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GENERAL WALL CROSSING FORMULA

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Introduction

Let M be an oriented four-manifold. Given a Riemannian metric g and a spin^c structure $\mathcal{L}$ on M, there are associated spin^c bundles S_+ and S_- and a canonical isomorphism

$$\tau : \mathrm{End}(S_+) \longrightarrow \Lambda_+ \otimes \mathbf{C}.$$

The Seiberg-Witten equations [W] are equations for a pair (A, ψ), where A is a connection on $L = \det(S_+)$ and ψ is a section of S_+. These equation read

$$D_A\psi = 0 \quad \text{and} \quad P_+F_A = \frac{1}{4}\tau(\psi \otimes \psi^*)$$

where $D_A : \Gamma(S_+) \longrightarrow \Gamma(S_-)$ is the Dirac operator on the spin^c bundle and $P_+ : \Lambda^2T^*X \longrightarrow \Lambda_+$ is the orthogonal projection. It is quite useful to consider perturbations which are of the form

$$D_A\psi = 0 \quad \text{and} \quad P_+F_A = \frac{1}{4}\tau(\psi \otimes \psi^*) + \mu$$

where μ is a fixed, imaginary valued, anti-self-dual 2-form on M. (The notation is from [T].)

The Seiberg-Witten invariant $SW(\mathcal{L})$ for the given spin^c structure $\mathcal{L}$ is obtained by making a suitable count of solutions. The group $C^\infty(M; S^1)$ naturally acts on the space of solutions and acts freely at solutions where ψ is not identically zero. The quotient is the moduli space and is denoted by $\mathcal{M}$. Fix a base point in M and let $C_0^\infty(M, S^1)$ denote the group of maps which map said point to 1. Let $\mathcal{M}^0$ denote the quotient of the space of solutions by $C_0^\infty(M, S^1)$. $\mathcal{M}^0$ is called the based moduli space. When

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$\mathcal{M}$ is a smooth manifold, the projection $\mathcal{M}^0 \longrightarrow \mathcal{M}$ defines a principle S^1-bundle.

By a theorem of Uhlenbeck ([FU]), when $b_2^+ \geq 1$, the space of solutions will contain no points where $\psi \equiv 0$ for a generic metric or choice of μ as long as $c_1(L)$ is not rationally trivial (here generic means off of a set of codimension b_2^+). For a Baire subset of μ in $C^\infty(\Lambda_+)$, $\mathcal{M}$ will be a compact smooth orientable manifold of dimension $d = \frac{1}{4}c_1(L)^2 - \frac{1}{4}(2\chi(M)+3\sigma(M))$. For such a μ, the Seiberg-Witten invariant $SW(\mathcal{L})$ is defined as follows: 1. if $d < 0$, then the SW invariant is zero; 2. if $d = 0$, then $\mathcal{M}$ is a finite union of signed points and the SW invariant is the sum of the corresponding of ± 1's; 3. if $d > 0$, the SW invariant is obtained by pairing the fundamental class of $\mathcal{M}$ with the maximal cup product of the Euler class of the S^1-bundle $\mathcal{M}^0 \longrightarrow \mathcal{M}$.

So, when $b_2^+ > 1$, the value of $SW(\mathcal{L})$, a cobordism invariant of $\mathcal{M}$, is independent of the choice of metric and perturbing form μ. It only depends on $\mathcal{L}$ up to isomorphism. However, when $b_2^+ = 1$, for two generic pairs $(g,\mu)_0$ and $(g,\mu)_1$, it is possible that a generic smooth path of pairs $(g,\mu)_t$ with these two pairs as end points cannot avoid a bad pair $(g,\mu)_{cr}$ where $\psi \equiv 0$ solutions occur. So singularity occurs in the cobordism and breaks the invariance of $SW(\mathcal{L})$. But it is believed that the jump of $SW(\mathcal{L})$ can be analyzed and there should be some wall crossing formulas.

The wall crossing formula of Seiberg-Witten invariants for four-manifolds with $b_2^+ = 1$, $b_1 = 0$ and zero-dimensional moduli spaces was given by Kronheimer and Mrowka [KM] in their proof of the Thom conjecture. In this paper, we prove the general wall crossing formula for four-manifolds with $b_2^+ = 1$. Many interesting applications will appear in [LL].

