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ON THE EXISTENCE OF HIGH MULTIPLICITY INTERFACES

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Abstract. In many singularly perturbed Ginzburg–Landau type partial differential equations, such as the Allen–Cahn equation, the nonlocal Allen–Cahn equation, and the Cahn–Hilliard equation, the question arises whether or not the limiting interfaces can have high multiplicity. In other words, do there exist solutions of these PDE's with many transition layers (where the solution passes rapidly between ± 1) which are so close to each other that they collapse to one interface in the limit. In this paper we prove that there exist interfaces with arbitrarily high multiplicity by studying the radially symmetric Allen-Cahn equation. We adapt the energy method of Bronsard-Kohn [BK].

1. Introduction

To motivate the results presented in this paper, we will first consider two volume preserving gradient flows for the following energy functional:

$$E_\varepsilon[u^\varepsilon] := \int_\Omega \left(\frac{\varepsilon}{2} |\nabla u^\varepsilon|^2 + \frac{1}{\varepsilon} W(u^\varepsilon) \right) dx,$$

where W is a smooth symmetric double well potential with strict minima at ± 1 ($W''(\pm 1) > 0$) and ε is a small parameter. The first one is the Cahn–Hilliard flow, which is obtained by taking the gradient flow with respect to the H^{-1} inner product, and the second is the so–called nonlocal Allen–Cahn flow, obtained by using the L^2 inner product. The associated PDE's are respectively:

$$\text{(CH)} \qquad \varepsilon u_t^\varepsilon - \Delta(-\varepsilon \Delta u^\varepsilon + \frac{1}{\varepsilon} W'(u^\varepsilon)) = 0,$$

$$\text{(nlAC)} \qquad \varepsilon u_t^\varepsilon - \varepsilon \Delta u^\varepsilon + \frac{1}{\varepsilon} W'(u^\varepsilon) - \frac{1}{\varepsilon} \frac{1}{|\Omega|} \int_\Omega W'(u^\varepsilon)\, dx = 0.$$

In the radially symmetric case, the singular limit as $\varepsilon \to 0$ of the Cahn–Hilliard flow was studied rigorously by Stoth [S], while the singular limit of the nonlocal Allen-Cahn flow was studied rigorously by Bronsard–Stoth [BS]. These results

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show that the geometric evolution equation satisfied by the limiting interfaces is of the form:

$$\alpha\dot{R}(t) + \frac{(n-1)}{R(t)} = \nu\psi(t),$$

where α is a constant, $R(t)$ is the position of a limiting interface, $\nu \in \{0, \pm 1\}$, and ψ is a function which can be derived explicitly for each flow. For example, in the case of the equation (nlAC), the function $\psi(t) \equiv 0$ in the two dimensional case, if there is an even number of interfaces. In higher dimension, ψ depends on all interfaces.

However, some information is lost when only considering the limiting geometric flow. Indeed, it is possible that several transition layers for u^ε collapse in the limit $\varepsilon \to 0$ to one interface. This interface would then have a multiplicity m given by the number of transition layers which are collapsing to that interface. In addition, if an even number of transition layers collapse, then the limiting interface no longer separates two different phases; this corresponds to $\nu = 0$. This case was introduced by Bronsard–Stoth in [BS], where such interfaces were called "phantom" interfaces. The natural question then is whether or not these high multiplicity interfaces do exist.

The limiting geometric flow contains some information on the multiplicity m. Indeed, the results of Stoth [S], and Bronsard–Stoth [BS] show that both

$$\alpha\dot{R}(t) + \frac{(n-1)}{R(t)} = \nu\psi(t)$$

and

$$m(\alpha\dot{R}(t) + \frac{(n-1)}{R(t)}) = \nu\psi(t)$$

must hold. This means that when $\psi \neq 0$, either $m = 1$ or $\nu = 0$, i.e. either $m = 1$ or m is even. In particular, high multiplicity interfaces must be "phantoms". In addition, for the Cahn–Hilliard flow, the kinetic coefficient α vanishes, so that both ν and ψ are nonzero and hence phantom interfaces do not exist, in other words all interfaces are "true" interfaces with $m = 1$. However for the nonlocal Allen–Cahn flow (where $\alpha \neq 0$), or when $\psi = 0$, such as for the Allen–Cahn equation, the limiting geometric flow cannot be used to obtain information about these high multiplicity interfaces.

