

HARNACK INEQUALITY FOR NON-SELF-ADJOINT EVOLUTION EQUATIONS

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Many years ago, S. Y. Cheng, P. Li and myself [1] gave an estimate of the heat kernel for the Laplacian. This was later improved by P. Li and myself in [2]. The key point of this later paper was a parabolic Harnack inequality that generalized an elliptic Harnack inequality that I developed more than twenty years ago.

Since many evolution equations in applied mathematics are not self-adjoint and have a convection term, I propose here to generalize the parabolic Harnack inequality that Li and I developed to cover the non-self-adjoint case. The Harnack inequality is considerably stronger than the estimates of the heat kernel. I hope that it will be useful for questions in control theory and fluid dynamics. The applications will be studied on a later occasion.

1. Harnack inequality for non-self-adjoint equation

Let u be a positive solution of the equation

$$(1.1) \quad \frac{\partial u}{\partial t} = \Delta u + \sum f_i u_i + V u$$

where f_i and V may depend on t .

Let $\varphi = -\log u$. Then

$$(1.2) \quad \varphi_t = \Delta \varphi - |\nabla \varphi|^2 + \sum f_i \varphi_i + V.$$

Let

$$(1.3) \quad \psi = \varphi_t + |\nabla \varphi|^2 - \sum f_i \varphi_i - V - \alpha(t) \sqrt{|\nabla \varphi|^2 + \beta(t)} - c$$

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where α, β, c are to be chosen later.

Then

$$\begin{aligned}
 \psi_t &= \varphi_{tt} + 2 \sum \varphi_j \varphi_{jt} - \sum (f_i)_t \varphi_i - \sum f_i \varphi_{ti} - V_t \\
 &\quad - \alpha' \sqrt{|\nabla \varphi|^2 + \beta} - \frac{\alpha \sum \varphi_i \varphi_{ti}}{\sqrt{|\nabla \varphi|^2 + \beta}} - \frac{\alpha \beta'}{\sqrt{|\nabla \varphi|^2 + \beta}} - c_t \\
 \Delta \psi &= (\Delta \varphi)_t + 2 \sum \varphi_{ij}^2 + 2 \sum \varphi_j (\Delta \varphi)_j \\
 (1.4) \quad &+ 2 \sum R_{ij} \varphi_i \varphi_j - \sum R_{ij} f_i \varphi_j \\
 &- 2 \sum f_{i,j} \varphi_{ij} - \sum f_i (\Delta \varphi)_i - \sum (\Delta f_i) \varphi_i \\
 &- \Delta V - \frac{\alpha \sum \varphi_{ij}^2}{\sqrt{|\nabla \varphi|^2 + \beta}} + \frac{\alpha \sum_j (\sum_i \varphi_i \varphi_{ji})^2}{(|\nabla \varphi|^2 + \beta)^{3/2}} \\
 &- \frac{\alpha \sum \varphi_i (\Delta \varphi)_i}{\sqrt{|\nabla \varphi|^2 + \beta}} - \Delta c
 \end{aligned}$$

where R_{ij} is the Ricci curvature of the manifold (which is zero if defined on Euclidean space).

Hence

$$\begin{aligned}
 \psi_t - \Delta \psi &= -(|\nabla \varphi|^2 - \sum f_i \varphi_i - V)_t \\
 &\quad - 2 \sum \varphi_{ij}^2 - 2 \sum \varphi_j [|\nabla \varphi|^2 - \sum f_i \varphi_i - V]_j \\
 &\quad + 2 \sum f_{i,j} \varphi_{ij} + \sum_i f_i [|\nabla \varphi|^2 - \sum f_j \varphi_j - V]_i \\
 &\quad + \sum_i [-(f_i)_t + \Delta f_i] \varphi_i - 2 \sum R_{ij} \varphi_i \varphi_j \\
 (1.5) \quad &+ \sum R_{ij} f_i \varphi_j + \Delta V - V_t \\
 &+ \frac{\alpha \sum_i \varphi_i [|\nabla \varphi|^2 - \sum f_j \varphi_j - V]_i}{\sqrt{|\nabla \varphi|^2 + \beta}} \\
 &+ \frac{\alpha \sum \varphi_{ij}^2}{\sqrt{|\nabla \varphi|^2 + \beta}} - \frac{\alpha \sum_i (\sum_j \varphi_j \varphi_{ji})^2}{(|\nabla \varphi|^2 + \beta)^{3/2}} \\
 &- \alpha' \sqrt{|\nabla \varphi|^2 + \beta} - \frac{\alpha \beta'}{2\sqrt{|\nabla \varphi|^2 + \beta}} - c_t + \Delta c
 \end{aligned}$$