To state our results, we need to introduce more notations. Let $\mathcal{G}$ denote the product of the space of metrics on M and $C^\infty(\Lambda_+)$. Let $r(t)$ denote a path in $\mathcal{G}$ with two end points r_0 and r_1. Suppose r_{cr} is the only point on the path such that $\psi \equiv 0$ solutions occur. We first prove the following proposition (Prof. Taubes kindly informed us that Proposition 1.1 was known to Kronheimer and Mrowka, we only include it here for completeness):

Proposition 1.1. *Let M be an oriented four-manifold with $b_2^+ = 1$ and $b_1 = 0$, then the SW invariant of a nonnegative dimensional moduli space jumps by ± 1 when it crosses the wall,*

$$SW(\mathcal{L}, r_1) - SW(\mathcal{L}, r_0) = \pm 1.$$

Then we prove our main results in this paper.

Theorem 1.2. *Let M be an oriented four-manifold with $b_2^+ = 1$ and b_1 even, and $\mathcal{L} \in H^2(M;\mathbf{Z})$ a spin^C structure with* $\dim \mathcal{M}(\mathcal{L}) \geq 0$*, then after crossing a wall, $SW(\mathcal{L})$ changes by*

$$\pm(\frac{1}{2}(\Omega^2 \cdot \mathcal{L})[M])^{b_1/2}/(b_1/2)! \ \ [T^{b_1}]$$

where

$$\Omega = c_1(\mathcal{U}) = \sum_i x_i \cdot y_i$$

$\mathcal{U}$ is the universal flat line bundle over $T^{b_1} \times M$, $\{y_i\}$ is any basis of $H^1(M;\mathbf{Z})$ modulo torsion, and $\{x_i\}$ is the dual basis in $H^1(T^{b_1};\mathbf{Z})$.

Corollary 1.3. *Let M be as in Theorem 1.2. Then there exists a basis $\{y_i\}$ depending on $\mathcal{L}$ such that the $b_1 \times b_1$ matrix G with entries given by $(y_i y_j \frac{\mathcal{L}}{2})[M]$ has the form*

$$\begin{pmatrix} 0 & d_1 & \cdots & & \\ -d_1 & 0 & d_2 & \cdots & \\ 0 & -d_2 & 0 & \cdots & \\ \cdots & \cdots & \cdots & & \end{pmatrix}$$

and $SW(\mathcal{L})$ changes by $\pm \prod_{i=1}^{b_1} d_i$.

Corollary 1.4. *Let M be an S^2-bundle over a Riemann surface Σ of genus g or some multiple blowup, and $E \in H^2(M;\mathbf{Z})$ with* $\dim \mathcal{M}(K^{-1}+2E) \geq 0$*. Then, after crossing a wall, $SW(E)$ changes by*

$$\pm\left(\frac{C_1(M) + 2E}{2}[S^2]\right)^g$$

where $[S^2]$ is the homology class represented by the fiber.

Combined with Taubes's results on Seiberg-Witten invariants of symplectic manifolds ([T]), we get

Corollary 1.5. *If M is a symplectic four-manifold with $b_2^+ = 1$ and has metrics of positive scalar curvature, then all the d_i for the anticanonical bundle associated to the symplectic structure are ± 1.*

Corollary 1.6. *If M is a symplectic four-manifold with $b_2^+ = 1$ and there is a $y \in H^1(M;\mathbf{R})$ which annihilates $H^1(M;\mathbf{R})$ by cup product, then M does not have any metric of positive scalar curvature.*

Notice that under the assumption of Corollary 1.5, the wall crossing number is zero for all line bundles, hence Seiberg-Witten invariants are still smooth invariants of the underlying manifolds.

Proof of wall crossing formula

In the case $b_1 = 0$, the wall crossing formula was proved for dimension zero classes by Kronheimer and Mrowka [KM]. We first adapt their argument to prove the wall crossing formula of higher dimensional classes. Then, we are going to prove the wall crossing formula when $b_1 \neq 0$.

In discussing the wall crossing formula, we need to know what happens near the reducible solution. The basic fact is the following proposition of local model.

Proposition 2.1. *If (A,ψ) is a solution of the SW equation over a four-manifold M, a neighborhood of (A,ψ) in $\mathcal{M}$ is modelled on a quotient $\phi^{-1}(0)/\Gamma_{(A,\psi)}$ where*

$$\phi : \operatorname{Ker}\delta \oplus \operatorname{Ker} D_A \longrightarrow \operatorname{Coker} P_+ d \oplus \operatorname{Coker} D_A$$

is a $\Gamma_{A,\psi}$-equivariant map, $\delta = d^* + P_+ d$ and $\Gamma_{A,\psi}$ is the isotropy group of the solution.

