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CONVEXIFIABILITY AND SUPPORTING FUNCTIONS IN $\mathbb{C}^2$

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Let p be a point on the boundary of a smooth domain $\Omega \subset \mathbb{C}^2$. A holomorphic function defined in a neighbourhood U of p, which satisfies

$$\{q \in U \mid f(q) = 0\} \cap \bar{\Omega} = \{p\}$$

is called a supporting function at p. The reciprocal of such function provides a function locally defined on Ω which blows up precisely at p. There is no such function if the Levi form at p is negative.

A related problem is to find a holomorphic function in a neighbourhood of p whose absolute value on $\bar{\Omega} \cap U$ attains its maximum at p (a holomorphic peak function). More generally a C^k-peak function, for $k = 0, 1, \ldots, \infty$, is defined to be a holomorphic function in $\Omega \cap U$ which belongs to $C^k(\bar{\Omega} \cap U)$ and satisfies $f(p) = 1$ and $|f(q)| < 1$ for $q \in \bar{\Omega} \cap U \setminus \{p\}$. We will use the term smooth peak function when $k = \infty$.

By local convexifiability we will mean the existence of local holomorphic coordinates in a full neighbourhood of p such that $b\Omega$ is convex in the induced linear space.

One of the basic properties of strongly pseudoconvex domains is that in a neighbourhood of any boundary point there is a biholomorphism which transforms the domain into a strongly convex domain. In particular it gives immediately a supporting function and a holomorphic peak function.

In this paper we consider some weakly pseudoconvex domains and two related questions: local convexifiability and existence of a supporting function. Since it is not any more difficult, we formulate the results also for peak functions. The passage from supporting functions to smooth peak functions is due to a result of Bloom in [B].

It follows from the Kohn-Nirenberg example that a weakly pseudoconvex domain need not be locally convexifiable and need not have a supporting function at a boundary point where the Levi form degenerates (see [KN]).

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The same domain, by a result of Fornæss, does not admit a C^1-peak function (see [F]). One can contrast this fact with the result of Bedford and Fornæss which gives a continuous peak function for weakly pseudoconvex domains in $\mathbb{C}^2$ (see [BF] and also [FM]).

The Kohn-Nirenberg example is the domain

$$\{(z,w) \in \mathbb{C}^2 \mid Re\ w > |z|^8 + \frac{15}{7}|z|^2 Re\ z^6\},$$

where the point in question is the origin. In this example, 0 is a point of finite type 8.

We will assume that $\Omega \subseteq \mathbb{C}^2$ is a pseudoconvex domain with C^∞ boundary and $p \in b\Omega$ is a point of finite type (see [K] for the original definition in $\mathbb{C}^2$ and [D] for a general definition). Recall that p is a point of type k if in suitable local holomorphic coordinates (z,w) centered at p the boundary is described by

$$Re\ w = P(z,\bar{z}) + o(|z|^k, Im\ w), \tag{1}$$

where

$$P(z,\bar{z}) = |z|^k + \sum_{j=2,4,\dots,k-2} |z|^{k-j} Re(a_j z^j) \tag{2}$$

for some $a_j \in \mathbb{C}$. The model domain at p is the domain

$$\{(z,w) \in \mathbb{C}^2 \mid Re\ w > P(z,\bar{z})\}.$$

One possible question is whether the model domain admits a C^∞-peak function and a supporting function or is convexifiable. Another question is how much the model domain determines the properties of $b\Omega$ at p. For supporting functions and smooth peak functions, the model domain almost always decides whether they exist at p on Ω. We prove the following sufficient condition:

Proposition 1. *Let the model domain at p be given by (2). If*

$$\sum_{j \nmid k} |a_j| + \sum_{j|k} \frac{k^2 - j^2}{k^2}|a_j| < 1, \tag{3}$$

then there is a supporting function and a smooth peak function at p.

On the other hand, convexity of the model domain is not sufficient to guarantee that $b\Omega$ is locally convexifiable; here the higher order terms also play a role. We give a sufficient condition for convexity of the model domain:

Proposition 2. *If*

$$\sum_{j^2 \geq 3k-2} \frac{j^2-k}{k}|a_j| + \sum_{j^2<3k-2} \sqrt{\frac{(4k-4)(k^2-j^2)}{(4k-j^2-4)k^2}}|a_j| < 1,$$

then P is convex.

