Mathematical Research Letters **2**, 339–344 (1995)

CLASSIFICATION OF CLOSED NONORIENTABLE 4-MANIFOLDS WITH INFINITE CYCLIC FUNDAMENTAL GROUP

Zhenghan Wang

Abstract. Closed, connected, nonorientable, topological 4-manifolds with infinite cyclic fundamental group are classified. The classification is an extension of results of Freedman and Quinn and of Kreck. The stable classification of such 4-manifolds is also obtained.

In principle, it is a consequence of Freedman's work on topological 4-manifolds that one can classify topological 4-manifolds with "good" fundamental group. Roughly, a "good" group is a group for which topological surgery in dimension 4 works ([F2]). For example, by results of Freedman, all groups of polynomial growth are "good". However, the full classification for closed, orientable 4-manifolds has only been obtained for manifolds with cyclic fundamental groups ([F1][FQ], [HK1][HK2], [FQ][K][SW][Wa]). For closed, nonorientable 4-manifolds the classification has only been given for fundamental group Z_2 ([HKT]). In this note, we give the classification for closed, nonorientable 4-manifolds with infinite cyclic fundamental group.

Our main result is the following theorem:

Theorem 1. *(1) Existence: Suppose (H, λ) is a nonsingular ω_1-hermitian form on a finitely generated free $Z[Z]$-module, $k \in Z_2$, and if λ is even then we assume $k = [\lambda] \in L_4(Z^-)$. Then there is a closed, connected, nonorientable 4-manifold with $\pi_1 = Z$, intersection form λ and Kirby-Siebenmann invariant k.*

(2) Uniqueness: Suppose M and N are closed 4-manifolds with $\pi_1 = Z$, not orientable but locally oriented, $h : H_2(M; Z[Z]) \longrightarrow H_2(N; Z[Z])$ is a $Z[Z]$-isomorphism which preserves intersection forms, and $ks(M) = ks(N)$. Then there is a homeomorphism $f : M \longrightarrow N$ which induces the identification of fundamental groups, preserves local orientations, and with $f_* = h$.

Received March 29, 1995.

1991 Mathematics Subject Classification: Primary 57N13, Secondary 57N35, 57N75.

Supported by a Sloan Dissertation Fellowship.

Before the breakthrough by Freedman ([F1]), there was a "stablized" theory of 4-manifolds ([CS], [FQ1], [L], [Q]). A stabilization of a 4-manifold M is $M\sharp k(S^2\times S^2)$, where k is a nonnegative integer. Let M and N be two closed, locally oriented 4-manifolds, M is said to be stably homeomorphic to N if there exist nonnegative integers r and s such that $M\sharp r(S^2\times S^2)$ is homeomorphic to $N\sharp s(S^2\times S^2)$. Here the connected sum is to be performed compatibly with the local orientations.

Theorem 2. *Let M be a closed, connected, nonorientable, locally oriented 4-manifold with $\pi_1 = Z$.*

(1) If $\tilde{M}$ is not spin, then M is stably homeomorphic to one of the following manifolds: $S^1\tilde{\times}S^3\sharp CP^2$, $S^1\tilde{\times}S^3\sharp * CP^2$, $S^1\tilde{\times}S^3\sharp CP^2\sharp CP^2$ or $S^1\tilde{\times}S^3\sharp CP^2\sharp{*}CP^2$. So the stable homeomorphism type of M is determined by its Kirby-Siebenmann invariant ks and mod 2 Euler number.

(2) If $\tilde{M}$ is spin, then M is stably homeomorphic to $S^1\tilde{\times}S^3$ or $S^1\tilde{\times}S^3\sharp E_8$ where E_8 is the closed, 1-connected, topological 4-manifold with intersection form E_8. So the stable homeomorphism type of M is determined by $ks(M) = [\lambda] \in L_4(Z^-)$, where $[\lambda]$ is the value of the intersection form on $\pi_2(M)$ as an element in the Wall group $L_4(Z^-) = Z_2$ [W].

(3) $\pi_2(M)$ is a finitely generated free $Z[Z]$-module.

(4) The intersection form on $\pi_2(M)$ is a nonsingular ω_1-hermitian form.

