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HODGE INDEX THEOREM FOR ARITHMETIC CYCLES OF CODIMENSION ONE

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0. Introduction

Let $f : X \to \mathrm{Spec}(\mathbb{Z})$ be a $(d+1)$-dimensional regular arithmetic variety over $\mathrm{Spec}(\mathbb{Z})$, i.e. X is regular, X is projective and flat over $\mathrm{Spec}(\mathbb{Z})$ and $d = \dim f$. Let H be an f-ample line bundle on X and k a Hermitian metric of H. Here we consider a homomorphism

$$L : \widehat{\mathrm{CH}}^p(X)_{\mathbb{R}} \to \widehat{\mathrm{CH}}^{p+1}(X)_{\mathbb{R}}$$

defined by $L(x) = x \cdot \widehat{c}_1(H, k)$. In [GS], H. Gillet and C. Soulé conjectured that

Arithmetic Analogues of Grothendieck's Standard Conjectures. *For a suitable choice of k, if $2p \leq d+1$, then*

(a) *The homomorphism $L^{d+1-2p} : \widehat{\mathrm{CH}}^p(X)_{\mathbb{R}} \to \widehat{\mathrm{CH}}^{d+1-p}(X)_{\mathbb{R}}$ is bijective, and*
(b) *If $x \in \widehat{\mathrm{CH}}^p(X)_{\mathbb{R}}$, $x \neq 0$ and $L^{d+2-2p}(x) = 0$, then $(-1)^p \widehat{\deg}(x \cdot L^{d+1-2p}(x)) > 0$.*

For example, K. Künnemann [Ku] proved that if X is a projective space, then the conjecture is true. Here we fix a notation. We say a Hermitian line bundle (H, k) on X is arithmetically ample if (1) H is f-ample, (2) the Chern form $c_1(H_\infty, k_\infty)$ is positive definite on the infinite fiber X_∞, and (3) there is a positive integer m_0 such that, for any integer $m \geq m_0$, $H^0(X, H^m)$ is generated by the set $\left\{ s \in H^0(X, H^m) \mid \|s_\infty\|_{\sup} < 1 \right\}$. Note that by virtue of [Zh], the third condition can be replaced by a numerical condition : (3)' for every irreducible horizontal subvariety Y (i.e. Y is flat over $\mathrm{Spec}(\mathbb{Z})$), the height $\widehat{c}_1((H,k)|_Y)^{\dim Y}$ of Y is positive. In this note, we would like to prove the following partial answer of the above conjecture for general regular arithmetic varieties.

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Theorem A. *Assume that $d \geq 1$ and (H, k) is arithmetically ample. Then we have the following:*

(1) $L^{d-1} : \widehat{\mathrm{CH}}^1(X)_{\mathbb{R}} \to \widehat{\mathrm{CH}}^d(X)_{\mathbb{R}}$ *is injective.*
(2) *If* $x \in \widehat{\mathrm{CH}}^1(X)_{\mathbb{R}}$, $x \neq 0$ *and* $L^d(x) = 0$, *then* $\widehat{\deg}(xL^{d-1}(x)) < 0$.

Theorem A is a consequence of the following higher dimensional generalization of Faltings-Hriljac's Hodge index theorem on arithmetic surfaces (cf. [Fa] and [Hr]).

Theorem B. *Assume that $d \geq 1$ and (H, k) is arithmetically ample. Let* $X \xrightarrow{f'} \mathrm{Spec}(O_K) \to \mathrm{Spec}(\mathbb{Z})$ *be the Stein factorization of* $f : X \to \mathrm{Spec}(\mathbb{Z})$ *and X_K the generic fiber of f', where O_K is the ring of integers of an algebraic number field K. Let* $z : \widehat{\mathrm{CH}}^1(X) \to \mathrm{CH}^1(X)$ *be the canonical homomorphism defined by $z(D, g) = D$. If* $x \in \widehat{\mathrm{CH}}^1(X)$ *and* $\left(z(x)|_{X_K} \cdot \left(H|_{X_K}\right)^{d-1} \right) = 0$, *then*

$$\widehat{\deg}(x^2 \cdot \widehat{c}_1(H, k)^{d-1}) \leq 0.$$

Moreover, equality holds if and only if there are a positive integer n and $y \in \widehat{\mathrm{CH}}^1(\mathrm{Spec}(O_K))$ *such that* $nx = f'^*(y)$.

