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ON THE COHOMOLOGY OF PARABOLIC LINE BUNDLES

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1. Introduction

Let X be a smooth projective variety over $\mathbb{C}$ of dimension n. Let $D = \sum_{i=1}^{d} D_i$ be a divisor of normal crossing with decomposition into irreducible components. Fix rational numbers $\{\alpha_1, \ldots, \alpha_d\}$ with $0 < \alpha_i < 1$. Assume that the Poincaré dual of the $\mathbb{Q}$-divisor $\sum_{i=1}^{d} \alpha_i D_i$ is in the image of $H^2(X, \mathbb{Z})$ in $H^2(X, \mathbb{Q})$. Such a data constitutes a parabolic bundle in the sense of [MY]. Let $P(X)$ be a component of the moduli space of parabolic bundles of parabolic degree zero (which simply is a component of the Picard group of X consisting of line bundles with first Chern class $-\sum_{i=1}^{d} \alpha_i [D_i]$, where $[D_i]$ is the Poincaré dual of D_i).

Let $\mathrm{Pic}^0(X)$ be the abelian variety consisting of isomorphism classes of topologically trivial line bundles. The group $\mathrm{Pic}^0(X)$ acts on $P(X)$ using tensor product, and $P(X)$ is an affine group for $\mathrm{Pic}^0(X)$.

Define the subvariety

$$T^i_m := \{L \in P(X) \mid \dim H^i(X, L) \geq m\} \subset P(X).$$

We prove the following theorem.

Theorem A. *Any irreducible component of T^i_m is a translation of an abelian subvariety of $\mathrm{Pic}^0(X)$ by a point of $P(X)$ for the above action.*

The special case of the above theorem where D is empty was proved in [GL2].

Let Y be smooth variety on which a finite group G acts, such that the quotient, Y/G, is a smooth variety. We first observe that if we consider G-invariant part of the cohomology, the result in [GL2] easily extends to the case of the moduli space of the group of topologically trivial line bundles on Y equipped with a lift of the action of G.

We now describe the main theme of this work. Using the "covering lemma" of Y. Kawamata, the moduli space $P(X)$ can be identified with the moduli space G-equivariant line bundles of the above type for some suitable Y and G.

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Using the above identification we deduce Theorem A from the corresponding result on G-equivariant bundles.

2. Equivariant line bundles and parabolic line bundles

2a. Group action on a line bundle. Let Y be a connected smooth projective variety over $\mathbb{C}$ of dimension n. The group of automorphisms of Y is denoted by $\mathrm{Aut}(Y)$. Let G be a finite group acting faithfully on Y. In other words,

$$\rho\colon G \longrightarrow \mathrm{Aut}(Y)$$

is a monomorphism of a finite group G into $\mathrm{Aut}(Y)$.

Definition 2.1. An *orbifold line bundle* on Y is a line bundle L on Y together with a lift of action of G, which means that G acts on the total space of L, and for any $g \in G$, the action of g on L is an isomorphism between L and $\rho(g^{-1})^*L$.

Remark. A line bundle L with the property that for any $g \in G$, the bundle L is isomorphic to $\rho(g^{-1})^*L$, need not have an orbifold structure. For an orbifold bundle L', clearly there is a natural lift of action of G on $H^0(Y, L')$. But an abelian variety together with a power of a principal polarization constitutes an example where a finite group of symmetry (the Heisenberg group) of the line bundle does not lift to the space of sections [M].

Let $\mathrm{Pic}^0(Y)$ be the abelian variety parametrizing holomorphic isomorphism classes of topologically trivial line bundles on Y. The group G acts on $\mathrm{Pic}^0(Y)$ by $g \circ L = \rho(g^{-1})^*L$. Let $\mathrm{Pic}_G(Y) \subset \mathrm{Pic}^0(Y)$ be the set of fixed points of this action of G. Note that $\mathrm{Pic}_G(Y)$ is a complex submanifold of $\mathrm{Pic}^0(Y)$; moreover, it is a closed subgroup of the abelian variety $\mathrm{Pic}^0(Y)$.