Proof. In principle, it is possible to obtain the stable classification of 4-manifolds with any fundamental group using Kreck's work ([K]). For details, see [K][T]. We will use the method and terminology of [K][T].

(1) In this case, the 1-universal fibration is given by $\xi = \rho \oplus \eta : BSO \oplus B\pi_1 \longrightarrow BO$, where η is a line bundle with $\omega_1(\eta) = \omega_1(M)$. There is a spectral sequence with E_2-term $H_p(B\pi_1; \tilde{\Omega_q^{SO}})$ converging to $\Omega_{p+q}(\xi)$, where $\tilde{()}$ denotes the twisted coefficient. Here the coefficient of any abelian group A is twisted by $\omega_1 : \pi_1 \longrightarrow Z_2$ via the inclusion of Z_2 into $Aut(A)$ by the action $a \to -a$. The proof is then a straightforward computation.

(2) In this case, the 1-universal fibration is given by $\xi = \rho \oplus \eta : BSpin \oplus B\pi_1 \longrightarrow BO$, where η is a line bundle with $\omega_1(\eta) = \omega_1(M)$ and $\omega_2(\eta) = 0$. There is a spectral sequence with E_2-term $H_p(B\pi_1; \tilde{\Omega_q^{TOPSpin}})$ converging to $\Omega_{p+q}(\xi)$. The proof of the first part is then a spectral sequence computation. The Kirby-Siebenmann invariant and the value of the intersection form in $L_4(Z^-)$ are both stable invariants. It is easy to verify that they agree on the generators.

(3) As a corollary of Seshadr's theorem ([B]), a finitely generated $Z[Z]$-module is free if and only if it is projective. So it is sufficient to prove that

$\pi_2 M$ is stably free since stably free implies projective. As each of the six stable homeomorphism types has free π_2, the proof is completed.

(4) The following exact sequence comes from the spectral sequence of the universal covering of M with $\Lambda = Z[\pi_1]$ coefficients:

$$0 \longrightarrow H^2(\pi_1;\Lambda) \longrightarrow H^2(M;\Lambda) \longrightarrow Hom_\Lambda(\pi_2 M,\Lambda) \longrightarrow H^3(\pi_1;\Lambda) \longrightarrow 0$$

Since $H^2(\pi_1;\Lambda) = 0$ and $H^3(\pi_1;\Lambda) = 0$,

$$b : H^2(M,\Lambda) \longrightarrow Hom_\Lambda(\pi_2 M,\Lambda)$$

is an isomorphism. Note that the intersection form λ is the composition of b and Poincaré duality with Λ-coefficient:

$$\pi_2(M) = H_2(M;\Lambda) \to H^2(M;\Lambda).$$

Therefore, λ is a nonsingular ω_1-hermitian form. □

Suppose we have a locally flat embedding $S^1 \to M$ of a circle in M. Then a regular neighborhood of this circle is homeomorphic to either $S^1 \times D^3$ or $S^1 \tilde{\times} D^3$, the twisted D^3 bundle over S^1, depending on whether or not the circle is orientable. Suppose there are embeddings of S^1 in both M^4 and W^4 on which ω_1 takes the same value. The sum of M and W along S^1, $M\sharp_{S^1}W$, is the 4-manifold obtained by deleting the interiors of the disk bundle neighborhoods of S^1 and identifying the boundaries.

Theorem 3. *Suppose M is a closed locally oriented 4-manifold with $\pi_1 = Z$, W^4 has "good" fundamental group π and no 2-torsion in $ker(\omega_1)$, $Z \longrightarrow \pi$ is injective, and ω_1 of the two manifolds takes the same value on the generator of Z and its image in π. Suppose further that*

$$H_2(M;Z[Z]) \otimes_{Z[Z]} Z[\pi] \longrightarrow H_2(W;Z\pi)$$

is a $Z[\pi]$-monomorphism that preserves λ and $\tilde{\mu}$, and that either $\omega_2 = 0$ on $\pi_2(W)$ or ω_2 does not vanish on the subspace perpendicular to the image. Then there is a decomposition $W \cong M\sharp_{S^1}W'$ inducing the given homomorphism to $\pi_2 W$. If $\omega_2 \neq 0$ does vanish on the perpendicular subspace, then exactly one of W or $*W$ decomposes.