This proposition is the analogue of Proposition 4.2.23 in [DK] for ASD connections and is used by Kronheimer and Mrowka in their proof of the Thom conjecture. If $b_1(M) = 0$, the idea is to separate the PDE system $P_+F_A = \frac{1}{4}\tau(\psi\otimes\psi^*) + r\mu$, $D_A\psi = 0$ into two parts in the Banach space, then after cut by the first part (transcendental part), the PDEs can be reduced to equations in finite dimensional spaces.

We are mainly interested in the case where $\psi = 0$, $\Gamma_{A,0} = S^1$. The local model of the cobordism near the reducible solutions looks like $\phi^{-1}(0)/S^1$ where

$$\phi : R \oplus H^1(M;\mathbf{R}) \oplus \operatorname{Ker} D_A \longrightarrow \operatorname{Coker} P_+ d \oplus \operatorname{Coker} D_A$$

is S^1-equivariant and R is the parameter direction (note that this map is not exactly the map ϕ in Proposition 2.1, nevertheless, we still denote it by ϕ). As discussed in KM's paper, we can merely perturb the cobordism (the deformation) such that the R factor maps to $\operatorname{Coker} P_+ d \cong R$ with nonzero differential. Therefore, for our purpose we can assume ϕ is a map from $H^1(M;\mathbf{R}) \oplus \operatorname{Ker} D_A$ to $\operatorname{Coker} D_A$. Reducible solutions are parametrized by (not canonically) the torus $H^1(M;\mathbf{R})/H^1(X;\mathbf{Z})$ and have S^1 as stabilizer, while irreducible solutions have trivial stabilizer.

Now, we prove Proposition 1.1.

Proposition 1.1. *Let M be an oriented four-manifold with $b_2^+ = 1$ and $b_1 = 0$, then the SW invariant of a nonnegative dimensional moduli space jumps by ± 1 when it crosses the wall,*

$$SW(\mathcal{L}, r_1) - SW(\mathcal{L}, r_0) = \pm 1.$$

Proof. Since the zero-dimensional case was exclusively dealt with in [KM], we assume that the dimension of the moduli space d is positive. Let us denote the SW moduli space of class E with deformation parameter r by $\mathcal{M}(E, r)$, and the based moduli space by $\mathcal{M}^0(E, r)$. If there is no wall, then $\mathcal{B} = \coprod_{r_0 \leq r \leq r_1} \mathcal{M}(E, r)$ forms a smooth cobordism. Let e be the Euler class of the S^1-bundle; the invariant is defined to be

$$SW(E, r_i) = \int_{\mathcal{M}(E, r_i)} e^k,$$

where $2k = \dim \mathcal{M}(E, r)$. As observed by several people, Stokes theorem shows the invariance of SW.

On the other hand, if the singular point (unique) actually shows up, then $\mathcal{B}$ is no longer a smooth manifold. Let $\mathcal{B}^0 = \coprod_{r_0 \leq r \leq r_1} \mathcal{M}^0(E, r)$, then $\mathcal{B}(E, r)$ can be described as the S^1 quotient of the based cobordism $\mathcal{B}^0(E, r)$. The neighborhood of the unique reducible solution in the based cobordism locally looks like $\mathbf{C}^{k+1}$ and therefore after taking S^1 quotient the punctured neighborhood, we obtain CP^k. This implies

$$SW(E, r_1) - SW(E, r_0) = \pm 1.$$

The proof of Proposition 1.1 is complete. $\square$

Remark. The above calculation can be interpreted homologically via the map $\mathcal{M} \longrightarrow \mathcal{A} \times (\Gamma(E \oplus K^{-1} \otimes E) - 0)/S^1$.

In general, if $b_1 \neq 0$, the reducible solutions are parametrized by the torus $H^1(M; \mathbf{R})/H^1(M; \mathbf{Z})$, where $H^1(M; \mathbf{Z})$ is naturally isomorphic to the group of components of the $U(1)$ gauge transformations. In this case, we must glue the local pictures from points to points, and finally the cobordism near the reducible solutions can be described by the following data.

$$\begin{array}{ccc} \operatorname{Ker} D_A & \xrightarrow{\phi} & \operatorname{Coker} D_A \\ \searrow & & \swarrow \\ & H^1(M; \mathbf{R})/H^1(M; \mathbf{Z}) & \end{array} \qquad *$$

Then $\phi^{-1}(0)$ is the neighborhood of T^{b_1} (the reducible solutions) in the cobordism.