Take any $L \in \mathrm{Pic}_G(Y)$. The above remark indicates that L need not have an orbifold structure. We want to identify the obstruction to having an orbifold structure. For $g \in G$ fix an isomorphism

$$\psi_g\colon L \longrightarrow \rho(g^{-1})^*L.$$

So for $g, h \in G$, $\psi_g \circ \psi_h \circ \psi_{(gh)^{-1}}$ is an automorphism of L, and hence is a nonzero scalar. The map $G \times G \to \mathbb{C}^*$ defined by

$$(g, h) \longmapsto \psi_g \circ \psi_h \circ \psi_{(gh)^{-1}}$$

gives a 2-cocycle, which we denote by ψ_L. It is easy to check that the cohomology class $\bar{\psi}_L$ represented by ψ_L, does not depend upon the choices of ψ_g, $g \in G$. The line bundle L has an orbifold structure if and only if $\bar{\psi}_L = 0$.

Let $\mathrm{Pic}_G(Y)' \subset \mathrm{Pic}^0(Y)$ be the subset consisting of all those line bundles which admit an orbifold structure. So we have $\mathrm{Pic}_G(Y)' \subset \mathrm{Pic}_G(Y)$.

For two orbifold bundles $(L, \bar{\rho})$ and $(L', \bar{\rho}')$, there is an obvious orbifold structure on $L \otimes L'$. So $\mathrm{Pic}_G(Y)'$ is a subgroup of $\mathrm{Pic}_G(Y)$. The following sequence of abelian groups is exact

$$0 \longrightarrow \mathrm{Pic}_G(Y)' \longrightarrow \mathrm{Pic}_G(Y) \longrightarrow H^2(G, \mathbb{C}^*)$$

where the last homomorphism is given by $L \longmapsto \bar{\psi}_L$ described earlier. So $\mathrm{Pic}_G(Y)'$ is a both open and closed subset of $\mathrm{Pic}_G(Y)$, *i.e.*, a union of some components of $\mathrm{Pic}_G(Y)$.

Now we want to determine how many distinct orbifold structures a given line bundle admits. Let $(L, \bar{\rho})$ be an orbifold line bundle and λ a character of G. Then we can construct a new orbifold structure, (L, l), on the line bundle L using the following action of G : for any $g \in G$ and $v \in L$

$$l(g)(v) = \lambda(g).\bar{\rho}(g)(v).$$

Clearly for two different characters λ and λ' the corresponding orbifold structures l and l' on L are different. It can be checked that any orbifold structure on L is gotten this way.

The set of all isomorphism classes of orbifold bundles of the form $(L, \bar{\rho})$, where $L \in \mathrm{Pic}^0(Y)$, is denoted by $P'_G(Y)$. Let $\hat{G}$ be the group of characters of G. We put down the summary of the previous discussions in the form of the following:

Lemma 2.2. *The finite group of characters $\hat{G}$ acts freely on the group $P'_G(Y)$, with the quotient being $\mathrm{Pic}_G(Y)'$. The Lie group $\mathrm{Pic}_G(Y)'$ is a finite index subgroup of the abelian group $\mathrm{Pic}_G(Y)$.*

Now we want to determine the tangent space of $P'_G(Y)$. From the above lemma it follows that for $(L, \bar{\rho}) \in P'_G(Y)$, the tangent space $T_{(L,\bar{\rho})} P'_G(Y)$ is canonically isomorphic to $T_L \,\mathrm{Pic}_G(Y)' = T_L \,\mathrm{Pic}_G(Y)$.

There is an obvious lift of the action of G to the trivial bundle $Y \times \mathbb{C}$. Let $H^1(Y, \mathcal{O})^G$ be the space of invariants of $H^1(Y, \mathcal{O})$, *i.e.*, the subspace of $H^1(Y, \mathcal{O})$ on which G acts trivially.

Lemma 2.3. *The Lie algebra of the complex abelian Lie group $P'_G(Y)$ is canonically isomorphic to $H^1(Y, \mathcal{O})^G$.*

Proof. The action of the group G on Y induces a homomorphism

$$\hat{\rho} \colon G \longrightarrow \mathrm{Aut}(\mathrm{Pic}^0(Y));$$

$\mathrm{Aut}(\mathrm{Pic}^0(Y))$ is the group of all automorphisms of the group $\mathrm{Pic}^0(Y)$. Now $\mathrm{Pic}_G(Y) \subset \mathrm{Pic}^0(Y)$ is the subgroup which is pointwise invariant under $\hat{\rho}(G)$. The Lie algebra of $\mathrm{Pic}^0(Y)$ is $H^1(Y, \mathcal{O})$. So the Lie algebra of $\mathrm{Pic}_G(Y)$ is $H^1(Y, \mathcal{O})^G$. But the Lie algebras of $\mathrm{Pic}_G(Y)$ and $P'_G(Y)$ are isomorphic (follows from Lemma 2.2). This completes the proof. □

The following restriction is imposed on the group action:

Assumption. The quotient $X := Y/G$ is a smooth variety. The quotient map $Y \longrightarrow X$, which is a morphism between smooth varieties, is denoted by π.