$$\begin{aligned}
&= -2 \sum \varphi_i \psi_i - 2\alpha \sum \varphi_i (\sqrt{|\nabla\varphi|^2 + \beta})_i - 2 \sum \varphi_i c_i \\
&\quad + \sum f_i \psi_i + \alpha \sum f_i (\sqrt{|\nabla\varphi|^2 + \beta})_i + \sum f_i c_i \\
&\quad - 2 \sum \varphi_{ij}^2 + 2 \sum f_{i,j} \varphi_{ij} + \sum \Delta f_i \varphi_i \\
&\quad + \Delta V + \alpha \frac{\sum_i \varphi_i [|\nabla\varphi|^2 - \sum f_j \varphi_j - V]_i}{\sqrt{|\nabla\varphi|^2 + \beta}} \\
&\quad - 2 \sum R_{ij} \varphi_i \varphi_j + \sum R_{ij} f_i \varphi_j + \frac{\alpha \sum \varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2 + \beta}} \\
&\quad - \frac{\alpha \sum_i (\sum_j \varphi_j \varphi_{ji})^2}{(|\nabla\varphi|^2 + \beta)^{3/2}} - \alpha' \sqrt{|\nabla\varphi|^2 + \beta} - \frac{\alpha\beta'}{2\sqrt{|\nabla\varphi|^2 + \beta}} - c_t + \Delta c \\
&= -2 \sum \varphi_i \psi_i + \sum f_i \psi_i - 2 \sum \varphi_{ij}^2 + 2 \sum f_{i,j} \varphi_{ij} + \sum (\Delta f_i) \varphi_i + \Delta V \\
&\quad - \frac{(\alpha \sum f_{i,j} \varphi_i \varphi_j + \alpha \sum \varphi_i V_i)}{\sqrt{|\nabla\varphi|^2 + \beta}} - 2 \sum R_{ij} \varphi_i \varphi_j \\
&\quad + \sum R_{ij} f_i \varphi_j + \frac{\alpha \sum \varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2 + \beta}} - \frac{\alpha \sum_i (\sum_j \varphi_j \varphi_{ji})^2}{(|\nabla\varphi|^2 + \beta)^{3/2}} \\
&\quad - \alpha' \sqrt{|\nabla\varphi|^2 + \beta} - \frac{\alpha\beta'}{2\sqrt{|\nabla\varphi|^2 + \beta}} - c_t + \Delta c - 2 \sum \varphi_i c_i + \sum f_i c_i.
\end{aligned}$$

Suppose $\psi < 0$ when $t = 0$. Assume that at $t_0 > 0$, ψ becomes zero at some point in the interior of the manifold and $\psi < 0$ for $t < t_0$. Then $\psi_t \geq 0$, $\nabla\psi = 0$ and $\Delta\psi \leq 0$ at this point. Hence

$$\begin{aligned}
(1.6) \quad &\left(2 - \frac{\alpha}{\sqrt{|\nabla\varphi|^2 + \beta}}\right) \sum_{i,j} [\varphi_{ij} - \frac{1}{2} \left(2 - \frac{\alpha}{\sqrt{|\nabla\varphi|^2 + \beta}}\right)^{-1} (f_{i,j} + f_{j,i})]^2 \\
&\leq \frac{1}{4} \left(2 - \frac{\alpha}{\sqrt{|\nabla\varphi|^2 + \beta}}\right)^{-1} \sum_{i,j} (f_{i,j} + f_{j,i})^2 \\
&\quad + \sum (\Delta f_i - 2c_i) \varphi_i + \Delta V - \frac{-\alpha(\sum f_{i,j} \varphi_i \varphi_j + \sum \varphi_i V_i)}{\sqrt{|\nabla\varphi|^2 + \beta}} \\
&\quad - 2 \sum R_{ij} \varphi_i \varphi_j + \sum R_{ij} f_i \varphi_j - \frac{(\alpha \sum_i (\sum_j \varphi_j \varphi_{ji})^2)}{(|\nabla\varphi|^2 + \beta)^{3/2}} \\
&\quad - \alpha' \sqrt{|\nabla\varphi|^2 + \beta} - \frac{\alpha\beta'}{2\sqrt{|\nabla\varphi|^2 + \beta}} - c_t + \Delta c + \sum f_i c_i.
\end{aligned}$$

By the arithmetic-geometric inequality, the lefthand side is greater than

$$(1.7) \quad \left(2 - \frac{\alpha}{\sqrt{|\nabla\varphi|^2 + \beta}}\right) n^{-1} [\Delta\varphi - \left(2 - \frac{\alpha}{\sqrt{|\nabla\varphi|^2 + \beta}}\right)^{-1} \sum f_{i,i}]^2.$$

Note that at $\psi = 0$, $\Delta\varphi = \alpha\sqrt{|\nabla\varphi|^2 + \beta} + c$.

