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WEAK COVERING WITHOUT COUNTABLE CLOSURE

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Theorem 0.1. *Suppose that there is no inner model with a Woodin cardinal. Suppose that Ω is a measurable cardinal. Let K be the Steel core model as computed in V_Ω. Let $\kappa \geq \omega_2$ and $\lambda = (\kappa^+)^K$. Then cf(λ) $\geq$ card(κ).*

The main result of [MiSchSt] is that Theorem 0.1 holds under the additional assumption that card(κ) is countably closed. But often, in applications, countable closure is not available. Theorem 0.1 also builds on the earlier covering theorems of Jensen, Dodd and Jensen, and Mitchell; some of the relevant papers are [DeJe], [DoJe1], [DoJe2], [Mi1], [Mi2], and [Je]. The results for smaller core models do not require the existence of a measurable cardinal; it is not known if the large cardinal hypothesis on Ω can be eliminated completely from Theorem 0.1 (see [Sch2]).

In this paper, we outline a proof of Theorem 0.1. By K^c, we mean Steel's background certified core model. We shall reduce what we must prove to some iterability properties for K^c (labeled "facts" in the proof). In turn, Steel has shown that K^c is sufficiently iterable, using the methods in [St, §9]. The proof of Theorem 0.1 is very closely tied to the proof in [MiSchSt], to which we shall refer freely.

1. An internally approachable chain

Our proof of Theorem 0.1 begins much as the proof of Jensen's covering theorem for L, with an internally approachable chain. Fix Ω, κ, and λ as in the statement Theorem 0.1, and assume for contradiction that $\mathrm{cf}(\lambda) < \mathrm{card}(\kappa)$. Let ε be a regular cardinal with $\mathrm{cf}(\lambda) < \varepsilon$ and $\omega_2 \leq \varepsilon \leq \mathrm{card}(\kappa)$. Though $\varepsilon = \mathrm{cf}(\lambda)^+$ would do, we prefer to work in slightly more generality.

Let $\langle\, X_i \mid i < \varepsilon \,\rangle$ be a continuous chain of elementary substructures of $V_{\Omega+1}$ such that for all $j < \varepsilon$, $\langle\, X_i \mid i \leq j \,\rangle \in X_{j+1}$, and $X_j \cap \varepsilon \in \varepsilon$, and $\mathrm{card}(X_j) = \mathrm{card}(X_j \cap \varepsilon)$. Assume also that $\kappa \in X_0$. For $i < \varepsilon$, let $\varepsilon_i = X_i \cap \varepsilon$. Note that $\langle\, \varepsilon_i \mid i < \varepsilon \,\rangle$ is a normal sequence converging to ε.

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For $i < \varepsilon$, let $\pi_i \colon N_i \longrightarrow V_{\Omega+1}$ be the uncollapse of X_i. So $\mathrm{crit}(\pi_i) = \varepsilon_i$. We call a partial function F on ε a *choice function* if and only if $F(i) \in X_i$ for all $i \in \mathrm{dom}(F)$.

Lemma 1.1. *Suppose that F is a choice function and that $dom(F)$ is stationary in ε. Then there is a stationary $S \subseteq dom(F)$ on which F is constant. Moreover, if this constant value is an ordinal $\geq \varepsilon$, then the map $i \mapsto (\pi_i)^{-1}(F(i))$ is strictly increasing on S.*

Proof. Let $\langle\, G_i \mid i < \varepsilon \,\rangle$ be a sequence, strictly increasing and continuous with respect to inclusion, such that for all $i < \varepsilon$, G_i is a function from ε_i onto X_i. Let $C = \{\, i < \varepsilon \mid \varepsilon_i = i \,\}$. Then C is club and if $i \in C$, then $\mathrm{crit}(\pi_i) = \varepsilon_i = i$. Define H on $\mathrm{dom}(F) \cap C$ by $H(i) = (G_i)^{-1}(F(i))$. Then $H(i) < \varepsilon_i = i$ for all $i \in \mathrm{dom}(H)$. By Fodor's lemma, there is a stationary set $S \subseteq \mathrm{dom}(H)$ on which H is constant. Suppose that $i, j \in S$ and $i < j$. Then $F(i) = G_i(H(i)) = G_j(H(i)) = G_j(H(j)) = F(j)$. Therefore, F is constant on S.

Suppose that F maps into the ordinals. It is clear that $i \mapsto (\pi_i)^{-1}(F(i))$ is nondecreasing on S. Suppose that $i < j$ are both in S and that $F(i) = F(j) \geq \varepsilon$. Then $(\pi_i)^{-1}(F(i)) < \mathrm{OR}^{N_i} < \mathrm{crit}(\pi_j) = \varepsilon_j = (\pi_j)^{-1}(\varepsilon) \leq (\pi_j)^{-1}(F(j))$. $\square$

Notation 1.2. Suppose that $\mathcal{T}$ is an iteration tree. We shall write $\mathcal{M}(\mathcal{T}, \eta)$ for $\mathcal{M}^{\mathcal{T}}_\eta$ and $E(\mathcal{T}, \eta)$ for $E^{\mathcal{T}}_\eta$. If $\mathcal{T}$ has successor length, then we denote the last model of $\mathcal{T}$ by $\mathcal{M}_\infty(\mathcal{T})$.

Notation 1.3. Suppose that N and M are transitive and $\pi \colon N \longrightarrow M$ is sufficiently elementary. Suppose that $\overline{\kappa} < \mathrm{OR}^N$. Let E be the long extender of length $\pi(\overline{\kappa})$ derived from π. Suppose that $\mathcal{P}$ is a premouse with $\overline{\kappa} < \mathrm{OR}^{\mathcal{P}}$ and that E is a long extender over $\mathcal{P}$. If $\mathcal{P}$ is a set premouse and, for some $n < \omega$, $\rho^{\mathcal{P}}_{n+1} \leq \overline{\kappa}$, then we set $n(\mathcal{P}, \overline{\kappa})$ equal to the least such n. Also, if $n = n(\mathcal{P}, \overline{\kappa})$, then we write $\mathrm{ult}(\mathcal{P}, \pi, \overline{\kappa})$ for $\mathrm{ult}_n(\mathcal{P}, E)$. If, on the other hand, $\mathcal{P}$ is a weasel, then we write $\mathrm{ult}(\mathcal{P}, \pi, \overline{\kappa})$ for $\mathrm{ult}_0(\mathcal{P}, E)$.

Fix an inaccessible cardinal $\Gamma < \Omega$ such that $\Gamma > \lambda$. Let W be the canonical very soundness witness for $\mathcal{J}^K_\Gamma$. We assume that $\Gamma \in X_0$.

In [MiSchSt], a single hull $X \prec V_{\Omega+1}$ was considered; N was the transitive collapse of X, and various objects related to the coiteration of (W^N, W) were identified. Here we have a chain of ε-many hulls X_i. We shall use a subscript or a superscript i on the name of the object identified in [MiSchSt] to indicate that it corresponds to the hull X_i.

Notation 1.4.

(a) Let $W^i = W^{N_i}$.

