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LIOUVILLE PROPERTIES OF HARMONIC MAPS

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Introduction

In this paper, we will prove the following Liouville-type results for harmonic maps:

Let M be a manifold which is quasi-isometric to a complete noncompact manifold with nonnegative Ricci curvature. Let N be a Cartan-Hadamard manifold (a CH manifold for simplicity). That is to say, N is a complete simply connected manifold with nonpositive curvature.

(1) *Let $o \in M$ be a fixed point. There exist constants $0 < \gamma \leq 1$ and $C > 0$ depending only on M and the dimension of N, such that for all $0 < r < R$*

$$s(r) \leq C \left(\frac{r}{R}\right)^{\gamma} \cdot s(R),$$

where $s(r) = \sup_{x \in B_o(r)} d_N\left(u(x), u(o)\right)$. In particular, if u is a harmonic map from M into N such that

$$d_N\left(u(x), u(o)\right) = o\left(\left(d_M(x,o)\right)^{\gamma}\right),$$

then u must be constant. Here d_M and d_N are the distance functions of M and N respectively, and $B_o(r)$ is the geodesic ball with center at o and radius r.

(2) *Suppose in addition that the sectional curvature of N is bounded above by $-a^2$ for some $a > 0$, then any harmonic map from M into N with image lying inside a horoball of N must be constant.*

The study of Liouville properties of harmonic maps has a long history. It is well-known that there is no nonconstant positive harmonic function on

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$\mathbf{R}^n$. It was proved by Yau in [Y], that there is no nonconstant positive harmonic function on a complete noncompact manifold with nonnegative Ricci curvature. In particular, there is no nonconstant bounded harmonic functions on such a manifold. In [C], Cheng generalized the result to bounded harmonic maps. He proved that there is no nonconstant harmonic map with bounded image from a complete noncompact manifold with nonnegative Ricci curvature into a CH manifold. Cheng's result can be extended to a very large class of manifolds. In many cases, if a property is satisfied for harmonic functions on a manifold M, then a similar property will also be satisfied for harmonic maps from M into a CH manifold. In fact, Kendall [Ke] proved that if a complete manifold supports no nonconstant bounded harmonic function, then it will also support no nonconstant harmonic map into a CH manifold with bounded image. See [S-T-W] and [C-T-W] for more results in this direction. Kendall's result implies the result of Cheng. However, the proof of [Ke] is by contradiction, and does not give any useful estimate for the maps. In contrast, sharp gradient estimates for harmonic functions on manifolds with Ricci curvature bounded from below have been obtained by Cheng and Yau [C-Y]. In [C], Cheng also obtained an estimate for the energy density of a harmonic map into a CH manifold. An immediate consequence of those estimates is that on a complete noncompact manifold with nonnegative Ricci curvature, there is no nonconstant harmonic function, or harmonic map into a CH manifold, which is of sublinear growth. Cheng's result on harmonic maps was generalized in [Ch] by allowing the image to lie strictly inside a regular geodesic ball of a complete manifold.

Liouville's theorem on harmonic functions on $\mathbf{R}^n$ has also been generalized in another direction. In [M], Moser proved that there is no nonconstant positive solution of a uniformly elliptic equation on $\mathbf{R}^n$. He also showed that a nonconstant solution to a uniformly elliptic equation must grow by at least some power of r, where r is the distance to the origin. In the last few years, there are many results on uniformly elliptic equations on a manifold. For example, it was proved in [SC1, Gr] that on a complete noncompact manifold M which is quasi-isometric to a complete manifold with nonnegative Ricci curvature, the Liouville theorem for positive harmonic functions in [Y, C-Y] is still true. More importantly, estimates and results for harmonic functions similar to those in [M] are still true on such a manifold. Note that the Laplacian equation on M can be considered as a uniformly elliptic equation on a complete manifold with nonnegative Ricci curvature. Motivated by the works [SC1, Gr], it is natural to conjecture that similar results should be true for harmonic maps. In fact it was asked by Cheng whether one can estimate the oscillation of a harmonic map on a geodesic ball in M into a CH manifold N, and whether a nonconstant harmonic map from M into N should grow fast enough. The main result (1) mentioned above gives a positive answer to Cheng's question.