Notice that the dimensions of $\operatorname{Ker} D_A$ and $\operatorname{Coker} D_A$ can jump. Later we will use a somewhat standard method to deal with this phenomenon. Unlike the corresponding case of $b_1 = 0$, the torus is not always imbedded in the cobordism. It often happens that only some subvariety (possibly singular) in T^{b_1} actually contributes to wall crossing. Especially when b_1 is large, the geometry can be extremely complicated. It is very hard to understand how the torus "interacts" with the cobordism and calculate the contribution directly.

We will not argue the wall crossing formula directly case by case. Instead, we are going to propose a wall crossing formula which holds in special cases, and later we will show it is true in general.

The following is our plan:

1. Assume that the "index bundle" does not jump. Under this assumption, we propose a wall crossing formula which would be true if the dimension of the cobordism is big enough and the torus is imbedded into the cobordism.
2. Show that the formula still holds without the assumption that the torus imbeds into the cobordism.
3. Remove the assumption that the "index bundle" is of constant rank.

In the following, we will denote $\operatorname{Ker} D_A$ by V_+ and $\operatorname{Coker} D_A$ by V_-. The simplist case is that $V_- = 0$. In this case, (*) tells us that the torus itself is imbedded in the cobordism as the zero section. Removing the zero section and dividing by the S^1-action, we find that the new end $\mathcal{E}$ of the cobordism is a projective bundle $P(V_+)$ over the torus and the Euler class of the S^1-bundle comes from the Euler class of the tautological line bundle of the projective bundle.

$$
\begin{array}{ccccccc}
S & \hookrightarrow & \pi_+^*(V_+) & \xrightarrow{\phi} & \pi_+^*(V_-) \\
 & \swarrow & \downarrow & \swarrow & \downarrow \\
V_+ & & P(V_+) & & V_- \\
 & \overset{\pi_+}{\searrow} & \Big\downarrow \pi_+ & \overset{\pi_-}{\swarrow} & \\
 & & T^{b_1} & &
\end{array}
$$

If $V_- \equiv 0$, then the torus (reducible solutions) imbeds into the whole cobordism as a singular set. As can be seen easily, if in this case T^{b_1} imbeds in the cobordism with trivial normal bundle, then $\int_{\mathcal{E}} e^{\dim \mathcal{E}/2} = 0$ and the invariant does not jump. Then the most important question is to ask when the normal bundle is trivial. The normal bundle is V_+ in this case since $V_- \equiv 0$. We can calculate the Chern classes (see Lemma 2.5); let us see what this tells us. The new end of the cobordism looks like a complex projective space bundle over T^{b_1}, and the invariant jump is calculated by $\pm \int_{P(V_+)} e^{b_1/2+\dim_C V_+ -1}$. Since in this case the SW moduli spaces must be of positive dimensions and we only care about the change up to sign, it is the same as $\int_{P(V_+)} H^{b_1/2+\dim_C V_+ -1}$ where H is the hyperplane line bundle on the projective bundle $P(V_+)$. It is fairly easy to calculate it using the universal relation ([BT]).

Now, let us take the bundle V_- into the picture. The easy case is that ϕ maps surjectively onto V_-. Then $\operatorname{Ker}\phi$ is an honest bundle which is equivalent to $V_+ - V_-$ and we reduce to the previous calculation. In general, V_- is viewed as the obstruction bundle. The strategy is to cut the fattened moduli space by the zero set of the obstruction bundle, as was done in gauge theory, symplectic geometry or Mirror symmetry. More precisely, cut $P(V_+)$ by a generic section of $\pi_+^* V_- \otimes_C H$. If the zero set is zero-dimensional, we count the signed points. If it is of positive dimension, we calculate $\int_{\mathcal{E}} e^{\dim \mathcal{E}/2}$, because $\mathcal{E}$ imbeds into $P(V_+)$ and the Euler class e of the S^1 bundle comes from the pull back of $-c_1(H)$. Moreover, the zero section is Poincare dual to the Euler class of $\pi_+^* V_- \otimes H$ in $P(V_+)$. So we propose the formula of invariant jump $\pm \int_{P(V_+)} H^{b_1/2+p-1-q} c_q(V_- \otimes H)$. We absorb all the signs into a single sign in front of the integral.