(b) Let $(\overline{\mathcal{T}}^i, \mathcal{T}^i)$ be the pair of iteration trees resulting from the coiteration of (W^i, W).
(c) Let $\theta^i + 1$ be the common length of $\mathcal{T}^i$ and $\overline{\mathcal{T}}^i$.
(d) Let $\Gamma^i = (\pi_i)^{-1}(\Gamma)$.
(e) Let $\vec{\kappa}^i = \langle\, \kappa^i_\alpha \mid \alpha \leq \gamma^i \,\rangle$ be the increasing list of cardinals of $\mathcal{M}_\infty(\overline{\mathcal{T}}^i)$ up to and including Γ^i. That is, $\vec{\kappa}^i$ is the initial segment of the $\aleph$-function up to and including Γ^i in the last model of $\overline{\mathcal{T}}^i$. (In fact, $\gamma^i = \Gamma^i$.)
(f) For $\alpha \leq \gamma^i$, let λ^i_α be the successor cardinal of κ^i_α in $\mathcal{M}_\infty(\overline{\mathcal{T}}^i)$. So $\lambda^i_\alpha = \kappa^i_{\alpha+1}$ whenever $\alpha < \gamma^i$. Put $\vec{\lambda}^i = \langle\, \lambda^i_\alpha \mid \alpha < \gamma^i \,\rangle$ (the sequence of length γ^i).
(g) For $\alpha \leq \gamma^i$, let $\eta^i(\alpha)$ be the least $\eta < \theta^i$ such that $E(\mathcal{T}^i, \eta)$ has generators $\geq \kappa^i_\alpha$, if such an η exists, and put $\eta^i(\alpha) = \theta^i$ if no such η exists.
(h) Let $(\vec{\mathcal{P}}^i, \vec{\lambda}^i)$ be the phalanx of length $\gamma^i + 1$ derived from $\mathcal{T}^i$. This means that for every $\alpha \leq \gamma^i$, $\mathcal{P}^i_\alpha$ is the longest initial segment of $\mathcal{M}(\mathcal{T}^i, \eta^i(\alpha))$ with just the subsets of κ^i_α constructed before λ^i_α. $\mathcal{P}^i_\alpha$ might be a set premouse, or it might be a weasel; we cannot rule out either case.
(i) For $\alpha < \gamma^i$, let $\mathcal{R}^i_\alpha = \mathrm{ult}(\mathcal{P}^i_\alpha, \pi_i, \kappa^i_\alpha)$. This definition assumes that the $(\varepsilon_i, \pi_i(\kappa^i_\alpha))$ long extender derived from π_i measures sets in $\mathcal{P}^i_\alpha$ (which would follow from hypothesis $(1)^i_\alpha$ of Definition 1.5 below). We allow for the possibility that $\mathcal{R}^i_\alpha$ is ill-founded. Even if $\mathcal{R}^i_\alpha$ is well-founded, it seems possible that $\mathcal{R}^i_\alpha$ is not a potential premouse (ppm), as $\dot{F}^{\mathcal{R}^i_\alpha}$, the last predicate of $\mathcal{R}^i_\alpha$, might code an extender fragment, rather than a total extender, over $\mathcal{R}^i_\alpha$.
(j) Let $\pi^i_\alpha : \mathcal{P}^i_\alpha \longrightarrow \mathcal{R}^i_\alpha$ be the ultrapower map.
(k) In [MiSchSt], a premouse $\mathcal{S}^i_\alpha$ is defined from $\mathcal{T}^i \restriction (\eta^i(\alpha)+1)$ and π_i. When $\mathcal{R}^i_\alpha$ is a premouse, then $\mathcal{S}^i_\alpha = \mathcal{R}^i_\alpha$; but otherwise, $\mathcal{S}^i_\alpha \neq \mathcal{R}^i_\alpha$. $\mathcal{S}^i_\alpha$ substitutes for $\mathcal{R}^i_\alpha$ in many roles. The most important difference is that $\mathcal{S}^i_\alpha$ is a premouse, even if $\mathcal{R}^i_\alpha$ is not a premouse.
(l) Let $\mathcal{Q}^i_\alpha$ be the structure defined from $\mathcal{P}^i_\alpha$ by analogy with how $\mathcal{S}^i_\alpha$ was defined from $\mathcal{R}^i_\alpha$. In fact, $\mathcal{S}^i_\alpha = \mathrm{ult}(\mathcal{Q}^i_\alpha, \pi_i, \kappa^i_\alpha)$.
(m) Let $\Lambda^i_\alpha = \sup(\pi_i``\lambda^i_\alpha)$. Then $\Lambda^i_\alpha = (\pi_i(\kappa^i_\alpha)^+)^{\mathcal{S}^i_\alpha}$.

Definition 1.5. For each $i < \varepsilon$ and $\alpha < \gamma^i$, we name the following six properties:

$(1)^i_\alpha$ if $\eta \leq \theta^i$ and $E(\overline{\mathcal{T}}^i, \eta) \neq \emptyset$, then $\mathrm{lh}(E(\overline{\mathcal{T}}^i, \eta)) > \lambda^i_\alpha$;
$(2)^i_\alpha$ $((W, \mathcal{S}^i_\alpha), \pi^i_\alpha(\kappa^i_\alpha))$ is an iterable phalanx;
$(3)^i_\alpha$ $((W^i, \mathcal{Q}^i_\alpha), \kappa^i_\alpha)$ is an iterable phalanx;

$(4)^i_\alpha$ $((\vec{\mathcal{P}}^i \restriction \alpha, W^i), \vec{\lambda}^i \restriction \alpha)$ is an iterable phalanx;
$(5)^i_\alpha$ $((\vec{\mathcal{R}}^i \restriction \alpha, W), \vec{\Lambda}^i \restriction \alpha)$ is an iterable phalanx;
$(6)^i_\alpha$ $((\vec{\mathcal{S}}^i \restriction \alpha, W), \vec{\Lambda}^i \restriction \alpha)$ is an iterable phalanx.

Lemma 1.6. *Consider any $i < \varepsilon$.*

(a) If $(1)^i_\alpha$–$(6)^i_\alpha$ hold for every $\alpha < \gamma^i$, then Theorem 0.1 holds.
(b) If π_i is continuous at ordinals of countable cofinality, then the following implications hold for any $\alpha < \gamma^i$.

$$(6)^i_\alpha \implies (5)^i_\alpha \implies (4)^i_\alpha \implies (1)^i_\alpha$$

$$\forall\ \beta < \alpha\ (4)^i_\beta \implies (3)^i_\alpha$$

$$\forall \beta < \alpha\ (2)^i_\beta \implies (6)^i_\alpha$$

Lemma 1.6 was proved in [MiSchSt], where it was also argued that if ${}^\omega X_i \subset X_i$, then $(3)^i_\alpha \Longrightarrow (2)^i_\alpha$ for every $\alpha < \gamma^i$, and consequently, Theorem 0.1 holds. We shall show that a weaker closure condition on X_i suffices, and holds for a stationary set of $i < \varepsilon$. In light of Lemma 1.6(a) and our denial of Theorem 0.1, we may make the following definition.

Definition 1.7. For any $i < \varepsilon$, define α^i to be the least α such that at least one of $(1)^i_\alpha$–$(6)^i_\alpha$ fails.