In this paper, we study this question for the radially symmetric Allen–Cahn equation in the unit ball in $\mathbf{R}^n$ (this flow is with respect to the H^1 inner product):

$$\varepsilon u_t^\varepsilon - \varepsilon\left(u_{rr}^\varepsilon + \frac{(n-1)}{r}u_r^\varepsilon\right) + \frac{1}{\varepsilon}W'(u^\varepsilon) = 0, \qquad u^\varepsilon(r,t)|_{r=1} = -1,$$

since this flow gives simple and clear answers. We show that there exist interfaces of arbitrarily high multiplicity by adapting the energy method introduced by Bronsard–Kohn [BK]. More precisely, we show that u^ε can have any given number of transition layers which are $O(\varepsilon^\alpha)$ apart $(0 < \alpha \leq \frac{1}{3})$ for times of order one. We note that this method applies directly to the one–dimensional Allen–Cahn equation.

2.

Since the limiting interface will be moving by the mean curvature flow, we introduce a moving coordinate system in which the solution u^ε will be asymptotically stationary. Let $\rho_0 \in (\frac{1}{2}, 1)$ and let $\rho(t)$ be the solution to the mean curvature flow

$$\dot\rho(t) = \frac{-(n-1)}{\rho(t)}, \quad \rho(0) = \rho_0,$$

i.e. $\rho(t) = \sqrt{\rho_0^2 - 2(n-1)t}$. In addition, define T_1 by $\rho(T_1) = \frac{1}{2}$. Since the distance to the moving sphere of radius $\rho(t)$ is given by $R = r - \rho(t)$, we define $w(R, \tau) = w^\varepsilon(R, \tau) = u^\varepsilon(R + \rho(\tau), \tau)$. The equation for w becomes

$$\varepsilon w_\tau - \varepsilon \left(w_{RR} - \frac{(n-1)R}{\rho(\tau)(R + \rho(\tau))} w_R \right) + \frac{1}{\varepsilon} W'(w) = 0$$

$$-\rho(\tau) \le R \le 1 - \rho(\tau),$$

with boundary conditions

$$w_R(-\rho(\tau), \tau) = 0, \quad w(1 - \rho(\tau), \tau) = -1.$$

If we let $\varphi = \varphi(R, \tau)$ solve

$$\varphi_R = \frac{-(n-1)R}{\rho(\tau)(R + \rho(\tau))}\, \varphi,$$

then the equation for w becomes

$$\varepsilon w_\tau - \varepsilon \frac{1}{\varphi} \left(\varphi w_R \right)_R + \frac{1}{\varepsilon} W'(w) = 0.$$

We choose, as in the paper of Bronsard–Kohn [BK],

$$\varphi(R, \tau) = e^{-(n-1)R/\rho(\tau)} \left(1 + \frac{R}{\rho(\tau)} \right)^{(n-1)}.$$

This function has the following properties:

$$0 \le \varphi \le 1, \qquad \varphi(-\rho(\tau), \tau) = 0, \qquad \varphi(0, \tau) = 1, \qquad \varphi_\tau \le 0.$$

In addition, using the Taylor expansion of φ, we see that

$$\varphi(R, \tau) \ge 1 - \frac{(n-1)^2}{\rho(T_1)^2} R^2 \ge \frac{1}{2}, \qquad \text{for } |R| \le a \text{ and } \tau \le T_1,$$

if we define $a = \frac{\rho(T_1)}{\sqrt{2}(n-1)} = \frac{1}{2\sqrt{2}(n-1)}$.