Let $(L, \bar{\rho})$ be an orbifold bundle on Y. Consider the direct image sheaf $\pi_* L$. Since π is finite and flat, $\pi_* L$ is a locally free $\mathcal{O}_X$-coherent sheaf. The action $\bar{\rho}$ on L induces a homomorphism of G into $\mathrm{Aut}(\pi_* L)$, the automorphism group of the bundle $\pi_* L$. This homomorphism is denoted by ρ'. Let $L^G \subset \pi_* L$ be the space of invariants, *i.e.*, the subsheaf on which G acts trivially. Clearly L^G is a $\mathcal{O}_X$ submodule of $\pi_* L$. The homomorphism

$$v \longmapsto \frac{1}{\# G} \sum_{g \in G} \rho'(g)(v) \in \pi_* L$$

defines a projection $\phi \colon \pi_* L \longrightarrow L^G$. In particular, the following exact sequence of $\mathcal{O}_X$-coherent sheaves on X

$$0 \longrightarrow L^G \longrightarrow \pi_* L \longrightarrow \pi_* L / L^G \longrightarrow 0$$

splits, and L^G is a line subbundle of $\pi_* L$.

The higher direct images of π vanish and $H^i(Y, L)$ is canonically isomorphic to $H^i(X, \pi_* L)$. Also any ith cocycle of $\pi_* L$ is a sum of cocycles of L^G and $\ker(\phi)$. Let $H^i(Y, L)^G \subset H^i(Y, L)$ be the space of invariants. We have proved the following:

Lemma 2.4. *The inclusion of sheaves $L^G \longrightarrow \pi_* L$ induces an isomorphism between $H^i(X, L^G)$ and $H^i(Y, L)^G$.*

2b. Parabolic line bundles. Let X be a connected smooth projective variety over $\mathbb{C}$ of dimension n. Let D be a divisor of normal crossing on X. By this we mean that D is a reduced effective divisor and each irreducible component of D is smooth and they intersect transversally. Let $D = \sum_{i=1}^{d} D_i$ be the decomposition into irreducible components. Following [MY] we define

Definition 2.5. A *parabolic line bundle* on on (X, D) is a pair of the form

$$(L, \{\alpha_1, \ldots, \alpha_i, \ldots, \alpha_d\})$$

where L is a holomorphic line bundle on X and any $0 \leq \alpha_i < 1$ is a real numbers.

Assumptions. The weights $\{\alpha_1, \ldots, \alpha_d\}$ are fixed once and for all, and they are assumed to be nonzero rational numbers; in particular $\alpha_i = m_i/N$ for some integer N (independent of i) and $1 \leq m_i < N$. For a divisor $D \subset X$

let $[D] \in H^2(X, \mathbb{Z})$ denote the Poincaré dual of D. It is assumed that the element $\sum_{i=1}^d \alpha_i[D_i] \in H^2(X, \mathbb{Q})$ belongs to the image of $H^2(X, \mathbb{Z})$.

Notation. Let $P(X)$ denote a component of the moduli space of holomorphic isomorphism classes of line bundles on X with first Chern class $\sum_{i=1}^d -\alpha_i[D_i]$. (From the assumption and Lefschetz 1-1 theorem it follows that $P(X)$ is non-empty.)

The "Covering Lemma" (Theorem 1.1.1 of [KMM], Theorem 17 of [K]) says that there is a connected smooth projective variety Y and a finite Galois morphism

$$\pi\colon Y \longrightarrow X$$

with Galois group $G = Gal(Rat(Y)/Rat(X))$ such that $\tilde{D} := (\pi^*D)_{red}$ is a divisor of normal crossing on Y and $\pi^*D_i = k_iN(\pi^*D_i)_{red}$, $1 \leq i \leq d$, where k_i are positive integers.