Both conditions follow from exact conditions which are computable for model domains of the Kohn-Nirenberg type, defined by

$$P_a^{k,l}(z,\bar{z}) = |z|^k + a|z|^{k-l} Re\ z^l, \tag{4}$$

where $l < k$ is an even integer and $a > 0$.

In part 2 we also formulate an exact condition for convexifiability of domains defined by (4) when $l \nmid k$, and compare it with the conditions for the existence of a supporting function.

1. Supporting functions

Let $M_a^{k,l}$ denote the domain in $\mathbb{C}^2$ defined by

$$\{(z,w) \in \mathbb{C}^2 \ \mid \ u > P_a^{k,l}(z)\},$$

where $w = u + iv$. The Levi form on $bM_a^{k,l}$ is equal to

$$\frac{1}{4}\Delta(P_a^{k,l})(z) = k^2|z|^{k-2} + a(k-l)(k+l)|z|^{k-l-2} Re\ z^l.$$

It follows that $M_a^{k,l}$ is pseudoconvex for

$$a \leq \frac{k^2}{k^2-l^2}. \tag{5}$$

If $a < 1$, then $P_a^{k,l}(z) > \epsilon|z|^k$ and there is a linear supporting function at the origin. By a result due to Bloom, pseudoconvexity is in this case enough to give also a smooth peak function at p (see [B]).

For $a > 1$, we first consider the case when $l \nmid k$. A simple modification of the proof of the main proposition in [F] (for peak functions) and of Proposition 28.1 in [FS] (for supporting functions) gives

Lemma 1. *Suppose that the model domain at $p \in b\Omega$ is $M_a^{k,l}$, and $l \nmid k$. If $a > 1$, then there is no C^1-peak function and no supporting function at p.*

(See [Ko] for more details.) In the remaining case, when $a = 1$, there is a supporting function on $M_1^{k,l}$ at 0, e.g., $f(z,w) = w + z^{kl}$, and a smooth peak function $g(z,w) = \exp(-w - z^{kl} + w^2)$. However, if $M_1^{k,l}$ is a model domain at p, then in general there need not be a smooth peak function at p. An example due to Bloom shows that there need not be a C^{13}-peak function (see [B]).

Now we consider the case when $l \mid k$.

Lemma 2. *If the model domain at p is given by $M_a^{k,l}$ and $l \mid k$, then there exists a supporting function and a C^∞-peak function at p.*

Proof. We need to consider $a \geq 1$. In polar coordinates, (1.4) becomes $u = r^k(1 + a\cos l\theta)$. Let $k = m\ l$. The pseudoconvexity condition is now $a \leq \frac{m^2}{m^2-1}$. For a real number c, we define new coordinates by $w^* = w + \frac{c}{2}z^k,\ z^* = z$. In such coordinates, after dropping stars,

$$u = r^k(1 + a\cos l\theta + c\cos k\theta).$$

We will show that there is a constant c_0 such that $1+a\cos l\theta+c_0\cos k\theta > 0$. It is enough to consider $a = \frac{m^2}{m^2-1}$. Denote $\phi = l\theta$, and put $c = (-1)^m\frac{1}{m^2-1}$. We will prove that the function

$$f(\phi) = 1 + \frac{m^2}{m^2-1}\cos\phi + (-1)^m\frac{1}{m^2-1}\cos m\phi$$

is nonnegative. Note that c is chosen so that $f(\pi) = 0$. We calculate the minimum of f. If f has minimum at ϕ, then

$$-\frac{m^2}{m^2-1}\sin\phi + (-1)^{m+1}\frac{m}{m^2-1}\sin m\phi = 0,$$

i.e., $m\sin\phi = (-1)^{m+1}\sin m\phi$. Concavity of the function $\sin\phi$ in the interval $(0,\pi)$ implies that $\sin\phi > \frac{1}{m}\sin m\phi$ when $\phi \in (0,\pi)$, while the reverse holds for $\phi \in (\pi, 2\pi)$. So the only extrema of f are at $\phi = 0$ and $\phi = \pi$, namely minimum at π and maximum at 0. Since $f(\pi) = 0$, we have $f(\phi) > 0$ for $\phi \neq \pi$.