Using the properties of φ, Bronsard–Kohn [BK] have shown that the weighted energy functional

$$E_\varphi[w^\varepsilon](\tau) := \int_{-\rho(\tau)}^{1-\rho(\tau)} \varphi(R,\tau)\left[\frac{\varepsilon}{2}|w_R^\varepsilon|^2 + \frac{1}{\varepsilon}W(w^\varepsilon)\right] dR,$$

is a Lyapunov functional. More precisely, they have shown that

$$\text{(EE)} \qquad \frac{d}{d\tau}E_\varphi[w^\varepsilon](\tau) \le -\varepsilon \int_{-\rho(\tau)}^{1-\rho(\tau)} \varphi(R,\tau)|w_\tau^\varepsilon|^2\, dR.$$

An important property of this weighted energy is that the energy associated to an interface is now given by a constant instead of being given in terms of the position of the interface. In other words, the problem is now closer to the one dimensional Allen–Cahn equation, where the energy only counts the number of interfaces and is independent of their location. This is due to the use of the moving coordinate frame.

Next we introduce some definitions. First, for some $0 < \alpha < 1$, we let

$$v_\varepsilon(R) := \begin{cases} -1, & R < -\varepsilon^\alpha, R > \varepsilon^\alpha \\ 1, & -\varepsilon^\alpha < R < \varepsilon^\alpha. \end{cases}$$

In addition, let $g(s) := \int_0^s \sqrt{2W(\lambda)}\, d\lambda$, so that $g(1) > 0$, $g(-1) = -g(1)$ and we set $g_0 = g(1) - g(-1)$. This function is very important in the energy approach we use and the constant g_0 can be thought of as the surface tension of an interface, and represents the energy associated to the presence of one interface. We are now ready to state our main Theorem.

Theorem. *Let $0 < \alpha \le \frac{1}{3}$, and assume that $E_\varphi[w^\varepsilon](0) \le 2g_0 + c_1\varepsilon^{2\alpha}$ for some constant $c_1 > 0$, independent of ε, and that for a as above,*

$$\int_{-a}^{a} |g(w^\varepsilon(R,0)) - g(v^\varepsilon)|\, dR < \frac{g(1)}{4}\varepsilon^\alpha.$$

Let T_ε be the first time that

$$\int_{-a}^{a} |g(w^\varepsilon(R,T_\varepsilon)) - g(w^\varepsilon(R,0))|\, dR = \frac{g(1)}{4}\varepsilon^\alpha.$$

Then $T_\varepsilon \ge \min(T_1, C)$ for some positive constant C independent of ε, i.e. T_ε is of order one in ε. In particular, $w^\varepsilon(R,\tau)$ has exactly two transitions layers in $(-a,a)$, which are located in $(-2\varepsilon^\alpha, 2\varepsilon^\alpha)$ for all $\tau \le T_\varepsilon$.

Remark. A more accurate statement of the last part of the Theorem is the following: for all $\tau \le T_\varepsilon$, there are points

$$-2\varepsilon^\alpha < R_1 < R_2 < 0 < R_3 < R_4 < 2\varepsilon^\alpha$$

with the property that

$$w^\varepsilon(R_1,\tau) < -1 + c_4\varepsilon^{\frac{1-\alpha}{2}}, \quad w^\varepsilon(R_2,\tau) > 1 - c_4\varepsilon^{\frac{1-\alpha}{2}},$$
$$w^\varepsilon(R_3,\tau) > 1 - c_4\varepsilon^{\frac{1-\alpha}{2}}, \quad w^\varepsilon(R_4,\tau) < -1 + c_4\varepsilon^{\frac{1-\alpha}{2}},$$

for some positive constant c_4 defined explicitly in the Lemma below. In addition $w^\varepsilon(\cdot,\tau)$ is positive in (R_2,R_3) and negative in $(-\rho+\delta(\varepsilon),R_1)\cup(R_4,1-\rho)$, for some $\delta(\varepsilon)\to 0$ with ε.