So far, we have been focusing on Liouville's theorem for harmonic maps which are either bounded or grow at most at a certain rate. One would like to consider harmonic maps which are analogues to positive harmonic functions without any assumption on the growth rate. A natural setup is to consider harmonic maps into a horoball of a CH manifold. Recall that $\gamma : (-\infty, \infty) \to N$ is said to be a *line* if γ is parametrized by arc length and for all $a < b$, $\gamma|_{[a,b]}$ is a minimizing geodesic from $\gamma(a)$ to $\gamma(b)$. A *ray* is just a half line, that is, γ is defined on $[0, \infty)$ with the same properties. It is well-known that on a CH manifold, a ray can always be extended to a line. Let γ be a ray in N, then the *Busemann function* with center at $\gamma(\infty)$ is defined to be $B_\gamma(z) = \lim_{t\to\infty}(t - d_N(z, \gamma(t)))$, $z \in N$. On a CH manifold, B_γ is a C^2 function, see [H-H]. A *horoball* is a set of the form $\{B_\gamma > 0\}$, for some γ. Under this setting, Shen [Sh] proved that a harmonic map from a complete manifold M with nonnegative Ricci curvature into a CH manifold N will be constant, provided its image lies inside a horoball and that the sectional curvature of N is bounded from above by a negative constant $-a^2$. The assumption on the sectional curvature of N cannot be relaxed by allowing $a = 0$. Counterexamples have been constructed in [Sh]. The result in [Sh] can be considered as a generalization to harmonic maps of Yau's theorem for positive harmonic functions. The main result (2), mentioned above, that we are going to prove is a further generalization of the result in [Sh].

In fact, (1) and (2) above are still true for a more general class of manifolds. More precisely, they are true for manifolds which satisfy volume doubling and certain Poincaré inequalities. See section 2 for details. Such a class of manifolds have been studied extensively by [Gr, SC2].

1. Harmonic maps into CH manifolds

Let (M^m, g) and (N^n, h) be two Riemannian manifolds. Let u be a map from M into N. The energy density $e(u)$ of u is defined to be the trace with respect to g of the tensor u^*h. In local coordinates

$$e(u)(x)=\sum_{i,j=1}^{m}\sum_{\alpha,\beta=1}^{n}g^{ij}(x)h_{\alpha\beta}(u(x))\frac{\partial u^\alpha}{\partial x^i}\frac{\partial u^\beta}{\partial x^j}.$$

where x^i, $1\le i\le m$, and u^α, $1\le\alpha\le n$ are local coordinates in M and N respectively. $g=\sum_{i,j=1}^m g_{ij}dx^idx^j$, $h=\sum_{\alpha,\beta=1}^n h_{\alpha,\beta}du^\alpha du^\beta$, and (g^{ij}) is the inverse of the matrix (g_{ij}). u is said to be a harmonic map if u is a critical point of the functional $\int_M e(u)$.

Now let (M^m,g) be a complete noncompact manifold, such that g is quasi-isometric to a complete metric $\tilde g$ on M, whose Ricci curvature is bounded from below by $-K$ for some $K\ge 0$. Hence, there is $\alpha>1$ such that

$$\alpha^{-1}g\le\tilde g\le\alpha g. \tag{1.1}$$

We want to study the behavior of harmonic maps from M into a CH manifold. One ingredient in the proofs of the results in this paper is to make use of Green's functions to solve some boundary value problem. The method has been used in [Ga-H] in the study of regularity of harmonic maps, see [S1]. We will also need Poincaré inequalities in [K-S], see also [Gr], for maps from manifolds with Ricci curvature bounded from below. We begin with the following estimates on Green's functions on geodesic balls.

Lemma 1.1. *Let (M^m,g) be a complete manifold, so that g is quasi-isometric to a complete metric $\tilde g$, whose Ricci curvature is bounded from below by $-K$ for some constant $K\ge 0$. Let $\alpha>1$ be the constant in (1.1). Let $o\in M$ and let G_R be the positive Green's function on $B_o(R)$ with Dirichlet boundary data. Then for all $x\in B_o(\frac{R}{5})$*

$$G_R(o,x)\ge C\int_{r(x)}^R\frac{t}{V_o(t)}dt$$

for some positive constant $C>0$ depending only on m, $\sqrt{K}R$, and α, where $r(x)$ is the distance from x to o, and $V_o(t)$ is the volume of $B_o(t)$.

Proof. By [L-T1, Corollary 2.4], and volume comparison [B-C]:

$$\int_r^R\frac{t}{V_o(t)}dt\le C_1\sup_{\partial B_o(r)}G_R(o,\cdot),$$

for some constant $C_1 > 0$ depending only on m, $\sqrt{K}R$ and α. Note that $\partial B_o(r)$ may not be connected. However, using the method of proof of Lemma 3.2 in [L-T2], one can apply the Harnack inequality for the positive harmonic function in [SC1, Theorem 5.3] to show that for $0 < r \leq \frac{R}{5}$, and for all $x,\ y \in \partial B_o(r)$,

$$G_R(o,x) \leq C_2 G_R(o,y),$$

for some constant C_2 depending only on m, $\sqrt{K}R$ and α. The lemma follows. □