If we assume that for some a , the following inequality holds:

$$(1.8) \quad 2\sum R_{ij}x_i x_j - \sum R_{ij}f_i x_j - \sum (\Delta f_i - (\sum f_{j,j})_i)x_i \geq -a$$

for all x , then we can take $\alpha = 0$ in the above inequalities and take

$$c = \frac{n}{2t} + \frac{1}{2} \sum f_{j,j} + b$$

to conclude

$$(1.9) \quad \begin{aligned} \frac{2}{n} \left(\frac{n}{2t} + b\right)^2 &\leq \frac{1}{8} \sum (f_{i,j} + f_{j,i})^2 + a + \Delta V \\ &+ \frac{n}{2t^2} - \frac{1}{2} (\sum f_{j,j})_t + \Delta \left(\frac{1}{2} \sum f_{j,j}\right) + \sum f_i \left(\frac{1}{2} \sum f_{j,j}\right)_i. \end{aligned}$$

This inequality is violated if

$$(1.10) \quad \begin{aligned} \frac{2b}{t} + \frac{2}{n}b^2 &> \frac{1}{8} \sum (f_{i,j} + f_{j,i})^2 + a + \Delta V \\ &- \frac{1}{2} (\sum f_{j,j})_t + \Delta \left(\frac{1}{2} \sum f_{j,j}\right) + \sum f_i \left(\frac{1}{2} \sum f_{j,j}\right)_i. \end{aligned}$$

On a noncompact manifold, ψ may not achieve its maximum in the interior of the manifold. In that case, we choose a nonnegative function ρ which tends to infinity when r tends to infinity. If ψ is bounded, the function $\psi - \epsilon\rho$ then attains a maximum somewhere for $t \leq t_0$. At such a point, $\Delta\psi \leq \epsilon\Delta\rho$ and $\nabla\psi = \epsilon\nabla\rho$. Hence

$$(1.11) \quad \begin{aligned} -\epsilon\Delta\rho &\leq -2\epsilon \sum \varphi_i \rho_i - \epsilon \sum f_i \rho_i - 2 \sum \varphi_{ij}^2 + 2 \sum f_{i,j} \varphi_{ij} \\ &+ \sum (\Delta f_i) \varphi_i + \Delta V - \frac{(\alpha(\sum f_{ij} \varphi_i \varphi_j + \sum \varphi_i V_i))}{\sqrt{|\nabla\varphi|^2 + \beta}} \\ &- 2 \sum R_{ij} \varphi_i \varphi_j + \frac{\alpha \sum \varphi_{ij}^2}{\sqrt{|\nabla\varphi|^2 + \beta}} - \frac{\alpha \sum_i (\sum_j \varphi_j \varphi_{ji})^2}{(|\nabla\varphi|^2 + \beta)^{3/2}} \\ &- \alpha' \sqrt{|\nabla\varphi|^2 + \beta} - \frac{\alpha\beta'}{2\sqrt{|\nabla\varphi|^2 + \beta}} - c_t + \Delta c \\ &- 2 \sum \varphi_i c_i + \sum f_i c_i. \end{aligned}$$

In this case we can take $c = \frac{n}{2t} + \frac{1}{2} \sum f_{j,j} + b$. Then under the assumption that

$$(1.12) \quad 2 \sum R_{ij} x_j x_j - \sum R_{ij} f_i x_j - \sum (\Delta f_i - (\sum f_{j,j})_i - 2\epsilon \rho_i) x_i \geq -a$$

for all x_i . We find

$$(1.13) \quad \begin{aligned} -\epsilon \Delta \rho + \frac{2}{n} \left(\frac{n}{2t} + b \right)^2 &\leq -\epsilon \sum f_i \rho_i + \frac{1}{8} \sum (f_{i,j} + f_{j,i})^2 + a + \Delta V \\ &+ \frac{n^2}{4t^2} - \frac{1}{2} (\sum f_{i,i})_t + \Delta \left(\frac{1}{2} \sum f_{i,i} \right) + \sum f_i \left(\frac{1}{2} \sum f_{j,j} \right)_i. \end{aligned}$$

We can choose ρ to be of the form $\log(d^2 + 1)$ where d is the distance function. Hence if we assume that $(d^2 + 1)^{-1} (\sum f_i^2)$ is bounded for all t , we can let $\epsilon \rightarrow 0$ in (1.13) and obtain

$$(1.14) \quad \begin{aligned} \frac{2}{n} \left(\frac{n}{2t} + b \right)^2 &\leq \frac{1}{8} \sum (f_{i,j} + f_{j,i})^2 + a + \Delta V \\ &+ \frac{n}{2t^2} - \frac{1}{2} (\sum f_{i,i})_t + \Delta \left(\frac{1}{2} \sum f_{i,i} \right) + \sum f_i \left(\frac{1}{2} \sum f_{j,j} \right)_i. \end{aligned}$$

If we also assume that $(d^2 + 1)^{-1} |\nabla \varphi|^2$ is bounded (in space) for all $t > 0$, then we can let $\epsilon \rightarrow 0$.

