} \cap \tilde{\Omega}_\delta\} \subset \Omega_\delta,$$

where $\tilde{\Omega}_\delta = \{z : r(z) < -\frac{3}{4}\delta\}$, for some constant $c > 0$ depending only on the lower bound of $|\frac{\partial f}{\partial w_n}|$ in U. Thus

$$C\delta^{-2} M_{\tilde{\Omega}_\delta \cap S_p}(p) \ge M_{\Omega_\delta}(p).$$

The comparability of $M_{\tilde{\Omega}_\delta \cap S_p}(p)$ and $M_{\Omega \cap S_p}(p)$ is an open question (for general Ω); see [D-O] for results related to this question.

Note added in proof

I would like to mention several papers, which contain results relevant to the material in this paper, that I, regretfully, omitted from the introduction. The very interesting paper by Berndtsson, [B2], gives a short proof of the Ohsawa-Takegoshi extension theorem based on the differential identity mentioned in the introduction. Diederich, [D], obtains estimates for (1.1) on strongly pseudoconvex domains equivalent to those in [H] already mentioned and also obtains results on certain related, differentiated versions of the extreme value problem. The papers [BSY], [DH1], and [He] contain extensions of results mentioned earlier. Finally, I point out the paper by Diederich-Herbort, [DH2], and its bibliography, as a guide to some of the issues behind these results and for further references.

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