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CHARACTERISTIC CLASSES AND MULTIDIMENSIONAL RECIPROCITY LAWS

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1. Introduction

In this note, we formulate and prove a non-commutative generalization of the reciprocity laws of Parshin and Kato in higher dimensional class field theory [Kat][P1][P4]. This can be regarded as a geometric realization of the reciprocity laws implicit in the Gersten-Quillen complex in algebraic K-theory [Q]. We expect this work to be related to the "higher dimensional Langlands program", which has yet to be formulated precisely (although see [Kap]).

Let X be an k-dimensional complex analytic space (possibly singular). Let $F := (C_0 \subset C_1 \subset \cdots \subset C_k = X)$ be a flag of irreducible subspaces in X, with $dim\ C_i = i$ (we often omit C_k from the notation). For any analytic open set $V \subset X$, we constructed in [Br-M1] a homology class $\kappa_{F,V} \in H_k(V;\mathbb{Z})$ and showed that it satisfies the homological reciprocity law described in Theorem 2.1. Let $\hat{c}_p$ denote the characteristic class of Deligne-Beilinson [Be] which refines the usual p-th Chern class c_p. In section 3, we define the non-commutative symbol $< \hat{c}_{k+1} >_{F,V}$ at the "place" F, by pulling back $\hat{c}_{k+1}$ via the evaluation map $BGL(n,\mathcal{O}(V)) \times V \to BGL(n,\mathbb{C})$ and taking the "slant-product" of the resulting class with $\kappa_{F,V}$. Here $\mathcal{O}(V)$ denotes the algebra of holomorphic functions on V and we are viewing $GL(n,\mathcal{O}(V))$ as a discrete group. It is important to note that the non-commutative symbol is not just a number, rather it is the cohomology class in $H^{k+1}(BGL(n,\mathcal{O}(V)),\mathbb{C}^*)$ of a degree $k+1$ group cocycle on $GL(n,\mathcal{O}(V))$. Since the homology class $\kappa_{F,V}$ satisfies a reciprocity law, we obtain

Theorem 1.1. *Let V be an analytic open subset of X. Fix a partial flag $C_0 \subset C_1 \subset \cdots \subset \hat{C}_j \subset \cdots \subset C_k = X$ of irreducible complex-analytic*

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spaces, where $\hat{C}_j$ *means that* C_j *is omitted. Then*

$$\prod_{C_j} < \hat{c}_{k+1} >_{(C_0,\dots,C_{k-1}),V} = 1$$

in $H^{k+1}(BGL(n,\mathcal{O}(V));\mathbb{C}^*)$*, where the product is taken over all irreducible subvarieties* C_j *lying in the chain* $C_0 \subset \cdots \subset C_j \subset \cdots \subset C_k$ *and the product is finite.*

We can pass to the limit over all analytic open subsets V, using a description of $\langle \hat{c}_{k+1}\rangle_{F,V}$ on the level of complexes. Then we can regard the symbol as a degree $k+1$ group cohomology class for the group of invertible $n \times n$ matrices over the function field $\mathbb{C}(X)$. We then obtain reciprocity laws which take place in $H^{k+1}_{group}(GL(n,\mathbb{C}(X));\mathbb{C}^*)$.

In the case where X has dimension 1, the non-commutative symbol can be viewed as a 2-group cocycle for a central extension of the loop group of $GL(n,\mathbb{C})$ (see Remark 3.3). Theorem 1.1 then reduces to the reciprocity law for loop groups [A-D-K][W]. This was used by Segal [Se] in his construction of the conformal blocks and underlies the proof of the Verlinde formula [T-U-Y].

In the case of $GL(1,\mathbb{C})$, one replaces $\hat{c}_{k+1}$ by $\hat{c}_1^{k+1}$-the $k+1$-st power of the universal first Chern class. We explain in section 4 how to obtain the Parshin-Kato symbol from $< \hat{c}_1^{k+1} >_{F,V}$. This recovers our geometric proof of the higher-dimensional reciprocity laws of class field theory given in [Br-M 1]. When $k = 1$, we obtain the usual tame symbol

$$\{f,g\}_p := (-1)^{\nu_p(f)\nu_p(g)}[\frac{f^{\nu_p(g)}}{g^{\nu_p(f)}}](p)$$

associated to the two meromorphic functions f, g at the point p [D].

