Mathematical Research Letters **3**, 185–189 (1996)

THE DIMENSION OF THE FIXED POINT SET ON AFFINE FLAG MANIFOLDS

Roman Bezrukavnikov

Let G be a semisimple simply-connected algebraic group over $\mathbb{C}$, $\mathfrak{g}$ its Lie algebra. Also, $F = \mathbb{C}((\varepsilon))$ is the field of formal Laurent series, $A = \mathbb{C}[[\varepsilon]]$ is the ring of integers in F. Set $\hat{\mathfrak{g}} = \mathfrak{g} \otimes F$, $\mathfrak{g}_A = \mathfrak{g} \otimes A$ and $\hat{G} = G(F)$.

Let $\mathcal{B}$ be the set of all Iwahori subalgebras in $\hat{\mathfrak{g}}$, and X the set of all subalgebras in $\hat{\mathfrak{g}}$ which are $\hat{G}$-conjugate to $\mathfrak{g}_A$. Then $\mathcal{B}$ and X have the structure of Ind-algebraic varieties over $\mathbb{C}$ (they are unions of increasing system of ordinary projective algebraic varieties over $\mathbb{C}$). They are called the affine flag variety and the affine Grassmanian of G respectively. We have $X = \hat{G}/G(A)$ and $\mathcal{B} = \hat{G}/I$, where I is an Iwahori subgroup.

For any $N \in \hat{\mathfrak{g}}$ let $\mathcal{B}_N \subset \mathcal{B}$ (respectively $X_N \subset X$) be the set of all Iwahori subalgebras (respectively, subalgebras conjugate to $\mathfrak{g}_A$) which contain N. Clearly, $\mathcal{B}_N$ (X_N) is a closed subvariety of the Ind-variety $\mathcal{B}$ (respectively X).

The varieties $\mathcal{B}_N$, X_N were studied by Kazhdan and Lusztig in [KL]. Following their paper let us suppose that N is *topologically nilpotent* (nil-element in the terminology of [KL]), i.e., $\mathrm{ad}(N)^r \to 0$ in $\mathrm{End}_F(\hat{\mathfrak{g}})$ when $r \to \infty$. (The topology on $\mathrm{End}_F(\hat{\mathfrak{g}})$ arises from the obvious topology on F.) It was shown in *loc. cit.* that the Ind-varieties $\mathcal{B}_N$, X_N are finite dimensional iff the element N is regular semisimple. We will assume from now on that this is the case. Then $\mathcal{B}_N$ and X_N are locally finite unions of finite dimensional projective varieties. Moreover, all components of $\mathcal{B}_N$ have the same dimension, which coincides with the dimension of X_N. A precise formula for the dimension of $\mathcal{B}_N$ was stated in [KL] as a conjecture. The aim of the present note is to give a proof of this conjecture.

Let O be the subset of X_N defined as follows: if $\hat{\mathfrak{p}} \in X_N$ is a subalgebra conjugate to $\mathfrak{g}_A$, then $\hat{\mathfrak{p}} \in O$ iff the image of N in $\mathfrak{g} \widetilde{=} \hat{\mathfrak{p}}/\varepsilon\hat{\mathfrak{p}}$ is a regular nilpotent.

Let $Z(N)$ be the centralizer of N in $\hat{G}$; let $\mathfrak{z}(N)$ be the centralizer of N in $\hat{\mathfrak{g}}$. We also fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and denote by W the Weyl group. The result containing the formula for the dimension of $\mathcal{B}_N$ is the following:

Received December 3, 1995.

Proposition.

a) *O is an orbit of the group $Z(N)$.*

b) *The dimension of this orbit is given by the formula (cf. [KL], p.130)*

$$\dim(O) = 1/2(\delta(N) - \mathrm{rk}(\mathfrak{g}) + \dim(\mathfrak{h}^w))$$

where $w \in W$ is such that $\mathfrak{z}(N)$ is of type w (see [KL], §1); $\mathfrak{h}^w$ denotes the w-invariants in $\mathfrak{h}$ and $\delta(N)$ is the valuation of

$$\det(\mathrm{ad}\, N \colon \hat{\mathfrak{g}}/\mathfrak{z}(N) \longrightarrow \hat{\mathfrak{g}}/\mathfrak{z}(N)).$$

c) $\dim(\mathcal{B}_N) = \dim(X_N) = \dim(O)$.