Since $(-1)^m\cos m\pi > 0$, it follows that if $\epsilon > 0$ is small enough, then for $c_0 = c + \epsilon$

$$1 + a\cos l\theta + c_0\cos k\theta > 0. \tag{6}$$

Therefore, in the new coordinates, the function $f(z,w) = w$ is a supporting function at 0. Since (6) implies that $u \geq \epsilon|z|^k$ on $b\Omega$, the existence of a C^∞-peak function follows again from [B]. $\square$

Proof of Proposition 1. We rewrite P as

$$P(z) = \sum_j P_j(z) + (1-A)|z|^k$$

where $P_j(z) = \frac{k^2-j^2}{k^2}|a_j||z|^k + |z|^{k-j}Re\ a_jz^j$ if $j \mid k$ and $P_j(z) = |a_j||z|^k + |z|^{k-j}Re\ a_jz^j$ if $j \nmid k$, and where A is the sum on the left hand side of (3).

For $j \mid k$, by Lemma 2, there exists a $c_j \in \mathbb{C}$ such that $P_j(z) + Re\ c_j z^k \geq 0$, while if $j \nmid k$, then already $P_j \geq 0$. So if we take $c = \sum_{j|k} c_j$, then

$$P(z) + Re\ cz^k > (1 - A)|z|^k.$$

That gives a supporting function. The existence of a C^∞-peak function follows again from [B]. □

2. Local convexifiability

Now we turn to the question about convexity of model domains, and convexifiability of Ω near p. First we calculate the conditions for convexity of $M_a^{k,l}$.

Lemma 3. *If $l^2 \geq 3k - 2$, then $M_a^{k,l}$ is convex for*

$$a \leq \frac{k}{l^2 - k}.$$

If $l^2 < 3k - 2$, then $M_a^{k,l}$ is convex for

$$a^2 \leq \frac{(4k - l^2 - 4)k^2}{(4k - 4)(k^2 - l^2)}.$$

Proof. Let us calculate the real Hessian of $P_a^{k,l}$ on the unit circle (and use homogeniety of $P_a^{k,l}$). We will express the values of $P_a^{k,l}$ and its derivatives in terms of polar coordinates (r, θ) and calculate the Hessian at a point $e^{i\theta}$ with respect to the rotated basis $\frac{\partial}{\partial n}(\theta), \frac{\partial}{\partial t}(\theta)$ formed by the unit outer normal vector and the unit tangent vector to the unit circle at $(1, \theta)$.

Denote $F = P_a^{k,l}$, i.e.,

$$F(r, \theta) = r^k(1 + a \cos l\theta).$$

The relation between F_{nn}, F_{tt}, F_{nt} and the derivatives of F with respect to polar coordinates follows from the formulas

$$\frac{\partial^2}{\partial n^2} = \frac{\partial^2}{\partial r^2}$$
$$\frac{\partial^2}{\partial t^2} = \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}$$
$$\frac{\partial^2}{\partial n \partial t} = \frac{1}{r}\frac{\partial^2}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial}{\partial \theta}.$$

From this we obtain the entries of the Hessian:

$$
\begin{aligned}
F_{nn} &= k(k-1)(1 + a\cos l\theta)\\
F_{tt} &= k + a(k-l^2)\cos l\theta\\
F_{nt} &= a(l-lk)\sin l\theta.
\end{aligned}
\tag{7}
$$

So $F_{nn} \geq 0$ for $a \leq 1$, and $F_{tt} \geq 0$ for

$$a \leq \frac{k}{|l^2-k|}. \tag{8}$$

The determinant of the Hessian, $F_{nn}F_{tt} - F_{nt}^2$, is a quadratic function in $\cos l\theta$. Let $x = \cos l\theta$, and let $p(x) = Ax^2 + Bx + C$ be this function. Its coefficients are

$$
\begin{aligned}
A &= a^2(k-1)(k^2-l^2)\\
B &= a\ k(k-1)(2k-l^2)\\
C &= k^2(k-1) - a^2(k-1)^2 l^2.
\end{aligned}
$$