In case X has dimension 2, there is a geometric construction of κ_F, which allows one to give a formula for the non-commutative symbol as a 3-cocycle on the double loop group $LL\ GL(n,\mathbb{C})$. We also apply Theorem 3.1 to construct a central extension of $GL(n,\mathcal{O}(V))$ for a Zariski open set V §5. Our construction is really an application of the simplicial structure on the two-dimensional adeles [P3], although we do not use this language. It would be fasinating to explore the significance of this central extension in non-commutative class field theory.

2. The homology class associated to a flag

This is a brief summary of the material in §3 of [Br-M1]. Fix a complete flag $F = (C_0 \subset C_1 \subset \cdots \subset C_{k-1} \subset C_k = X)$. We say that an open set

$U \subset X$ is analytic if its complement is a closed analytic subset. Let S_j be the set of analytic open subsets of X satisfying:

(a) $U \cap C_j$ is non-empty (hence dense in C_j);

(b) $U \cap C_{j-1}$ is empty.

We say that two open sets U_1, U_2 in S_j are j-equivalent (written $U_1 \cong_j U_2$), if $U_1 \cap C_j = U_2 \cap C_j$.

Let T_j be the set of $2j+2$-tuples of open sets $(U_0, V_0, ..., U_j, V_j)$ satisfying the conditions:

(c) U_l, V_l belong to S_l;

(d) $U_l \cong_l V_l$;

(e) $V_{l-1} \setminus (V_{l-1} \cap C_{l-1}) \subseteq U_l$, for $1 \leq l \leq j$.

Using the exact sequence associated to a triple, it is easy to see that for each l, these conditions guarantee the existence of a boundary map

$$\delta_l : H_k(U_{l-1}, U_{l-1} \setminus (U_{l-1} \cap C_{l-1})) \to H_{k-1}(U_l, U_l \setminus (U_l \cap C_l)).$$

Now we can define the homology class $\kappa_{F,V}$ associated to the flag F and any $V \in S_k$. Let us denote the point C_0 by p. Note that S_0 consists of those analytic open sets U containing p, and for each such U, we have the fundamental class $o_p \in H_{2k}(U, U \setminus \{p\})$. It is not hard to see that there exists an element $(U_0, V_0, ..., U_k, V_k)$ in T_k, with $V_k = V$. By composition of the boundary maps δ_l, we then obtain the class $\kappa_{F,V} := \delta_k \cdot .... \cdot \delta_0(o_p)$ in $H_k(V; \mathbb{Z})$. In Proposition 3.5 and Theorem 3.6 of [Br-M1], we proved the homological reciprocity law:

Theorem 2.1. *Let V be an analytic open subset of X and let $C_0 \subset C_1 \subset \cdots \subset C_{j-1} \subset C_{j+1} \subset \cdots \subset C_{k-1}$ be a partial flag in X. Then*

$$\sum_{C_j} \kappa_{F,V} = 0$$

in $H_k(V; \mathbb{Z})$, $0 \leq j \leq k-1$, where the sum is taken over all j-dimensional irreducible subspaces lying in the chain $C_0 \subset \cdots \subset C_j \subset \cdots \subset C_k$ and the sum is finite.

When X is one-dimensional, a flag F is just a point p and κ_F can be realized as the homology class of a small loop encircling p.

In case X has dimension 2, there is also a geometric construction of $\kappa_{F,V}$, which was used by Parshin in his work on two-dimensional residues [P2]. In this situation, a flag F is a pair (p, C) consisting of a point p on an irreducible curve C in X. Let $B_\epsilon(p)$ be a ball of radius ϵ centered at p. Then for ϵ small enough, $\partial B_\epsilon(p)$ intersects C transversally in a link K_ϵ, with one component of the link for each branch of C through p. Now take a δ-neighbourhood $N_\delta(K_\epsilon)$ of the link in $\partial B_\epsilon(p)$. For $\delta << \epsilon$, the boundary of $N_\delta(K_\epsilon)$ is a link of disjoint tori $\phi_i : S^1 \times S^1 \to X \setminus C$. By choosing ϵ, δ small enough, we can arrange for each ϕ_i to lie in any open set $V \in S_k$. It is not hard to see that the homology class in $H_2(V;\mathbb{Z})$ determined by $\sum_i \phi_i$ coincides with $\kappa_{F,V}$. This type of description generalizes to higher dimensions.