By a parametrix argument, if both $(d^2 + 1)^{-1} \sum f_i^2$ and $(d^2 + 1)^{-1} |V|$ in (1.12) are bounded, then both $(d^2 + 1)^{-1} |\nabla \varphi|^2$ and ψ are bounded for small time. Hence we have proved the following

Theorem 1. *Let u be a positive solution of (1.1) so that $(d^2 + 1)^{-1} \sum f_i^2$ and $(d^2 + 1)^{-1} V$ are bounded. Suppose there is a constant a so that*

$$(1.15) \quad 2 \sum R_{ij} x_i x_j - \sum R_{ij} f_i x_j - \sum (\Delta f_i - (\sum f_{i,j})_i) x_j \geq -a$$

for all x_i . Suppose that for some constant b

$$(1.16) \quad \begin{aligned} \frac{2}{n} b^2 + \frac{2b}{t} &> \frac{1}{8} \sum (f_{i,j} + f_{j,i})^2 + a + \Delta V \\ &- \frac{1}{2} (\sum f_{i,i})_t + \Delta \left(\frac{1}{2} \sum f_{i,i} \right) + \sum f_i \left(\frac{1}{2} \sum f_{j,j} \right)_i. \end{aligned}$$

Then $-(\log u)_t + |\nabla \log u|^2 + \sum f_i (\log u)_i - V - \frac{n}{2t} - b - \frac{1}{2} \sum f_{i,i}$ is nonpositive for all $t > 0$ if it is nonpositive for $0 \leq t \leq \epsilon$ for some $\epsilon > 0$.

Note that the last assumption can be checked by the parametrix argument.

Let us now consider the case when the manifold is compact with boundary. In this case, we shall assume that the vector field $F = \sum f_i \frac{\partial}{\partial x_i}$ is tangential to the boundary. We require that u is constant on the boundary.

If ψ achieves its maximum on the boundary, then $\frac{\partial \psi}{\partial \nu} \geq 0$ at that point. On the other hand, u is constant implies that $\frac{d\varphi}{dt} = 0$ on the boundary. Hence $\frac{d}{dt} \frac{\partial \varphi}{\partial \nu} = 0$ and

$$(1.17) \quad \begin{aligned} 0 &\leq \frac{\partial \psi}{\partial \nu} = \frac{d}{dt} \frac{\partial \varphi}{\partial \nu} + \frac{\partial}{\partial \nu} (|\nabla \varphi|^2) - \frac{\partial}{\partial \nu} F(\varphi) - \frac{\partial}{\partial \nu} (V) - \frac{\partial}{\partial \nu} (c) \\ &= \frac{\partial}{\partial \nu} (|\nabla \varphi|^2) - \frac{\partial}{\partial \nu} (V + \frac{1}{2} \sum_i f_{i,i}). \end{aligned}$$

Let us now assume the boundary of the manifold has positive mean curvature. Then $\frac{\partial}{\partial \nu} (|\nabla \varphi|^2) \leq 0$. Hence if we assume $\frac{\partial}{\partial \nu} (V + \frac{1}{2} \sum_i f_{i,i}) \geq 0$, we can prove the following

Theorem 2. *Let u be a positive solution of (1.1) which satisfies the condition that $u = \text{constant}$ on the boundary. Assume that the boundary has positive mean curvature and $\frac{\partial}{\partial \nu} (V + \frac{1}{2} \sum_i f_{i,i}) \geq 0$. Assume that (1.15) and (1.16) hold for some constants a and b . Then*

$$(1.18) \quad -(\log u)_t + |\nabla \log u|^2 + \sum f_i (\log u)_i - V - \frac{n}{2t} - b - \frac{1}{2} \sum f_{i,i} \leq 0.$$

Let us now explain (1.15) and (1.16). If f_i is a gradient vector field and $R_{ij} = 0$ (as in Euclidean space, (1.15) automatically holds. It is remarkable that in (1.16), only ΔV appears. In particular, if V is linear for each time and $\Delta V = 0$, the theorem applies.

