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EXTENDING CR FUNCTIONS FROM MANIFOLDS WITH BOUNDARIES

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Introduction

The extendibility of CR functions on smooth real manifolds in $\mathbf{C}^N$ has been extensively studied by many specialists (see, e.g., [Bo] for references). In this paper we generalize the results of [T1] and [T2] on the extendibility of CR functions to manifolds with boundaries and edges.

Recall that a generic manifold $M \subset \mathbf{C}^N$ is said to be minimal at $p \in M$ if there is no proper CR submanifold $S \subset M$ passing through p with the same CR dimension as M. According to [T1], if M is minimal at p, then all CR functions on M extend to the same full dimensional wedge with edge M near p. Conversely, Baouendi and Rothschild [BR] show that if M is not minimal at p, then there are CR functions in a neighborhood of p on M that do not extend to any full dimensional wedge with edge M. Nevertheless, all CR functions on M generally still extend to a larger manifold [BP] [T2]. This manifold has the form of a wedge of dimension lower than the dimension of the ambient space. We call it a *manifold with edge* M, reserving the term *wedge* for full dimensional sets in $\mathbf{C}^N$. Since the natural domains of extended CR functions are manifolds with edges, it is appropriate to consider CR functions initially defined on such sets.

The simplest case of a manifold with edge is a manifold with boundary. Let $M \subset \mathbf{C}^N$ be a manifold with generic boundary M_0. Suppose M is minimal at all interior points. Then by [T1] all CR functions on M extend over a wedge W with edge M. A natural question is the boundary behavior of this wedge at M_0. We show that W approaches M_0 as a wedge with edge M_0 (Corollary 1.3).

In the general case, we introduce the defect of a manifold M with edge M_0 at a point $p \in M_0$ and show that unless the defect is maximal, all CR

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functions on M extend to a larger manifold W with the same edge M_0. The dimension of W is related to the defect of M at p. In particular, we introduce the notion of a minimal point on the edge of the manifold. If M is minimal at $p \in M_0$, we show that all CR functions on M extend over a full dimensional wedge with edge M_0 near p (Corollary 1.2).

The paper is organized as follows. In Section 1 we formulate the main result and some consequences. Section 2 includes some preliminaries. In Sections 3 and 4 we prove the main result. The exposition in this paper is self-contained except that we refer to [T1] for the proof of technical Lemma 2.3.

1. Main results

Let M be a smooth real manifold in $\mathbf{C}^N$. Let $T^c_p(M) = T_p(M) \cap JT_p(M)$ be the maximal complex subspace of the tangent space $T_p(M)$ at a point $p \in M$, where J is the operator of multiplication by the imaginary unit in $\mathbf{C}^N$. Recall that the manifold M is called a *CR manifold* if all the spaces $T^c_p(M)$, $p \in M$, have the same dimension. This dimension is called the *CR dimension* of M and denoted by $\mathrm{CRdim}(M)$ here. Recall that a manifold M is called *generic* if $T_p(M) + JT_p(M) = T_p(\mathbf{C}^N) \simeq \mathbf{C}^N$, $p \in M$. A generic manifold $M \subset \mathbf{C}^N$ is always a CR manifold. We denote the real codimension of M in $\mathbf{C}^N$ by $\mathrm{codim}(M) = 2N - \dim(M)$. For a generic manifold $M \subset \mathbf{C}^N$, we have $\mathrm{CRdim}(M) + \mathrm{codim}(M) = N$.

Recall that a smooth, complex valued function on M is called a *CR function* if its differential is $\mathbf{C}$-linear on $T^c(M)$. A continuous function is called a CR function if the last condition holds in the sense of distribution theory. We denote by $\mathrm{CR}(M)$ the set of all continuous CR functions on M.

Let M and M_0 be manifolds in $\mathbf{R}^n$, and $M_0 \subset \bar{M}$. We call M a *manifold with edge* M_0, if for any point $p \in M_0$, there exist open sets $\Omega \subset \mathbf{R}^n$, $U \subset M_0 \times \mathbf{R}^l$ (where $l = \dim(M) - \dim(M_0)$), an open convex cone $\Gamma \subset \mathbf{R}^l$, and a smooth embedding $F : U \to \mathbf{R}^n$ such that $p \in \Omega$, $(p, 0) \in U$, $\Omega \cap M = F(U \cap (M_0 \times \Gamma))$. Since we are interested in local questions, we assume that Γ does not depend on p.

For $l = 1$, M is a manifold with boundary M_0. For $l = n - \dim(M_0)$, M is a full-dimensional wedge with edge M_0.

Let $C^{k,\alpha}$ denote the space of functions with derivatives up to order k satisfying a Lipschitz condition with exponent α. We say that the manifold M with edge M_0 is $C^{k,\alpha}$ smooth (where $k \geq 1$ and $0 < \alpha < 1$) if the defining mapping F belongs to $C^{k,\alpha}$.

As we mentioned in the introduction, we consider CR functions defined on manifolds with edges. We say that f is a CR function on a CR manifold M with edge M_0 and write $f \in \mathrm{CR}(M \cup M_0)$ if $f \in C(M \cup M_0) \cap \mathrm{CR}(M)$.

The substance of this paper is the following theorem.