3. The non-commutative symbol

Recall the Deligne complex of sheaves [Be] on a complex manifold X:

$$\mathbb{Z}(p)_D := \mathbb{Z}(p) \hookrightarrow \mathcal{O} \to \underline{\Omega}^1 \to \dots \to \underline{\Omega}^{p-1},$$

where $\mathcal{O}$ denotes the sheaf of holomorphic functions, $\underline{\Omega}^i$ denotes the sheaf of holomorphic i-forms and $\mathbb{Z}(p) := (2\pi i)^p\mathbb{Z}$. The hypercohomology $H^*(X;\mathbb{Z}(p)_D)$ of this complex is called the Deligne cohomology. It can be defined for any simplicial complex-analytic manifold. The Deligne-Beilinson cohomology is defined for algebraic manifolds in a similar but more delicate way, using holomorphic forms with logarithmic poles along the divisor at infinity of a suitable compactification [Be].

The exponential map induces an isomorphism

$$H^i(-;\mathbb{Z}(p)_D) \cong H^{i-1}(-;\mathcal{O}^* \xrightarrow{dlog} \underline{\Omega}^1 \to \cdots \to \underline{\Omega}^{p-1}).$$

Moreover, for any k-dimensional complex-analytic manifold X, exterior differentiation induces an isomorphism

$$H^k(X;\mathcal{O}^* \to \underline{\Omega}^1 \to \dots \to \underline{\Omega}^k) \xrightarrow{\cong} H^k(X;\mathbb{C}^*) \qquad (3-1)$$

between the Deligne cohomology group $H^{k+1}(X;\mathbb{Z}(k+1)_D)$ and the usual cohomology group $H^k(X;\mathbb{C}^*)$.

Using Deligne-Beilinson cohomology and mixed Hodge theory on the simplicial scheme $BGL(n,\mathbb{C})_\bullet$, Beilinson [Be] has introduced Chern classes $\hat{c}_p \in H^{2p}(BGL(n,\mathbb{C})_\bullet;\mathbb{Z}(p)_D)$, which map to the usual Chern classes under the projection $\mathbb{Z}(p)_D \to \mathbb{Z}(p)$.

Let X be a k-dimensional complex-analytic space and let $F = (C_0 \subset C_1 \subset \cdots \subset C_k)$ be a fixed flag. For any smooth open set $V \in S_k$, let $\kappa_{F,V}$ be the corresponding homology class in $H_k(V;\mathbb{Z})$ constructed in section 2. There is a natural morphism of schemes $ev : V \times BGL(n,\mathcal{O}(V))_\bullet \to BGL(n,\mathbb{C})_\bullet$, which in degree d is given by the obvious evaluation map $ev_d : V \times GL(n,\mathcal{O}(V))^d \to GL(n,\mathbb{C})^d$. Here we view $GL(n,\mathcal{O}(V))$ as a *discrete* group, so that $V \times GL(n, \mathcal{O}(V))^d$ is a disjoint union of complex manifolds, each of which is isomorphic to V.

We now describe a natural pairing (a sort of slant-product):

$$H_k(V, \mathbb{Z}) \otimes H^m(V \times BGL(n, \mathcal{O}(V))_\bullet, \mathbb{Z}(k+1)_D) \to$$
$$H^{m-k}(BGL(n, \mathcal{O}(V))_\bullet, \mathbb{Z}(1)_D).$$

First we map the Deligne complex $\mathbb{Z}(k+1)_D$ to the smooth Deligne complex [Br] $\mathbb{Z}(k+1)_D^\infty$, in which holomorphic differential forms are replaced by smooth ones.

Let $\mathcal{U} = (U_i)$ be an open cover of V by contractible Stein open sets all of whose partial intersections are empty or contractible. Given a smooth map $\sigma : \Delta^q \to V$ with $q \leq k$, there is a natural map (integration over the simplex σ)

$$C^q(\mathcal{U} \times GL(n, \mathcal{O}(V))^r, \mathbb{Z}(k+1)_D^\infty) \to \Gamma(GL(n, \mathcal{O}(V))^{r-1}, \mathbb{C}^*),$$

where C^q is the degree q-term of a Cech double complex. We next introduce the complex $K^\bullet$ of V such that $K^{-q} = \mathbb{Z}[Map_{sm}(\Delta^q, V)]$ when $0 \leq q \leq k$ and 0 otherwise. We have a triple complex with (p, q, r)-term $K^{-q} \otimes C^p(\mathcal{U} \times GL(n, \mathcal{O}(V))^r, \mathbb{Z}(k+1)_D)$. We consider the associated single complex and construct a morphism of complexes

$$\oplus_{p-q+r=m} K^{-q} \otimes C^p(\mathcal{U} \times GL(n, \mathcal{O}(V))^r, \mathbb{Z}(k+1)_D^\infty) \to$$
$$\Gamma(GL(n, \mathcal{O}(V))^{m-1}, \mathbb{C}^*).$$