So $A > 0$, and $p(x)$ as a function on $(-\infty, +\infty)$ has minimum at $x = -\frac{B}{2A}$, i.e., at

$$x = \frac{k(2k-l^2)}{2a(k^2-l^2)}.$$

There are two possibilities, either $-\frac{B}{2A} \in (-1, 1)$, or the minimum of $p(x)$ on $[-1, 1]$ is attained at 1 or -1. The first possibility occurs when

$$a > \frac{k|2k-l^2|}{2(k^2-l^2)}. \tag{9}$$

When condition (8) is stricter than (9), i.e., if

$$\frac{k|2k-l^2|}{2(k^2-l^2)} > \frac{k}{|l^2-k|}, \tag{9a}$$

then $-\frac{B}{2A}$ is not in $(-1, 1)$ for $a \leq \frac{k}{l^2-k}$ and the determinant is nonnegative on $(-1, 1)$ if and only if it is nonnegative at $+1$ and -1, i.e., when $F_{nn},\ F_{tt} \geq 0$. Condition (9a) simplifies to

$$l^2 > 3k - 2,$$

and if this condition is satisfied, then automatically $\frac{k}{l^2-k} < 1$. In other words, $M_a^{k,l}$ is convex if and only if $F_{tt} \geq 0$, i.e., for $a \leq \frac{k}{l^2-k}$. If $l^2 < 3k-2$, then there are values of a for which $F_{nn},\ F_{tt} \geq 0$, but $p(x)$ is negative somewhere inside of $(-1,1)$. Then the condition for convexity comes from the discriminant of $p(x)$. $M_a^{k,l}$ is convex if and only if $B^2 - 4AC \leq 0$. This simplifies to the condition

$$a^2 \leq \frac{(4k-l^2-4)k^2}{(4k-4)(k^2-l^2)} \quad . \quad \square$$

Proof of Proposition 2. We split P in the same way as in the proof of Proposition 1. By Lemma 3, each of the summands is convex. So their sum, $P(z)$, is also convex. $\square$

If the model domain at p is $M_a^{k,l}$ and $l \nmid k$, then its convexity gives a necessary condition for convexifiability of $b\Omega$ around p.

Proposition 3. *(a) Let the model domain at p be $M_a^{k,l}$, where $l \nmid k$ and $l^2 \geq 3k-2$. If $a > \frac{k}{l^2-k}$, then Ω is not convex in any holomorphic coordinates around p.*

(b) Let the model domain at p be $M_a^{k,l}$, where $l \nmid k$ and $l^2 < 3k-2$. If $a^2 > \frac{(4k-l^2-4)k^2}{(4k-4)(k^2-l^2)}$, then Ω is not convex in any holomorphic coordinates around p.

Proof. Let (z,w) be local holomorphic coordinates in which $b\Omega$ is described by

$$u = F(z,\bar{z},v),$$

where F is a C^∞ function vanishing together with its 1-st partial derivatives at p (the general case is obtained by an affine change of coordinates). A usual argument (contained, e.g., in [Ko]) shows that either

$$F(z,\bar{z},v) = Re\ \alpha z^j + O(|z|^{j+1}, v), \tag{10}$$

where $2 \leq j \leq k-1$ and $\alpha \in \mathbb{C} \setminus \{0\}$, or

$$F(z,\bar{z},v) = \tilde{P}(z,\bar{z}) + O(|z|^{k+1}, v), \tag{11}$$

where $\tilde{P}$ is a homogeneous nonharmonic polynomial of degree k, and that in the second case $\tilde{P}$ is obtained from $P_a^{k,l}$ by a transformation $z^* = \alpha z,\ w^* = w + \beta z^k$, where $\alpha, \beta \in \mathbb{C}$, $\alpha \neq 0$. If F has form (10), $b\Omega$ is not convex around p. Let F have form (11). We have, up to a linear transformation of the z variable, $\tilde{P}(z) = P_a^{k,l}(z) + Re\ cz^k$ for some $c \in C$. By Lemma 3, $P_a^{k,l}$ is not convex. We need to show that $\tilde{P}$ is not convex. Consider again the unit circle, parametrized by θ. Let $D^2 f(\theta,\xi)$ denote the value of the Hessian of a function f at θ in the direction ξ, and let $P = P_a^{k,l}$. From (7) it follows that