Theorem 1.1. *Let $M \subset \mathbf{C}^N$ be a $C^{k,\alpha}$ smooth generic manifold with generic edge M_0, where $k \geq 2$ and $0 < \alpha < 1$. Then for every point $p \in M_0$, there exists an integer $d = \mathrm{def}(p) = \mathrm{def}(p, M_0, M)$, $0 \leq d \leq \mathrm{codim}(M)$, called the* defect *of M at p such that the following two statements* (A) *and* (B) *hold:*

- (A) *$\exists$ a manifold W with edge M_0 such that $\dim(W) = 2N - d$ and all CR functions on $M \cup M_0$ extend to be CR on $W \cup M_0$. The manifold W has almost the same Lipschitz smoothness as M. Precisely, for every $0 < \beta < \alpha$, there exists a neighborhood $U \subset \mathbf{C}^N$ of p such that $W \cap U$ is of class $C^{k,\beta}$.*
- (B) *$\exists$ a neighborhood $U \subset \mathbf{C}^N$ of p and a closed CR submanifold $S \subset M \cap U$ with edge $S_0 = M_0 \cap \bar{S}$ such that $p \in S_0$, $\dim(S) = \dim(M) - d$ and $\mathrm{CRdim}(S) = \mathrm{CRdim}(M)$. The manifold S is $C^{k,\beta}$-smooth for all $0 < \beta < \alpha$.*

Note.

1. The assumption that M is generic can be dropped here. Indeed, since M_0 is generic, M is automatically generic near M_0.
2. M_0 is of class $C^{k,\alpha}$ (by our definition of the manifold with edge of class $C^{k,\alpha}$).
3. The manifold M_0 in Theorem 1.1 may be totally real, in which case $\mathrm{CRdim}(M_0) = 0$. On the contrary, $\mathrm{CRdim}(M) \geq 1$ because M_0 is generic.
4. The conditions ($\mathrm{CRdim}(S) = \mathrm{CRdim}(M)$ and M_0 is generic) imply that $\bar{S}$ is transversal to M_0 in $\bar{M}$. Therefore, $S_0 = M_0 \cap \bar{S}$ is automatically a smooth manifold, $\dim(M_0) - \dim(S_0) = \dim(M) - \dim(S)$ and $\mathrm{CRdim}(S_0) = \mathrm{CRdim}(M_0)$. In particular, if M is a manifold with boundary M_0, then S is a manifold with boundary S_0.

Our Theorem 1.1 is a direct generalization of the results of [T1] and [T2] on the extendibility of CR functions from manifolds without edges. Recall that a CR manifold $M \subset \mathbf{C}^N$ is *minimal* at a point $p \in M$ if there is no CR submanifold $S \subset M$ such that $p \in S$, $\mathrm{CRdim}(S) = \mathrm{CRdim}(M)$, but $\dim(S) < \dim(M)$ [T1]. Likewise, we say that a manifold M with edge M_0 is *minimal* at $p \in M_0$ if there is no *closed* submanifold $S \subset M$ in a neighborhood of p such that $p \in \bar{S}$, $\mathrm{CRdim}(S) = \mathrm{CRdim}(M)$, but $\dim(S) < \dim(M)$.

Corollary 1.2. *Let M be a manifold with the generic edge M_0. Assume that M is minimal at $p \in S_0$. Then all CR functions on $M \cup M_0$ extend to be holomorphic in the same full-dimensional wedge with edge M_0 near p.*

By (A), $d = \mathrm{def}(p) > 0$ if CR functions on $M \cup M_0$ do not extend into any full-dimensional wedge of edge M_0 near $p \in M_0$. Then (B) contradicts the assumption that M is minimal at p. We note that Corollary 1.2 gives new information only if the edge M_0 itself is not minimal at p. For instance, this is the case when M_0 is totally real.

If the manifold M with edge M_0 is minimal at all interior points, it is certainly minimal as a manifold with edge whence the following:

Corollary 1.3. *Let M be a manifold with the generic edge M_0. Assume that M is minimal at all interior points. Then all CR functions on $M \cup M_0$ extend to be holomorphic in the same full-dimensional wedge with edge M_0.*

We give another consequence of the main theorem that relates the existence of submanifolds $S_0 \subset M_0$ such that $\mathrm{CRdim}(S_0) = \mathrm{CRdim}(M_0)$, and the extendibility of CR functions defined on the manifold (without edge) M_0.

Corollary 1.4. *Let M_0 be a generic manifold in $\mathbf{C}^N$ and $p \in M_0$. Assume there is no CR submanifold $S_0 \subset M_0$ such that $p \in M_0$,*

$$\mathrm{CRdim}(S_0) = \mathrm{CRdim}(M_0) \quad \text{and}$$
$$\dim(M_0) - d' < \dim(S_0) < \dim(M_0) - d'',$$

where $0 \le d'' < d' \le \mathrm{codim}(M_0)$. Then if a CR function f on M_0 extends to be CR on a manifold M' with edge M_0 such that $p \in \bar{M}'$ and $\dim(M') > 2N - d'$, then f automatically extends to be CR on another manifold M'' with edge M_0 such that $p \in \bar{M}''$ and $\dim(M'') = 2N - d''$.