On cohomology, this induces a pairing

$$H_q(V, \mathbb{Z}) \otimes H^p(V \times BGL(n, \mathcal{O}(V)), \mathbb{Z}(k+1)_D) \to$$
$$H^{p-q-1}(BGL(n, \mathcal{O}(V)), \mathbb{C}^*).$$

This is analogous to similar constructions of regulators in [Be] and in [So].

We can now define the *non-commutative symbol*

$$< \hat{c}_{k+1} >_{F,V} \in H^{k+1}(BGL(n, \mathcal{O}(V))_\bullet, \mathbb{C}^*)$$

at the place F. It is equal to the slant product of $\kappa_{F,V} \in H_k(V, \mathbb{Z})$ and of the pull-back class

$$ev^* \hat{c}_{k+1} \in H^{2k+2}(V \times BGL(n, \mathcal{O}(V))_\bullet; \mathbb{Z}(k+1)_D).$$

It is an important feature of our approach that the symbol is a group cohomology class for the discrete group $GL(n, \mathcal{O}(V))$. Theorem 2.1 immediately implies Theorem 1.1 in the Introduction.

Remarks 3.2.

(a) When X is algebraic, Theorem 1.1 can be viewed as a geometric realization of the reciprocity laws encoded in the following piece of the Gersten-Quillen complex in algebraic K-theory [Q]:

$$\oplus_{x\in X^{(k-j-1)}} K_{j+1}(\mathbb{C}(x)) \xrightarrow{\partial} \oplus_{x\in X^{(k-j)}} K_j(\mathbb{C}(x)) \xrightarrow{\partial} \oplus_{x\in X^{(k-j+1)}} K_{j-1}(\mathbb{C}(x)),$$

followed by an iterated residue map. Here $X^{(i)}$ denotes the set of irreducible subvarieties of codimension i. This relation between algebraic K-theory and the non-commutative symbol can be seen by comparing our constructions to Soulé's approach to the regulator map [So].

(b) The local symbol can be extended to a cohomology class $< \hat{c}_{k+1} >^{\infty}_{F,V}$ of the group $Map(V, GL(n, \mathbb{C}))$ of smooth maps $V \to GL(n, \mathbb{C})$. To see this, one uses throughout the smooth Deligne complex $\mathbb{Z}(k)^{\infty}_{D}$. The details are left to the reader. The cohomology class $< \hat{c}_{k+1} >^{\infty}_{F,V}$ is called the *smooth symbol.*

(c) When $k = 1$, the class $\kappa_{F,V}$ is represented by a loop $\phi : S^1 \to V$ encircling the point $p = F$. By naturality, the smooth symbol at the place p can be computed by pulling back via ϕ to the circle. The local symbol is then a 2-group cocycle for the loop group $L\ GL(n, \mathbb{C})$ and therefore represents a central extension $\widetilde{LGL(n, \mathbb{C})}$ of $L\ GL(n, \mathbb{C})$ by $\mathbb{C}^*$. Theorem 1.1 implies that this extension has the following property: for any Riemann surface Σ whose boundary $\partial\Sigma$ is a disjoint union of circles, the extension $\widetilde{Map(\partial\Sigma, GL(n, \mathbb{C}))}$ induced by Baer multiplication of the extensions $\widetilde{LGL(n, \mathbb{C})}$ on each boundary component, *splits* when pulled back to $Hol(\Sigma, GL(n, \mathbb{C}))$–the group of holomorphic maps of Σ to $GL(n, \mathbb{C})$. This is the notion of reciprocity used in conformal field theory [A-D-K][Se][W][Br-M2].