As the reader may notice, there is no reason to assume that the new corner in the cobordism is as simple as we just described. In fact, if d is zero, the cobordism locally looks like several line segments ending at the big torus, therefore the new corner will not cover the whole torus (by the projection π). In general, the description of the local geometry around the touching points is quite hard and delicate. However, in the following "localization lemma", we are going to show that under the assumption that V_+ and V_- are of constant rank we can compute the final result without knowing in detail what they really look like.

Lemma 2.2. *(Localization lemma) If V_+ and V_- have constant ranks p and q,*

$$\int_{P(V_+)} H^{b_1/2+p-1-q} c_q(V_- \otimes H) = \int_{\mathcal{E}} H^{b_1/2+p-q}.$$

Proof. Let S be the tautological line bundle in $\pi_+^* V_+$ over $P(V_+)$. ϕ shall still represent the pull back bundle map. Let $\mathcal{E}$ denote the new corner in the cobordism. $\mathcal{E}$ is smooth though the image of $\mathcal{E}$ in T^{b_1} is singular in general. $\mathcal{E}$ imbeds into $P(V_+)$. If $\dim \mathcal{E}$ is zero, we count the signed points, if $\dim \mathcal{E}$ is positive, we calculate $\int_{\mathcal{E}} e^{\dim \mathcal{E}/2}$. Since counting points can be thought as formal integration over oriented zero-dimensional manifolds, the following calculation is true regardless of the dimension.

Recall $\mathcal{E} = \phi^{-1}(0)/S^1$. The key observation is that the zero set of a section of the bundle map by ϕ can be naturally identified as zeros of a section of $S^* \otimes \pi_+^* V_-$, which is Poincare dual to the Euler class of $S^* \otimes \pi_+^* V_-$ which is $c_q(S^* \otimes \pi_+^* V_-)$. But S^* is just the hyperplane bundle H, so

$$\int_{\mathcal{E}} e^{\dim \mathcal{E}/2} = \int_{\mathcal{E}} (-H)^{\dim \mathcal{E}/2} = \int_{P(V_+)} (-H)^{\dim \mathcal{E}/2} c_q(H \otimes \pi_+^* V_-)$$

Formally, this picture is consistent with KM's calculation. In their case, $b_1 = 0$, $p - q = 1$ and the dimension of the moduli space is zero. Our formula gives $\pm \int_{P(V_+)} c_q(V_- \otimes H)$. Notice that in this case, the torus is a single point and $P(V_+)$ is simply a projective space, and V_+ is a vector space. Then $\pm \int_{P(V_+)} c_q(V_- \otimes H) = \pm \int_{CP^q} H^q = \pm 1$. $\square$

Next we prove the stabilization lemma, which tells us that we can assume that V_- is trivial.

Lemma 2.3. *(Stablization lemma) If V_+ and V_- have constant rank p and q, and $\tilde{V}_-$ is a $\tilde{q}$ dimensional complex vector bundle on T^{b_1} such that $V_- \oplus \tilde{V}_-$ is trivial, then*

$$\int_{P(V_+)} H^{\frac{b_1}{2}+p-1-q} c_q(V_- \otimes H) = \int_{P(V_+ \oplus \tilde{V}_-)} H^{\frac{b_1}{2}+p-1-q} c_{q+\tilde{q}}(\mathbf{C}^{q+\tilde{q}} \otimes H).$$

Proof. $P(V_+)$ is Poincare dual to $c_{\tilde{q}}(\tilde{V}_- \otimes H)$ in $P(V_+ \oplus \tilde{V}_-)$. Also note that c_q and $c_{\tilde{q}}$ are top Chern classes of V_- and $\tilde{V}_-$ respectively, so

$$c_{q+\tilde{q}}((V_- \oplus \tilde{V}_-) \otimes H) = c_q(V_- \otimes H) c_{\tilde{q}}(\tilde{V}_- \otimes H)$$

and the proof follows. $\square$

The tautological line bundle on $T^{b_1} \times M$ has first Chern class $\Omega = \sum_i x_i y_i$ where $\{y_i\}$ is a basis of $H^1(M, \mathbf{Z})$ modulo torsion and $\{x_i\}$ is the dual basis in $H^1(T^{b_1}, \mathbf{Z})$. T^{b_1} is the Albanese torus $H^1(M; \mathbf{R})/H^1(M; \mathbf{Z})$. Taking the powers of Ω produces $\prod y_{i_1} \cdots y_{i_k}$. The following simple lemma shows that for our purpose, the only even power of y_i which contributes to Chern character is 2.