1. Proof of Theorem B

In this section, we would like to give the proof of Theorem B. An advantage to use arithmetical ampleness of the Hermitian line bundle (H, k) is the following arithmetic Bertini's theorem.

Arithmetic Bertini's Theorem. (cf. [Mo2, Theorem 4.2 and Theorem 5.2]) *Let* $f : X \to \mathrm{Spec}(\mathbb{Z})$ *be an arithmetic variety, and (H, k) an arithmetically ample Hermitian line bundle on X. If $x_1, \ldots, x_s$ are points (not necessarily closed) on X, then, for a sufficiently large integer m, there is a section ϕ of $H^0(X, H^m)$ such that*

(1) $\mathrm{div}(\phi)$ *is smooth over* $\mathbb{Q}$,
(2) $\phi(x_i) \neq 0$ *for all* $1 \leq i \leq s$, *and*
(3) $\|\phi_\infty\|_{\sup} < 1$.

By the above theorem, we can proceed to induction on $d = \dim f$. However, regularity of X doesn't preserve by induction step in general. Here we consider the following weaker version on general arithmetic varieties.

Theorem 1.1. *Let K be an algebraic number field and O_K the ring of integers of K. Let $f : X \to \operatorname{Spec}(O_K)$ be an arithmetic variety such that $d = \dim f \geq 1$ and X_K is smooth and geometrically irreducible. Let (H, k) be an arithmetically ample Hermitian line bundle on X. Let D be a Cartier divisor on X and g_σ a Green current of D_σ on each $\sigma \in K(\mathbb{C})$. If $(D_K \cdot H_K^{d-1}) = 0$, then*

$$\widehat{\deg}\left(\left(D, \sum g_\sigma\right)^2 \cdot \widehat{c}_1(H,k)^{d-1}\right) \leq 0.$$

Moreover, if equality holds, then there are a positive integer n, a Cartier divisor Z on X and constants $\{g'_\sigma\}_{\sigma \in K(\mathbb{C})}$ such that the support of Z is vertical and the class of $(Z, \sum g'_\sigma)$ is equal to the class of $n(D, \sum g_\sigma)$ in $\widehat{\mathrm{CH}}^1(X)$. In particular, if equality holds, then $\mathcal{O}_{X_K}(D_K)$ is a torsion of $\operatorname{Pic}(X_K)$.

Proof. First of all, we prepare two lemmas.

Lemma 1.1.1. *Let X be a d-dimensional compact Kähler manifold with a Kähler form Φ and φ a real valued smooth function on X. Then,*

$$\int_X \varphi dd^c(\varphi)\Phi^{d-1} \leq 0.$$

Moreover, equality holds if and only if φ is a constant.

Proof. Since $dd^c = \dfrac{\sqrt{-1}}{2\pi}\partial\bar{\partial}$ and $d(\varphi\bar{\partial}(\varphi)) = \partial(\varphi)\bar{\partial}(\varphi) + \varphi\partial\bar{\partial}(\varphi)$, by Stokes' theorem, we have

$$\int_X \varphi dd^c(\varphi)\Phi^{d-1} = -\frac{\sqrt{-1}}{2\pi}\int_X \partial(\varphi)\bar{\partial}(\varphi)\Phi^{d-1}.$$

Here let $\theta^1, \ldots, \theta^d$ be a local unitary frame of Ω^1_X with $\Phi = \sqrt{-1}\sum_i \theta^i \wedge \bar{\theta}^i$. We set $\partial(\varphi) = \sum_i a_i\theta^i$. Then, $\bar{\partial}(\varphi) = \overline{\partial(\varphi)} = \sum_i \bar{a}_i\bar{\theta}^i$. Therefore,

$$-\frac{\sqrt{-1}}{2\pi}\partial(\varphi)\bar{\partial}(\varphi)\Phi^{d-1} = \frac{-1}{2\pi}\sum_{i=1}^{d}|a_i|^2\Phi^d.$$

Thus, we have

$$\int_X \varphi dd^c(\varphi)\Phi^{d-1} \leq 0.$$

Moreover, equality hold if and only if $\partial(\varphi) = 0$. Here, since φ is real valued, $\partial(\varphi) = 0$ implies that φ is a constant. $\square$

Lemma 1.1.2. *Let X be a d-dimensional Kähler manifold with a Kähler form Φ and ω a smooth $(1,1)$-form on X such that $\bar{\omega} = -\omega$ and $\omega \wedge \Phi^{d-1} = 0$. Then, there is a real valued smooth function u on X with the following properties.*

(1) $\omega^2 \wedge \Phi^{d-2} = u\Phi^d$.
(2) $u(x) \leq 0$ *for all* $x \in X$.
(3) $u(x) = 0$ *for all* $x \in X$ *if and only if* $\omega = 0$.