4. Recovering the Parshin-Kato symbol

First recall that $\mathbb{Z}(p)_D$ comes equipped with a cup-product [Be]

$$\mathbb{Z}(p)_D \otimes \mathbb{Z}(q)_D \to \mathbb{Z}(p+q)_D,$$

which refines the usual cup-product in ordinary cohomology. We may therefore consider the $(k+1)$-st power of the universal first Chern class;

$$\hat{c}_1^{k+1} \in H^{2k+2}(B\mathbb{C}^*_\bullet; \mathbb{Z}(k+1)_D) \cong H^{2k+1}(B\mathbb{C}^*_\bullet; \mathcal{O}^* \to \underline{\Omega}^1 \to .... \to \underline{\Omega}^k).$$

For any simplicial space $X_\bullet$, let $X_{\bullet \geq k}$ denote its truncation in degrees $\geq k$. Since the first Chern class c_1 really lies in $H^2(B\mathbb{C}^*_{\bullet\geq 1}, \mathbb{Z}(1)_D)$, its $(k+1)$-st power lives in $H^{2k+2}(B\mathbb{C}^*_{\bullet\geq k+1}; \mathbb{Z}(k+1)_D)$. Notice that as the truncated simplicial manifold $B\mathbb{C}^*_{\bullet\geq k+1}$ starts with $(\mathbb{C}^*)^{k+1}$, we have an edge homomorphism

$$e : H^{2k+2}(B\mathbb{C}^*_{\bullet\geq k+1}; \mathbb{Z}(k+1)_D) \to H^{k+1}((\mathbb{C}^*)^{k+1}, \mathbb{Z}(k+1)_D),$$

arising in the spectral sequence for hypercohomology of a simplicial manifold.

To give a concrete description of the class, let $(w_1, ..., w_{k+1})$ denote the standard coordinates on the $(k+1)$-fold Cartesian product $\mathbb{C}^* \times \cdots \times \mathbb{C}^*$. In view of the isomorphism $H^0(-; \mathcal{O}^*) \cong H^1(-; \mathbb{Z}(1)_D)$, we can regard each w_i as a 1-cocycle for $\mathbb{Z}(1)_D$. Taking the cup-product $w_1 \cup .... \cup w_{k+1}$ then defines a class in $H^{k+1}((\mathbb{C}^*)^{k+1}; \mathbb{Z}(k+1)_D)$, which we denote by $(w_1, ..., w_{k+1}]$. We have

Proposition 4.1. *The class* $(w_1, ..., w_{k+1}] \in H^{k+1}((\mathbb{C}^*)^{k+1}; \mathbb{Z}(k+1)_D)$ *is the image of* $\hat{c}_1^{k+1} \in H^{2k+2}(B\mathbb{C}^*_{\bullet\geq k+1}; \mathbb{Z}(k+1)_D)$ *under the edge homomorphism* $e : H^{2k+2}(B\mathbb{C}^*_{\bullet\geq k+1}; \mathbb{Z}(k+1)_D) \to H^{k+1}((\mathbb{C}^*)^{k+1}, \mathbb{Z}(k+1)_D)$.

It is easy to see that the "slant product" of $ev^* \hat{c}_1^{k+1}$ with $\kappa_{F,V}$ is determined by its image under e. This leads to...

Theorem 4.2. *Let* F *be a fixed flag in* X *and* V *an analytic open subset of the complement of* F. *For any* $k+1$*-tuple* $(f_1, ..., f_{k+1})$ *of invertible holomorphic functions on* V, *we may regard the cup-product* $(f_1, ..., f_{k+1}]$ *as a class in* $H^k(V; \mathbb{C}^*)$ *using the isomorphism (3-1). Let* $[(f_1, ..., f_{k+1}], \kappa_{F,V}]$ *denote the non-zero complex number obtained by pairing this cohomology class with the homology class* $\kappa_{F,V}$ *of §2. Then the function*

$$\mathcal{O}(V)^* \times .... \times \mathcal{O}(V)^* \to \mathbb{C}^*$$

defined by

$$(f_1, ..., f_{k+1}) \longmapsto < (f_1, ..., f_{k+1}], \kappa_{F,V} >$$

is a group $(k+1)$*-cocycle, whose cohomology class in* $H^{k+1}(B\mathcal{O}(V)^*; \mathbb{C}^*)$ *coincides with the symbol* $< \hat{c}_1^{k+1} >_{F,V}$.

Now we recall the Parshin-Kato symbol [P4]. Suppose for simplicity that the flag $F = C_0 \subset C_1 \subset \cdots \subset C_k = X$ is such that C_i is irreducible along C_{i-1} for all i. Introduce valuations $v_1(f), ..., v_k(f)$ associated to a meromorphic function f as follows; $v_1(f)$ is just the order of f along C_{k-1}. To define v_2, choose a meromorphic function z_1 with $v_1(z_1) = 1$.