Corollary. *There can exist limiting interfaces of arbitrarily high multiplicity.*

This follows since the above Theorem holds true if for example we let v_ε be a step function with any given number $N \geq 2$ of "jumps" or interfaces around $R=0$ located at distances $2\varepsilon^\alpha$ apart. In this case, we must assume that $E_\varphi[w^\varepsilon](0) \leq Ng_0 + c_1\varepsilon^{2\alpha}$.

We note that there are initial data which satisfy the conditions of the Theorem. Indeed, let $H^\varepsilon(R) := \Xi(R)\tanh(\frac{R+\varepsilon^\alpha}{\varepsilon}) + (1-\Xi(R))\tanh(\frac{R-\varepsilon^\alpha}{\varepsilon})$, where supp $\Xi(1-\Xi) \subset (-\frac{\varepsilon^\alpha}{2},\frac{\varepsilon^\alpha}{2})$. This function has two transition layers around $\pm\varepsilon^\alpha$ in the sense that it is converging (in fact exponentially) to ± 1 on the appropriate sides of $\pm\varepsilon^\alpha$. In addition, its weighted energy is bounded by $2g_0 + C\varepsilon^{2\alpha}$. (In fact, the error is exponentially small.)

To prove this Theorem we need the following Lemma.

Lemma. *If for some smooth function h*

$$\int_{-a}^{a} |g(h) - g(v^\varepsilon)| \leq \frac{g(1)}{2}\varepsilon^\alpha \qquad \text{and} \qquad E_\varphi[h](\tau) \leq C_1,$$

then

$$E_\varphi[h](\tau) \geq 2g_0 - c_2\varepsilon^{1-\alpha} - c_3\varepsilon^{2\alpha},$$

for $\tau \leq T_1$, where $c_2 = 4c_4^2\|g''\|_{C^0([-1,1])}$ with $c_4 = 4\sqrt{C_1}\sqrt{\frac{1}{W''(1)}}$, and $c_3 = 32g_0(n-1)^2$. If in addition $E_\varphi[h](\tau) \leq 2g_0 + c_1\varepsilon^{2\alpha}$ as in the Theorem, then h has exactly two transition layers in $(-a,a)$, which are located in $(-2\varepsilon^\alpha, 2\varepsilon^\alpha)$, in the sense of the Remark above.

This Lemma is variational in character: it has nothing to do with the particular solution we are studying. It basically says that if a function is sufficiently close to v_ε and its energy is bounded, then it must have two transitions layers and its energy must be close to the energy associated to functions with two transition layers.

Proof of the Lemma. By the assumptions of the Lemma and the definition of v_ε, we know that

$$(*) \qquad \int_{-a}^{-\varepsilon^\alpha} |g(h) - g(-1)|\, dR \leq \frac{g(1)}{2}\varepsilon^\alpha.$$

Let $S^+ := \{R | h(R) \geq 0\}$ and $S^- := \{R | h(R) < 0\}$. It follows from $(*)$ that

$$|S^+ \cap (-a, -\varepsilon^\alpha)| \leq \frac{1}{2}\varepsilon^\alpha.$$

Indeed for $R \in S^+$, we have $g(h(R)) > 0$. Hence

$$|S^+ \cap (-a, -\varepsilon^\alpha)| \leq \frac{1}{g(1)} \int_{S^+ \cap (-a, -\varepsilon^\alpha)} |g(h) - g(-1)| \, dR \leq \frac{\varepsilon^\alpha}{2},$$

since $g(-1) = -g(1)$. This means that

$$(**) \qquad |S^- \cap (-2\varepsilon^\alpha, -\varepsilon^\alpha)| \geq \frac{\varepsilon^\alpha}{2}.$$