Proof. Let $\theta^1, \ldots, \theta^d$ be a local unitary frame of Ω^1_X with $\Phi = \sqrt{-1}\sum_i \theta^i \wedge \bar{\theta}^i$. We set $\omega = \sum_{i,j} a_{ij}\theta^i \wedge \bar{\theta}^j$. Then, $\bar{\omega} = -\omega$ implies that $a_{ji} = -\bar{a}_{ij}$. Moreover, since

$$\omega \wedge \Phi^{d-1} = -\sqrt{-1}(a_{11} + \cdots + a_{dd})\Phi^d,$$

we have $a_{11} + \cdots + a_{dd} = 0$. On the other hand, by an easy calculation,

$$d(d-1)\omega^2 \wedge \Phi^{d-2} = \left(\sum_{i,j} a_{ij}a_{ji} - a_{ii}a_{jj} \right) \Phi^d.$$

Therefore, we get

$$\omega^2 \wedge \Phi^{d-2} = \frac{-1}{d(d-1)} \left(\sum_{i,j} |a_{ij}|^2 \right) \Phi^d.$$

Hence, if we set $u = \dfrac{-1}{d(d-1)} \displaystyle\sum_{i,j} |a_{ij}|^2$, the lemma is obtained because $\sum_{i,j} |a_{ij}|^2$ is independent of the choice of $\theta^1, \ldots, \theta^d$. □

Let us start of the proof of Theorem 1.1. We will prove it by induction on d. First, we consider the case $d = 1$. In this case, taking a desingularization of X, we may assume that X is regular. Thus, our theorem can be derived from Faltings-Hriljac's Hodge index theorem (cf. [Fa] and [Hr]).

Next, we assume $d \geq 2$. We set $x = (D, \sum g_\sigma)$ and $L = \mathcal{O}_X(D)$. Let h_σ be an Einstein-Hermitian metric of L_σ with respect to $c_1(H_\sigma, k_\sigma)$. Let s be a rational section of L with $\mathrm{div}(s) = D$. Here we consider an arithmetic cycle

$$y = \left(D, \sum_{\sigma \in K(\mathbb{C})} - \log(h_\sigma(s_\sigma, s_\sigma)) \right).$$

Since g_σ and $-\log(h_\sigma(s_\sigma, s_\sigma))$ are Green currents of the same D_σ, there is a real valued smooth function ϕ_σ on each X_σ such that $x = y + a(\sum_{\sigma \in K(\mathbb{C})} \phi_\sigma)$ in $\widehat{\mathrm{CH}}^1(X)$. Then, it is easy to see that

$$\widehat{\deg}(x^2 \cdot \widehat{c}_1(H,k)^{d-1}) = \widehat{\deg}(y^2 \cdot \widehat{c}_1(H,k)^{d-1}) + \frac{1}{2} \sum_{\sigma \in K(\mathbb{C})} \int_{X_\sigma} \phi_\sigma dd^c(\phi_\sigma) c_1(H_\sigma, k_\sigma)^{d-1}$$

because $c_1(L_\sigma, h_\sigma) c_1(H_\sigma, k_\sigma)^{d-1} = 0$. Therefore, by Lemma 1.1.1,

$$\widehat{\deg}(x^2 \cdot \widehat{c}_1(H,k)^{d-1}) \le \widehat{\deg}(y^2 \cdot \widehat{c}_1(H,k)^{d-1})$$

and equality holds if and only if ϕ_σ is a constant for each $\sigma \in K(\mathbb{C})$. On the other hand, by virtue of arithmetic Bertini's theorem, for a sufficiently large m, there is a section $t \in H^0(X, H^m)$ with the following properties:

i) $\mathrm{div}(t)_K$ is smooth and geometrically irreducible.

ii) If $\mathrm{div}(t) = Y + a_1 F_1 + \cdots + a_s F_s$ is the irreducible decomposition such that Y is horizontal and F_i's are vertical, then F_i's are smooth fibers.

iii) D and $\mathrm{div}(t)$ has no common irreducible component.

iv) $\sup_{x \in X_\sigma} \left(\|t_\sigma\|_{k_\sigma^m}(x) \right) < 1$ for all $\sigma \in K(\mathbb{C})$.