This is our main technical theorem and is a version of Theorem 10.7B in Section 10.7 of [FQ]. The statement there is not correct. This has been corrected in a more general setting by Stong ([St1] [St2]). For simplicity we assume there is no 2-torsion in $ker(\omega_1)$. For a proof, see [St1][Wa].

Proof of Theorem 1.
Existence: If λ is even, $L_4(1) = Z \longrightarrow L_4(Z^-) = Z_2$ is onto ([W]). It follows that λ is stably isomorphic to $(E_8 \bigotimes_Z \Lambda) \bigoplus \cdots \bigotimes (E_8 \bigotimes_Z \Lambda)$. This form is the intersection form of $W = S^1 \tilde{\times} S^3 \sharp E_8 \cdots \sharp E_8$. Thus there are nonnegative integers r and s such that $\lambda \oplus H(\Lambda)^r$ is the intersection form of $W \sharp s(S^2 \times S^2)$, where $H(\Lambda)$ is the hyperbolic form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. However $H(\Lambda)^r$ can be realized in $W \sharp s(S^2 \times S^2)$ by a topological embedding of $\sharp r(S^2 \times S^2) \backslash int D^4$ ([F1], [FQ]). Therefore, λ is realized as the intersection form of the surgered 4-manifold.

If λ is odd, it follows from $L^0(1) = Z \longrightarrow L^0(Z^-) = Z_2$ is onto ([MR]) that λ is stably isomorphic to $\oplus p(1) \oplus q(-1)$. In this stabilization, metabolic forms are used instead of hyperbolic ones. For odd forms, stabilization by direct sum with metabolic forms is the same as by hyperbolic ones (see the following lemma). Therefore, there are nonnegative integers r and s such that $\lambda \oplus H(\Lambda)^r$ is the intersection form of $W = S^1 \tilde{\times} S^3 \sharp pCP^2 \sharp q\overline{CP^2} \sharp s(S^2 \times S^2)$. Again $H(\Lambda)^r$ can be realized by a topological embedding of $\sharp r(S^2 \times S^2) \backslash int D^4$ in W. By surgering this out, we have the desired manifold.

For each form λ, we have constructed one 4-manifold. If the form λ is odd, another 4-manifold with opposite Kirby-Siebenmann invariant ks can be obtained by the $*-$operation ([FQ], [St3]).
Uniqueness: Regarding the isomorphism $h : H_2(M, Z[Z]) \longrightarrow H_2(N, Z[Z])$ as an injection in Theorem 3, we obtain a decomposition of $N = M \sharp_{S^1} P$ or $*N = M \sharp_{S^1} P$ realizing the injection h. The piece P_S is a manifold with $\pi_1 = Z$, and the $Z[Z]$-homology of $S^1 \tilde{\times} D^3$. By Theorem 11.5 of [FQ], such 4-manifolds are unique relative to the boundary. Therefore, either $N = M$ or $*N = M$. Since the *-operation changes the ks of a manifold if the *-manifold is a different one, it follows from $ksM = ksN$ that $M = N$. Thus there is a homeomorphism $f : M \longrightarrow N$ inducing h.

To tie up the loose end in the existence proof, we prove the following lemma.

Lemma. *In matrix notation, if λ is odd, then stabilization by direct sum with $\begin{pmatrix} 0 & I \\ I & A \end{pmatrix}$ is the same as direct sum with $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$.*

Proof. Let $A = B + \bar{B} + D$, where $D = \begin{pmatrix} 0 & 0 \\ 0 & \epsilon \end{pmatrix}$ and $\epsilon = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$; we have the following identity:

$$\begin{pmatrix} I & 0 \\ -B & I \end{pmatrix} \cdot \begin{pmatrix} 0 & I \\ I & A \end{pmatrix} \cdot \begin{pmatrix} I & -\bar{B} \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & D \end{pmatrix}.$$