Lemma 2.4. *For a four-manifold M with $b_2^+ = 1$, $y_{i_1} y_{i_2} y_{i_3} y_{i_4} = 0$.*

Proof. We can assume i_1, i_2, i_3, i_4 are distinct. Without loss of generality, let us assume the four y's that violate the equality are y_1, y_2, y_3, y_4. Then $y_i y_j \neq 0$ for all six combinations and furthermore they are linearly independent. If not, then $\sum c_{ij} y_i y_j = 0$ and some $c_{ij} \neq 0$. By permutation, we can assume $c_{12} \neq 0$. Wedging with $y_3 y_4$, it is easy to see $c_{12} = 0$. Contradiction! So $y_i y_j$, $1 \leq i, j \leq 4$, forms a six-dimensional subspace in $H^1(M; \mathbf{R})$ and the quadratic form on this subspace has signature 0. Since $b_2^+ = 1$, we reach a contradiction. $\square$

Still assume that $V_+ = \operatorname{Ker} D$, $V_- = \operatorname{Coker} D$ form two vector bundles over T^{b_1}. Let $c_i = c_i(V_+ - V_-)$, then we have

Lemma 2.5. *The Chern classes are given by* $c_i = \frac{1}{i!} c_1^i$.

Proof. By the family index theorem,

$$\begin{aligned}
\operatorname{ch}(\operatorname{ind}_a D) &= \operatorname{ch}(\mathcal{U})[\hat{A}(M) e^{\mathcal{L}/2}][M] \\
&= \operatorname{ch}(\operatorname{ind}_a D)_0 + (\Omega + \frac{1}{2}\Omega^2 + \sum_{j \geq 3} \frac{1}{j!}\Omega^j)(1 + \mathcal{L}/2 + \frac{1}{2}(\mathcal{L}/2)^2)[M] \\
&= \operatorname{ch}(\operatorname{ind}_a D)_0 + \frac{\Omega^2}{2}\mathcal{L}/2[M].
\end{aligned}$$

So

$$c_1(\operatorname{ind}_a D) = \frac{\Omega^2}{2}\frac{\mathcal{L}}{2}[M],$$

and for $i \geq 2$,

$$\operatorname{ch}(\operatorname{ind}_a D)_{[i]} = 0.$$

Written in terms of formal Chern roots z_j, we get, for $i \geq 2$,

$$\sum_j \frac{z_j^i}{i!} = 0,$$

i.e., the Newton polynomial of Chern roots s_i vanishes for $i \geq 2$. Using the relation $c_i = \dfrac{s_1^i}{i!} + f(s_2, s_3, \cdots)$, the lemma is proved. $\square$

The following reduction lemma tells us that under the assumption that V_- is trivial, our formula gives the correct answer.

Lemma 2.6. *(Reduction Lemma) If V_- is a trivial bundle, then*

$$\int_{P(V_+)} H^{b_1/2+p-1-q} c_q(V_- \otimes H) = \pm \int_{T^{b_1}} c_{\frac{b_1}{2}}(V_+ - V_-).$$

Proof. Since V_- is trivial, $c_q(V_- \otimes H) = H^q$. Hence the left-hand side is $\int_{P(V_+)} H^{b_1/2+p-1}$. On $P(V_+)$, we have the universal relation

$$\sum_{j\geq 0}^{b_1/2} H^{p-j} c_j(V_+) = 0.$$

We would like to inductively prove the following formula:

$$H^{p-1+l} = \sum_{i=l}^{b_1/2} \Big(\sum_{0\leq j\leq l-1} (-1)^j \binom{i}{j} \Big) \frac{c_1^i}{i!} H^{p-1-i+l}.$$