(Note that $H^1(X_K, H_K^{-m}) = 0$ guarantees geometrical irreducibility of $\mathrm{div}(t)_K$.) Since $(D|_{F_i}^2 \cdot H|_{F_i}^{d-2}) \le 0$ by the geometric Hodge index theorem, we obtain

$$\begin{aligned}
\widehat{\deg}(y^2 \cdot \widehat{c}_1(H^m, k^m)^{d-1}) &= \widehat{\deg}(y|_Y^2 \cdot \widehat{c}_1((H^m, k^m)|_Y)^{d-2}) + \\
&\quad \sum a_i m (D|_{F_i}^2 \cdot H|_{F_i}^{d-2}) - \\
&\quad \sum_{\sigma \in K(\mathbb{C})} \int_{X_\sigma} \log(\|t_\sigma\|_{k_\sigma^m}) c_1(L_\sigma, h_\sigma)^2 c_1(H_\sigma^m, k_\sigma^m)^{d-2} \\
&\le \widehat{\deg}(y|_Y^2 \cdot \widehat{c}_1((H^m, k^m)|_Y)^{d-2}) - \\
&\quad \sum_{\sigma \in K(\mathbb{C})} \int_{X_\sigma} \log(\|t_\sigma\|_{k_\sigma^m}) c_1(L_\sigma, h_\sigma)^2 c_1(H_\sigma^m, k_\sigma^m)^{d-2}.
\end{aligned}$$

Since (L_σ, h_σ) is Einstein-Hermitian, by Lemma 1.1.2, there is a real-valued smooth function u_σ on X_σ with the following properties:

(1) $c_1(L_\sigma, h_\sigma)^2 c_1(H_\sigma, k_\sigma)^{d-2} = u_\sigma c_1(H_\sigma, k_\sigma)^d$.

(2) $u_\sigma(x) \le 0$ for all $x \in X_\sigma$.

(3) $u_\sigma(x) = 0$ for all $x \in X_\sigma$ if and only if (L_σ, h_σ) is flat.

Therefore, we have

$$\widehat{\deg}(y^2 \cdot \widehat{c}_1(H,k)^{d-1}) \leq \widehat{\deg}((y|_Y)^2 \cdot \widehat{c}_1((H,k)|_Y)^{d-2}).$$

Hence, by hypothesis of induction, we get our inequality.

Finally, we consider the equality condition. We assume $\widehat{\deg}(x^2 \cdot \widehat{c}_1(H,k)^{d-1}) = 0$. Then, if we trace back the above proof carefully, we can see

(a) ϕ_σ is a constant for each $\sigma \in K(\mathbb{C})$.
(b) (L_σ, h_σ) is flat for each $\sigma \in K(\mathbb{C})$.
(c) $L_K|_{Y_K}$ is a torsion of $\mathrm{Pic}(Y_K)$.

By (b), $L_\mathbb{C}$ is given by a representation $\rho : \pi_1(X_\mathbb{C}) \to \mathbb{C}^*$ of the fundamental group of $X_\mathbb{C}$. (c) implies that the image of $\pi_1(Y_\mathbb{C}) \to \pi_1(X_\mathbb{C}) \to \mathbb{C}^*$ is finite. On the other hand, by Lefschetz theorem (cf. Theorem 7.4 in [Mi]), $\pi_1(Y_\mathbb{C}) \to \pi_1(X_\mathbb{C})$ is surjective. Thus, the image of $\rho : \pi_1(X_\mathbb{C}) \to \mathbb{C}^*$ is also finite. Therefore, there is a positive integer n with $L_\mathbb{C}^n \simeq \mathcal{O}_{X_\mathbb{C}}$. Thus,

$$\dim_K H^0(X_K, L_K^n) = \dim_\mathbb{C} H^0(X_K, L_K^n) \otimes \mathbb{C} = \dim_\mathbb{C} H^0(X_\mathbb{C}, L_\mathbb{C}^n) = 1.$$

Hence, since $(L_K \cdot H_K^{d-1}) = 0$, we have $L_K^n \simeq \mathcal{O}_{X_K}$. Thus, there is a rational section s' of L^n with $s'_K = 1$. We set $Z = \mathrm{div}(s')$ and $g'_\sigma = -\log(h_\sigma^n(s', s')) + n\phi_\sigma$. Then, the support of Z is vertical. Moreover, since h_σ^n is a flat metric of $\mathcal{O}_{X_\sigma}$, $h_\sigma^n(s', s')$ must be a constant. Therefore, $(Z, \sum g'_\sigma)$ is our desired cycle. □

Proof of Theorem B. Since $f'_* \mathcal{O}_X = O_K$, X_K is geometrically irreducible. So the inequality is an immediate consequence of Theorem 1.1.