Let us now consider the case when (1.15) does not hold for any choice of a . We shall take $c = \frac{n}{2t} + b$ where b is a positive constant to be determined. We shall choose $\beta = d + \frac{\varepsilon}{t}$ where d and e are positive constants to be chosen. Choose $d \geq \alpha^2$ so that $\alpha(\sqrt{|\nabla \varphi|^2 + \beta})^{-1} \leq 1$.

The previous argument requires that the following inequality be violated:

$$\begin{aligned}
(1.19) \quad & \left(2 - \frac{\alpha}{\sqrt{|\nabla\varphi|^2 + \beta}}\right) n^{-1} \left[\alpha\sqrt{|\nabla\varphi|^2 + \beta} + \frac{n}{2t} + b \right. \\
& \quad \left. - \left(2 - \frac{\alpha}{\sqrt{|\nabla\varphi|^2 + \beta}}\right)^{-1} \sum f_{i,i} \right]^2 \\
& \leq \frac{1}{4} \left(2 - \frac{\alpha}{\sqrt{|\nabla\varphi|^2 + \beta}}\right)^{-1} \sum_{i,j} (f_{i,j} + f_{j,i})^2 + \sum (\Delta f_i)\varphi_i + \Delta V \\
& \quad - \alpha \frac{(\sum f_{i,j}\varphi_i\varphi_j + \sum \varphi_i V_i)}{\sqrt{|\nabla\varphi|^2 + \beta}} - 2 \sum R_{ij}\varphi_i\varphi_j + 2 \sum R_{ij}f_i\varphi_j \\
& \quad \quad \quad + \frac{\alpha e}{2t^2\sqrt{|\nabla\varphi|^2 + \beta}} + \frac{n}{2t^2}.
\end{aligned}$$

The righthand side is not greater than

$$\begin{aligned}
(1.20) \quad & \frac{1}{4} \sum_{i,j} (f_{i,j} + f_{j,i})^2 + \Delta V - \alpha|\nabla V| + \epsilon|\nabla\varphi|^2 + \frac{1}{4\epsilon} \sum_i (\Delta f_i)^2 \\
& + \epsilon|\nabla\varphi|^2 + \frac{\alpha^2}{16\epsilon} \sum_{i,j} (f_{i,j} + f_{j,i})^2 + \epsilon|\nabla\varphi|^2 + \frac{1}{\epsilon} \sum_j \left(\sum_i R_{ij}f_i\right)^2 \\
& - 2 \sum R_{ij}\varphi_i\varphi_j + \frac{\alpha e}{2t^2\sqrt{|\nabla\varphi|^2 + \beta}} + \frac{n}{2t^2}
\end{aligned}$$

where $\epsilon > 0$ is arbitrary. If $b \geq |\sum f_{i,i}|$, the lefthand side is not less than

$$\begin{aligned}
(1.21) \quad & \frac{\alpha^2}{n} (|\nabla\varphi|^2 + \beta) + \frac{n}{2t^2} - \frac{\alpha n}{4t^2\sqrt{|\nabla\varphi|^2 + \beta}} \\
& \quad \frac{\alpha\sqrt{|\nabla\varphi|^2 + \beta}}{t} + (b - \sum f_{i,i})^2.
\end{aligned}$$

By choosing $e \geq \frac{n}{2}$, we can arrange

$$(1.22) \quad \frac{\alpha\sqrt{|\nabla\varphi|^2 + \beta}}{t} - \frac{\alpha n}{4t^2\sqrt{|\nabla\varphi|^2 + \beta}} \geq \frac{\alpha e}{2t^2\sqrt{|\nabla\varphi|^2 + \beta}}.$$

Hence we simply choose α and b so that

$$(1.23) \quad \frac{\alpha^2}{n} |\nabla\varphi|^2 \geq 3\epsilon|\nabla\varphi|^2 - 2 \sum R_{ij}\varphi_i\varphi_j$$

$$(1.24) \quad \begin{aligned} (b - \sum f_{i,i})^2 &> \left(\frac{1}{4} + \frac{\alpha^2}{16\epsilon}\right) \sum_{i,j} (f_{i,j} + f_{j,i})^2 + \Delta V - \alpha |\nabla V| \\ &+ \frac{1}{4\epsilon} \sum_i (\Delta f_i)^2 + \frac{1}{\epsilon} \sum_j \left(\sum_i R_{ij} f_i\right)^2 \end{aligned}$$

$$(1.25) \quad b \geq \left| \sum_i f_{i,i} \right|.$$

Theorem 3. *Let u be a positive solution of (1.1) so that $(d^2 + 1)^{-1} \sum_i f_i^2$ and $(d^2 + 1)^{-1} V$ are bounded. Suppose that we can choose constants α and b so that (1.23) and (1.24) hold. Then*

$$(1.26) \quad \begin{aligned} & -(\log u)_t + |\nabla \log u|^2 + \sum f_i (\log u)_i - V \\ & \leq \alpha \sqrt{|\nabla(\log u)|^2 + d + \frac{e}{t} + \frac{n}{2t}} + b \end{aligned}$$

where $d \geq \alpha^2$ and $e \geq \frac{n}{2}$.