If $\mathrm{cf}(i) > \omega$, then π_i is continuous at ordinals of countable cofinality, and so Lemma 1.6 implies that $(1)^i_{\alpha^i}$ and $(3)^i_{\alpha^i}$–$(6)^i_{\alpha^i}$ hold, while $(2)^i_{\alpha^i}$ fails. We shall use the following notation:

$$\kappa^i = \kappa^i_{\alpha^i} \qquad \eta^i = \eta^i(\alpha^i) \qquad \mathcal{P}^i = \mathcal{P}^i_{\alpha^i} \qquad \Lambda^i = \Lambda^i_{\alpha^i} \qquad \mathcal{R}^i = \mathcal{R}^i_{\alpha^i}$$

$$\lambda^i = \lambda^i_{\alpha^i} \qquad \mathcal{Q}^i = \mathcal{Q}^i_{\alpha^i} \qquad \mathcal{S}^i = \mathcal{S}^i_{\alpha^i}$$

From now on, we shall write $\vec{\mathcal{P}}^i$ when we mean $\vec{\mathcal{P}}^i \restriction \alpha^i$. As we shall never again refer to coordinates of $\vec{\mathcal{P}}^i$ beyond α^i, there is no ambiguity. The same goes for $\vec{\mathcal{Q}}^i$, $\vec{\kappa}^i$, $\vec{\lambda}^i$, $\vec{\mathcal{R}}^i$, $\vec{\mathcal{S}}^i$, and $\vec{\Lambda}^i$.

By Lemma 1.1, there is a stationary set $S \subseteq \{\, i < \varepsilon \mid \mathrm{cf}(i) > \omega \wedge \varepsilon_i = i \,\}$ on which the following choice functions are constant:

$$i \mapsto \pi_i(\alpha^i) \qquad i \mapsto \pi_i(\kappa^i) \qquad i \mapsto \pi_i(\eta^i) \qquad i \mapsto n(\mathcal{P}^i, \kappa^i)$$

$$i \mapsto \pi_i(\lambda^i) \qquad i \mapsto n(\mathcal{Q}^i, \kappa^i)$$

Then $i \mapsto \alpha^i$ and $i \mapsto \eta^i$ are non-decreasing on S, while $i \mapsto \kappa^i$ and $i \mapsto \lambda^i$ are strictly increasing on S. (Note that $\kappa^i \geq \varepsilon_i$, since $((W, W), \varepsilon_i)$ is iterable; hence $\pi_i(\kappa^i) \geq \varepsilon$. Apply Lemma 1.1.)

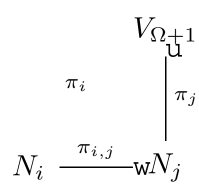
2. A pull-back $\mathcal{Q}^*$ of $\mathcal{Q}^j$

Let S be the stationary set from §1. For the rest of this paper, fix $j \in S \cap \lim(S)$. Since $j \in S$, $(2)^j_{\alpha^j}$ fails. Let $\mathcal{U}$ be an ill behaved iteration tree on $((W, \mathcal{S}^j), \pi_j(\kappa^j))$. We include here the possibility that $\mathcal{S}^j$ itself is ill-founded, which would mean that $\mathcal{U}$ is the empty tree.

Let $\psi\colon M \longrightarrow V_{\Omega+1}$ be elementary with M countable and transitive with everything relevant in the range of ψ. Say $\mathcal{U} = \psi(\mathcal{U}')$, $W = \psi(W')$, $\mathcal{S}^j = \psi(\mathcal{S}')$, and $\pi_j(\kappa^j) = \psi(\kappa')$. $\mathcal{U}'$ is a countable, ill behaved iteration tree on $((W', \mathcal{S}'), \kappa')$, and ψ"$(\mathcal{U}')$, the copy of $\mathcal{U}$ by ψ is a countable, ill behaved iteration tree on $((W, \mathcal{S}^j), \pi_j(\kappa^j))$.

We remark that in [MiSchSt, 3.13], the countable completeness of the extender E_π derived from π was used to find maps from $((W', \mathcal{S}'), \kappa')$ into $((W^j, \mathcal{Q}^j), \kappa^j)$. These maps were then used to copy $\mathcal{U}'$ to an ill behaved iteration tree on $((W^j, \mathcal{Q}^j), \kappa^j)$, thereby contradicting $(3)^j_{\alpha^j}$. But here, E_π is not countably complete.

For the rest of this paper, fix $i \in S$ such that $i < j$ and $\operatorname{ran}(\psi) \cap X_j \subset X_i$. This is possible since $j \in \lim(S)$ and $\operatorname{cf}(j) > \omega$. Let $\pi_{i,j}\colon N_i \longrightarrow N_j$ be the natural embedding, that is, the uncollapse of $(\pi_j)^{-1}(X_i)$. We have the following commutative diagram.



By a standard fine structural construction, we shall define a "pull-back" of $\mathcal{Q}^j$ to a premouse $\mathcal{Q}^*$ that agrees with $\mathcal{Q}^i$ below λ^i. This is done in two cases, depending on whether or not $\mathcal{Q}^j$ is a proper class. In both cases, $\mathcal{Q}^*$ ends up being an appropriate hull in $\mathcal{Q}^j$ of $\pi_{i,j}$"κ^i and a parameter (part of what we need to show is that no new ordinals $< \kappa^j$ get into this hull).

Lemma 2.1. *Suppose that $\mathcal{Q}^j$ is a set premouse. Let $n = n(\mathcal{Q}^j, \kappa^j)$. There is a premouse $\mathcal{Q}^*$ with the following properties:*

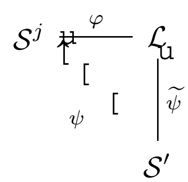
- *(a) $\mathcal{Q}^*$ and W^i agree below λ^i;*
- *(b) $\lambda^i = (\kappa_i^+)^{\mathcal{Q}^*}$;*
- *(c) $\mathcal{Q}^*$ is κ^i-sound;*
- *(d) $n(\mathcal{Q}^*, \kappa^i) = n$;*
- *(e) $\mathcal{Q}^j = ult(\mathcal{Q}^*, \pi_{i,j}, \kappa^i)$;*
- *(f) the ultrapower map $\pi^*\colon \mathcal{Q}^* \longrightarrow \mathcal{Q}^j$ is an n-embedding such that*

$$\pi^* \restriction J^{\mathcal{Q}^*}_{\lambda^i} = \pi_{i,j} \restriction J^{W^i}_{\lambda^i}.$$

Sketch. Recall that $\mathcal{Q}^j$ is κ^j-sound in this case, with $\lambda^j = ((\kappa^j)^+)^{\mathcal{Q}^j}$, and that $\mathcal{Q}^j$ and W^j agree below λ^j. Also, recall that $\mathcal{S}^j = \mathrm{ult}(\mathcal{Q}^j, \pi_j, \kappa^j)$ is $\pi_j(\kappa^j)$-sound, and $\Lambda^j = \sup(\pi``\lambda^j)$. The following claim implies that $\pi_{i,j}$ is continuous at $\lambda^i$; that is, $\lambda^j = \pi_{i,j}(\lambda^i) = \sup(\pi_{i,j}``\lambda^i)$.