We need to consider the precise equality condition. Clearly, if there are a positive integer n and $y \in \widehat{\mathrm{CH}}^1(\mathrm{Spec}(O_K))$ such that $nx = f'^*(y)$, then $\widehat{\deg}(x^2 \cdot \widehat{c}_1(H,k)^{d-1}) = 0$. Conversely we assume $\widehat{\deg}(x^2 \cdot \widehat{c}_1(H,k)^{d-1}) = 0$. Then, by Theorem 1.1, there are a positive integer n_1 and an arithmetic cycle $(Z, \sum_{\sigma \in K(\mathbb{C})} g_\sigma)$ such that Z is vertical with respect to f', g_σ's are constant and $n_1 x$ is equal to the class of $(Z, \sum_{\sigma \in K(\mathbb{C})} g_\sigma)$ in $\widehat{\mathrm{CH}}^1(X)$. Then,

$$\widehat{\deg}((n_1 x)^2 \cdot \widehat{c}_1(H,k)^{d-1}) = (Z^2 \cdot H^{d-1}) = 0.$$

Here, we need the following lemma.

Lemma 1.3. *Let X be a regular scheme, R a discrete valuation ring, $f : X \to \mathrm{Spec}(R)$ a projective morphism with $f_* \mathcal{O}_X = R$, and H an f-ample line bundle on X. Let X_o be the central fiber of f and $(X_o)_{\mathrm{red}} =$*

$X_1 + \cdots + X_n$ *the irreducible decomposition of* $(X_o)_{\text{red}}$. *We consider a vector space* $V = \bigoplus_{i=1}^n \mathbb{Q}X_i$ *generated by* X_i*'s and the natural pairing* $(\ ,\ ) : V \times V \to \mathbb{Q}$ *defined by*

$$(D_1, D_2) = (D_1 \cdot D_2 \cdot H^{d-1}),$$

where $d = \dim f$ *and* $\cdot$ *is the intersection product. Then, we have* $(D, D) \leq 0$ *for all* $D \in V$ *and equality holds if and only if* $D \in \mathbb{Q}X_o$.

Proof. For example, see (i)' of Lemma (2.10) in Chap. I of [BPV]. □

By the above lemma, there is a positive integer n_2 and a cycle T on $\mathrm{Spec}(\mathcal{O}_K)$ such that $n_2 Z = f'^*(T)$. Therefore, if we set $y = (T, \sum_{\sigma \in K(\mathbb{C})} n_2 g_\sigma)$, then $n_1 n_2 x = f'^*(y)$. □

2. Proof of Theorem A

Let us begin the proof of Theorem A. This is an easy corollary of Theorem B.

(1) Let us see that (2) implies (1). Assume that $L^{d-1}(x) = 0$. Then, $L^d(x) = 0$. Thus if $x \neq 0$, then $\widehat{\deg}(xL^{d-1}(x)) < 0$ by (2). This is a contradiction. Therefore, $x = 0$.

(2) Let $X \xrightarrow{f'} \mathrm{Spec}(O_K) \to \mathrm{Spec}(\mathbb{Z})$ be the Stein factorization of $f : X \to \mathrm{Spec}(\mathbb{Z})$. In the following arguments, the subscript K means the restriction to the generic fiber of f'.

Since x can be approximated by points $y \in \widehat{\mathrm{CH}}^1(X)_{\mathbb{Q}}$ with $L^d(y) = 0$, we may assume that $x \in \widehat{\mathrm{CH}}^1(X)_{\mathbb{Q}}$. Let t be a rational number with $(z(x)_K + tH_K \cdot H_K^{d-1}) = 0$. Replacing x by mx, we may assume that $x \in \widehat{\mathrm{CH}}^1(x)$ and $t \in \mathbb{Z}$. We set $y = x + t\widehat{c}_1(H, k)$. Then, $(z(y)_K \cdot H_K^{d-1}) = 0$. Thus, by Theorem B, we have $\widehat{\deg}(y^2 \cdot \widehat{c}_1(H, k)^{d-1}) \leq 0$. Therefore, since $L^d(x) = 0$, we get

$$\widehat{\deg}(x^2 \cdot \widehat{c}_1(H, k)^{d-1}) + (t)^2 \widehat{\deg}(\widehat{c}_1(H, k)^{d+1}) \leq 0.$$