Proof. a) is equivalent to the following:

Claim. *Suppose that N , N' are nil-elements in $\mathfrak{g}_A$, which lie in the same $\hat{G}$ orbit, and the images of N and N' in $\mathfrak{g} = \mathfrak{g}_A/\varepsilon\mathfrak{g}_A$ are regular nilpotents. Then N and N' lie in one $G(A)$ orbit.*

Proof of the claim. Since the regular nilpotents in $\mathfrak{g}$ form one conjugacy class we can assume that $N \mod \varepsilon\mathfrak{g}_A = N' \mod \varepsilon\mathfrak{g}_A = n$, where $n \in \mathfrak{g}$ is a regular nilpotent.

Let V be any complement to the image of $\mathrm{ad}(n)$ in $\mathfrak{g}$, and let $v_1, \ldots, v_r$ be a basis of V.

Lemma 1.

a) *Assume that $x \in \mathfrak{g}_A$ is such that $x \in n + \varepsilon V \otimes A + \varepsilon^i\mathfrak{g}_A$ for some $i \geq 1$. Then there exists $g \in G(A)$ such that $g = 1 \mod \varepsilon^i$ and $\mathrm{ad}(g)(x) \in n + \varepsilon V \otimes A + \varepsilon^{i+1}\mathfrak{g}_A$.*

b) *Any such x is $G(A)$-conjugate to an element lying in $n + \varepsilon V \otimes A$.*

Proof. To prove a) let us write x as $x = n+v+[n, \varepsilon^i y]$ for some $v \in \varepsilon V \otimes A$, $y \in \mathfrak{g}_A$. (This is possible because $\varepsilon^i\mathfrak{g}_A \subset \varepsilon V \otimes A + \mathrm{ad}(n)(\varepsilon^i\mathfrak{g}_A)$ as follows from the definition of V). It is enough to take $g = \exp(-\varepsilon^i y)$.

To prove b) note that by the statement a) there exists a sequence of elements $g_i \in G(A)$ such that $g_i = g_{i+1} \mod \varepsilon^i$ and $\mathrm{ad}(g_i)(x) \in n + \varepsilon V \otimes A + \varepsilon^i\mathfrak{g}_A$. It is obvious that $g := \lim g_i$ exists, lies in $G(A)$ and satisfies

$$\mathrm{ad}(g)(x) \in n + \varepsilon V \otimes A. \quad \square$$

Now we see that N (respectively N') is $G(A)$-conjugate to an element of the form $N_0 = n + \Sigma a_i v_i$ (respectively $N'_0 = n + \Sigma a'_i v_i$), where $a_i, a'_i \in \varepsilon A$. By the theorem of Kostant (see [K], Theorem 0.10) there exists a set of generators $Q_1, .., Q_r$ of the ring of invariant polynomials on $\mathfrak{g}$ such that

$$Q_k(n + \Sigma a_i v_i) = a_k.$$

Since N' , N'_0 , N , N_0 lie in one $\hat{G}$ orbit, we have $Q_i(N_0) = Q_i(N'_0)$, hence $a_i = a'_i$ and $N_0 = N'_0$. The claim is proved. □

c) It is clear that O is open; O is nonempty by [KL], §4, Corollary 1. The natural projection $\pi\colon \mathcal{B}_N \longrightarrow X_N$ is 1-1 over O. Since $\pi^{-1}(O)$ is open in $\mathcal{B}_N$ and all components of $\mathcal{B}_N$ have the same dimension ([KL], §4, Proposition 1) we see that $\dim(O) = \dim(\pi^{-1}(O)) = \dim(\mathcal{B}_N) = \dim(X_N)$. (The last equality is Corollary 2, §4 of [KL].)

b) For split N the formula for the dimension of X_N follows from [KL], §5 (and coincides with the formula b) above) so by the statement (c) we are done. The general case can be reduced to the case of split N by the next two lemmas.