Remark 1.1. *The lemma is still true if we only assume that M has volume doubling and that the Harnack inequality holds for positive harmonic functions. To be precise, suppose that $V_x(2r) \leq AV_x(r)$ for some constant A and for all x, $r > 0$, and that there is a constant A' such that $\sup_{B_x(r)} u \leq A' \inf_{B_x(r)} u$ for all positive harmonic function u on $B_x(2r)$, for all x and r. Then Lemma 1.1 is still true, with the constant $C > 0$ depending only on A and A'. We will use this fact in the next section.*

Lemma 1.2. *Let M be a complete manifold, and let $o \in M$. Let $p \geq 1$, and suppose for some $k \geq 1$ the Poincaré inequality holds for functions on $B_o(kR)$ for p. That is to say, there is a constant A such that for all smooth functions on $B_o(kR)$,*

$$\inf_{a \in \mathbf{R}} \int_{B_o(R)} |f-a|^p \leq AR^p \int_{B_o(kR)} |\nabla f|^p.$$

Then for any smooth map u from $B_o(kR)$ into a CH manifold N^n, we have

$$\int_{B_o(R)} d_N^p\,(u(x), u_0) \leq CAR^p \int_{B_o(kR)} e^{\frac{p}{2}}(u),$$

for some point $u_0 \in N$, for some constant C depending only on n and p, where $e(u)$ is the energy density of u and d_N is the distance function of N.

Remark 1.2. *If M satisfies the Poincaré inequality for functions in the lemma, then it is easy to see that*

$$\int_{B_o(R)} |f-\overline{f}|^p \leq CAR^p \int_{B_o(kR)} |\nabla f|^p$$

for some constant C depending only on A and p, where $\overline{f}$ is the average of f over $B_o(R)$.

Proof of the lemma. Let u be a smooth map from $B_o(kR)$ into N and let f be the function on N defined by

$$f(w) = \int_{B_o(R)} d_N^2(u(x), w)dV_M(x).$$

Since $f(w) \to \infty$ as $w \to \infty$, $f(w)$ attains a minimum at some point $u_0 \in N$. Let $y = (y^1, \dots, y^n)$ be the normal coordinates with origin at u_0. At u_0, $\nabla_N f = 0$, and so

$$\int_{B_o(R)} \nabla'_N d_N^2(u(x), u_0)dV_M(x) = 0. \tag{1.2}$$

where $\nabla'_N d_N^2(u(x), u_0)$ is the gradient on N of the function $d_N^2(u(x), \cdot)$ with $u(x)$ fixed. Suppose $u(x) = u_0$, then $\nabla'_N d_N^2(u(x), u_0) = 0$. Suppose $u(x) \neq u_0$, then $\nabla'_N d(u(x), u_0)$ is the tangent vector at u_0 of the minimal geodesic parametrized by arc-length from $u(x)$ to u_0. Since $(y \circ u)(x)$ is the normal coordinates of $u(x)$ with origin at u_0, it is not hard to see from the definition of normal coordinates, that

$$\nabla'_N d(u(x), u_0) = -\frac{1}{|(y \circ u)(x)|} \sum_{i=1}^{n} (y^i \circ u)(x) \frac{\partial}{\partial y^i}\bigg|_{u_0}.$$

However, $|y \circ u(x)| = d_N(u(x), u_0)$, hence

$$\nabla'_N d_N^2(u(x), u_0) = -2 \sum_{i=1}^{n} (y^i \circ u)(x) \frac{\partial}{\partial y^i}\bigg|_{u_0}.$$

By (1.2), we have

$$\int_{B_o(R)} (y^i \circ u)(x)dV_M(x) = 0, \tag{1.3}$$

for $1 \leq i \leq n$. Since N has nonpositive curvature, by the triangle comparison theorem, for any point w_1, w_2 on N, we have $|y(w_1) - y(w_2)| \leq d_N(w_1, w_2)$, where $y(w_j)$ are the normal coordinates of w_j. From this, it is not hard to see that $|\nabla(y \circ u)| \leq \sqrt{e(u)}$, where $\nabla(y \circ u)$ is the gradient of the vector-valued function $y \circ u$. From (1.3) and the Poincaré inequality for functions, we have

$$\begin{aligned} \int_{B_o(R)} |y \circ u|^p &\leq C_3 A R^p \int_{B_o(kR)} |\nabla(y \circ u)|^p \\ &\leq C_3 A R^p \int_{B_o(kR)} (e(u))^{\frac{p}{2}} \end{aligned} \tag{1.4}$$

for some constant C_3 depending only on n. Since y are the normal coordinates with center at u_0, $|(y \circ u)(x)| = d_N(u(x), u_0)$. The lemma follows from (1.4). □

Note that if M satisfies a Poincaré inequality for some $p \geq 1$ for functions as stated in the previous lemma, then any complete manifold which is quasi-isometric to M also satisfies the Poincaré inequality for $p \geq 1$, with possibly different A and k. Poincaré inequalities for manifolds with Ricci curvature bounded from below have been obtained by [L-Y1, B]. From their results, we have the following, which is a special case of [K-S], see also [Gr].