Hence, $\widehat{\deg}(x^2 \cdot \widehat{c}_1(H, k)^{d-1}) \leq 0$. Here, we assume that $\widehat{\deg}(x^2 \cdot \widehat{c}_1(H, k)^{d-1}) = 0$. Then, $t = 0$. Thus, $(z(x)_K \cdot H_K^{d-1}) = 0$. So, by Theorem B, there is a positive integer n and $u \in \widehat{\mathrm{CH}}^1(\mathrm{Spec}(O_K))$ such that $nx = f'^*(u)$. We know $nx \cdot \widehat{c}_1(H, k)^d = 0$, which implies $u \cdot f'_*(\widehat{c}_1(H, k)^d) = 0$. Therefore, $u = 0$ in $\widehat{\mathrm{CH}}^1(\mathrm{Spec}(O_K))_{\mathbb{Q}}$ because $f'_*(\widehat{c}_1(H, k)^d) = (H_K^d)[\mathrm{Spec}(O_K)]$. Thus, $x = 0$ in $\widehat{\mathrm{CH}}^1(X)_{\mathbb{Q}}$. This is a contradiction. Hence, we get $\widehat{\deg}(x^2 \cdot \widehat{c}_1(H, k)^{d-1}) < 0$.

3. Variants of Theorem B (non-abelian case)

In this section, we will study variants of Theorem B or Theorem 1.1. The following theorem is a generalization of Theorem 1.1 to a higher rank vector bundle.

Theorem 3.1. *Let K be an algebraic number field and O_K the ring of integers. Let $f : X \to \operatorname{Spec}(O_K)$ be an arithmetic variety and (H, k) an arithmetically ample Hermitian line bundle on X. Assume that $d = \dim f \geq 1$ and X_K is smooth and geometrically irreducible. Let (E, h) be a Hermitian vector bundle on X such that $E_{\overline{\mathbb{Q}}}$ is semi-stable with respect to $H_{\overline{\mathbb{Q}}}$ and $(c_1(E_K) \cdot c_1(H_K)^{d-1}) = 0$. Then, we have*

$$\widehat{\deg}\left(\widehat{\mathrm{ch}}_2(E,h) \cdot \widehat{c}_1(H,k)^{d-1}\right) \leq 0.$$

Moreover, if the equality holds, then h_σ is Einstein-Hermitian with respect to a Kähler form $\Omega_\sigma = c_1(H_\sigma, k_\sigma)$ and E_σ is flat for every $\sigma \in K(\mathbb{C})$.

Proof. Let r be the rank of E. Since

$$\widehat{\mathrm{ch}}_2(E,h) = \frac{1}{2}\widehat{c}_1(E,h)^2 - \widehat{c}_2(E,h),$$

we have

$$\begin{aligned}\widehat{\mathrm{ch}}_2(E,h) \cdot \widehat{c}_1(H,k)^{d-1} &= \frac{1}{2r}\widehat{c}_1(E,h)^2 \cdot \widehat{c}_1(H,k)^{d-1} \\ &\quad - \left\{\widehat{c}_2(E,h) - \frac{r-1}{2r}\widehat{c}_1(E,h)^2\right\} \cdot \widehat{c}_1(H,k)^{d-1}.\end{aligned}$$

By Lemma 8.2 of [Mo1], E_σ is semistable with respect to H_σ. Thus the main theorem in [Mo2] implies that

$$\widehat{\deg}\left(\left\{\widehat{c}_2(E,h) - \frac{r-1}{2r}\widehat{c}_1(E,h)^2\right\} \cdot \widehat{c}_1(H,k)^{d-1}\right) \geq 0.$$

On the other hand, by Theorem 1.1, $\widehat{\deg}\left(\widehat{c}_1(E,h)^2 \cdot \widehat{c}_1(H,k)^{d-1}\right) \leq 0$. Therefore, we have $\widehat{\deg}\left(\widehat{\mathrm{ch}}_2(E,h) \cdot \widehat{c}_1(H,k)^{d-1}\right) \leq 0$.

Next we consider equality condition.

We assume that $\widehat{\deg}\left(\widehat{\mathrm{ch}}_2(E,h) \cdot \widehat{c}_1(H,k)^{d-1}\right) = 0$. First of all, by equality condition of the main theorem of [Mo2], E_σ is flat for every $\sigma \in K(\mathbb{C})$.