Then $f \cdot z_1^{-v_1(f)}$ can be restricted to a meromorphic function on C_{k-2} and $v_2(f)$ is just the order of this function along C_{k-2}. To define v_3, choose z_2 satisfying $v_1(z_2) = 0$ and $v_2(z_2) = 1$, so that $f \cdot z_1^{-v_1(f)} \cdot z_2^{-v_2(f)}$ can be restricted to a meromorphic function on C_{k-3}. Set $v_3(f)$ equal to the order of this function along C_{k-3}. Proceeding in this manner, we obtain k valuations $v_1, ..., v_k$. These are not intrinsic; different choices for the functions z_i satisfying $v_j(z_i) = \delta_{ij}$ (Kronecker delta) induce "gauge transformations" $v_m \to v_m + l_1 \cdot v_1 + ... + l_{m-1} \cdot v_{m-1}$, for some integers l_i. However, if we set

$$K(f_1, ..., f_{k+1})_F := \sum_{I \subset \{1,...,k\}} (\sum_{i \in I} v_i(f_1))(\sum_{i \in I} v_i(f_2)) \cdots (\sum_{i \in I} v_i(f_{k+1})),$$

then it is easy to verify that the sign $(-1)^{K_F}$ is independent of the choices of the z_i.

Definition 4.3. *Let $f_1, ..., f_{k+1}$ be $k+1$ meromorphic functions on X. The Parshin-Kato symbol $\{f_1, ..., f_{k+1}\}_F$ is defined by the expression*

$$[(-1)^K \prod_i f_i^{(-1)^{i+1}\nu(f_1,...,\hat{f}_i,...,f_{k+1})}]_F,$$

where $\nu(g_1, ..., g_k)$ is defined to be the determinant of the $k \times k$-matrix $(v_i(g_j))$. The expression inside the square brackets has all valuations $v_1, . .., v_k$ equal to zero and can therefore be evaluated at the point C_0 of the flag. The square brackets are used to denote this evaluation. If the flag F is not irreducible, then the symbol is defined by taking a product of such expressions over all the irreducible components of F. When $k = 1$, Definition 4.3 reduces to the usual formula for the tame symbol given in the introduction. In [Br-M1], we showed the following.

Theorem 4.4. *Fix a flag F and $k+1$ meromorphic functions $f_1, ..., f_{k+1}$ on X. Let V be an analytic open set in the complement of C_{k-1} and of the divisors of the functions f_i. Then the Parshin-Kato symbol at the place F is equal to the pairing $< (f_1, ..., f_{k+1}], \kappa_{F,V} >$.*

The reciprocity laws of higher dimensional class field theory [Kat][P4] now follow directly, up to a root of unity, from Theorem 2.1.. Using Theorem 4.2 we may conclude:

Theorem 4.5. *Let V be an analytic open subset of the complement of C_{k-1} and of the divisors of the meromorphic functions $f_1, ..., f_{k+1}$. The assignment*

$$(f_1, ..., f_{k+1}) \longmapsto \{f_1, ..., f_{k+1}\}_F \in \mathbb{C}^*$$

determines a cohomology class in $H^{k+1}(B\mathcal{O}(V)^*; \mathbb{C}^*)$ which coincides with $< \hat{c}_1^{k+1} >_{F,V}$.

5. The two-dimensional theory

Suppose that X is an irreducible complex analytic space of dimension 2, so that a flag F in X is a pair (p, C) consisting of a point p on an irreducible curve C. Let V be an analytic open subset of the complement of F. We now give an explicit formula for the local symbol $< \hat{c}_3 >_{F,V}$. Suppose for simplicity that C is locally irreducible in some neighbourhood of p. We saw in §2 that the homology class $\kappa_{F,V} \in H_2(V; \mathbb{Z})$ can then be represented by a map $\phi : S^1 \times S^1 \to V$. By naturality, we may pull back to the torus via the map ϕ, so that the local symbol can be realized as a group 3-cocycle on the double loop group $LL\ GL(n, \mathbb{C})$. We have Beilinson's third Chern class $\hat{c}_3 \in H^6(BGL(n, \mathbb{C})_\bullet; \mathbb{Z}(3)_D)$. Let $g_i(s,t), i = 1, 2, 3$ be elements of $LL\ GL(n, \mathbb{C}) = Map(\mathbb{R}^2/\mathbb{Z}^2, GL(n, \mathbb{C}))$. We assume $n \geq 3$ and we work with the homogeneous space $GL(n, \mathbb{C})/GL(2, \mathbb{C})$. Note that $GL(n, \mathbb{C})/GL(2, \mathbb{C})$ is 4-connected and we have

$$H^5(GL(n, \mathbb{C})/GL(2, \mathbb{C}), \mathbb{Z}) = H^5(GL(n, \mathbb{C}), \mathbb{Z}) = \mathbb{Z}.$$