Define $\tilde{D}_i := (\pi^*D_i)_{red}$; so $\pi^*D_i = k_iN\tilde{D}_i$. The divisor π^*D_i is obviously invariant under the action of the Galois group G on Y, and hence, the reduced divisor $\tilde{D}_i$ is also invariant under the action. In particular, the line bundle $\mathcal{O}(\tilde{D}_i)$ has an orbifold structure. For any $k \in \mathbb{Z}$ the bundle $\mathcal{O}(k\tilde{D}_i)$ has an induced orbifold structure.

Let $\xi \in P(X)$. The pull-back bundle $\pi^*\xi$ has an obvious orbifold structure. Define

$$L \,:=\, \pi^*(\xi) \otimes \mathcal{O}(\sum_{i=1}^d k_im_i\tilde{D}_i). \tag{2.6}$$

This line bundle L has an orbifold structure

$$c_1(L) = \pi^*c_1(\xi) + \sum_{i=1}^d k_im_i[\tilde{D}_i] = \pi^*c_1(\xi) + \sum_{i=1}^d \frac{m_i}{N}[k_iN\tilde{D}_i].$$

By definition, $[k_iN\tilde{D}_i] = \pi^*[D_i]$, so

$$c_1(L) \,=\, \pi^*c_1(\xi) + \sum_{i=1}^d \frac{m_i}{N}\pi^*[D_i] \,=\, 0.$$

Hence, $L \in \mathrm{Pic}^0(Y)$. For a general point p of $\tilde{D}_i$, the isotropy group is the cyclic group $\mathbb{Z}/(k_iN)$. The action of any $n \in \mathbb{Z}/(k_iN)$ on the fiber L_y is multiplication by $\exp(2\pi\sqrt{-1}nm_i/N)$.

Let $\hat{D} \subset Y$ be the reduced effective divisor consisting of all the points $y \in Y$ such that the isotropy group of y for the G action is nontrivial. That $\hat{D}$ is a divisor follows from the assumption that X is smooth. (The bundle map $d\pi\colon \pi^*\Omega^1_X \longrightarrow \Omega^1_Y$ fails to be an isomorphism precisely over $\hat{D}$.) So $\tilde{D}$ is contained in $\hat{D}$.

Recall the group $P'_G(Y)$ defined in Section 2a. Let $P_G(Y) \subset P'_G(Y)$ be the set of all orbifold bundles $\bar{L}$ such that for a general point p of $\tilde{D}_i$, the action of $n \in \mathbb{Z}/(k_iN)$ (the group $\mathbb{Z}/(k_iN)$ is the isotropy group of p) on the fiber $\bar{L}_p$ is multiplication by $\exp(2\pi\sqrt{-1}nm_i/N)$, and on a general point, y, of any other component of $\hat{D}$ (not in $\tilde{D}$) the action of the isotropy group of y on $\bar{L}_y$ is trivial. From rigidity of the representations of a finite group it follows that $P_G(Y)$ is both open and closed in $P'_G(Y)$.

Define the morphism $F\colon P(X) \longrightarrow P_G(Y)$ using the correspondence $\xi \mapsto L$ obtained above.

Theorem 2.7. *The morphism $F\colon P(X) \longrightarrow P_G(Y)$ is an isomorphism.*

Proof. For $L \in P_G(Y)$ let L^G be the line bundle on X gotten by taking the invariant direct image (as done in Section 2a).

Let $U := \{z \in \mathbb{C} \,|\, |z| < 1\}$ be the open disk and $U \times \mathbb{C}$ be the trivial line bundle on U. Let the group $\mathbb{Z}/(mn)$ act on U by $\alpha \circ z = \exp(2\pi\sqrt{-1}\alpha/(mn))z$, where $\alpha \in \mathbb{Z}/(mn)$ and $z \in U$, and let $\mathbb{Z}/(mn)$ act on $U \times \mathbb{C}$ by

$$\alpha \circ (z, c) = (\exp(2\pi\sqrt{-1}\alpha/(mn))z, \exp(2\pi\sqrt{-1}\alpha/m)c).$$