But by hypothesis, $E_\varphi[h](\tau) \leq C_1$, and hence

$$\int_{S^- \cap (-2\varepsilon^\alpha, -\varepsilon^\alpha)} \varphi \frac{1}{\varepsilon} W(h) \, dR \leq C_1.$$

Therefore, using $(**)$ and that $\varphi(R, \tau) \geq \frac{1}{2}$ for $|R| \leq a$ and $\tau \leq T_1$, we conclude that there exists

$$R_1 \in S^- \cap (-2\varepsilon^\alpha, -\varepsilon^\alpha),$$

such that

$$\frac{1}{2} \frac{\varepsilon^\alpha}{2} W(h(R_1)) \leq C_1 \varepsilon.$$

This means that (for sufficiently small ε)

$$h(R_1) \leq -1 + c_4 \varepsilon^{\frac{1-\alpha}{2}},$$

where $c_4 = 4\sqrt{C_1}\sqrt{\frac{1}{W''(-1)}}$. Similarly, there exists points $R_2 \in (-\varepsilon^\alpha, 0)$, $R_3 \in (0, \varepsilon^\alpha)$ and $R_4 \in (\varepsilon^\alpha, 2\varepsilon^\alpha)$ such that

$$h(R_2) \geq 1 - c_4 \varepsilon^{\frac{1-\alpha}{2}}, \quad h(R_3) \geq 1 - c_4 \varepsilon^{\frac{1-\alpha}{2}}, \quad h(R_4) \leq -1 + c_4 \varepsilon^{\frac{1-\alpha}{2}}.$$

In consequence, we obtain

$$
\begin{aligned}
&E_\varphi[h](\tau) \geq \\
&\int_{R_1}^{R_2} \varphi\left[\frac{\varepsilon}{2}|h_R|^2 + \frac{1}{\varepsilon}W(h)\right] dR + \int_{R_3}^{R_4} \varphi\left[\frac{\varepsilon}{2}|h_R|^2 + \frac{1}{\varepsilon}W(h)\right] dR \\
&\geq \int_{R_1}^{R_2} \varphi\sqrt{2}|h_R|\sqrt{W(h)}dR + \int_{R_3}^{R_4} \varphi\sqrt{2}|h_R|\sqrt{W(h)}dR \\
&= \int_{R_1}^{R_2} \varphi|\partial_R g(h)|dR + \int_{R_3}^{R_4} \varphi|\partial_R g(h)|dR \\
&\geq \varphi(R_1,\tau)\left(g(h(R_2)) - g(h(R_1))\right) + \varphi(R_4,\tau)\left(g(h(R_3)) - g(h(R_4))\right) \\
&\geq \varphi(R_1,\tau)\left[g_0 - (g(1) - g(h(R_2))) - (g(h(R_1)) - g(-1))\right] \\
&\quad + \varphi(R_4,\tau)\left[g_0 - (g(1) - g(h(R_3))) - (g(h(R_4)) - g(-1))\right] \\
&\geq \varphi(-2\varepsilon^\alpha,\tau)\left[g_0 - ||g''||_{C^0([-1,1])}\left((1-h(R_2))^2 + (h(R_1)+1)^2\right)\right] \\
&\quad + \varphi(-2\varepsilon^\alpha,\tau)\left[g_0 - ||g''||_{C^0([-1,1])}\left((1-h(R_3))^2 + (h(R_4)+1)^2\right)\right] \\
&\geq \varphi(-2\varepsilon^\alpha,\tau)\left(2g_0 - 4||g''||_{C^0([-1,1])}c_4^2\varepsilon^{1-\alpha}\right) \\
&\geq 2g_0 - c_2\varepsilon^{1-\alpha} - 32g_0(n-1)^2\varepsilon^{2\alpha},
\end{aligned}
$$

since $\varphi(-2\varepsilon^\alpha,\tau) \geq 1 - \frac{(n-1)^2}{\rho(T_1)^2}(2\varepsilon^\alpha)^2$, for $\tau \leq T_1$.