Corollary 1.3. *Let M^m be as in Lemma 1.1 and N^n be a CH manifold. For all $p \geq 1$, there is a constant $k \geq 1$ depending only on α, and a constant C, depending only on m, n, α, p and $\sqrt{K}R$, such that for all $x \in M$, for all $R > 0$, and for all smooth maps from $B_o(kR)$ into N, there is $u_0 \in N$ such that*

$$\int_{B_o(R)} d_N^p \left(u(x), u_0\right) \leq C R^p \int_{B_o(kR)} e^{\frac{p}{2}}(u),$$

where d_N is the distance function on N.

In the proof of [K-S], one needs the comparison theorem for the volume elements in a manifold with Ricci curvature bounded from below. However, in section 2, we will need Lemma 1.2 to study situations which do not make assumtions on the curvature of M. We should emphasis that in [K-S], N can be any complete metric space. In this respect, their result is more general.

Lemma 1.4. *Let M^m and N^n be as in Corollary 1.3. Let $o \in M$. There exists a constant $0 < \delta < 1$ depending only on m, n, α and $\sqrt{K}R$, such that for any smooth harmonic map u from $B_o(R)$ of M into N, we have*

$$s(\delta R) \leq \frac{1}{2} s(R),$$

where $s(r) = \sup_{x \in B_o(r)} d\left(u(x), u(o)\right)$.

Proof. Let $e(u)$ be the energy density of u. Let $f(x) = d_N^2\left((u(x), u(o)\right)$. By the Hessian comparison theorem [G-W], the fact that N is a CH manifold, and that u is harmonic, we have (see for example [C]):

$$\Delta f \geq 2e(u). \tag{1.5}$$

Let h be the function such that

$$\Delta h = -\Delta f$$

in $B_o(R)$, and $h = 0$ on $\partial B_o(R)$. Then $h + f$ is harmonic with boundary value f. By the maximum principle, $h + f \leq \sup_{\partial B_o(R)} f \leq (s(R))^2$. In the following, C_i will always denote positive constants depending only on m, n, α, and $\sqrt{K}R$. Let G_R be the positive Green function on $B_o(R)$ with Dirichlet boundary data. By the estimate of G_R in Lemma 1.1, we have

$$\begin{aligned}
(s(R))^2 &\geq h(o) \\
&= \int_{B_o(R)} G_R(o,y)\Delta f(y)dy \\
&\geq 2\int_{B_o(R)} G_R(o,y)e(u)(y)dy \\
&\geq 2\int_{B_o(\frac{R}{5})} G_R(o,y)e(u)(y)dy \\
&= 2\int_0^{\frac{R}{5}} \left(\int_{\partial B_o(r)} G_R e(u)\right) dr \\
&\geq C_4 \int_0^{\frac{R}{5}} \left(\left(\int_r^R \frac{t}{V_o(t)}dt\right) \int_{\partial B_o(r)} e(u)\right) dr \\
&= C_4 \left\{ \left(\int_r^R \frac{t}{V_o(t)}dt\right) \left(\int_{B_o(r)} e(u)\right) \Bigg|_0^{\frac{R}{5}} \right. \\
&\qquad \left. + \int_0^{\frac{R}{5}} \left(\frac{r}{V_o(r)} \int_{B_o(r)} e(u)\right) dr \right\},
\end{aligned}$$

for some constant $C_4 > 0$. Since

$$\lim_{r \to 0} \left(\int_r^R \frac{t}{V_o(t)}dt\right) \left(\int_{B_o(r)} e(u)\right) = 0,$$

we have

$$(s(R))^2 \geq C_4 \int_0^{\frac{R}{5}} \left(\frac{r}{V_o(r)} \int_{B_o(r)} e(u) \right) dr.$$