2. Estimate on the time derivative

In this section, we shall make use of the estimate of the previous section. Let

$$(2.1) \quad \psi = \varphi + t\varphi_t + F,$$

where F is a function to be determined.

Then

$$(2.2) \quad \begin{aligned} \frac{\partial \psi}{\partial t} - \Delta \psi &= (\varphi_t - \Delta \varphi) + \varphi_t + t(\varphi_t - \Delta \varphi)_t + F_t - \Delta F \\ &= (-|\nabla \varphi|^2 + \sum f_i \varphi_i + V) + \varphi_t \\ &\quad + t(-|\nabla \varphi|^2 + \sum f_i \varphi_i + V)_t + F_t - \Delta F \\ &= -2\varphi_i (t\varphi_t + \varphi + F)_i \\ &\quad + \sum f_i (t\varphi_t + \varphi + F)_i + \varphi_t \\ &\quad + |\nabla \varphi|^2 + \sum f_i \varphi_i + V + tV_t + 2 \sum F_i \varphi_i \\ &\quad + t \sum (f_i)_t \varphi_i - \sum f_i F_i + F_t - \Delta F. \end{aligned}$$

The first simple case is to assume f_i is a gradient vector field and $R_{ij} = 0$. In this case we choose F so that $t(f_i)_t + f_i = -2F_i$, a can be chosen to be zero in (1.8) and $b > 0$ can be chosen so that

$$(2.3) \quad \begin{aligned} \frac{2}{n}b^2 &= \frac{1}{8} \sum_{i,j} (f_{i,j} + f_{j,i})^2 + \Delta V \\ &+ \frac{1}{2} \Delta \left(\sum_i f_{i,i} \right) - \frac{1}{2} \left(\sum_i f_{i,i} \right)_t + \sum_i f_i \left(\frac{1}{2} \sum_j f_{j,j} \right)_i. \end{aligned}$$

Then by Theorem 1,

$$(2.4) \quad \varphi_t + |\nabla \varphi|^2 - \sum_i f_i \varphi_i - V - \frac{n}{2t} - b - \frac{1}{2} \sum_i f_{i,i} \leq 0.$$

Hence

$$(2.5) \quad \begin{aligned} &\varphi_t + |\nabla \varphi|^2 + V + tV_t + \sum_i (2F_i + t(f_i)_t + f_i) \varphi_i - \sum_i f_i F_i + F_t - \Delta F \\ &\leq 2V + \frac{n}{2t} + b + \sum_i f_{i,i} + tV_t + \frac{1}{4} t \left(\sum_i f_i^2 \right)_t + \frac{1}{2} \sum_i f_i^2 + F_t. \end{aligned}$$

If we replace F by $-\frac{n}{2} \log t + F$, we obtain the following:

Theorem 4. *Assume that in Theorem 1, $R_{ij} = 0$ and (f_i) is a gradient vector field. Then if u is a positive solution of (1.1) and $b > 0$ is defined by (2.3), then*

$$-t(\ln u)_t - \log u - \frac{n}{2} \ln t - ct + F \leq 0$$

such that $t(f_i)_t + f_i = -2F_i$, where

$$(2.6) \quad c = \sup \left(2V + b + \frac{1}{2} \sum_i f_{i,i} + tV_t + \frac{1}{2} \sum_i f_i^2 + \frac{1}{4} t \left(\sum_i f_i^2 \right)_t + F_t \right),$$

and

$$(2.7) \quad \lim_{t \rightarrow 0} \left(-t(\ln u)_t - \log u - \frac{n}{2} \ln t + F \right) \leq 0.$$

Note that when $f_i = 0$, we need only make the assumption $R_{ij} \geq 0$ instead of $R_{ij} = 0$.

Corollary. *Under the assumption of Theorem 4,*

$$(2.8) \quad t \log u - \lim_{\tau \rightarrow 0} \tau \log u(\tau) \geq -\frac{n}{2}t \log t + \frac{n}{2}t - \frac{ct^2}{2} + \int_0^t F$$

and

$$(2.9) \quad u \geq t^{-n/2} \exp\left[\frac{1}{t} \int_0^t F + \frac{n}{2} + \frac{1}{t} \lim_{\tau \rightarrow 0} (\tau \log u(\tau)) - \frac{ct}{2}\right].$$

When f_i is not a gradient vector field, the estimate is more complicated. However, the opposite case when $f_{i,j} + f_{j,i} = 0$ is quite simple. (This corresponds to the vector field defined by a one-parameter family of orthogonal matrices.)