Hence it is sufficient to prove that

$$\lambda \oplus \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \cong \lambda \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Since λ is odd, we can find an element v such that $\lambda(v, v) = x + \bar{x} + 1$ for some x. After stabilizations by direct sum with hyperbolic forms, there is some element w such that $w \perp v$ and $\lambda(w, w) = -x - \bar{x}$. Hence there is some element u such that $\lambda(u, u) = 1$. Let C be the orthogonal complement of u , then $\lambda \oplus \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = < 1 > \oplus C$. We have the following identity:

$$< 1 > \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong < 1 > \oplus \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Geometrically, this is the identity: $CP^2 \sharp (S^2 \times S^2) = CP^2 \sharp \overline{CP^2} \sharp CP^2$. So

$$\lambda \oplus \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \oplus \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \cong C \oplus < 1 > \oplus \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\cong C \oplus < 1 > \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cong \lambda \oplus \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

This completes the proof of the lemma, □

and thus the theorem. □

Acknowledgments

I am grateful to P. Teichner for pointing out the identity $ks(M) = [\lambda]$ in Theorem 2 (2).

This work forms a part of the author's doctoral dissertation for the University of California at San Diego. The author would like to thank Mike Freedman, Richard Stong and Peter Teichner for their encouragement and assistance.

References

[B] H. Bass, *Algebraic K-theory*, Benjamin, New York, 1968.

[CS] S. E. Cappell and J. L. Shaneson, *On four-dimensional surgery and its applications*, Comment. Math. Helv., **46** (1971), 500–528.

[F1] M. H. Freedman, *The topology of 4-manifolds,* J. Diff. Geom. **17** (1982), 357–453.

[F2] ———, *The disk theorem for four-dimensional manifolds,* in "Proc. Int. Congress of Math. Warsaw 1983," pp. 647–663.

[FQ] M.H. Freedman and F. Quinn, *Topology of 4-manifolds,* Princeton Univ. Press, 1990.

[FQ1] ———, *A quick proof of the 4-dimensional stable surgery theorem,* Comm. Math. Helv. **55** (1980), 668–671.

[HK1] I. Hambleton and M. Kreck, *On the classification of topological four-manifolds with finite fundamental group,* Math. Ann. **280** (1986), 85–104.

[HK2] ———, *Cancellation, elliptic surfaces and the topology of certain four-manifolds,* J. Reine Angew. Math. **444** (1993), 79–100.

[HKT] I. Hambleton, M. Kreck, and P. Teichner, *Nonorientable four-manifolds with fundamental group of order 2,* Tran. of AMS **344** (1994), 649–665.

[K] M. Kreck, *Surgery and Duality*, to appear as a book in the Vieweg-Verlag, Wiesbaden. As a preprint of the Johannes-Gutenberg-Universitat Mainz 1985 available under the title: *An extension of results of Browder, Novikov and Wall about surgery on compact manifolds.*

[T] T. Lawson, *Trivializing h-cobordism by stabilization,* Math. Z. **156** (1977), 211–215.

[MR] J. Milgram and A. Ranicki, *The L-theory of Laurent polynomial extension and genus 0 function fields,* J. Reine Angew. Math. **406** (1990), 121–166.

[Q] F. Quinn, *The stable topology of 4-manifolds,* Top. and Appl. **15** (1983), 71–77.

[St1] R. Stong, *Existence of π_1-negligible embeddings in 4-manifolds: a correction to Theorem 10.5 of Freedman and Quinn,* Proc. of AMS **120** (1994), 1309–1314.

[St2] ———, *Uniqueness of π_1-negligible embeddings in 4-manifolds: a correction to Theorem 10.5 of Freedman and Quinn,* Topology **32** (1993), 677–699.

[St3] ———, *Uniqueness of connected sum decompositions in dimension 4,* Top. and Appl. **56** (1994), 277–291.

[SW] R. Stong and Z. Wang, *Self-homeomorphisms of 4-manifolds with fundamental group Z,* preprint.

[T] P. Teichner, *Topological 4-manifolds with finite fundamental group,* Ph.D thesis, Universitat Mainz, 1992.

[W] C. T. C. Wall, *Surgery on compact manifolds*, Academic Press, New York, 1970.

[Wa] Z. Wang, *Classification of topological four manifolds with infinite cyclic fundamental group,* Ph.D thesis, UCSD, 1993.

Department of Mathematics, University of Michigan, Ann Arbor, MI 48109
E-mail address: zwang@math.lsa.umich.edu