This corollary gives new information only if there is a submanifold $S_0 \subset M_0$ such that $p \in S_0$, $\mathrm{CRdim}(S_0) = \mathrm{CRdim}(M_0)$, $\dim(S_0) < \dim(M_0) - d''$ (or $\dim(S_0) \le \dim(M_0) - d'$, which is the same). Otherwise, the conclusion of the corollary follows immediately from the result of [T2] on the extendibility of CR functions on manifolds without edges. In the special case in which $d' = \operatorname{codim}(M_0)$ and $d'' = 0$, any manifold $S_0 \subset M_0$ with $\operatorname{CRdim}(S_0) = \operatorname{CRdim}(M_0)$ must be complex. The corollary then asserts that if a CR function f on M_0 extends to be CR on a manifold M with boundary M_0, then f extends to be holomorphic in a full-dimensional wedge W of edge M_0. It is of interest to find out if the tangent cone to W depends on the direction of M in the normal space of M_0, i.e., $T_p(M)/T_p(M_0)$.

We note that Corollary 1.4 is related to a result of [T3] (Theorem 6.1). In that theorem, instead of nonexistence of S_0 with the indicated properties, it is assumed that the connection that governs the propagation of extendibility of CR functions has sufficiently rich holonomy.

Proof of Corollary 1.4. By statement (A), it suffices to show that $d = \operatorname{def}(p, M_0, M') \le d''$. By statement (B), there is a submanifold $S \subset M'$ with edge $S_0 \subset M_0$ such that $p \in S_0$, $\operatorname{CRdim}(S) = \operatorname{CRdim}(M')$, $\dim(S) = \dim(M') - d$. Then $\dim(S) \ge 2\operatorname{CRdim}(S) = 2\operatorname{CRdim}(M') = 2(\dim(M') - N)$. Hence $d \le 2N - \dim(M')$. Using $\dim(M') > 2N - d'$, we get $d < d'$. The edge S_0 has dimension $\dim(S_0) = \dim(M_0) - d > \dim(M_0) - d'$. Therefore, by the assumptions of the corollary, $\dim(S_0) \ge \dim(M_0) - d''$, whence $d \le d''$. $\square$

2. Analytic discs and their deformations

Let $M \subset \mathbf{C}^N$ be a generic manifold with generic edge M_0. Assume that M and M_0 are $C^{k,\alpha}$-smooth, where $k \ge 1$, $0 < \alpha < 1$.

An *analytic disc* in $\mathbf{C}^N$ is a continuous mapping $A : \bar{\Delta} \to \mathbf{C}^N$ holomorphic in the unit disc Δ. We say that A is attached to $M \cup M_0$ if $A(b\Delta) \subset M \cup M_0$.

Let $p \in M_0$. Let $\mathcal{A}_p$ be the set of all small $C^{k,\alpha}$-smooth discs attached to $M \cup M_0$ passing through p, that is

(2.1)
$$\mathcal{A}_p = \{A \in \mathcal{O}(\Delta) \cap C^{k,\alpha}(\bar{\Delta}) : ||A||_{C^{k,\alpha}} \le r_0, A(1) = p, A(b\Delta) \subset M \cup M_0\},$$

where $\mathcal{O}(\Delta)$ denotes all holomorphic functions in Δ, and $r_0 > 0$ is small.

For a disc $(\zeta \mapsto A(\zeta)) \in \mathcal{A}_p$, the vector $-\partial A(1)/\partial\zeta$ determines the direction of the disc A at p. Let V'_p be the vector subspace of $T_p(\mathbf{C}^N)/T_p(M_0)$ spanned by the above direction vectors, that is

$$V'_p = \operatorname{Span}\{-\partial A(1)/\partial\zeta \bmod T_p(M_0) : A \in \mathcal{A}_p\}. \tag{2.2}$$

Proposition 2.1. *Let* $\dim(V_p') = r$. *Then there is a manifold* W *with edge* M_0 *of dimension* $\dim(M_0) + r$ *such that all continuous CR functions on* $M \cup M_0$ *extend to be CR on* W. *For every* $0 < \beta < \alpha$, *there exists a neighborhood* $U \subset \mathbf{C}^N$ *of* p *such that* $W \cap U$ *is of class* $C^{k,\beta}$.

Proof. The proof is a combination of the approximation theorem by Baouendi and Treves [BT], the edge-of-the-wedge theorem by Ayrapetian and Henkin [A], and a simple fact on the deformation of discs [T3]. Indeed, let $A_1, \dots, A_r \in \mathcal{A}_p$ be the discs such that $v_j = -\partial A_j(1)/\partial\zeta (1 \le j \le r)$ span V_p'. By deforming these discs, one obtains manifolds $M_1, \dots, M_r$ with the same boundary M_0 such that each $T_p(M_j)$ is spanned by $T_p(M_0)$ and v_j. (See [T3, Proposition 1.3] for details.) Therefore $\dim(\sum_j T_p(M_j)) = \dim(M_0) + r$. Since each M_j is a union of discs attached to $M \cup M_0$, by the Baouendi-Treves approximation theorem, all CR functions on $M \cup M_0$ extend to be CR on M_j. The proposition now follows by the edge-of-the-wedge theorem. $\square$

Remark. In our statement regarding the smoothness of W, the neighborhood U depends on β. This has happened because of the repeated use of Bishop's equation. We use it for the first time to construct M_j-s. By the result of [T4] on the regularity of Bishop's equation, the manifolds M_j-s are of class $C^{k,\beta}$ for all $0 < \beta < \alpha$. However, the $C^{k,\beta}$ norms of defining functions of M_j-s blow up as $\beta \to \alpha$. We use Bishop's equations one more time to construct W to be a union of discs attached to $\bigcup M_j$ (see [A]). Applying the result of [T4] in this situation yields shrinking U as $\beta \to \alpha$. The author does not know whether it is possible to choose U independent of β.