Claim 2.1.1. *If $\mathcal{Q}^j$ is a set premouse, then the map $\psi\colon M \longrightarrow V_{\Omega+1}$ is cofinal in Λ^j.*

Suppose, to the contrary, that $\mathrm{ran}(\psi) \cap \Lambda^j$ is bounded in Λ^j. We can use the condensation theorem, [Sch1, 2.8], to find a proper initial segment $\mathcal{L}$ of $\mathcal{S}^j$, an almost Σ_{n+1}-embedding φ, and a Σ_{n+1}-elementary embedding $\widetilde{\psi}$ such that following diagram commutes.



Moreover, we may arrange that $\rho_{n+1}(\mathcal{L}) = \pi_j(\kappa^j)$ and $\sup(\mathrm{ran}(\psi) \cap \Lambda^j) < (\pi_j(\kappa^j)^+)^{\mathcal{L}} = \mathrm{crit}(\varphi) < \Lambda^j$. This allows us to use the pair of maps $((\psi \restriction W), \widetilde{\psi})$ to copy $\mathcal{U}'$ to an ill behaved iteration tree on $((W, \mathcal{L}), \pi_j(\kappa^j))$. Since $\mathcal{S}^j$ and W agree below Λ^j, $\mathcal{L}$ is a proper initial segment of W. Since W is iterable, $((W, \mathcal{L}), \pi_j(\kappa^j))$ is iterable. This contradiction completes the sketch of Claim 2.1.1.

Because $\mathcal{Q}^j$ is κ^j-sound, there is a directed system $D \subset J^{\mathcal{Q}^j}_{\lambda^j}$ such that $\mathcal{Q}^j$ is the direct limit of D. We take D to have as structures, premice of the form:

$$\mathcal{H}^{\mathcal{Q}^j \restriction \xi}_{n+1}(\kappa^j \cup p(\mathcal{Q}^j, \kappa^j))$$

for $\xi < \mathrm{OR}^{\mathcal{Q}^j}$. The maps of D are the natural Σ_n-elementary maps between the structures of D.

Let D^* be the directed system whose structures are of the form: $\pi^{-1}_{i,j}(\mathcal{H})$ for some structure $\mathcal{H}$ of D with $\mathcal{H} \in \mathrm{ran}(\pi_{i,j})$. Likewise for the maps of D^*. Then $D^* \subset J^{\mathcal{Q}^i}_{\lambda^i}$. Let $\mathcal{Q}^*$ be the direct limit of D^*, and let $\pi^*\colon \mathcal{Q}^* \longrightarrow \mathcal{Q}^j$ be the natural map. Clearly, π^* is Σ_n-elementary. But from Claim 2.1.1, it follows that π^* is cofinal, and therefore Σ_{n+1}-elementary. The lemma now follows by standard calculations. □

Lemma 2.2. *Suppose that $\mathcal{Q}^j$ is a weasel. There is a set premouse $\mathcal{Q}^*$ such that, if we set $\lambda^* = ((\kappa^i)^+)^{\mathcal{Q}^*}$, then the following hold:*

> *(a)* $\mathcal{Q}^*$ *and* W^i *agree below* λ^**;*
> *(b)* $\lambda^* \leq \lambda^i$*;*

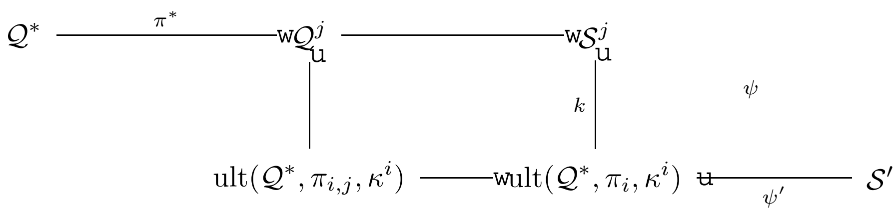
(c) *there is an elementary embedding* $\pi^*\colon \mathcal{Q}^* \longrightarrow \mathcal{Q}^j$ *such that*

$$\pi^* \upharpoonright J_{\lambda^*}^{\mathcal{Q}^*} = \pi_{i,j} \upharpoonright J_{\lambda^*}^{\mathcal{Q}^i}.$$

Sketch. Let D be the directed system consisting of transitive premice of the form $\mathcal{H}_\omega^{\mathcal{Q}^j}(\kappa^j \cup \{x\})$ with $x \in \operatorname{ran}(\psi) \cap |\mathcal{Q}^j|$. Pull D back using $\pi_{i,j}$ to a system D^*. Let $\mathcal{Q}^*$ be the direct limit and let $\pi^*\colon \mathcal{Q}^* \longrightarrow \mathcal{Q}^j$ be the natural elementary embedding. □

Lemma 2.3. $((W, ult(\mathcal{Q}^*, \pi_i, \kappa^i)), \pi_i(\kappa^i))$ *is not iterable.*

Proof. We have the following commutative diagram.



And, $\operatorname{crit}(k) > \pi_i(\kappa^i)$. So we can use the pair of embeddings (ψ, ψ') to copy $\mathcal{U}'$ to an ill behaved iteration tree on $((W, \operatorname{ult}(\mathcal{Q}^*, \pi_i, \kappa^i)), \pi_i(\kappa^i))$ □

It is worth noting that the map from $\operatorname{ult}(\mathcal{Q}^*, \pi_{i,j}, \kappa^i)$ into $\mathcal{Q}^j$ in the diagram above is elementary and has critical point strictly greater than κ^j. In fact, if $\mathcal{Q}^j$ is a set premouse, then the map is the identity.

Definition 2.4.

(a) A premouse $\mathcal{M}$ is ∞-*bad* iff $((W, \operatorname{ult}(\mathcal{M}, \pi_i, \kappa^i)), \pi_i(\kappa^i))$ is a phalanx, but is not iterable.
(b) $\mathcal{M}$ is j-*bad* iff $((W^j, \operatorname{ult}(\mathcal{M}, \pi_{i,j}, \kappa^i)), \kappa^j)$ is a phalanx, but is not iterable.

Corollary 2.5.