Let $0 < \epsilon < \frac{1}{5}$,

$$\begin{aligned} C_4^{-1} (s(R))^2 &\geq \int_{\epsilon R}^{\frac{R}{5}} \left(\frac{r}{V_o(r)} \int_{B_o(r)} e(u) \right) dr \\ &= \int_{\epsilon R}^{\frac{R}{5}} \frac{1}{r} \left(\frac{r^2}{V_o(r)} \int_{B_o(r)} e(u) \right) dr \\ &\geq \log\left(\frac{1}{5\epsilon}\right) \inf_{\epsilon R \leq r \leq \frac{R}{5}} \left(\frac{r^2}{V_o(r)} \int_{B_o(r)} e(u) \right). \end{aligned}$$

From this and the Poincaré inequality Corollary 1.3, there is a $\epsilon R \leq r_0 \leq \frac{R}{5}$, and a point $u_0 \in N$, such that

(1.6)
$$\begin{aligned} \frac{1}{V_o(r_0)} \int_{B_o(\frac{1}{k} r_0)} d_N^2(u, u_0) &\leq \frac{C_5 r_0^2}{V_o(r_0)} \int_{B_o(r_0)} e(u) \\ &\leq C_4 C_3^{-1} (s(R))^2 \left(\log\left(\frac{1}{5\epsilon}\right) \right)^{-1}, \end{aligned}$$

for some constants $k \geq 1$ and C_5, which depend only on m, n, α, and $\sqrt{K}R$. Let $r_1 = r_0/k$. By (1.6) and volume comparison, for $x \in B_o(\frac{r_1}{2})$, we have

$$\begin{aligned} \frac{1}{V_x(\frac{r_1}{2})} \int_{B_x(\frac{r_1}{2})} d_N^2(u, u_0) &\leq \frac{C_6}{V_o(r_0)} \int_{B_o(r_1)} d_N^2(u, u_0) \\ &\leq C_7 \left(\log\left(\frac{1}{5\epsilon}\right) \right)^{-1} (s(R))^2, \end{aligned}$$

where C_6 and C_7 are constants depending only on m, n, and α, and $\sqrt{K}R$. By the mean value inequality in [SC1], see also [L-S], and the fact that $d_N^2(u(x), u_0)$ is subharmonic in $B_o(R)$, we have

$$d_N^2(u(x), u_0) \leq C_8 \left(\log\left(\frac{1}{5\epsilon}\right) \right)^{-1} (s(R))^2 \tag{1.7}$$

for all $x \in B_o(\frac{r_1}{2})$, for some constant C_8 depending only on m, n, and α. Since $\epsilon R/k \le r_0/k = r_1$, by (1.7) and the triangle inequality,

$$\left(s\left(\frac{\epsilon R}{2k}\right)\right)^2 \le C_9\left(\log\left(\frac{1}{5\epsilon}\right)\right)^{-1}(s(R))^2$$

for some constant C_9 depending only on m, n, α and $\sqrt{K}R$. Fix ϵ such that $\log(1/5\epsilon) \ge 4C_9$ and let $\delta = \epsilon/2k$. The lemma follows. □

By Lemma 1.4, and a standard iteration argument, we have

Theorem 1.5. *Let M^m and N^n be as in Lemma 1.4. Let $o \in M$. There exists a constant $0 < \gamma \le 1$, and a constant C, depending only on m, n, α and $\sqrt{K}R$, such that if u is a harmonic map from $B_o(R)$ into N then*

$$s(r) \le C\left(\frac{r}{R}\right)^\gamma \cdot s(R)$$

for all $r \le R$, where $s(r) = \sup_{B_o(r)} d_N(u(x), u(o))$. In particular, if M is quasi-isometric to a complete manifold with nonnegative Ricci curvature, that is, $K = 0$, and if u is a harmonic map from M into N such that $d_N(u(x), u(o)) = o\left((r(x)^\gamma)\right)$, then u must be a constant. Here $r(x)$ is the distance from x to o.

Using similar methods, we want to generalize a result of Shen [Sh]. Let M^m be a complete noncompact manifold with nonnegative Ricci curvature and N^n be a CH manifold with sectional curvature bounded from above by $-a^2$ for some $a > 0$. It was proved in [Sh] that if u is a harmonic map from M into N such that the image of u lies a horoball of N, then u must be a constant. This is an analogue of the Liouville theorem for positive harmonic functions by Yau [Y]. Since Yau's theorem has been generalized to manifolds which are quasi-isometric to complete manifolds with nonnegative Ricci curvature by [SC1, Gr], it is reasonable to expect that Shen's result is still true under a quasi-isometric transformation on M. We will prove that in fact this is true.