We will use the Bishop equation [B] to describe the set $\mathcal{A}_p$. In a suitable holomorphic local system of coordinates ($z = x + iy \in \mathbf{C}^m, w \in \mathbf{C}^n$) in $\Omega \subset \mathbf{C}^N$ with origin at $p = 0$, the manifolds M and M_0 can be defined as

$$\begin{aligned} M_0 \cap \Omega &= \{(x+iy, w) : x = h(y, w, 0)\}, \\ M \cap \Omega &= \{(x+iy, w) : x = h(y, w, t), t \in \Gamma\}. \end{aligned} \tag{2.3}$$

where h is a smooth real $\mathbf{R}^m$-valued function in a neighborhood of zero in $\mathbf{R}^m \times \mathbf{C}^n \times \mathbf{R}^l$ such that h and the partial derivatives h_y and h_w vanish at zero while h_t has the maximum rank l at zero, that is

$$h(0) = 0, \qquad h_y(0) = 0, \qquad h_w(0) = 0, \qquad \operatorname{rank} h_t(0) = l \tag{2.4}$$

and $\Gamma \subset \mathbf{R}^l$ is an open convex cone. In this notation, $\mathrm{CRdim}(M_0) = n$, $\mathrm{codim}(M_0) = m$ and $\dim(M) = \dim(M_0) + l$.

Along with M, we consider $\tilde{M}$, the continuation of M across the edge M_0, obtained by dropping the condition that $t \in \Gamma$ in (2.1), that is

$$\tilde{M} = \{(x + iy, w) \in \Omega : x = h(y, w, t), t \in \mathbf{R}^l\}.$$

Let $q = (x + iy, w) \in \tilde{M}$. Then there is a unique $t \in \mathbf{R}^l$ such that $x = h(y, w, t)$. We call such t the *t-component* of q.

We denote by $\tilde{\mathcal{A}}_p$ the set of all small analytic discs A attached to $\tilde{M}$ with $A(1) = 0$.

Proposition 2.2. *There exists a unique disc $A \in \tilde{\mathcal{A}}_p$ with given sufficiently small w- and t-components respectively $(\zeta \mapsto w(\zeta)) \in C^{k,\alpha}(\bar{\Delta}) \cap \mathcal{O}(\Delta)$ and $(\zeta \mapsto \tau(\zeta)) \in C^{k,\alpha}(b\Delta)$ such that $w(1) = 0$, $\tau(1) = 0$. Furthermore, $A \in \mathcal{A}_p$ (2.1) if and only if $\tau(\zeta) \in \Gamma \cup 0$ for $\zeta \in b\Delta$.*

Proof. Let T_1 denote the harmonic conjugation operator on $b\Delta$ normalized by the condition $(T_1\phi)(1) = 0$. That is $T_1\phi = T\phi - (T\phi)(1)$ where T is the standard Hilbert transform.

Let $A(\zeta) = (x(\zeta) + iy(\zeta), w(\zeta))$ and let $\zeta \mapsto \tau(\zeta)$ be the t-component of $A|_{b\Delta}$, that is $x(\zeta) = h(y(\zeta), w(\zeta), \tau(\zeta))$ for $\zeta \in b\Delta$. Since the x- and y-components of A are harmonic conjugates, and $y(1) = 0$, the function $\zeta \mapsto y(\zeta)$ satisfies the Bishop equation

$$y = T_1 h(y, w, \tau). \tag{2.5}$$

For given $\zeta \mapsto w(\zeta)$ and $\zeta \mapsto \tau(\zeta)$ small in the $C^{k,\alpha}$ norm, this equation has a unique solution $\zeta \mapsto y(\zeta)$ in $C^{k,\alpha}$ (see [T4]). Once (2.5) is solved, $\zeta \mapsto x(\zeta) + iy(\zeta)$ is obtained by harmonically extending to Δ. The characterization of $\mathcal{A}_p$ by the condition $\tau(\zeta) \in \Gamma \cup 0$ is obvious. □

Proposition 2.2 identifies $\tilde{\mathcal{A}}_p$ with a neighborhood of zero in the Banach space

$$\begin{aligned}\tilde{\mathcal{B}} = \{(w, \tau) :\\ w \in C^{k,\alpha}(\bar{\Delta}, \mathbf{C}^n) \cap \mathcal{O}(\Delta), \tau \in C^{k,\alpha}(b\Delta, \mathbf{R}^l), w(1) = 0, \tau(1) = 0\},\end{aligned} \tag{2.6}$$

and $\mathcal{A}_p$ is represented by elements of

$$\mathcal{B} = \{(w, \tau) \in \tilde{\mathcal{B}} : \tau(\zeta) \in \Gamma \cup 0 \text{ for } \zeta \in b\Delta\}. \tag{2.7}$$

We write $A \leftrightarrow (w, \tau)$ if w and τ are the w- and t-components of A.

A *(infinitesimal tangential to $\tilde{M}$) deformation* of a disc $A \in \tilde{\mathcal{A}}_p$ is a continuous mapping $\dot{A} : \bar{\Delta} \to \mathbf{C}^N$ holomorphic in Δ such that $\dot{A}(\zeta) = (\dot{x}(\zeta) + i\dot{y}(\zeta), \dot{w}(\zeta)) \in T_{A(\zeta)}(\tilde{M})$ for $\zeta \in b\Delta$ and $\dot{A}(1) = 0$.