Let $N \in \hat{\mathfrak{g}}$ be any regular semisimple nil-element. Consider the field extension $\widetilde{F}/F$ of degree n, and the corresponding ring extension $\widetilde{A}/A$, such that N splits over $\widetilde{F}$. We have $\widetilde{F} \cong \mathbb{C}((\varepsilon^{1/n}))$; $\widetilde{A} \cong \mathbb{C}[[\varepsilon^{1/n}]]$. Let $\widetilde{X} = G(\widetilde{F})/G(\widetilde{A})$ be the corresponding affine Grassmanian, and $\widetilde{Z}(N)$ be the centralizer of N in $G(\widetilde{F})$.

For any $\hat{\mathfrak{p}} \in X$ consider the orbit of $Z(N)$ on X containing $\hat{\mathfrak{p}}$, and the orbit of $\widetilde{Z}(N)$ on $\widetilde{X}$ containing $\hat{\mathfrak{p}} \otimes \widetilde{A} \in \widetilde{X}$. They will be denoted by $O_{\hat{\mathfrak{p}}}$ and $\widetilde{O}_{\hat{\mathfrak{p}}}$ respectively. Also let $\hat{P} \subset \hat{G}$ (respectively $\widetilde{P} \subset G(\widetilde{F})$) denote the stabilizer of $\hat{\mathfrak{p}}$ (respectively the stabilizer of $\hat{\mathfrak{p}} \otimes \widetilde{A}$).

Let us call an element $x \in \hat{\mathfrak{g}}$ (respectively $y \in \mathfrak{g} \otimes \widetilde{F}$) integral if for any ad-invariant polynomial Q which is defined over $\mathbb{C}$ we have $Q(x) \in A$ (respectively $Q(y) \in \widetilde{A}$). It is easy to see that the integral elements in $\mathfrak{z}(N)$ form a lattice, provided N is regular semisimple. The exponent is a surjective homomorphism with discrete kernel from this lattice to the connected component of $Z(N)$.

Let M (respectively $\widetilde{M}$) be the lattice of integral elements in $\mathfrak{z}(N)$ (respectively in $\mathfrak{z}(N) \otimes \widetilde{F}$). The Killing form will be denoted by k.

Lemma 2. *For any* $\hat{\mathfrak{p}} \in X$ *we have*

$$\dim(\widetilde{O}_{\hat{\mathfrak{p}}}) = n\,[\dim(O_{\hat{\mathfrak{p}}}) + 1/2\; v(\det(k|M))]$$

where v_F *is the valuation of* F.

Proof. Indeed

$$(1)\qquad \begin{aligned}\dim(O_{\hat{\mathfrak{p}}}) &= \dim(Z(N)/(Z(N)\cap \hat{P})) = \dim(M/(M\cap\hat{\mathfrak{p}}))\\ &= 1/2[v_F(\det(k|(M\cap\hat{\mathfrak{p}}))) - v_F(\det(k|M))]\end{aligned}$$

and

$$\begin{aligned}\dim(\widetilde{O}_{\hat{\mathfrak{p}}}) &= \dim(\widetilde{Z}(N)/(\widetilde{Z}(N)\cap\widetilde{P})) = \dim(\widetilde{M}/(\widetilde{M}\cap(\hat{\mathfrak{p}}\otimes\widetilde{A}))) \\ &= 1/2[v_{\widetilde{F}}(\det\ (k|(\widetilde{M}\cap(\hat{\mathfrak{p}}\otimes\widetilde{A})))) - v_{\widetilde{F}}(det\ (k|\widetilde{M}))],\end{aligned} \tag{2}$$

where $v_{\widetilde{F}}$ is the valuation of $\widetilde{F}$. But

$$\widetilde{M}\cap(\hat{\mathfrak{p}}\otimes\widetilde{A}) = (M\otimes\widetilde{F})\cap\hat{\mathfrak{p}}\otimes\widetilde{A} = (M\cap\hat{\mathfrak{p}})\otimes\widetilde{A}.$$

So we see that

$$\begin{aligned}v_{\widetilde{F}}(\det\ (k|(\widetilde{M}\cap\hat{\mathfrak{p}}\otimes A)) &= v_{\widetilde{F}}(\det(k|\ (M\cap\hat{\mathfrak{p}})\otimes\widetilde{A})) \\ &= n\ v_F(\det(k|\ (M\cap\hat{\mathfrak{p}}))).\end{aligned} \tag{3}$$

Besides,

$$v_{\widetilde{F}}(\det(k|\widetilde{M})) = 0 \tag{4}$$

because N is split over $\widetilde{F}$, so $\widetilde{M}$ is $G(\widetilde{F})$-conjugate to $\mathfrak{h}\otimes\widetilde{A}$. (Recall that $\mathfrak{h}\subset\mathfrak{g}$ is a Cartan subalgebra.) Substituting (3), (4) in (2) and comparing the result with (1) we get the lemma. □

Lemma 3. $v_F(\det(k|M)) = \mathrm{rk}(\mathfrak{g}) - \dim(\mathfrak{h}^w)$.