If we now assume that $E_\varphi[h] \leq 2g_0 + c_1\varepsilon^{2\alpha}$, then the above inequality implies in addition that the energy corresponding to the complement of the transition region $C := (-\rho, 1-\rho)\backslash(R_1,R_2)\cup(R_3,R_4)$ is small

$$\int_C \varphi|\partial_R g(h)|\, dR \leq \int_C \varphi\left[\frac{\varepsilon}{2}|h_R|^2 + \frac{1}{\varepsilon}W(h)\right] dR \leq (c_1+c_3)\varepsilon^{2\alpha} + c_2\varepsilon^{1-\alpha}.$$

Since φ is strictly positive away from $R = -\rho$, we may conclude that h does not change sign in (R_2, R_3), $(R_4, 1-\rho)$ and in $(-\rho+\delta(\varepsilon), R_1)$ for some $\delta(\varepsilon) \to 0$ with ε. Since $R = -\rho$ corresponds to the origin, there might be at most a transition zone near the origin, which vanishes in the limit $\varepsilon \to 0$.

We are now ready for the

Proof of the Theorem.

If $T_\varepsilon \geq T_1$, there is nothing to prove. Suppose that $T_\varepsilon \leq T_1$. We recall that for all $\tau \leq T_1$, we have

$$(*) \qquad \min_{(-a,a)} \min_{\tau \leq T_\varepsilon} \varphi(R,\tau) \geq \frac{1}{2}.$$

From the assumptions of the Theorem, it follows that

$$\int_{-a}^{a} |g(w^\varepsilon(R,T_\varepsilon)) - g(v_\varepsilon)|\, dR \leq \frac{g(1)}{2}\varepsilon^\alpha.$$

In addition, since the weighted energy is a Lyapunov functional, we have

$$E_\varphi[w^\varepsilon](T_\varepsilon) \le E_\varphi[w^\varepsilon](0) \le 2g_0 + c_1\varepsilon^{2\alpha}.$$

Therefore the Lemma applies and using the definition of T_ε, inequality $(*)$ and the energy estimate (EE), we find

$$\begin{aligned}
&\frac{g(1)}{4}\varepsilon^\alpha = \\
&\int_{-a}^{a} |g(w^\varepsilon(R,T_\varepsilon)) - g(w^\varepsilon(R,0))| dR \\
&\le \int_{-a}^{a}\int_0^{T_\varepsilon} |\partial_\tau g(w^\varepsilon(R,\tau))| d\tau dR \\
&\le 2\int_{-a}^{a}\int_0^{T_\varepsilon} \varphi|\partial_\tau w^\varepsilon|\sqrt{2W(w^\varepsilon)} d\tau dR \\
&\le 2\left(\delta\varepsilon^{1-\alpha}\int_0^{T_\varepsilon}\int_{-a}^{a}\varphi|\partial_\tau w^\varepsilon|^2 dR d\tau + \frac{1}{\delta\varepsilon^{1-\alpha}}\int_0^{T_\varepsilon}\int_{-a}^{a}\varphi W(w^\varepsilon) dR d\tau\right) \\
&\le 2\left(\frac{\delta}{\varepsilon^\alpha}(E_\varphi[w^\varepsilon](0) - E_\varphi[w^\varepsilon](T_\varepsilon)) + \frac{\varepsilon^\alpha}{\delta}T_\varepsilon E_\varphi[w^\varepsilon](0)\right) \\
&\le 2\left(\frac{\delta}{\varepsilon^\alpha}(c_1\varepsilon^{2\alpha} + c_2\varepsilon^{1-\alpha} + c_3\varepsilon^{2\alpha}) + \frac{\varepsilon^\alpha}{\delta}T_\varepsilon E_\varphi[w^\varepsilon](0)\right).
\end{aligned}$$