Let $v = (\dot{x} + i\dot{y}, \dot{w}) \in T_q(\tilde{M})$ at $q = (x + iy, w) \in \tilde{M}$. Then $\dot{x} = h_y\dot{y} + h_w\dot{w} + h_{\bar{w}}\bar{\dot{w}} + h_t\dot{t}$, where the partial derivatives $h_y, \dots$ are evaluated at (y, w, t) such that $x = h(y, w, t)$. We call $\dot{t}$ the *t-component* of v.

Thus, the y-component of $\dot{A}$ must satisfy the linearized Bishop's equation

$$\dot{y} = T_1(h_y\dot{y} + h_w\dot{w} + h_{\bar{w}}\bar{\dot{w}} + h_t\dot{\tau}), \tag{2.8}$$

where $\dot{\tau}$ is the t-component of $\dot{A}|_{b\Delta}$. The partial derivatives $h_y, \dots$ are evaluated at $(y(\zeta), w(\zeta), \tau(\zeta))$, where τ is the t-component of $A|_{b\Delta}$.

We restrict to deformations $\dot{A}$ with the w- and t-components $\zeta \mapsto \dot{w}(\zeta)$ and $\zeta \mapsto \dot{\tau}(\zeta)$ in $C^{k,\alpha}$. In this case the equation (2.8) has a unique solution $\zeta \mapsto \dot{y}(\zeta)$ in $C^{k-1,\alpha}$. Thus, the set of deformations of $A \in \tilde{\mathcal{A}}_p$ is identified with $\tilde{\mathcal{B}}$. We write $\dot{A} \leftrightarrow (\dot{w}, \dot{\tau}) \in \tilde{B}$.

Following [T1], we associate to a disc $A \in \tilde{\mathcal{A}}_p$, a $C^{k-1,\alpha}$ real $m \times m$ matrix function $\zeta \mapsto G(\zeta)$ on $b\Delta$ as the unique solution to the equation

$$G = \mathbf{1} - T_1(Gh_y), \tag{2.9}$$

where h_y is evaluated at $(y(\zeta), w(\zeta), \tau(\zeta))$ as in (2.8) and $\mathbf{1}$ denotes the identity matrix. Since A is small, G is nondegenerate.

We introduce the following notation. For a $C^{1,\alpha}$ function ϕ on the unit circle with $\phi(1) = 0$, we write

$$\mathcal{J}(\phi) = \frac{1}{\pi}\int_0^{2\pi} \frac{\phi(e^{i\theta})\, d\theta}{|e^{i\theta} - 1|^2} = \frac{i}{\pi}\int_{b\Delta} \frac{\phi(\zeta)\, d\zeta}{(\zeta - 1)^2}, \tag{2.10}$$

where the integral is understood in the sense of principal value. Note that for any function $\phi \in C^{1,\alpha}(\bar{\Delta})$ holomorphic in Δ with $\phi(1) = 0$, we have

$$\mathcal{J}(\phi) = -\frac{\partial\phi(1)}{\partial\zeta}. \tag{2.11}$$

Lemma 2.3. *The solution of (2.8) has the form*

$$\dot{y} = (\mathbf{1} + h_y^2)^{-1}(G^{-1}T_1(G\phi) - h_y\phi), \tag{2.12}$$

where $\phi = h_w\dot{w} + h_{\bar{w}}\bar{\dot{w}} + h_t\dot{\tau}$. Moreover

$$\mathcal{J}(\dot{x}) = \mathcal{J}(G\phi), \tag{2.13}$$

where $\dot{x}$ is the x-component of $\dot{A}$.

The proof of this lemma is given in [T1].

3. Defect of discs and proof of (A)

Let M and M_0 be the same as in Section 2. In addition, assume $k \geq 2$. We define the defect of a disc $A \in \tilde{\mathcal{A}}_p$.

Let $\mathbf{R}^{m*}$ be the dual space to $\mathbf{R}^m$. To apply the matrix notation, we regard it as the space of row m-vectors. Let $A \in \tilde{\mathcal{A}}_p$. We set

(3.1)

$$V^*(A) = \{c \in \mathbf{R}^{m*} : cGh_w \text{ extends holomorphically to } \Delta, cGh_t = 0\},$$

$$\operatorname{def}(A) = \dim(V^*(A)),$$

where h_w and h_t are evaluated at $(y(\zeta), w(\zeta), \tau(\zeta))$ as in (2.8). Note that $\operatorname{rank}(Gh_t) = \operatorname{rank}(h_t) = l$. Therefore, $\operatorname{def}(A) \leq m - l = \operatorname{codim}(M)$. One can see that $\operatorname{def}(A)$ is the same as the defect of A as a disc attached to $\tilde{M}$ (see [T1] and [BRT]). We set

$$(3.2)\quad V'(A) = \operatorname{Span}\{-\partial\dot{A}(1)/\partial\zeta \bmod T_p(M_0) : \dot{\tau}(\zeta) \in \Gamma \cup 0 \text{ for } \zeta \in b\Delta\},$$

where $\dot{A}$ denotes a deformation of A and $\dot{\tau}$ is the t-component of $\dot{A}$.

Lemma 3.1. *For $A \in \mathcal{A}_p$, $\dim(V'(A)) = m - \operatorname{def}(A)$.*

Proof. The proof is very similar to that of a related result of [T1]. Since $T_p(M_0)$ is given by the equation $x = 0$, the normal space $T_p(\mathbf{C}^N)/T_p(M_0)$ is identified with the x-space $\mathbf{R}^m$. We show that $V^*(A) = V'(A)^\perp$.