In this case, let

$$(2.10) \quad \psi = \varphi + t\varphi_t - t \sum_i f_i \varphi_i - \frac{n}{2} \ln t + F(t).$$

Then

$$\begin{aligned} \psi_t - \Delta\psi &= \varphi_t - \Delta\varphi + \varphi_t + t(\varphi_t - \Delta\varphi)_t - \sum f_i \varphi_i - t \sum (f_i)_t \varphi_i \\ &\quad + t \sum R_{ij} f_i \varphi_j - t \sum f_i (\varphi_t - \Delta\varphi)_i \\ &\quad + 2t \sum f_{i,j} \varphi_{ij} + t \sum \Delta f_i \varphi_i - \frac{n}{2t} + F' \\ &= (-|\nabla\varphi|^2 + \sum f_i \varphi_i + V) + \varphi_t \\ &\quad + t(-|\nabla\varphi|^2 + \sum f_i \varphi_i + V)_t - \sum f_i \varphi_i \\ (2.11) \quad &- t \sum (f_i)_t \varphi_i + t \sum R_{ij} f_i \varphi_j \\ &- t \sum f_i (-|\nabla\varphi|^2 + \sum f_i \varphi_i + V)_i \\ &+ 2t \sum f_{i,j} \varphi_{ij} + t \sum \Delta f_i \varphi_i - \frac{n}{2t} + F' \\ &= \varphi_t - 2 \sum \varphi_i \psi_i + |\nabla\varphi|^2 - 2t \sum f_{i,j} \varphi_i \varphi_j \\ &- \sum f_i \varphi_i + V + tV_t + \sum f_i \psi_i + t \sum R_{ij} f_i \varphi_j \\ &+ 2t \sum f_{i,j} \varphi_{ij} + t \sum (\Delta f_i) \varphi_i - \frac{n}{2t} + F'. \end{aligned}$$

Note that $f_{ij} = -f_{j,i}$ implies $\sum f_{i,j} \varphi_i \varphi_j = 0 = \sum f_{i,j} \varphi_{ij}$. If $R_{ij} = 0$, $\Delta f_i = 0$ also. Theorem 1 says that

$$(2.12) \quad \varphi_t + |\nabla\varphi|^2 - \sum \varphi_i f_i - V - \frac{n}{2t} \leq \sqrt{\frac{n}{2}(\Delta V)_+}.$$

Hence when $\psi_i = 0$ (2.11) is less than

$$2V + tV_t + \sqrt{\frac{n}{2}(\Delta V)_+} + F'.$$

Theorem 5. Assume that in Theorem 1, $R_{ij} = 0$ and $f_{i,j} = -f_{j,i}$. Let u be a positive solution of (1.1). Let F be a function of t so that

$$(2.13) \quad 2V + tV_t + \sqrt{\frac{n}{2}(\Delta V)_+} + F' \leq 0.$$

and $\lim_{t \rightarrow 0} [-t(\log u)_t - \log u + \sum f_i(\log u)_i - \frac{n}{2} \log t + F] \leq 0$. Then

$$(2.14) \quad -t(\log u)_t - \log u + \sum f_i(\log u)_i - \frac{n}{2} \log t + F \leq 0 \quad \text{for all } t.$$

If f_i is neither a gradient vector field nor a Killing field (i.e., $f_{i,j} = -f_{j,i}$), we proceed as follows. Let

$$(2.15) \quad \psi = \varphi + t\varphi_t - \gamma(t) \sqrt{|\nabla\varphi|^2 + d + \frac{e}{t}} - \frac{n}{2} \ln t + F(t).$$