$$D^2 P(\theta,\xi) = b + c\cos l\theta + d\sin l\theta, \tag{12}$$

where b,c,d are functions of the coordinates of ξ with respect to the basis $\frac{\partial}{\partial n}(\theta), \frac{\partial}{\partial t}(\theta)$. Let $\tilde{\theta}, \tilde{\xi}$ be such that $D^2 P(\tilde{\theta},\tilde{\xi}) < 0$ and let $\tilde{\xi} = \xi_1 \frac{\partial}{\partial n}(\tilde{\theta}) + \xi_2 \frac{\partial}{\partial t}(\tilde{\theta})$. If $\theta^j = \tilde{\theta} + \frac{2\pi j}{l}, \quad j = 0,...,l-1$, then by (12),

$$D^2 P(\theta^j,\xi^j) = D^2 P(\tilde{\theta},\tilde{\xi}),$$

where $\xi^j = \xi_1 \frac{\partial}{\partial n}(\theta^j) + \xi_2 \frac{\partial}{\partial t}(\theta^j)$. Denote $h_c = Re\; cz^k$. Again from (7), we get

$$D^2 h_c(\theta,\xi) = b\cos k\theta + c\sin k\theta,$$

where b,c are functions of the coordinates of ξ with respect to the basis $\frac{\partial}{\partial n}(\theta), \frac{\partial}{\partial t}(\theta)$. It follows that

$$\sum_{j=0}^{l-1} D^2 h_c(\theta^j,\xi^j) = 0,$$

so

$$\sum_{j=0}^{l-1} D^2 \tilde{P}(\theta^j,\xi^j) < 0, \tag{13}$$

and $\tilde{P}$ is not convex. Therefore, by (13) and (11), if $r_0 > 0$ is sufficiently small, then for every $r < r_0$ there exists a j such that

$$D^2 F(r\; e^{i\theta^j},\xi^j) < -\epsilon r^{k-2},$$

and F is not convex in any neighbourhood of p. $\square$

To compare the conditions for supporting functions and for local convexifiability, take as an example

$$P_a^{8,6}(z,\bar{z}) = |z|^8 + a|z|^2 Re\; z^6.$$

$M_a^{8,6}$ is pseudoconvex for $a \le \frac{16}{7}$. When $a = \frac{15}{7}$, it gives the Kohn-Nirenberg example. If $1 < a \le \frac{16}{7}$, then there is no supporting function and no C^1-peak function at 0. If $\frac{2}{7} < a \le 1$, then there is a supporting function and a smooth peak function at 0, but there are no local holomorphic coordinates in which the boundary is convex. If $a \le \frac{2}{7}$, then the domain is convex.

References

[BF] E. Bedford and J. E. Fornæss, *A construction of peak functions on weakly pseudoconvex domains*, Ann. of Math. **107** (1978), 555–568.

[B] T. Bloom, *C^∞-peak function for pseudoconvex domains*, Duke Math. J. **45** (1978), 133–147.

[D] J. D'Angelo, *Real hypersurfaces, orders of contact, and applications*, Ann. of Math. **115** (1982), 615–637.

[F] J. E. Fornæss, *Peak points on weakly pseudoconvex domains*, Math. Ann. **227** (1977), 173–175.

[FM] J. E. Fornæss and J. McNeal, *A construction of peak functions on some finite type domains*, American J. of Math. **116** (1994), 737–755.

[FS] J. E. Fornæss and B. Stensønes, *Lectures on counterexamples in several complex variables*, Math. notes, Princeton University Press, 1987.

[K] J. J. Kohn, *Boundary behaviour of $\bar\partial$ on weakly pseudoconvex manifolds of dimension two*, J. Diff. Geometry **6** (1972), 523–542.

[KN] J. J. Kohn and L. Nirenberg, *A pseudoconvex domain not admitting a holomorphic support function*, Math. Ann. **201** (1973), 265–268.

[Ko] M.Kolář, PhD thesis, Princeton University, 1995.

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