Let $c \in V'(A)^\perp$. Let $\dot{A} = (\dot{x} + i\dot{y}, \dot{w})$ be a deformation of A with the t-component $\dot{\tau}$ taking values in $\Gamma \cup 0$. By (2.11), the x-component of $-\partial\dot{A}(1)/\partial\zeta$ is $\mathcal{J}(\dot{x})$. Thus, $c\mathcal{J}(\dot{x}) = 0$. By (2.13),

$$0 = c\mathcal{J}(\dot{x}) = \mathcal{J}(cGh_w\dot{w}) + \mathcal{J}(cGh_{\bar{w}}\bar{\dot{w}}) + \mathcal{J}(cGh_t\dot{\tau}).$$

This holds for every pair $(\dot{w}, \dot{\tau}) \in \mathcal{B}$ (2.7), in particular $(\dot{w}, 0)$, $(i\dot{w}, 0)$ and $(0, \dot{\tau})$. Therefore,

$$\mathcal{J}(cGh_w\dot{w}) = 0, \qquad \mathcal{J}(cGh_t\dot{\tau}) = 0.$$

Using the holomorphic form of $\mathcal{J}$ in (2.10), we see that cGh_w must extend holomorphically to Δ. Since $\dot{\tau}$ is any function with $\dot{\tau}(\zeta) \in \Gamma \cup 0$ and $\dot{\tau}(1) = 0$, we conclude that $cGh_t = 0$ identically. Hence $c \in V^*(A)$. The converse is obvious. $\square$

Lemma 3.2. *For $A \in \mathcal{A}_p$, $V'(A) \subset V'_p$.*

Proof. The proof is a simple check of the definitions (2.2) and (3.2). Indeed, let $\dot{A}$ be a deformation of A with the t-component $\dot{\tau}$ with $\dot{\tau}(\zeta) \in \Gamma \cup 0$. We need to show that $-\partial \dot{A}(1)/\partial\zeta \bmod T_p(M_0) \in V'_p$.

Let w and τ be the w- and t-components of A, and let $\dot{w}$ and $\dot{\tau}$ be the w- and t-components of $\dot{A}$. Using Proposition 2.2, we construct the family of discs $\zeta \mapsto A(\zeta, s)$ (where $s > 0$), having the w- and t-components $w + s\dot{w}$ and $\tau + s\dot{\tau}$. Since Γ is convex, $\tau(\zeta), \dot{\tau}(\zeta) \in \Gamma \cup 0$ and $s > 0$, the discs belong to $\mathcal{A}_p$. Then $\dot{A}(\zeta) = \frac{d}{ds}\big|_{s=0} A(\zeta, s)$. Therefore,

$$-\partial \dot{A}(1)/\partial\zeta \bmod T_p(M_0) = \frac{d}{ds}\Big|_{s=0}(-\partial A(1,s)/\partial\zeta \bmod T_p(M_0)).$$

Since the expression in parentheses is in V'_p, the derivative is also in V'_p, which is what we need. (We use the fact that M is $C^{k,\alpha}$ with $k \geq 2$.) $\square$

We now define the defect of M at p.

$$\text{(3.3)} \qquad \operatorname{def}(p, M_0, M) = \liminf \operatorname{def}(A) \qquad \text{as } A \in \mathcal{A}_p, \quad ||A||_{C^{k,\alpha}} \to 0.$$

End of the proof of (A). Let $d = \operatorname{def}(p, M_0, M)$. Then there exists $A \in \mathcal{A}_p$ with $\operatorname{def}(A) = d$. By Lemmas 3.1 and 3.2, $\dim(V'_p) \geq m - d$. The statement (A) now follows by Proposition 2.1. $\square$

4. Proof of (B)

Let M, M_0 and p be the same as in the previous section. We construct the submanifold S as a union of boundaries of analytic discs attached to $M \cup M_0$. We consider the evaluation map

$$\text{(4.1)} \qquad \mathcal{F} : \tilde{\mathcal{A}}_p \to \tilde{M}, \qquad \mathcal{F} : A \mapsto A(-1).$$

Since we identify $\tilde{\mathcal{A}}_p$ with a neighborhood of zero in $\tilde{B}$ (2.6), we also regard $\mathcal{F}$ as a mapping on this neighborhood. According to results on the regularity of Bishop's equations [T2] [T4], $\mathcal{F}$ is $C^{k,\beta}$ for every $0 < \beta < \alpha$.

Let $A \in \tilde{\mathcal{A}}_p$, $q = \mathcal{F}(A) = A(-1)$. Let $L(A) = \mathcal{F}'(A)\tilde{\mathcal{B}} \subset T_q(\tilde{M})$, the image of the derivative of $\mathcal{F}$ at A.

Proposition 4.1.

(i) $\dim L(A) = \dim(M) - \operatorname{def}(A)$;
(ii) $T^c_q(\tilde{M}) \subset L(A)$.

Proof. To prove (i), we construct an isomorphism between $V^*(A)$ (3.1) and $L(A)^\perp \subset T_q^*(\tilde{M})$. To prove (ii), we show that $L(A)^\perp$ annihilates $T_q^c(\tilde{M})$.