Then

$$(2.16) \quad \begin{aligned} \psi_t - \Delta\psi &= \varphi_t - \Delta\varphi + \varphi_t + t(\varphi_t - \Delta\varphi)_t \\ &\quad - \gamma' \sqrt{|\nabla\varphi|^2 + d + \frac{e}{t}} + F' - \gamma \left(\sum R_{ij} \varphi_i \varphi_j \right) \left(|\nabla\varphi|^2 + d + \frac{e}{t} \right)^{-\frac{1}{2}} \\ &\quad - \gamma \left[\sum_i \varphi_i (\varphi_t - \Delta\varphi)_i - \frac{e}{t^2} \right] \left(|\nabla\varphi|^2 + d + \frac{e}{t} \right)^{-1/2} \\ &\quad - \gamma \left(\sum \varphi_{ij}^2 \right) \left(|\nabla\varphi|^2 + d + \frac{e}{t} \right)^{-1/2} \\ &\quad + \gamma \sum_j \left(\sum_i \varphi_i \varphi_{ij} \right)^2 \left[|\nabla\varphi|^2 + d + \frac{e}{t} \right]^{-3/2} - \frac{n}{2t} \\ &= -|\nabla\varphi|^2 + \sum f_i \varphi_i + V + \varphi_t \\ &\quad + t(-|\nabla\varphi|^2 + \sum f_i \varphi_i + V)_t - \gamma' \sqrt{|\nabla\varphi|^2 + d + \frac{e}{t}} + F' \\ &\quad - \gamma \left[\sum_i \varphi_i \left(-|\nabla\varphi|^2 + \sum f_i \varphi_i + V \right)_i - \frac{e}{t^2} \right] \left[|\nabla\varphi|^2 + d + \frac{e}{t} \right]^{-1/2} \\ &\quad - \gamma \left(\sum R_{ij} \varphi_i \varphi_j \right) \left(|\nabla\varphi|^2 + d + \frac{e}{t} \right)^{-1/2} \\ &\quad - \gamma \left(\sum \varphi_{ij}^2 \right) \left(|\nabla\varphi|^2 + d + \frac{e}{t} \right)^{-1/2} \\ &\quad + \gamma \sum_j \left(\sum_i \varphi_i \varphi_{ij} \right)^2 \left(|\nabla\varphi|^2 + d + \frac{e}{t} \right)^{-3/2} - \frac{n}{2t} \end{aligned}$$

$$\begin{aligned}
&= \varphi_t - 2 \sum_i \varphi_i \psi_i + |\nabla \varphi|^2 + V + tV_t + \sum f_i \psi_i + F' \\
&\quad - \gamma \left[\sum f_{i,j} \varphi_i \varphi_j + \sum \varphi_i V_i - \frac{e}{t^2} \right] (|\nabla \varphi|^2 + d + \frac{e}{t})^{-1/2} \\
&\quad - \gamma' \sqrt{|\nabla \varphi|^2 + d + \frac{e}{t}} \\
&\quad - \gamma \left[\sum R_{ij} \varphi_i \varphi_j + \sum \varphi_{ij}^2 \right] (|\nabla \varphi|^2 + d + \frac{e}{t})^{-1/2} \\
&\quad + \gamma \sum_j \left(\sum_i \varphi_i \varphi_{ij} \right)^2 (|\nabla \varphi|^2 + d + \frac{e}{t})^{-3/2} + t \sum (f_i)_t \varphi_i - \frac{n}{2t} \\
&\leq -2 \sum_i \varphi_i \psi_i + \sum f_i \psi_i + 2V + tV_t \\
&\quad + \frac{1}{2} \sum_i f_{i,i} + \sum (t(f_i)_t + f_i) \varphi_i + \alpha \sqrt{|\nabla \varphi|^2 + d + \frac{e}{t}} \\
&= b + F' + \gamma \left(\frac{1}{2} \sqrt{\sum (f_{i,j} + f_{j,i})^2} |\nabla \varphi| + |\nabla V| + \frac{\sqrt{e}}{t} \right) \\
&\quad - \gamma' \sqrt{|\nabla \varphi|^2 + d + \frac{e}{t}} - \gamma (R_{ij})_- |\nabla \varphi|,
\end{aligned}$$

where $(R_{ij})_-$ is the smallest eigenvalue of the Ricci curvature R_{ij} .

Theorem 6. *Using the notation of Theorem 3, we choose a function $\gamma(t)$ so that*

$$\begin{aligned}
(2.17) \quad &(-\gamma' + \alpha) \sqrt{|\nabla \varphi|^2 + d + \frac{e}{t}} + \sum (t(f_i)_t + f_i) \varphi_i \\
&+ \gamma \left(\frac{1}{2} \sqrt{\sum (f_{i,j} + f_{j,i})^2} |\nabla \varphi| + |\nabla V| + \frac{\sqrt{e}}{t} \right) - \gamma (R_{ij})_- |\nabla \varphi| \leq 0.
\end{aligned}$$

We also choose $F(t)$ so that

$$(2.18) \quad 2V + tV_t + b + F' \leq 0.$$

Then under the same assumption as in Theorem 3,

$$(2.19) \quad -\log u - t(\log u)_t - \gamma \sqrt{|\nabla \varphi|^2 + d + \frac{e}{t}} - \frac{n}{2} \ln t + F(t) \leq 0.$$

References

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