(a) $\mathcal{Q}^i$ *is* ∞*-bad.*
(b) $\mathcal{Q}^*$ *is* ∞*-bad.*
(c) $\mathcal{Q}^*$ *is not* j*-bad.*

Proof. By our choice of α^i, $(2)^i_{\alpha^i}$ fails. Therefore, clause (a) holds. Clause (b) follows from Lemma 2.3. Recall that $(3)^j_{\alpha^j}$ holds and asserts that $((W^j, \mathcal{Q}^j), \kappa^j)$ is an iterable phalanx. Since $\operatorname{ult}(\mathcal{Q}^*, \pi_{i,j}, \kappa^i)$ embeds into $\mathcal{Q}^j$ with critical point greater than κ^j, clause (c) holds. □

Lemma 2.6. *If* $\mathcal{M}$ *is* ∞*-bad and* $\mathcal{M} \in ran(\pi_j)$*, then* $\mathcal{M}$ *is* j*-bad. In particular,* $\mathcal{Q}^i$ *is* j*-bad.*

Proof. Since π_j is elementary and $\pi_j^{-1}(\pi_i \restriction \pi_i(\kappa^i)) = \pi_{i,j} \restriction \kappa^j$, we have that

$$N_j \models \text{“}((W^j, \mathrm{ult}(\pi_j^{-1}(\mathcal{M}), \pi_{i,j}, \kappa^i)), \kappa^j) \text{ is not iterable.”}$$

By absoluteness (using the generic branch formulation of iterability),

$$((W^j, \mathrm{ult}(\pi_j^{-1}(\mathcal{M}), \pi_{i,j}, \kappa^i)), \kappa^j)$$

is not iterable. By the shift lemma, we have a map k with $\mathrm{crit}(k) \geq \kappa^j$ so that the following diagram commutes:

$$\begin{array}{ccc}
\pi_j^{-1}(\mathcal{M}) & \xrightarrow{\quad\pi_j\quad} & \mathcal{M} \\
\downarrow & & \downarrow \\
\mathrm{ult}(\pi_j^{-1}(\mathcal{M}), \pi_{i,j}, \kappa^i) & \xrightarrow[k]{} & \mathrm{ult}(\mathcal{M}, \pi_{i,j}, \kappa^i)
\end{array}$$

An ill behaved iteration tree on $((W^j, \mathrm{ult}(\pi_j^{-1}(\mathcal{M}), \pi_{i,j}, \kappa^i)), \kappa^j)$ can be copied to an ill behaved iteration tree on $((W^j, \mathrm{ult}(\mathcal{M}, \pi_{i,j}, \kappa^i)), \kappa^j)$ using the pair $((\mathrm{id} \restriction W^j), k)$. So $\mathcal{M}$ is j-bad. $\square$

In light of Corollary 2.5(c) and Lemma 2.6, we would have a contradiction if we could show that $\mathcal{Q}^i$ embeds into $\mathcal{Q}^*$ with critical point at least κ^i. This is a first approximation to our general strategy.

Definition 2.7. A premouse $\mathcal{M}$ is *i-good* iff $((\vec{\mathcal{P}}^i, \mathcal{M}), \vec{\lambda}^i)$ is an iterable phalanx.

Fact 2.8. *(Steel). $\mathcal{Q}^i$ is i-good.*

The fact is proved using the methods of [St, §9]. Much of the rest of this section will be taken up with showing that $\mathcal{Q}^*$ is also i-good.

Definition 2.9. Let $\Lambda^{i,j}_\beta = \sup(\pi_{i,j}\text{“}\lambda^i)$, $\mathcal{R}^{i,j}_\beta = \mathrm{ult}(\mathcal{P}^i, \pi_{i,j}, \kappa^i_\beta)$, and $\mathcal{S}^{i,j}_\beta = \mathrm{ult}(\mathcal{Q}^i_\beta, \pi_{i,j}, \kappa^i_\beta)$, for any $\beta < \alpha^i$.

Lemma 2.10. *Let $\beta < \alpha^i$. There is an iteration tree $\mathcal{V}_\beta$ on W such that*

(a) $\mathcal{V}_\beta$ extends $\mathcal{T}^j \restriction (\eta^j(\pi_{i,j}(\beta)) + 1)$;
(b) $\mathcal{V}_\beta$ has a last model;
(c) there is $\mathcal{N}_\beta$, a premouse, and $\varphi_\beta \colon \mathcal{S}^{i,j}_\beta \longrightarrow \mathcal{N}_\beta$, an elementary embedding, such that $\mathcal{N}_\beta$ is an initial segment of $\mathcal{M}_\infty(\mathcal{V}_\beta)$, and crit$(\varphi_\beta) \geq \pi_{i,j}(\kappa^i_\beta)$.

Sketch. Fix $\beta < \alpha^i$. Intuitively, the idea is to compare $\mathcal{S}^{i,j}$ and $\mathcal{T}^j$. Suppose that

$$((\vec{\mathcal{P}}^j \restriction \pi_{i,j}(\beta), \mathcal{S}^{i,j}_\beta), \vec{\lambda}^j \restriction \pi_{i,j}(\beta))$$

and

$$((\vec{\mathcal{P}}^j \restriction \pi_{i,j}(\beta), \mathcal{P}^j_{\pi_{i,j}(\beta)}), \vec{\lambda}^j \restriction \pi_{i,j}(\beta))$$

are coiterable, and that $(\mathcal{U}, \mathcal{V})$ is the pair of iteration trees resulting from the coiteration. Then, by standard arguments, the iteration tree $\mathcal{V}$ can be rearranged as the iteration tree $\mathcal{V}_\beta$ that we are looking for, with the embedding along the branch from $\mathcal{S}^{i,j}_\beta$ to $\mathcal{M}_\infty(\mathcal{U})$ serving as φ_β. The details are like those in the proof of [MiSchSt, 3.14 and 3.15] (the lemmas that derive $(1)^j_\alpha$ from $(4)^j_\alpha$, for $\alpha = \pi_{i,j}(\beta)$). The second phalanx displayed above is iterable, since W is. The first phalanx is also iterable, as we now argue.

By a standard copying argument, it is enough to show that the phalanx

$$((\vec{\mathcal{R}}^j \restriction \pi_{i,j}(\beta), \mathcal{S}^i_\beta), \vec{\Lambda}^j \restriction \pi_{i,j}(\beta))$$

is iterable. Briefly, for each $\gamma < \pi_{i,j}(\beta)$, we can copy using the ultrapower map $\pi^j_\gamma \colon \mathcal{P}^j_\gamma \longrightarrow \mathcal{R}^j_\gamma = \mathrm{ult}(\mathcal{P}^j_\gamma, \pi_j, \kappa^j_\gamma)$ on $\mathcal{P}^j_\gamma$. And, we use the map from the diagram

$$\begin{array}{ccc}
\mathcal{Q}^i_\beta & \xrightarrow{\quad \pi^i_\beta \quad} & \mathcal{S}^i_\beta = \mathrm{ult}(\mathcal{Q}^i_\beta, \pi_i, \kappa^i_\beta) \\
 & \searrow^{\pi^{i,j}_\beta} & \Big\uparrow \text{identity} \\
 & \mathcal{S}^{i,j}_\beta & \longrightarrow \mathrm{ult}(\mathcal{S}^{i,j}_\beta, \pi_j, \pi_{i,j}(\kappa^i_\beta))
\end{array}$$

between the starting models $\mathcal{S}^{i,j}_\beta$ and $\mathcal{S}^i_\beta$. All the copying maps agree with π_j out to the appropriate ordinals.