Let $\omega = P\,dy + Q\,dw + \bar{Q}\,d\bar{w} + R\,dt \in L(A)^\perp$. This means precisely that for every deformation $\dot{A} \leftrightarrow (\dot{w}, \dot{\tau}) \in \tilde{\mathcal{B}}$,

$$(P\dot{y} + Q\dot{w} + \bar{Q}\bar{\dot{w}} + R\dot{\tau})(-1) = 0,$$

where $\dot{y}$ is the y-component of $\dot{A}$. Using (2.12), we get

$$(P(\mathbf{1} + h_y^2)^{-1}(G^{-1}T_1(G\phi) - h_y\phi) + Q\dot{w} + \bar{Q}\bar{\dot{w}} + R\dot{\tau})(-1) = 0,$$

where $\phi = h_w\dot{w} + h_{\bar{w}}\bar{\dot{w}} + h_t\dot{\tau}$. This holds for every pair $(\dot{w}, \dot{\tau}) \in \tilde{\mathcal{B}}$, in particular, $(\dot{w}, 0)$, $(i\dot{w}, 0)$ and $(0, \dot{\tau})$. Therefore,

$$\begin{aligned} &(P(\mathbf{1} + h_y^2)^{-1}(G^{-1}T_1(Gh_w\dot{w}) - h_yh_w\dot{w}) + Q\dot{w})(-1) = 0, \\ &(P(\mathbf{1} + h_y^2)^{-1}(G^{-1}T_1(Gh_t\dot{\tau}) - h_yh_t\dot{\tau}) + R\dot{\tau})(-1) = 0. \end{aligned} \tag{4.2}$$

Note that for every $\psi \in C^{0,\alpha}(b\Delta)$,

$$(T_1\psi)(-1) = \frac{2}{\pi}\int_{b\Delta} \frac{\psi(\zeta)\,d\zeta}{\zeta^2 - 1}.$$

Let $\dot{w}(-1) = 0$ and $\dot{\tau}(-1) = 0$. Then (4.2) yields

$$\int_{b\Delta} cGh_w \frac{\dot{w}(\zeta)}{\zeta^2 - 1} d\zeta = 0, \qquad \int_{b\Delta} cGh_t \frac{\dot{\tau}(\zeta)}{\zeta^2 - 1} d\zeta = 0, \tag{4.3}$$

where

$$c = (P(\mathbf{1} + h_y^2)^{-1}G^{-1})(-1). \tag{4.4}$$

Since (4.3) holds for every $(\dot{w}, \dot{\tau}) \in \tilde{\mathcal{B}}$ vanishing at -1, cGh_w must extend holomorphically to Δ, and cGh_t must vanish identically. Hence $c \in V^*(A)$.

Since $cGh_w\dot{w}$ is holomorphic and $\dot{w}(1) = 0$, we have $T_1(cGh_w\dot{w}) = -icGh_w\dot{w}$. Plugging this and $cGh_t = 0$ in (4.2), we get

$$\begin{aligned} &(P((h_y - i\mathbf{1})^{-1}h_w - Q)\dot{w}(-1) = 0, \\ &(P((\mathbf{1} + h_y^2)^{-1}h_yh_t - R)\dot{\tau}(-1) = 0. \end{aligned} \tag{4.5}$$

Since $\dot{w}(-1) \in \mathbf{C}^n$ and $\dot{\tau}(-1) \in \mathbf{R}^l$ are arbitrary,

$$Q = P(h_y - i\mathbf{1})^{-1}h_w, \qquad R = P(\mathbf{1} + h_y^2)^{-1}h_yh_t, \tag{4.6}$$

where $h_y \dots$ are evaluated at q. A simple check shows that (4.6) means precisely that $\omega \in T_q^c(\tilde{M})^\perp$, which proves (ii).

We claim that the mapping $\omega \mapsto c$ given by (4.4) is an isomorphism. Indeed, it is linear by (4.4). It is injective because Q and R are uniquely defined by P. It is also surjective, because, given $c \in V^*(A)$, we can use (4.4) and (4.6) to get $\omega \in L(A)^\perp$, which completes the proof of (i). The proposition is now proved. □

We fix a smooth *scalar* function ψ on $b\Delta$ such that $\psi(\zeta) > 0$ for every $\zeta \in b\Delta$ except that $\psi(1) = 0$. We take ψ with $\psi(-1) = 1$. For $\epsilon > 0$ we introduce the subspace

(4.7)
$$\mathcal{B}_\epsilon = \{(w,\tau) \in \tilde{\mathcal{B}} : \tau = \lambda\psi + \phi, \lambda \in \mathbf{R}^l, \phi(\zeta) = 0 \text{ if } |\zeta+1| < \epsilon \text{ or } |\zeta-1| < \epsilon\}.$$

We set $L_\epsilon(A) = \mathcal{F}'(A)(\mathcal{B}_\epsilon)$.

Lemma 4.2. *For every $A \in \mathcal{A}_p$ there exists $\epsilon > 0$ such that $L_\epsilon(A) = L(A)$.*

Proof. Since $L_\epsilon(A)$ and $L(A)$ are finite dimensional, it suffices to show that $\bigcup_{\epsilon>0} L_\epsilon(A) = L(A)$. By passage to orthogonal complements in $T_q^*(M)$, it reduces to $\bigcap_{\epsilon>0} L_\epsilon(A)^\perp = L(A)^\perp$.