Theorem 1.6. *Let M^m be a complete noncompact manifold which is quasi-isometric to a manifold with nonnegative Ricci curvature. Let N^n be a CH manifold with sectional curvature bounded from above by $-a^2$ for some $a > 0$. Let u be a harmonic map from M into N such that $u(M)$ is contained in a horoball. Then u must be constant.*

Proof. By assumption, the image of u lies inside a horoball with center at $\gamma(\infty)$ for some ray γ in N. Let $\mathcal{B}_\gamma(z) = \lim_{t\to\infty}\left(t - d_N\left(z, \gamma\left(t\right)\right)\right)$ be the

Busemann function for the ray γ. Let $\beta(x) = (B_\gamma \circ u)(x)$. Without loss of generality, we may assume that $\beta \geq 1$. Let $e(u)$ be the energy density of u. By the computation in [Sh], β is superharmonic, and

$$-\Delta\beta + |\nabla\beta|^2 \geq C_{10} e(u) \tag{1.8}$$

for some positive constant C_{10} depending only on a. Let $0 < \lambda < 1$ be a constant, such that $2 - \lambda < 1 + \frac{2}{m}$, where m is the dimension of M. From (1.8), the facts that $\beta \geq 1$, $-\Delta\beta \geq 0$, and that $0 < \lambda < 1$, we obtain

$$\begin{aligned} -\Delta\beta^\lambda &= -\lambda\beta^{\lambda-1}\Delta\beta - \lambda(\lambda-1)\beta^{\lambda-2}|\nabla\beta|^2 \\ &= \lambda\beta^{\lambda-2}\left(-\beta\Delta\beta + (1-\lambda)|\nabla\beta|^2\right) \\ &\geq \lambda(1-\lambda)\beta^{\lambda-2}\left(-\Delta\beta + |\nabla\beta|^2\right) \\ &\geq C_{10}\lambda(1-\lambda)\beta^{\lambda-2}e(u). \end{aligned} \tag{1.9}$$

Let $o \in M$. For $R > 0$, let

$$h(x) = -\int_{B_o(R)} G_R(x,y)\Delta\beta^\lambda(y)dy$$

for $x \in B_o(R)$, where G_R is the Green's function with Dirichlet boundary value on $B_o(R)$. Then $\Delta h = \Delta\beta^\lambda$ and $h = 0$ on $\partial B_o(R)$. Hence $h - \beta^\lambda$ is harmonic. Since $h - \beta^\lambda \leq 0$ on $\partial B_o(R)$, $h - \beta^\lambda \leq 0$ in $B_o(R)$. We proceed as in the proof of Lemma 1.4; given $0 < \epsilon < \frac{1}{5}$, we use Lemma 1.1 and (1.9) to get

$$\begin{aligned} \beta^\lambda(o) &\geq h(o) \\ &= -\int_{B_o(R)} G_R(o,y)\Delta\beta^\lambda(y)dy \\ &\geq C_{10}\lambda(1-\lambda)\int_{B_o(R)} G_R(o,y)\beta^{\lambda-2}e(u)(y)dy \\ &\geq C_{11}\log\left(\frac{1}{5\epsilon}\right)\inf_{\epsilon R \leq t \leq \frac{R}{5}}\left(\frac{t^2}{V_o(t)}\int_{B_o(t)}\beta^{\lambda-2}e(u)\right) \end{aligned}$$

for some positive constant $C_{11} > 0$ independent of R, u and ϵ. Hence there is a $\epsilon R \leq r_0 \leq \frac{R}{5}$, such that

$$\frac{r_0^2}{V_o(r_0)}\int_{B_o(r_0)}\beta^{\lambda-2}e(u) \leq C_{12}\left(\log\left(\frac{1}{5\epsilon}\right)\right)^{-1}\beta^\lambda(o) \tag{1.10}$$

for some constant C_{12} independent of R, u and ϵ. Since β is a positive superharmonic function, by the mean value inequality in [SC1] and the fact that $0 < 2-\lambda < 1+\frac{2}{m}$, we have

$$\frac{1}{V_o(r_0)}\int_{B_o(r_0)}\beta^{2-\lambda} \le C_{13}\beta^{2-\lambda}(o),$$

for some constant C_{13} independent of R, u and ϵ. Combining this with (1.10), we have

$$\begin{aligned}\left(\frac{r_0}{V_o(r_0)}\int_{B_o(r_0)}\sqrt{e(u)}\right)^2 &\le \left(\frac{r_0}{V_o(r_0)}\right)^2\left(\int_{B_o(r_0)}\beta^{\lambda-2}e(u)\right)\left(\int_{B_o(r_0)}\beta^{2-\lambda}\right)\\ &\le C_{13}\beta^{2-\lambda}(o)\left(\frac{r_0^2}{V_o(r_0)}\int_{B_o(r_0)}\beta^{\lambda-2}e(u)\right)\\ &\le C_{14}\beta^2(o)\left(\log\left(\frac{1}{5\epsilon}\right)\right)^{-1}\end{aligned}$$