Let ν be a closed $GL(n, \mathbb{C})$-invariant 5-form on $GL(n, \mathbb{C})/GL(2, \mathbb{C})$ whose cohomology class is the generator. Let $1 \in GL(n, \mathbb{C})/GL(2, \mathbb{C})$ denote the base point. For three elements (g_1, g_2, g_3) of $LL\ GL(n, \mathbb{C})$, we construct a smooth mapping $\sigma_{g_1,g_2,g_3} : T^2 \times \Delta^3 \to GL(n, \mathbb{C})/GL(2, \mathbb{C})$ as follows. For $g \in LL\ GL(n, \mathbb{C})$, and for $(s, t) \in T^2$, choose a path $\tau_g(s, t)$ from 1 to $g(s, t)$ inside $GL(n, \mathbb{C})/GL(2, \mathbb{C})$.

Since $GL(n, \mathbb{C})/GL(2, \mathbb{C})$ is 2-connected, we may assume that the paths $\tau_g(s, t)$ depend smoothly on the parameters s and t. For each pair (g_1, g_2), and for each fixed (s, t), the composition of paths $\tau_{g_1} * [g_1 \cdot \tau_{g_2}] * \tau_{g_1 g_2}^{-1}$ is a loop in $GL(n, \mathbb{C})/GL(2, \mathbb{C})$. Since $GL(n, \mathbb{C})/GL(2, \mathbb{C})$ is simply-connected, this loop bounds a 2-simplex $\eta_{g_1,g_2}(s, t)$. Since $GL(n, \mathbb{C})/GL(2, \mathbb{C})$ is in fact 3-connected, we may assume that $\eta_{g_1,g_2}(s, t)$ depends smoothly on (s, t). Now, for fixed (s, t), and for $(g_1, g_2, g_3) \in LL\ GL(n, \mathbb{C})$ there are four possible 2-simplices whose vertices are $1, g_1(s, t), g_2(s, t), g_3(s, t)$; these four simplices are the boundary of a tetrahedron $\sigma_{g_1,g_2,g_3}(s, t)$ in $GL(n, \mathbb{C})/GL(2, \mathbb{C})$. Since $GL(n, \mathbb{C})/GL(2, \mathbb{C})$ is 4-connected, we may pick $\sigma_{g_2,g_2,g_3}(s, t)$ to be a smooth function of (s, t). Then we view σ_{g_1,g_2,g_3} as a smooth map $S^1 \times S^1 \times \Delta^3 \to GL(n, \mathbb{C})/GL(2, \mathbb{C})$, where Δ^3 is the standard 3-simplex. Using to these constructions we can prove:

Theorem 5.1. *Let ν be an invariant 5-form on $GL(n, \mathbb{C})/GL(2, \mathbb{C})$ which transgresses to the third Chern class. The assignment*

$$(g_1(s,t), g_2(s,t), g_3(s,t)) \longmapsto exp(2\pi i \int_{S^1 \times S^1 \times \Delta^3} \sigma^*_{g_1,g_2,g_3} \nu)$$

defines a degree 3 group cocycle on the double loop group $LL\ GL(n,\mathbb{C})$ *with coefficients in* $\mathbb{C}^*$. *The corresponding cohomology class in* $H^3_{group}(LL\ GL(n,\mathbb{C});\mathbb{C}^*)$ *is independent of the choices involved in constructing* σ *and coincides with the local symbol* $< \hat{c}_3 >^{\infty}_{F,V}$.

Let us recall that for any discrete group G, the 3-dimensional cohomology group $H^3_{group}(G;\mathbb{C}^*)$ can be interpreted as the group of equivalence classes of *abstract kernels* [E-M]. An abstract kernel is a pair (θ, L) consisting of a non-abelian group L with center $\mathbb{C}^*$, together with a homomorphism $\theta : G \to Out(L)$. Given such a kernel (θ, L), the obstruction to the existence of an extension of G by L with "group of operators" θ, defines a class in $H^3_{group}(G;\mathbb{C}^*)$. Eilenberg and Mac Lane showed that all elements of $H^3_{group}(G;\mathbb{C}^*)$ arise as obstructions in this manner. If an extension exists corresponding to some abstract kernel, we will refer to it as an object of the kernel. If we have some extension $1 \to A \to E \to G \to 1$ corresponding to a given abstract kernel, then any other such extension is obtained by twisting E by a central extension of G by $\mathbb{C}^*$. There is a natural product of abstract kernels, as well as a notion of equivalence of kernels. The group of equivalence classes of abstract kernels identifies with the group $H^3_{group}(G;\mathbb{C}^*)$.