Let h' be an Einstein-Hermitian metric of E. Then, by Lemma 6.1 of [Mo1],

$$\widehat{\deg}\left((\widehat{\mathrm{ch}}_2(E,h) - \widehat{\mathrm{ch}}_2(E,h')) \cdot \widehat{c}_1(H,k)^{d-1}\right) = -\frac{(d-1)!}{4\pi} \sum_{\sigma \in K(\mathbb{C})} DL(E_\sigma, h_\sigma, h'_\sigma),$$

where DL is the Donaldson's Lagrangian. Therefore, we have

$$\sum_{\sigma \in K(\mathbb{C})} DL(E_\sigma, h_\sigma, h'_\sigma) \leq 0.$$

On the other hand, since h' is Einstein-Hermitian, we get $DL(E_\sigma, h_\sigma, h'_\sigma) \geq 0$ for all $\sigma \in K(\mathbb{C})$. Hence $DL(E_\sigma, h_\sigma, h'_\sigma) = 0$ for all $\sigma \in K(\mathbb{C})$. Thus h_σ is Einstein-Hermitian for all $\sigma \in K(\mathbb{C})$. $\square$

In the case where $\operatorname{rk} E = 1$, Theorem 1.1 says that if $\widehat{\deg}\left(\widehat{\mathrm{ch}}_2(E,h) \cdot \widehat{c}_1(H,k)^{d-1}\right) = 0$, then E_K is a torsion element of $\mathrm{Pic}^0(X_K)$. So we might expect a stronger property of (E,h) than flatness. Here we introduce one notation. Let M be a complex manifold and F a flat vector bundle of rank r on M. Let $\rho_F : \pi_1(M) \to \mathrm{GL}_r(\mathbb{C})$ be the representation of the fundamental group of M arising from the flat vector bundle F. F is said to be *of torsion type* if the image of ρ_F is finite.

Proposition 3.2. *Let K be an algebraic number field and O_K the ring of integers. Let $f : X \to \mathrm{Spec}(O_K)$ be an arithmetic variety, H an f-ample line bundle on X and k a Hermitian metric of H. Assume that $d = \dim f \geq 1$ and X_K is smooth and geometrically irreducible. Let (E,h) be a Hermitian vector bundle of rank r on X such that (E_σ, h_σ) is flat for each $\sigma \in K(\mathbb{C})$ and $\widehat{\deg}\left(\widehat{\mathrm{ch}}_2(E,h) \cdot \widehat{c}_1(H,k)^{d-1}\right) = 0$. Let $\rho_{E_\mathbb{C}} : \pi_1(X_\mathbb{C}) \to \mathrm{GL}_r(\mathbb{C})$ be the representation of the fundamental group of $X_\mathbb{C}$ arising from the flat vector bundle $E_\mathbb{C}$. If the image of $\rho_{E_\mathbb{C}}$ is abelian, then E_σ is of torsion type for all $\sigma \in K(\mathbb{C})$.*

Proof. We prove it by induction on $\dim X$. First, we consider the case $d = 1$. Since the representation $\rho_{E_\mathbb{C}}$ is abelian, we have the decomposition $\rho_{E_\mathbb{C}} = \rho_1 \oplus \cdots \oplus \rho_r$ such that $\dim \rho_i = 1$ for all i. Therefore, there are flat line bundles $L'_1, \ldots, L'_r$ on $X_\mathbb{C}$ such that $E_\mathbb{C} = L'_1 \oplus \cdots \oplus L'_r$. Thus, by an easy descent, we can find line bundles $L_1, \ldots, L_r$ on $X_{\overline{\mathbb{Q}}}$ such that $E_{\overline{\mathbb{Q}}} = L_1 \oplus \cdots \oplus L_r$ and $\deg(L_i) = 0$ for all i. Thus, by Proposition 10.8 in [Mo1], we have our assertion.

Next, we assume that $d \geq 2$. Replacing H by a higher multiple H^m of H, we may assume that there is a section $\phi \in H^0(X, H)$ with the following properties:

i) $\operatorname{div}(\phi)_K$ is smooth and geometrically irreducible.
ii) If $\operatorname{div}(\phi) = Y + a_1 F_1 + \cdots + a_s F_s$ is the irreducible decomposition such that Y is horizontal and F_i's are vertical, then F_i's are smooth fibers.