If $l = 1$, it is just the relation. Assume that it is proved for l, then, for $l+1$,

$$\begin{aligned} H^{p-1+l+1} &= H^{p-1+l} H \\ &= \sum_{i=l}^{b_1/2} \Big(\sum_{0\leq j\leq l-1} (-1)^j \binom{i}{j} \Big) \frac{c_1^i}{i!} H^{p-1-i+l} \\ &= \sum_{0\leq j\leq l-1} (-1)^j \binom{l}{j} \frac{c_1^l}{l!} H^p + \sum_{i=l+1}^{b_1/2} \Big(\sum_{0\leq j\leq l-1} (-1)^j \binom{i}{j} \Big) \frac{c_1^i}{i!} H^{p-1-i+l+1} \\ &= (-1)^{l-1}(-1) \Big(\sum_{r=1}^{b_1/2-l} \frac{(l+r)!}{l!r!} \Big) \frac{c_1^{l+r}}{(l+r)!} H^{p-r} \\ &\quad + \sum_{i=l+1}^{b_1/2} \Big(\sum_{0\leq j\leq l-1} (-1)^j \binom{i}{j} \Big) \frac{c_1^i}{i!} H^{p-1-i+l+1}. \end{aligned}$$

Set $r + l = i$ in the first term; the two terms in the last equality can be combined into

$$\sum_{i=l+1}^{b_1/2} \Big(\sum_{0\leq j\leq l} (-1)^j \binom{i}{j} \Big) \frac{c_1^i}{i!} H^{p-1-i+l+1}.$$

This is exactly what we want. Note that we have used

$$(-1)(-1)^l = \sum_{j=0}^{l-1} \binom{l}{j} (-1)^j,$$

which follows from $(1-1)^l = 0$.

When $l = b_1/2$,

$$H^{b_1/2+p-1} = \Big(\sum_{0 \le j \le b_1/2-1} (-1)^j \binom{b_1/2}{j} \Big) \frac{c_1^{b_1/2}}{b_1/2!} H^{p-1}.$$

Using $(1-1)^{b_1/2} = 0$, we get

$$H^{p+b_1/2-1} = (-1)^{b_1/2+1} \frac{c_1^{b_1/2}}{(b_1/2)!} H^{p-1}.$$

Now interpret H^{p-1} as cutting by $p-1$ hyperplanes along the fiber. We get

$$\int_{P(V_+)} H^{p+q-1} = \int_{T^{b_1}} (-1)^{b_1/2+1} \frac{c_1^{b_1/2}}{(b_1/2)!},$$

and the lemma is proved. □

We have assumed that the index virtual bundles are the difference of actual bundles V_+ and V_-. In fact, this kind of assumption is not always realistic. We are going to show that even though the kernel and cokernel do not form vector bundles (the dimensions jump), the general case can be reduced to the special case where kernel and cokernel do form vector bundles.

In the argument of local models, one actually splits the first SW equation into two parts; $0 = D_A\psi = P(D_A\psi) + (1-P)D_A\psi$, where P is the projection onto $\operatorname{Coker} D_A$. Now forget $PD_A\psi = 0$ first, cut the configuration space by two equations $P_+F_A = \frac{1}{4}\tau(\psi \otimes \psi^*) + r\mu$ and $(1-P)D_A\psi = 0$. The locus is a nonlinear object in the configuration space which we call the premoduli space. At a reducible solution a_0, $\operatorname{Ker} D_{a_0}$ is tangent to the premoduli space (up to first jet).

It is well known that ([AS], [BGV]) for a compact family of Dirac operators on compact manifolds, there exists some zeroth order perturbation such that the index bundle of the new operator has constant rank. But it is not enough for our purpose, though we still need a similar construction. First we add a trivial bundle to $\Gamma(M, S_+)$ to kill the cokernel.

Lemma 2.7 ([BGV]). *If the base manifold is compact, there exists an integer N and a map $\eta : \mathbf{C}^N \longrightarrow \Gamma(M, S_-)$ such that $D^\eta = D \oplus \eta : \Gamma(M, S_+) \oplus \mathbf{C}^N \longrightarrow \Gamma(M, S_-)$ is surjective at every point.*

The original SW equations can be changed to the following equations of triples $(A, \psi, w) \in \mathcal{A} \times \Gamma(M, S^+) \times \mathbf{C}^N$ (N as in Lemma 2.7) by adding dummy variables,

$$D_A\psi = 0$$

$$P_+F_A = \frac{1}{4}\tau(\psi \otimes \psi^*) + r\mu$$

$$x_i = 0, \qquad i = 1, \cdots, N$$

where x_i are the coordinate functions of $w \in \mathbf{C}^N$.