Let V be a fixed analytic open subset of the two-dimensional complex analytic space X. In [Br-M1], we showed that there are only finitely many flags $F = (p, C)$ for which the corresponding homology class $\kappa_{(p,C),V}$ is non trivial. Let S_0 be the set of points which occur in these flags, and let S_1 be the set of curves which occur. For simplicity, we fix a kernel (L_F, θ_F) for each flag $F = (p, C)$ where $p \in S_0$ and $C \in C_1$. The reciprocity laws of §3 admit the following strengthening:

Proposition 5.2.

(i) *For each* $p \in S_0$, *there exists a canonical object* E_p *of the product abstract kernel* $\prod_{C\ni p, C\in S_1} (L_{(p,C)}, \theta_{(p,C)})$.

(ii) *For each* $C \in S_1$, *there exists a canonical object* E_C *of the product abstract kernel* $\prod_{p\in C, p\in S_0} (L_{(p,C)}, \theta_{(p,C)})$.

The proof uses the language of simplicial gerbes [Br-M2] rather than the language of kernels.

We may then consider the abstract kernel given by the double product

$$\prod_{(p,C) p\in S_0, C\in S_1} (L_{(p,C)}, \theta_{(p,C)}).$$

There are now two ways to obtain an object of this product kernel. On one hand, we can consider the product $\prod_C E_C$ of the local objects E_C

corresponding to the curves C which belong to S_1. On the other hand, we can take the product $\prod_p E_p$, where the product ranges over all points p which belong to S_0. Since we have two objects of the *same* trivial kernel (with the same group of operators), it follows that they differ by a unique central extension of $GL(n, \mathcal{O}(V))$ by $\mathbb{C}^*$.

We briefly explain how to find a group 2-cocycle c for the central extension of $GL(n, \mathcal{O}(V))$. For each pair $(p, C) \in S_0 \times S_1$, we have a 2-torus $\phi_{(p,C)}$. For $g_1, g_2 \in LL\, GL(n, \mathbb{C})$ we have the map

$$\eta_{g_1,g_2}(p, C) : S^1 \times S^1 \to Map(\Delta^2, GL(n, \mathbb{C})/GL(2, \mathbb{C})).$$

By triangulating $S^1 \times S^1$, we obtain a 2-cycle (also called $\eta_{g_1,g_2}(p, C)$) in $Map(\Delta^2, GL(n, \mathbb{C})/GL(2, \mathbb{C}))$. Then for each $p \in S_0$ we construct a 3-chain $a_{g_1,g_2}(p)$ in $Map(\Delta^2, GL(n, \mathbb{C})/GL(2, \mathbb{C}))$ whose boundary is $\sum_{C \ni p} \eta_{g_1,g_2}(p, C)$; similarly for each $C \in S_1$ we construct a 3-chain $b_{g_1,g_2}(C)$ in $Map(\Delta^2, GL(n, \mathbb{C})/GL(2, \mathbb{C}))$ whose boundary is $\sum_{p \in C} \eta_{g_1,g_2}(p, C)$. The construction of these a and b's is similar to that done at the beginning of the section. Then

$$\upsilon(g_1, g_2) = \sum_{p \in S_0} a_{g_1,g_2}(p) - \sum_{p \in S_1} b_{g_1,g_2}(C)$$

is a 3-cycle in $Map(\Delta^2, GL(n, \mathbb{C})/GL(2, \mathbb{C}))$. Note we can integrate a 5-form on $GL(n, \mathbb{C})/GL(2, \mathbb{C})$ over υ_{g_1,g_2}. Then we put:

$$c(g_1, g_2) = exp(2\pi i \cdot \int_{\upsilon_{g_1,g_2}} \nu).$$

The construction of the central extension of $GL(n, \mathcal{O}(V))$ depends on the choice of the system of curves and of points. In fact, it is very likely that the central extension is trivial if the open set V is Stein; so the central extension should be viewed as a semi-local (or semi-global) invariant attached to a system of curves and points.

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