Next we indicate why it is enough to show that

$$((\vec{\mathcal{S}}^j \restriction \pi_{i,j}(\beta), \mathcal{S}^i_\beta), \vec{\Lambda}^j \restriction \pi_{i,j}(\beta))$$

is iterable. Recall [MiSchSt, 3.18], the lemma that says $(6)^j_\alpha \implies (5)^j_\alpha$ whenever $\alpha < \alpha^j$, in particular, when $\alpha = \pi_{i,j}(\beta)$. The proof involved a kind of enlargement that differed from the usual copying construction, that used the details of how each $\mathcal{S}^j_\gamma$ was obtained from $\mathcal{R}^j_\gamma$. It might be helpful to recall that the enlarged iteration tree had a different tree structure from the given iteration tree. Without giving the details, if we carry out the analogous enlargement construction here, we see how to reduce the iterability of $((\vec{\mathcal{R}}^j \restriction \pi_{i,j}(\beta), \mathcal{S}^i_\beta), \vec{\Lambda}^j \restriction \pi_{i,j}(\beta))$ to that of

$$((\vec{\mathcal{S}}^j \restriction \pi_{i,j}(\beta), \mathcal{S}^i_\beta), \vec{\Lambda}^j \restriction \pi_{i,j}(\beta)).$$

Now, we outline how to reduce the iterability of the last phalanx to that of a W-based phalanx. First, because $(1)^i_\beta$ and $(2)^i_\beta$ hold, [MiSchSt, 3.12] gives an iteration tree $\mathcal{Y}$ on W such that $\mathcal{Y}$ has a successor length, and all extenders used on $\mathcal{Y}$ have length at least Λ^i_β, and the corollary also gives an elementary embedding k from $\mathcal{S}^i_\beta$ into an initial segment $\mathcal{A}$ of $\mathcal{M}_\infty(\mathcal{Y})$, with $\mathrm{crit}(k) \geq \pi_i(\kappa^i_\beta)$. Similarly, for each $\gamma < \pi_{i,j}(\beta)$, because $(1)^j_\gamma$ and $(2)^j_\gamma$ hold, [MiSchSt, 3.12] gives an iteration tree $\mathcal{Y}_\gamma$ on W such that $\mathcal{Y}_\gamma$ has a last model, and all extenders used on $\mathcal{Y}_\gamma$ have length at least Λ^j_γ, and the corollary also gives an elementary embedding k_γ from $\mathcal{S}^j_\gamma$ into an initial segment $\mathcal{A}_\gamma$ of $\mathcal{M}_\infty(\mathcal{Y}_\gamma)$, with $\mathrm{crit}(k_\gamma) \geq \pi_j(\kappa^j_\gamma)$. Using the sequence of maps $(\langle\, k_\gamma \mid \gamma < \pi_{i,j}(\beta)\,\rangle,\ k)$ we can copy a putative iteration tree on

$$((\vec{\mathcal{S}}^j \restriction \pi_{i,j}(\beta), \mathcal{S}^i_\beta), \vec{\Lambda}^j \restriction \pi_{i,j}(\beta))$$

to an iteration tree on $((\langle \mathcal{A}_\gamma \mid \gamma < \pi_{i,j}(\beta)\rangle, \mathcal{A}), \vec{\Lambda}^j \restriction \pi_{i,j}(\beta))$. This last phalanx is W-based, and therefore iterable, by the main result in [St, §9].

There is a small subtlety in the last copying argument, since we must allow for the possibility that $\mathrm{crit}(k_\beta) = \pi_i(\kappa^i_\beta)$. It is the variation of the usual copying procedure, as explained in [St, §6], and also in the proof of [MiSchSt, 3.16] (deriving $(3)^j_\alpha$ from $(4)^j_\alpha$), that we have in mind. □

Lemma 2.11. *$\mathcal{Q}^*$ is i-good.*

Sketch. We must see that the phalanx $((\vec{\mathcal{P}}^i, \mathcal{Q}^*), \vec{\lambda}^i)$ is iterable. By the usual copying construction, it is enough to show that $((\vec{\mathcal{R}}^{i,j}, \mathcal{Q}^j), \vec{\Lambda}^{i,j})$ is iterable.

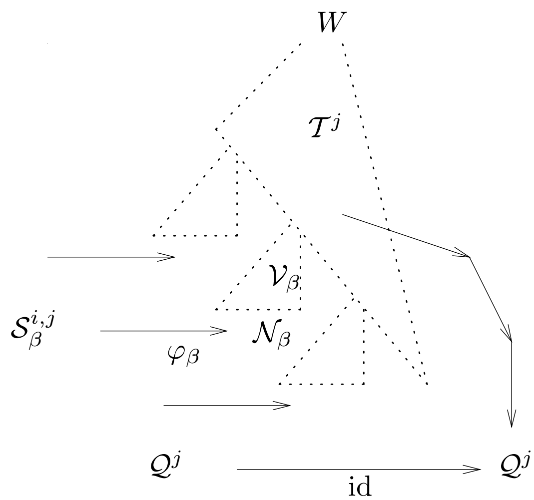
We remark that $\vec{\mathcal{S}}^{i,j}$ is obtained from $\vec{\mathcal{R}}^{i,j}$ as $\vec{\mathcal{Q}}^i$ was obtained from $\vec{\mathcal{P}}^i$. The proof is like that of the claim in the proof of [MiSchSt, 3.13].

Now recall the proof that $(6)^i_\beta \implies (5)^i_\beta$ for $\beta < \alpha^i$, that is, the proof of [MiSchSt, 3.18]. Using an enlargement similar to the one introduced there, we see that it is enough to show that $((\vec{\mathcal{S}}^{i,j}, \mathcal{Q}^j), \vec{\Lambda}^{i,j})$ is iterable.

For $\beta < \alpha^i$, let $\varphi_\beta \colon \mathcal{S}^{i,j}_\beta \longrightarrow \mathcal{N}_\beta$ be the map from Lemma 2.10. Then copying using $(\langle \varphi_\beta \mid \beta < \alpha^i\rangle, \mathrm{id} \restriction |\mathcal{Q}^j|)$ can be used to see that it is enough to show that

$$((\langle \mathcal{N}_\beta \mid \beta < \alpha^i\rangle, \mathcal{Q}^j), \vec{\Lambda}^{i,j})$$

is iterable. The following picture illustrates the situation.



In the last copying construction, we must allow for the possibility that $\text{crit}(\varphi_\beta) = \pi_{i,j}(\kappa^i_\beta)$. It is the variation of the usual copying procedure, as explained in [St, §6], and also in the proof of [MiSchSt, 3.16], that we have in mind.

Fact 2.11.1 *(Steel).* $((\langle\mathcal{N}_\beta \mid \beta < \alpha^i\rangle, \mathcal{Q}^j), \vec{\Lambda}^{i,j})$ *is iterable.*

The fact is proved by the methods of [St, §9], and the lemma follows. □

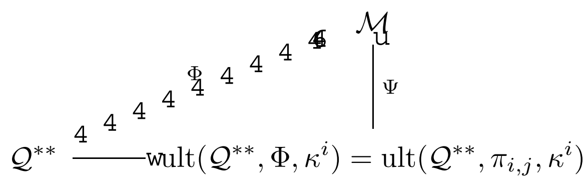
Recall that $\mathcal{Q}^*$ is not j-bad; this was Lemma 2.3(c). The following lemma is a strengthening of this fact. It might be read as saying that $\mathcal{Q}^*$ is "hereditarily not j-bad".