Denote T_η the projection onto $\operatorname{Ker} D^\eta_{a_0}$. $\operatorname{Ker} D^\eta_{a_0}$ is not tangent to the premoduli space. However, the crucial observation is that, by rescaling η, $\operatorname{Ker} D^\eta_{a_0}$ is very close to $\operatorname{Ker} D_{a_0}$ such that under the projection T_η, $\operatorname{Ker} D_{a_0}$ maps injectively into $\operatorname{Ker} D^\eta_{a_0}$. So by the inverse function theorem, (and notice that the torus of reducible solutions is compact) the fattened premoduli space can be viewed sitting inside $\operatorname{Ker} D^\eta_{a_0}$. To get the real moduli space, we need to cut the premoduli space by x_i. This is easily done by viewing the x_i as maps to the trivial bundle $\mathbf{C}^N$. Hence, the moduli space (or the relative cobordism) is the zero locus of a S^1 equivariant map ϕ from $\operatorname{Ker} D^\eta$ to $\mathbf{C}^N$. The proof of Theorem 1.2 is complete. $\square$

Proof of Corollary 1.3. Now

$$\begin{aligned} c_1(V_+ - V_-) &= \frac{1}{2}\sum_{i \neq j} x_i x_j (y_i y_j \mathcal{L}/2)[M] \\ &= \sum_{i<j} (y_i y_j \mathcal{L}/2)[M] x_i x_j \in H^2(T^{b_1}; \mathbf{R}) \end{aligned}.$$

Therefore, $c_1^{b_1/2}/(b_1/2)! = c_{b_1}$ is the Pfaffian of the $b_1 \times b_1$ matrix G with entries given by $(y_i y_j \frac{\mathcal{L}}{2})[M]$. Notice that $G \in M_{b_1}(\mathbf{Z})$, the ring of b_1 by b_1 integer matrices. By simultaneous row and column operations, we can change to a new basis over $\mathbf{Z}$ such that $\tilde{G}$ looks like

$$\begin{pmatrix} 0 & d_1 & \cdots & \\ -d_1 & 0 & d_2 & \cdots \\ 0 & -d_2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \end{pmatrix}.$$

Namely, there exists $\Gamma \in M_{b_1}(\mathbf{Z})$ invertible over $\mathbf{Z}$ such that $\Gamma G \Gamma^T = \tilde{G}$.

This just implies that if we change the basis of the free part of $H^1(M;\mathbf{Z})$ from $\{y_i\}$ to $\{\Gamma y_i\}$, the above matrix is skew-diagonal. We call $\{\Gamma y_i\}$ the symplectic basis (in general, the symplectic basis is not canonical up to $SP(b_1)$, it depends on $\mathcal{L}$). Clearly, $c_{\frac{b_1}{2}} = \prod d_i$. □

Proof of Corollary 1.4. For a ruled surface over a Riemann surface of genus $g \geq 1$, the entire H^1 comes from the base. In this case, the symplectic basis is canonical up to $SP(g)$. For the symplectic basis, the Pfaffian can be further reduced to $(\frac{c_1(M)+2E}{2}[S^2])^g$. □

Proof of Corollary 1.5. By Lemma 2.4 in [LL], the SW invariant of K^{-1} jumps by ± 1, so Corollary 1.5 follows from Corollary 1.3. □

Proof of Corollary 1.6. Under a change of a R-basis, the matrix has a trivial row, so the Pfaffian is zero. □

Notice that, by the hard Lefshetz theorem, the above assumption is never satisfied for Kahler surfaces. The corollary also implies that if a symplectic manifold has psc metric, then it is similar to Kahler manifolds in the above sense.

Remark. Up to now, we do not know any example of a non-Kahler symplectic manifold of the above type. For Kahler manifolds, ruled surfaces have psc metrics and the theorem constrains the canonical bundles of symplectic forms. For some elliptic surface with elliptic base, $b_2^+ = 1$ and $b_1 = 2$; the b_1 comes from the base. Because for elliptic surfaces, K^{-1} consists of a fiber class, which (after multiplying a large integer) comes from the base also, the wall crossing number is even, so it does not have psc metrics. Blowups of this manifold give examples of even type K^{-1}.

Remark. Based on some techniques of algebraic geometry, especially a theorem of Fulton on determinantal loci, we can describe (assuming the geometric objects are holomorphic) in some detail the local geometry. Due to the complexity of the singularities when $b_1 >> 0$, this description is far from complete. But we do see some patterns, and more importantly, for $b_1 \leq 6$, the description is more or less complete and leads to the direct calculation of the invariant jump.

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