Then the pull-back of the line bundle on $U/(\mathbb{Z}/(mn))$ to U, given by the $\mathbb{Z}/(mn)$-invariant sections, is generated as an $\mathcal{O}(U)$-module by the section (z, z^n) of $U \times \mathbb{C}$. This observation implies that L is isomorphic to $\pi^*(L^G) \otimes \mathcal{O}(\sum_{i=1}^d k_im_i\tilde{D}_i)$. So

$$c_1(\pi^*L^G) = c_1(L) - \sum_{i=1}^d k_im_i[\tilde{D}_i] = -\sum_{i=1}^d \frac{m_i}{N}\pi^*[D_i],$$

so $L^G \in P(X)$. Thus the correspondence $L \mapsto L^G$ gives a morphism

$$F'\colon P_G(Y) \longrightarrow P(X).$$

Assume that $\xi \in P(X)$ and L are related as in (2.6). The divisor $\tilde{D}_i$, $1 \le i \le d$, is effective, so there is an inclusion of sheaves $j\colon \pi^*(\xi) \longrightarrow \pi^*(\xi) \otimes \mathcal{O}(\sum_{i=1}^d k_im_i\tilde{D}_i)$. Moreover this homomorphism j commutes with the actions of G on $\pi^*\xi$ and $(\pi^*\xi) \otimes \mathcal{O}(\sum_{i=1}^d k_im_i\tilde{D}_i)$. So j induces a homomorphism $j'\colon \pi^*\xi \longrightarrow \pi^*L^G$. This homomorphism, being G-equivariant, induces a homomorphism $\bar{j}\colon \xi \longrightarrow L^G$. However,

$$c_1(\xi) \;=\; -\sum_{i=1}^d \frac{m_i}{N}[D_i] \;=\; c_1(L_G),$$

so $\bar{j}$ must be an isomorphism. Thus we have proved that $F' \circ F = Id$. We saw earlier that $L = \pi^*(L^G) \otimes \mathcal{O}(\sum_{i=1}^{d} k_i m_i \tilde{D}_i)$. This implies that $F \circ F' = Id$, completing the proof. □

Remark 2.8. $P(X)$ is irreducible. So Theorem 2.7 implies that $P_G(Y)$ is also irreducible. Neither $P_G(Y)$ nor $P(X)$ have Lie group structure. But they have affine group structure. The variety $P(X)$ is an affine group for the group $\mathrm{Pic}^0(X)$. Let $\bar{P}_G(Y) \subset P'_G(Y)$ be the subgroup of all orbifold bundles $\bar{L}$ such that for a general point p of $\tilde{D}_i$, the action of the isotropy group of p on the fiber $\bar{L}_y$ is trivial. Clearly $P_G(Y)$ is an affine group for the group $\bar{P}_G(Y)$, where the action is given by tensor product. From Theorem 2.7 it follows that the two abelian groups $\mathrm{Pic}^0(X)$ and $\bar{P}_G(Y)$ are canonically isomorphic. Using this isomorphism, the morphism F in Theorem 2.7 is an isomorphism of affine groups.

3. Proof of Theorem A

We continue with the notation of the previous section.

Define the subvariety $S^i_m := \{(L, \bar{\rho}) \in P_G(Y) \mid \dim H^i(Y, L)^G \geq m\}$. We want to prove the following:

Theorem 3.1. *Any irreducible component of S^i_m is a translation of an abelian subvariety of $\bar{P}_G(Y)$ by a point of $P_G(Y)$ for the action defined in Remark 2.8.*

The special case of the above theorem where $G = \{e\}$ was proved in [GL2]. It is easy to see that the proof in [GL2] goes through verbatim in our situation. (The Theorem 1.6 of [GL1] is one of the key points in the proof in [GL2]. The equivariant analogue of [Theorem 1.6 GL1] is also easily seen to be true.) We refrain from reproducing the argument in [GL2].

Define

$$T^i_m := \{L \in P(X) \mid \dim H^i(X, L) \geq m\}.$$

Lemma 2.4 implies that the isomorphism F between $P(X)$ and $P_G(Y)$, obtained in Theorem 2.7, identifies the subvariety S^i_m of $P_G(Y)$ with the subvariety T^i_m of $P(X)$.

After choosing a base point $q \in P(X)$ the variety $P(X)$ is identified with the abelian variety $\mathrm{Pic}^0(Y)$. Given a subvariety $V \subset P(X)$, whether it is a translation of an abelian subvariety does not depend upon the choice of the base point q. In Remark 2.8 we saw that the two affine group structures of $P_G(Y)$ and $P(X)$ coincide. So Theorem 3.1 gives the Thereon A stated in the introduction.

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