for some constant C_{14} independent of R, u and ϵ. Arguing as in the proof of Lemma 1.4, we can use the Poincaré inequality in Corollary 1.3 with $p = 1$ and prove that

$$\mathrm{osc}_{B_o(\frac{\epsilon}{2k}R)}u \le C_{15}\left(\log\left(\frac{1}{5\epsilon}\right)\right)^{-\frac{1}{2}}\beta(o) \tag{1.11}$$

for some constants $k \ge 1$ and C_{15} which are independent on R, u and ϵ. Here $\mathrm{osc}_{B_o(r)}u$ is defined to be

$$\mathrm{osc}_{B_o(r)}u = \sup_{x,y\in B_o(r)} d_N(u(x),u(y)).$$

For any fixed $r > 0$, and for $R > 0$, let $\epsilon = 2kr/R$ in (1.11), and let $R \to \infty$, we conclude that $\mathrm{osc}_{B_o(r)}u = 0$. Hence u must be a constant map. □

Similar to Theorem 1.5, we can also obtain another form of the Liouville theorem. In (1.11), the constant C_{15} is independent of R, ϵ, u and β. Also, (1.11) holds as long as $\beta \ge 1$ on $B_o(R)$. Note that, since N is a CH manifold, every ray can be extended to be a line. If $\mathcal{B}_\gamma$ is the Busemann function for some ray γ beginning at $\gamma(0)$, and if $\beta = \mathcal{B}_\gamma \circ u$, then $\mathcal{B}_\gamma - \inf_{B_o(R)} \beta + 1$ is the Busemann function for the ray $\gamma^*(t) = \gamma(t + \inf_{B_o(R)} \beta - 1)$. Using (1.11), we have

$$\begin{aligned}\operatorname{osc}_{B_o(\frac{\epsilon R}{2k})}\beta &\leq C_{15}\left(\log\left(\frac{1}{5\epsilon}\right)\right)^{-\frac{1}{2}}\left(\beta(o) - \inf_{B_o(R)}\beta + 1\right)\\ &\leq C_{15}\left(\log\left(\frac{1}{5\epsilon}\right)\right)^{-\frac{1}{2}}\left(\operatorname{osc}_{B_o(R)}\beta + 1\right)\end{aligned}$$

where we have used the fact that $|\nabla_N B_\gamma| \leq 1$. Hence if $\operatorname{osc}_{B_o(R)}\beta \geq 1$, then

$$\operatorname{osc}_{B_o(\frac{\epsilon R}{2k})}\beta \leq \frac{1}{2}\operatorname{osc}_{B_o(R)}\beta \tag{1.15}$$

where $\epsilon > 0$ is chosen such that $C_{15}\left(\log\left(\frac{1}{5\epsilon}\right)\right)^{-\frac{1}{2}} = 1/4$. Note that $\epsilon > 0$ depending only on m, n and α in (1.1). From this, and Theorem 1.6, we have:

Theorem 1.7. *Let M^m and N^n as in Theorem 1.6. There is a $0 < \delta \leq 1$ depending only on m, n, a and α, such that if u is a harmonic map from M into N, and if there is a ray γ in N, such that $|(\mathcal{B}_\gamma \circ u)(x)| = o\left((r(x))^\delta\right)$ where $\mathcal{B}_\gamma$ is the Busemann function for γ, then u must be constant.*

2. Some generalizations

Consider the following conditions on a complete noncompact manifold M:

(a) (Volume doubling) There exists a constant $A > 0$ such that for all $x \in M$ and for all $R > 0$

$$V_x(2R) \leq AV_x(R),$$

where $V_x(r)$ is the volume of the geodesic ball $B_x(r)$ of radius r with center at x.

(b_1) (Poincaré inequality, $p = 1$) There exist constants $\Gamma \geq 1$ and $a > 0$, such that for any function f which is C^∞ in $B_x(\Gamma R)$,

$$\frac{a}{R}\int_{B_x(R)}|f - f_{x,R}| \leq \int_{B_x(\Gamma R)}|\nabla f|,$$

where $f_{x,R}$ is the average of f on $B_x(R)$.