Using the same notation as in the proof of Proposition 4.1, let $\omega \in \bigcap_{\epsilon>0} L_\epsilon(A)^\perp$. We then find that (4.3) holds for all $\dot\tau$ vanishing near ± 1. This suffices to conclude that c given by (4.4) is in $V^*(A)$. We further get that since $\dot\tau(-1) = \lambda\psi(-1) \in \mathbf{R}^l$ is arbitrary, (4.5) still implies (4.6). Thus $\omega \in L(A)^\perp$. □

We now turn to the proof of (B). Let $d = \operatorname{def}(p, M_0, M)$ defined by (3.3). Let $A_0 \in \mathcal{A}_p$ be a disc with $\operatorname{def}(A_0) = d$. Let $A_0 \leftrightarrow (w_0, \tau_0)$. We can assume that $A_0(-1) = p = 0$, otherwise we replace A_0 by the disc $\zeta \mapsto A_0(\zeta^2)$, which has the same defect and is still small in $C^{k,\alpha}$. We can further assume that $\zeta = \pm 1$ are the only points on $b\Delta$ where $A_0(\zeta) \in M_0$ because small perturbations can only reduce the defect of A_0, which is already of minimum defect among sufficiently small discs in $\mathcal{A}_p$.

We fix $\epsilon > 0$ such that $L_\epsilon(A_0) = L(A_0)$. We choose $\dot A_j \leftrightarrow (\dot w_j, \dot\tau_j) \in \mathcal{B}_\epsilon$, $1 \le j \le r = \dim(M) - d$, such that $\mathcal{F}'(A_0)\dot A_j$ form a basis in $L_\epsilon(A_0)$ whence in $L(A_0)$. Specifically, we can take $\dot w_j = 0$, $\dot\tau_j = e_j\psi$ for $1 \le j \le l$, where e_j is the unit vector of the j-th coordinate in $\mathbf{R}^r$. Indeed, these $\dot A_j$, $1 \le j \le l$, have linearly independent t-components. We can further assume that

$$\dot\tau_j(\zeta) = 0 \qquad \text{if } |\zeta - 1| < \epsilon \text{ or } |\zeta + 1| < \epsilon \qquad \text{for } l < j \le r. \tag{4.8}$$

(Otherwise, for $j > l$, we replace $\dot{A}_j$ by $\dot{A}_j - \sum_{\nu=1}^{l} u_\nu \dot{A}_\nu$, where $(u_1, \ldots, u_l)^T = \dot{\tau}_j(-1)$, see (4.7).)

We restrict the evaluation map to the discs $A_s \in \tilde{\mathcal{A}}_p$, $s \in \mathbf{R}^r$, $A_s \leftrightarrow (w_s, \tau_s)$, where

$$w_s = w_0 + \sum_{j=1}^{r} s_j \dot{w}_j \quad \text{and}$$

$$\tau_s = \tau_0 + \sum_{j=1}^{r} s_j \dot{\tau}_j = \tau_0 + \sum_{j=1}^{l} s_j e_j \psi + \sum_{j=l+1}^{r} s_j \dot{\tau}_j. \tag{4.9}$$

Let $H : s \mapsto \mathcal{F}(A_s) = A_s(-1)$. The mapping H is $C^{k,\beta}$ in a neighborhood of zero $U \subset \mathbf{R}^r$ for all $0 < \beta < \alpha$. Since $\partial H(0)/\partial s_j = \mathcal{F}'(A_0)\dot{A}_j$ are linearly independent, H is an immersion if U is small.

Let $\tilde{S} = H(U)$. Since $\tau_s(-1) = (s_1, \ldots, s_l)^T$, we have $(s_1, \ldots, s_l)^T \in \Gamma$ iff $H(s) \in M$. Therefore, $S = \tilde{S} \cap M = H((\Gamma \times \mathbf{R}^{r-l}) \cap U)$ is a manifold with edge $S_0 = \tilde{S} \cap M_0 = H((0 \times \mathbf{R}^{r-l}) \cap U)$.

We now prove that $T^c(S) = T^c(M)|_S$. We first note that if $s \in \mathbf{R}^r$ is small, then $(s_1, \ldots, s_l)^T = \tau_s(-1) \in \Gamma$ implies that $\tau_s(\zeta) \in \Gamma$ for all $\zeta \in b\Delta$, $\zeta \neq 1$. Indeed, since Γ is convex and $\psi(\zeta) \geq 0$, by (4.8) the right-hand side of (4.9) is in Γ for ζ close to ± 1. If ζ is outside a neighborhood of ± 1, $\tau_s(\zeta) \in \Gamma$ holds for small s because $\tau_0(\zeta) \in \Gamma$ for $\zeta \in b\Delta$, $\zeta \neq \pm 1$.

Let $q \in S$, $q = H(s)$, $s \in (\Gamma \times \mathbf{R}^{r-l}) \cap U$. By the remark above, $A_s \in \mathcal{A}_p$, that is A_s is attached to $M \cap M_0$. Since H is obtained as a restriction of the evaluation map $\mathcal{F}$, we have $T_q(S) \subset L(A_s)$. Since $A_s \in \mathcal{A}_p$ is close to A_0, $\operatorname{def}(A_s) = d$. Therefore, by Proposition 4.1, $\dim T_q(S) = r = \dim(M) - d = \dim L(A_s)$. Hence, $T_q(S) = L(A_s) \supset T_q^c(M)$, which is what we need.

The proof of (B) is complete.

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