Since (E_σ, h_σ) is flat for each $\sigma \in K(\mathbb{C})$, we have $(\mathrm{ch}_2(E) \cdot F_i \cdot H^{d-2}) = 0$ and $\mathrm{ch}_2(E_\sigma, h_\sigma)$ is zero as differential form for every $\sigma \in K(\mathbb{C})$. Thus we have

$$\widehat{\deg}\left(\widehat{\mathrm{ch}}_2(E,h) \cdot \widehat{c}_1(H,k)^{d-1}\right) = \widehat{\deg}\left(\widehat{\mathrm{ch}}_2(\,(E,h)|_Y) \cdot \widehat{c}_1(\,(H,k)|_Y)^{d-2}\right).$$

Let $\rho_{E_{\mathbb{C}}|_{Y_{\mathbb{C}}}} : \pi_1(Y_{\mathbb{C}}) \to \mathrm{GL}_r(\mathbb{C})$ be the representation arising from $E_{\mathbb{C}}|_{Y_{\mathbb{C}}}$. Since $\rho_{E_{\mathbb{C}}|_{Y_{\mathbb{C}}}}$ is the composition of $\pi_1(Y_{\mathbb{C}}) \to \pi(X_{\mathbb{C}})$ and $\rho_{E_{\mathbb{C}}} : \pi_1(X_{\mathbb{C}}) \to \mathrm{GL}_r(\mathbb{C})$, the image of $\rho_{E_{\mathbb{C}}|_{Y_{\mathbb{C}}}}$ is also abelian. Thus, by hypothesis of induction, $E_\sigma|_{Y_\sigma}$ is of torsion type for every $\sigma \in K(\mathbb{C})$. On the other hand, by Lefschetz theorem, $\pi_1(Y_\sigma) \to \pi_1(X_\sigma)$ is surjective. Hence, E_σ is also of torsion type for every $\sigma \in K(\mathbb{C})$. □

Finally, we will pose two questions. Let $f : X \to \mathrm{Spec}(O_K)$ be a $(d+1)$-dimensional arithmetic variety, (H, k) an arithmetically ample Hermitian line bundle on X, and (E, h) a Hermitian vector bundle on X such that $E_{\overline{\mathbb{Q}}}$ is semistable with respect to $H_{\overline{\mathbb{Q}}}$ and $(c_1(E_K) \cdot c_1(H_K)^{d-1}) = 0$. An interesting problem is to find stronger equality conditions for

$$\widehat{\deg}\left(\widehat{\mathrm{ch}}_2(E,h) \cdot \widehat{c}_1(H,k)^{d-1}\right) \leq 0.$$

Theorem 3.1 says that if $\widehat{\deg}\left(\widehat{\mathrm{ch}}_2(E,h) \cdot \widehat{c}_1(H,k)^{d-1}\right) = 0$, then at least E_σ is flat for every $\sigma \in K(\mathbb{C})$. Optimistically, one may pose the following question:

Question 3.3. If $\widehat{\deg}\left(\widehat{\mathrm{ch}}_2(E,h) \cdot \widehat{c}_1(H,k)^{d-1}\right) = 0$, is E_σ of torsion type for every $\sigma \in K(\mathbb{C})$?

By Proposition 3.2, if $\pi_1(X_{\mathbb{C}})$ is abelian or $\mathrm{rk}\, E = 1$, we have an affirmative answer of the above question. Moreover, if we carefully trace back the proof in Proposition 3.2, Question 3.3 can be reduced to the case $d = 1$. So from now on, we assume that $d = 1$. Let $\overline{\mathbf{M}}_{X_K/K}(r, 0)$ be the moduli scheme of semistable vector bundles on X_K with rank r and degree 0. Let h be a height function on $\overline{\mathbf{M}}_{X_K/K}(r, 0)$ arising from some ample line bundle on $\overline{\mathbf{M}}_{X_K/K}(r, 0)$. Our next question is

Question 3.4. Are there constants A and B with the following properties?

(1) $A, B \in \mathbb{R}$ and $A > 0$.
(2) For all semistable Hermitian vector bundle (E, h) on X with rank r and degree 0, we have

$$h(E_K) \leq \frac{-A}{[K:\mathbb{Q}]} \widehat{\deg}\left(\widehat{\mathrm{ch}}_2(E,h)\right) + B$$

In some sense, Question 3.4 is related to Question 3.3. For, if $\widehat{\deg}\left(\widehat{\mathrm{ch}}_2(E,h)\right) = 0$ and Question 3.4 holds, then the height of E_K is bounded. So E_K should have some simple structure.

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