(b_2) (Poincaré inequality, $p = 2$) There exist constants $\Gamma \geq 1$ and $a > 0$, such that for any function f which is C^∞ in $B_x(\Gamma R)$

$$\frac{a}{R^2}\int_{B_x(R)}(f - f_{x,R})^2 \leq \int_{B_x(\Gamma R)}|\nabla f|^2.$$

Suppose M satisfies (a) and (b_1) [or (b_2)] for some $\Gamma > 1$ and $a > 0$, then it will satisfy (b_1) [or (b_2)] for $\Gamma = 1$ for some $a > 0$, by a covering arguments of Jerison [J]. Hence it is easy to see that (a) and (b_1) implies (b_2). By [Gr, SC2], if M satisfies conditions (a) and (b_2), then we have a Harnack inequality for positive solutions to the heat equation. Using the method in [L-Y2], we have the following estimate for the heat kernel on M, see [Gr]:

$$\frac{C_1}{V_x(\sqrt{t})}\exp\left(-C_2\frac{r^2(x,y)}{t}\right) \leq H(x,y,t) \leq \frac{1}{C_1 V_x(\sqrt{t})}\exp\left(-C_3\frac{r^2(x,y)}{t}\right),$$

for all $x,\ y \in M$, where $C_3 > 0$ is any number less than $\frac{1}{4}$, $C_2 > 0$ is a constant depending only on A, a, and Γ, $C_1 > 0$ is a constant depending only on A, a, Γ and C_3. It was also proved in [Gr] that (a) and (b_2) implies the $\lambda_1(B_x(R)) \geq C/R^2$, for some positive constant $C > 0$ depending only on A, a and Γ. Here $\lambda_1(B_x(R))$ is the first eigenvalue of $B_x(R)$ with Dirichlet boundary value. Using these and the method by [V], as in [SC1] and [L2], one can prove that there exist constants $\ell > 2$ and S depending only on A, a and Γ, such that

$$\left(\int_M |f|^{\frac{2\ell}{\ell-2}}\right)^{\frac{\ell-2}{\ell}} \leq SR^2\left(V_x(R)\right)^{-\frac{2}{\ell}}\int_M |\nabla f|^2, \tag{2.1}$$

for all smooth functions f on M with compact support in $B_x(R)$. Using (a), (2.1) and the argument by Moser's method of iteration, one can show that if u is a positive subharmonic function on $B_x(R)$, then for all $p > 0$,

$$u^p(x) \leq \frac{C}{V_x(R)}\int_{B_x(R)} u^p \tag{2.2}$$

for some constant C depending only on A, a, Γ and p. Using (a), (b_2) and (2.1), if u is positive superharmonic on $B_x(R)$, then for all $0 < p < 1 + \frac{2}{\ell}$

$$u^p(x) \geq \frac{C}{V_x(R)}\int_{B_x(R)} u^p \tag{2.3}$$

for some positive constant C depending only on A, a, Γ and p. See [S2], [L1] and [Gi-T] for details. From the proofs of the results in section 1, it is easy to see that the following are still true:

Theorem 2.1. *Let M be a complete noncompact manifold satisfying conditions (a) and (b_2). There exists a constant $0 < \delta \leq 1$ and a constant C both depending only on A, a, Γ and n, such that if u is a harmonic map from some geodesic ball $B_o(R)$ into a CH manifold N^n, then for all $0 < r < R$*

$$s(r) \leq C \left(\frac{r}{R}\right)^{\delta} s(R)$$

where $s(r) = \sup_{B_o(r)} d_N(u(x), u(o))$. In particular, if u is a harmonic map from M into N such that $d(u(x), u(o)) = o\left((r(x))^{\delta}\right)$, where $r(x) = d_M(x, o)$, then u must be constant.

Theorem 2.2. *Let M be a complete noncompact manifold satisfying conditions (a) and (b_1). Let N^n be a CH manifold with sectional curvature bounded from above by $-k^2$ for some $k > 0$.*

(1) *If u is a harmonic map from M into N so that the image $u(M)$ of u lies in a horoball of N, then u must be a constant map.*
(2) *There is a $0 < \delta \leq 1$ depending only on A, a, Γ, k and n, such that if u is a harmonic map from M into N, and if there is a ray γ in N, such that $|(\mathcal{B}_\gamma \circ u)(x)| = o\left((r(x))^{\delta}\right)$ where $\mathcal{B}_\gamma$ is the Busemann function for γ, then u must be constant.*

Proof of Theorem 2.1 and 2.2. Note that if f is a positive harmonic function on $B_x(2r) \subset M$, then by (2.2) and (2.3),

$$\sup_{B_x(r)} f \leq C \inf_{B_x(r)} f,$$

where C is a constant depending only on A, a, and Γ. In fact, this was proved in [Gr, SC2]. Hence Lemma 1.1 is still true on M, with the constant C depending only on A, a, and Γ. See Remark 1.1. Since M satisfies (a), (b_1) [or (b_2)], the corresponding Poincaré inequality for maps also holds, by Lemma 1.2. The rest of the proofs are similar to those of Theorems 1.5–1.7. □

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