Lemma 2.12. *Suppose that $\mathcal{I}$ is an iteration tree on $((\vec{\mathcal{P}}^i, \mathcal{Q}^*), \kappa^i)$ and that $\mathcal{Q}^{**}$ is an initial segment of $\mathcal{M}_\infty(\mathcal{I})$. Then $\mathcal{Q}^{**}$ is not j-bad.*

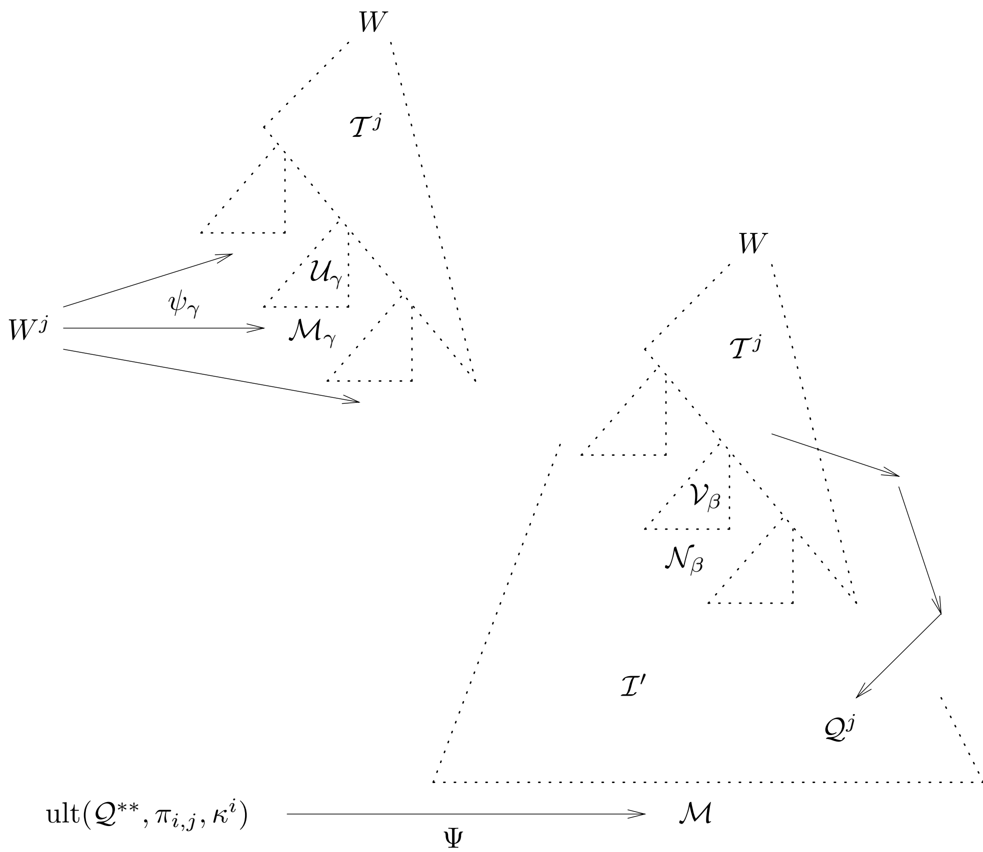
Sketch. Let $\mathcal{I}$ and $\mathcal{Q}^{**}$ be as in the statement of the lemma. We remark that there is no assumption on which model starts the main branch of $\mathcal{I}$. We must see that $((W^j, \text{ult}(\mathcal{Q}^{**}, \pi_{i,j}, \kappa^i)), \kappa^j)$ is an iterable phalanx.

First, let $\mathcal{I}'$ be the iteration tree on a phalanx with starting model $\mathcal{Q}^j$ and back-up models $\mathcal{N}_\beta$ for $\beta < \alpha^i$, that comes from $\mathcal{I}$ by the copying–enlarging–copying procedure done in the proof of Lemma 2.11. Let Φ be the map from $\mathcal{Q}^{**}$ into an initial segment $\mathcal{M}$ of $\mathcal{M}_\infty(\mathcal{I}')$ that comes from this procedure. Then $\Phi \restriction \kappa^j = \pi_{i,j} \restriction \kappa^j$. So we have a map Ψ with

$\mathrm{crit}(\Psi) \geq \kappa^j$, and the following commutative diagram:



Now consider any $\gamma < \alpha^j$. Since $(4)^j_\gamma$ holds, we can apply [MiSchSt, 3.14] to get an iteration tree $\mathcal{U}_\gamma$ extending $\mathcal{T}^j \restriction (\eta^j(\gamma) + 1)$, and an elementary embedding ψ_γ from W^j into an initial segment $\mathcal{M}_\gamma$ of $\mathcal{M}_\infty(\mathcal{U}_\gamma)$ such that $\mathrm{crit}(\psi_\gamma) \geq \kappa^j_\gamma$. The following picture illustrates the situation:



We can use the system of maps $(\langle \psi_\gamma \mid \gamma < \alpha^j \rangle, \Psi)$ to copy a putative iteration tree on $((W^j, \mathrm{ult}(\mathcal{Q}^{**}, \pi_{i,j}, \kappa^i)), \kappa^j)$ to a putative iteration tree on

$$((\langle \mathcal{M}_\gamma \mid \gamma < \alpha^j \rangle, \mathcal{M}), \vec{\lambda}^j).$$

Fact 2.12.1 *(Steel).* $((\langle \mathcal{M}_\gamma \mid \gamma < \alpha^j \rangle, \mathcal{M}), \vec{\lambda}^j)$ *is iterable.*

The fact is proved by the methods of [St, §9], and the lemma follows. $\square$

3. A minimal i-good, j-bad premouse $\mathcal{M}^*$

We continue the proof of Theorem 0.1. Our strategy is to find a premouse $\mathcal{M}^*$ which, like $\mathcal{Q}^i$, is both i-good and j-bad. Since $\mathcal{Q}^*$ is also i-good, we shall be able to coiterate the phalanxes $((\vec{\mathcal{P}}^i, \mathcal{M}^*), \vec{\lambda}^i)$ and $((\vec{\mathcal{P}}^i, \mathcal{Q}^*), \vec{\lambda}^i)$. We shall choose $\mathcal{M}^*$ so that this coiteration yields a map φ from $\mathcal{M}^*$ into the last model on the $\mathcal{Q}^*$-side, with $\mathrm{crit}(\varphi) \geq \kappa^i$. Using the next lemma, and Lemma 2.12, we shall derive a contradiction.

Lemma 3.1. *Let* $\varphi\colon \mathcal{M} \longrightarrow \widetilde{\mathcal{M}}$ *be an* n*-embedding with* $\mathrm{crit}(\varphi) \geq \kappa^i$.