Therefore

$$\begin{aligned}
T_\varepsilon &\ge \left(\frac{g(1)}{8}\varepsilon^\alpha - \delta(c_1\varepsilon^\alpha + c_2\varepsilon^{1-2\alpha} + c_3\varepsilon^\alpha)\right)\frac{\delta}{\varepsilon^\alpha E_\varphi[w^\varepsilon](0)} \\
&= \left(\frac{g(1)}{8} - \delta(c_2\varepsilon^{1-3\alpha} + c_1 + c_3)\right)\frac{\delta}{E_\varphi[w^\varepsilon](0)} \\
&\ge C(g_0, c_1, c_2, c_3),
\end{aligned}$$

as long as $\delta > 0$ is chosen sufficiently small and $1 - 3\alpha \ge 0$, i.e. $\alpha \le \frac{1}{3}$. This completes the proof of the Theorem.

Remark. By iterating this proof, we can show that the high multiplicity interface constructed above exists until $\rho(t)$ reaches the origin. Indeed, by repeating the proof a finite number of times, we can show that the interface exists up to time T_1. For that, we iterate our argument as follows. We start with the initial data $w^\varepsilon(R, T_\varepsilon)$ and we define the next time step $T_\varepsilon^{(2)}$ to be the first time such that

$$\int_{-a}^{a} |g(w^\varepsilon(R, T_\varepsilon^{(2)})) - g(w^\varepsilon(R, T_\varepsilon))|\, dR = 2\frac{g(1)}{4}\varepsilon^\alpha.$$

This means that we must replace $\frac{g(1)}{4}$ by $2\frac{g(1)}{4}$ in the statement of the Theorem and we must replace $\frac{g(1)}{2}$ by $g(1)$ in the statement of the Lemma. Going through the proof of the Lemma, we see that at the mth step $(**)$ must be replaced by

$$|S^- \cap (-(m+1)\varepsilon^\alpha, -\varepsilon^\alpha)| \geq m\frac{\varepsilon^\alpha}{2},$$

and that now $R_1 \in S^- \cap (-(m+1)\varepsilon^\alpha, -\varepsilon^\alpha)$. This means that c_4 becomes $\frac{c_4}{\sqrt{m}}$ and hence that c_2 must be replaced by $\frac{c_2}{m}$. In addition, since $\varphi(-(m+1)\varepsilon^\alpha, \tau) \geq 1 - \frac{(n-1)^2}{\rho(T_1)^2}((m+1)\varepsilon^\alpha)^2$, we see that c_3 becomes $c_3(\frac{m+1}{2})^2$. Therefore following the proof of the Theorem, we obtain that at the mth step

$$\begin{aligned} T_\varepsilon^{(m)} &\geq \left(m\frac{g(1)}{8} - \delta(\frac{c_2}{m}\varepsilon^{1-3\alpha} + c_1 + c_3(\frac{m+1}{2})^2) \right) \frac{\delta}{E_\varphi[w^\varepsilon](0)} \\ &\geq C'(g_0, c_1, c_2, c_3), \end{aligned}$$

if we choose $\delta = \delta' \frac{1}{m+1}$ with a suitably small $\delta' = \delta'(g_0, c_1, c_2, c_3) > 0$. In particular $T_\varepsilon^{(m)}$ is of order one at each step, hence we can reach T_1 in finitely many steps. If we now define times T_k by $\rho(T_k) = \frac{1}{2^k}$ we may reiterate our argument until $\rho(t)$ has reached the origin.

Remark. As another consequence of the Lemma we see that the limit as $\varepsilon \to 0$ of the energy of a phantom interface of multiplicity two is $2g_0$.

Conclusion

In conclusion, we have shown that there exist interfaces of arbitrarily high multiplicity by studying the radially symmetric Allen–Cahn flow and adapting the energy method introduced by Bronsard–Kohn [BK]. In particular, the limiting geometric curvature motion for the phase boundary does not retain the complete structure of the evolving interfaces for the PDE.

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