Proof. Since N is split over $\widetilde{F}$, there exists an inner automorphism of $\mathfrak{g}\otimes\widetilde{F}$ which induces an isomorphism

$$\widetilde{M}\cong\mathfrak{h}\otimes\widetilde{A}. \tag{5}$$

On the two sides of (5) there is a natural action of the Galois group of $\widetilde{F}/F$. Let ρ_l, ρ_r denote the corresponding actions. On the RHS we have an action of W (through $\mathfrak{h}$), which we denote by σ. Let s be a generator of $\mathrm{Gal}(\widetilde{F}/F)$. By the definition of w we have $\rho_l(s) = \rho_r(s)\sigma(w')$ for some element $w'\in W$ conjugate to w. (We identified the endomorphisms of the two sides of (5).)

Denote by q the n-th primitive root of unity which satisfies the equation $s(\varepsilon^{1/n}) = q\varepsilon^{1/n}$. For any $\lambda\in\mathbb{C}$ let $\mathfrak{h}_\lambda$ be the λ-eigenspace of w' acting on $\mathfrak{h}$. Then the A-module $M = (\widetilde{M})^{Gal}$ is the direct sum of its submodules

$$M_i\cong\mathfrak{h}_{q^{(n-i)}}\otimes\varepsilon^{(i/n)}A$$

for $i = 0, .., n-1$. Consider the restriction of the Killing form on M. Since the conjugation respects the Killing form, we see that the induced Khuri-Makdisi pairing $M_i\times M_j\to A$ is

nondegenerate mod ε	if $i = j = 0$
ε times the one nondegenerate mod ε	if $0 < i < n$, $i + j = n$
0	otherwise.

We see that $v_F(\det(k|M)) = \mathrm{rk}(\mathfrak{g}) - \dim(\mathfrak{h}^{w'}) = \mathrm{rk}(\mathfrak{g}) - \dim(\mathfrak{h}^w)$. The lemma is proved. □

Now we are ready to finish the proof of the Proposition. Applying Lemma 2 to $\hat{\mathfrak{p}} \in O$ we get (using Lemma 3):

$$\dim(\widetilde{O}) = n[\dim(O) + 1/2(\mathrm{rk}(\mathfrak{g}) - \dim(\mathfrak{h}^w))],$$

where $\widetilde{O}$ is the corresponding open orbit of $\widetilde{Z}(N)$ on $\widetilde{X}_N$; obviously $\hat{\mathfrak{p}} \otimes \widetilde{A} \in \widetilde{O}$. Since N is split in $\mathfrak{g} \otimes \widetilde{F}$ we know by [KL], §5 that:

$$\dim(\widetilde{O}) = \dim(\widetilde{X}_N) = \delta_{\widetilde{F}}(N)/2 = n\delta(N)/2,$$

where $\delta_{\widetilde{F}} := v_{\widetilde{F}}[\det(\mathrm{ad}\ N : \mathfrak{g} \otimes \widetilde{F}/\mathfrak{z}(N) \otimes_F \widetilde{F} \to \mathfrak{g} \otimes \widetilde{F}/\mathfrak{z}(N) \otimes_F \widetilde{F})]$. Comparing the last two formulas we get the statement b) of the Proposition. □

Acknowledgement

I express my gratitude to V. Ginzburg for stimulating attention to the work. The paper [KL] was brought to my attention by him. I thank N. Spaltenstein for encouragement in publishing the note.

References

[K] B. Kostant, *Lie group representations on polynomial rings*, Amer. J. Math., **85** (1963), 327–404.

[KL] D. Kazhdan and G. Lusztig, *Fixed points varieties on affine flag manifolds*, Israel J. of Math., **62** (1988), 129–168.

Mathematics Department, Tel-Aviv University, Tel-Aviv, Israel
E-mail address: roman@math.tau.ac.il