(a) If $\mathcal{M}$ *is* ∞*-bad, then* $\widetilde{\mathcal{M}}$ *is* ∞*-bad.*
(b) If $\mathcal{M}$ *is* j*-bad, then* $\widetilde{\mathcal{M}}$ *is* j*-bad.*

Proof. We prove (a), the proof of (b) being almost identical. Consider the diagram

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\quad\varphi\quad} & \widetilde{\mathcal{M}} \\ \big\downarrow & & \big\downarrow \\ \mathrm{ult}(\mathcal{M}, \pi_i, \kappa^i) & \xrightarrow[\quad k \quad]{} & \mathrm{ult}(\widetilde{\mathcal{M}}, \pi_i, \kappa^i) \end{array}$$

where

$$k([a,f]^{\mathcal{M}}_{E_{\pi_i}}) = [a, \varphi(f)]^{\widetilde{\mathcal{M}}}_{E_{\pi_i}}.$$

Then k is an n-embedding with $\mathrm{crit}(k) \geq \kappa^j$ (in particular, $\mathrm{ult}(\widetilde{\mathcal{M}}, \pi_i, \kappa^i)$ is a premouse). So, a copying argument using the pair of maps $(\mathrm{id} \restriction W^j, k)$ reduces the iterability of $((W^j, \mathrm{ult}(\mathcal{M}, \pi_i, \pi_i(\kappa^i)), \kappa^i), \pi_i(\kappa^i))$ to that of $((W^j, \mathrm{ult}(\widetilde{\mathcal{M}}, \pi_i, \pi_i(\kappa^i)), \kappa^i), \pi_i(\kappa^i))$. $\square$

Lemma 3.2. *There is an* i*-good,* ∞*-bad premouse* $\mathcal{M}^*$ *with the following properties. Suppose that* $\mathcal{I}$ *is an iteration tree of successor length on* $((\vec{\mathcal{P}}^i, \mathcal{M}^*), \vec{\lambda}^i)$*. Then*

(a) No proper initial segment of $\mathcal{M}_\infty(\mathcal{I})$ *is* ∞*-bad.*
(b) Suppose that $\mathcal{M}_\infty(\mathcal{I})$ *is* ∞*-bad. Then* $\mathcal{M}_\infty(\mathcal{I})$ *lies above* $\mathcal{M}^*$ *in the tree ordering of* $\mathcal{I}$ *and there is no dropping along the branch from* $\mathcal{M}^*$ *to* $\mathcal{M}_\infty(\mathcal{I})$*. That is,* $\mathrm{root}^{\mathcal{I}}(\mathit{lh}(\mathcal{I}) - 1) = \alpha^i$ *and*

$$\mathcal{D}^{\mathcal{I}} \cap (\alpha^i, (\mathit{lh}(\mathcal{I}) - 1)) = \emptyset.$$

Sketch. We define a sequence $\langle \mathcal{M}_0, \ldots, \mathcal{M}_n, \ldots \rangle$ by induction. Put $\mathcal{M}_0 = \mathcal{Q}^i$. Suppose that $\mathcal{M}_n$ has been defined, $\mathcal{M}_n$ is i-good, but there is an iteration tree $\mathcal{I}$ on $((\vec{\mathcal{P}}^i, \mathcal{M}_n), \vec{\lambda}^i)$ witnessing that (a) or (b) fail for $\mathcal{M}_n$. Let $\mathcal{I}_n$ be such an iteration tree $\mathcal{I}$. Let $\mathcal{M}_{n+1}$ be the shortest initial segment $\mathcal{N}$ of $\mathcal{M}_\infty(\mathcal{I}_n)$ such that $\mathcal{N}$ is ∞-bad.
Fact 3.2.1 *(Steel). $\mathcal{M}_{n+1}$ is i-good.*

Fact 3.2.2 *(Steel). For some n, $\mathcal{M}_{n+1}$ is not defined.*

Both facts are iterability properties of W (recall that $\mathcal{M}_0 = \mathcal{Q}^i$ is an initial segment of a model on $\mathcal{T}^i$, and that $\mathcal{T}^i$ is an iteration tree on W). They are proved by the methods of [St, §9]. □

We remark that Lemma 3.2 expresses what we mean by "minimal" in the title of this section. Fix some premouse $\mathcal{M}^*$ as in Lemma 3.2. By elementarity, we may choose $\mathcal{M}^*$ so that $\mathcal{M}^* \in \operatorname{ran}(\pi_j)$; in fact, the proof of Lemma 3.2 gives such an $\mathcal{M}^*$. By Lemma 2.6, $\mathcal{M}^*$ is also j-bad.

For the rest of this paper, let $(\mathcal{U}, \mathcal{V})$ be the pair of iteration trees resulting from the coiteration of $((\vec{\mathcal{P}}^i, \mathcal{M}^*), \vec{\lambda}^i)$ versus $((\vec{\mathcal{P}}^i, \mathcal{Q}^*), \vec{\lambda}^i)$.

Lemma 3.3. *$root^{\mathcal{U}}(lh(\mathcal{U}) - 1) = \alpha^i$ and $\mathcal{D}^{\mathcal{V}} \cap (\alpha^i, (lh(\mathcal{V}) - 1)) = \emptyset$.*

Proof. Suppose otherwise.

Case A. $\mathcal{Q}^j$ is a set premouse.

In Case A, we have available to us the κ^i-soundness of $\mathcal{Q}^*$. The usual fine structural considerations show that $\mathcal{Q}^* = \mathcal{M}_\infty(\mathcal{V})$ is an initial segment of $\mathcal{M}_\infty(\mathcal{U})$. Since $\mathcal{Q}^*$ is ∞-bad, Lemma 3.2(a) tells us that $\mathcal{Q}^*$ cannot be a proper initial segment of $\mathcal{M}_\infty(\mathcal{U})$. So $\mathcal{Q}^* = \mathcal{M}_\infty(\mathcal{U})$. Thus, $\mathcal{M}_\infty(\mathcal{U})$ is ∞-bad, contradicting Lemma 3.2(b).

Case B. $\mathcal{Q}^j$ is a weasel.

In Case B, we have available to us that $\mathcal{Q}^*$ is a model of ZFC. The usual fine structural considerations show that $\mathcal{M}_\infty(\mathcal{V})$ is an initial segment of $\mathcal{M}_\infty(\mathcal{U})$, and that $\operatorname{root}^{\mathcal{V}}(\operatorname{lh}(\mathcal{V}) - 1) = \alpha^i$. But then, by Lemma 3.1(a), $\mathcal{M}_\infty(\mathcal{V})$ is ∞-bad. By Lemma 3.2(a), we must have that $\mathcal{M}_\infty(\mathcal{V}) = \mathcal{M}_\infty(\mathcal{U})$. But this contradicts Lemma 3.2(b). □

Lemma 3.4. *$\mathcal{M}_\infty(\mathcal{U})$ is a initial segment of $\mathcal{M}_\infty(\mathcal{V})$.*

Proof. Otherwise, $\mathcal{M}_\infty(\mathcal{V})$ is a proper initial segment of $\mathcal{M}_\infty(\mathcal{U})$,

$$\text{root}^{\mathcal{V}}(\text{lh}(\mathcal{V})-1)=\alpha^i\ ,$$

and $\mathcal{D}^{\mathcal{V}}\cap(\alpha^i,(\text{lh}(\mathcal{V})-1))=\emptyset$. But then, by Lemma 3.1(b), $\mathcal{M}_\infty(\mathcal{V})$ is ∞-bad, which contradicts Lemma 3.2(a). □

By lemma 3.1(b), we have that $\mathcal{M}_\infty(\mathcal{U})$ is j-bad. It follows easily that $\mathcal{M}_\infty(\mathcal{V})$ is j-bad. But, by Lemma 2.12, $\mathcal{M}_\infty(\mathcal{V})$ is not j-bad. This contradiction completes the proof of Theorem 0.1.

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