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SMOOTH RIGIDITY OF RANK-1 LATTICE ACTIONS ON THE SPHERE AT INFINITY

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0. Introduction

Let Γ be a cocompact lattice in $SO(n,1)$. Γ acts naturally on the sphere S^{n-1} preserving the canonical conformal structure. Let us denote this action by $\rho_0 : \Gamma \to \text{Conf}\,(S^{n-1})$. Let $\rho : \Gamma \to \text{Diff}^1(S^{n-1})$ be a C^1-action of Γ on S^{n-1} which is sufficiently C^1-close to ρ_0 among a set of finitely many generators of Γ. Dennis Sullivan [S] proved that there exists a unique homeomorphism $h \in \text{Homeo}(S^{n-1})$ such that $\rho = h^{-1} \circ \rho_0 \circ h$. This result also follows from the structural stability theorem for Anosov flows via a suspension construction (see §1.3). The suspension construction works equally well for the canonical action of any cocompact lattice in a rank-1 semisimple lie group on the ideal boundary of the corresponding symmetric space. The purpose of this paper is to prove the following rigidity result.

Theorem 1. *Let Γ be a cocompact lattice in a rank-1 noncompact semisimple Lie group. Let ∂H be the sphere at infinity of the corresponding symmetric space and $\rho_0 : \Gamma \to Conf\,(\partial H)$ be the canonical action conformal with respect to the Carnot-Carathéodory metric. If $\rho : \Gamma \to Diff^\infty(\partial H)$ is a smooth action of Γ which is sufficiently C^1 close to ρ such that the conjugacy h exists and is absolutely continuous with respect to the Lebesgue measure. Then h is C^∞.*

We suspect that when $\dim(\partial H) > 1$, the condition that h is absolutely continuous is superfluous. What we really need in this paper is a condition weaker than the absolute continuity of h (see §2.2). Actually the theorem is true for C^k-actions ρ for some finite number k (see [BFL]). It is reasonable to make the following

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Conjecture. *If* $\dim(\partial H) > 1$*, then all* C^2*-actions* $\rho : \Gamma \to \mathit{Diff}^2(\partial H)$ *which are sufficiently* C^1*-close to* ρ_0 *must be conjugate to* ρ_0 *via a* C^2 *diffeomorphism.*

We should point out that the conjecture is false for C^1-actions. Indeed consider a closed Riemannian manifold (M, g_0) of constant curvature $K \equiv -1$. Perturb the metric g_0 to a nearly metric g of nonconstant curvature $K(g)$ which satisfies $|K(g) + 1| \le \epsilon$ for a small $\epsilon > 0$. The ideal boundary $\partial\widetilde{M}$ of the universal cover (M, g) carries a $C^{1+\alpha(\epsilon)}$ structure, where $\alpha(\epsilon) \to 1$ (as $\epsilon \to 0$). $\Gamma = \pi_1(M)$ acts on $\partial\widetilde{M}$ as a group of $C^{1+\alpha(\epsilon)}$-diffeomorphisms which is never C^1-conjugate to the canonical action.

The conjecture is also false if $\dim \partial H = 1$. Deformations corresponding to the Teichmüller space are non-smoothly conjugate to each other([Mo], p. 178). However, E. Ghys [G2] proved that for any C^∞-action ρ which is C^1-close to ρ_0, there exists a C^∞-diffeomorphism $h \in \mathrm{Diff}^\infty(S^1)$ such that $h^{-1} \circ \rho \circ h$ coincides with the action of a cocompact lattice $\Gamma_1 \subset PSL(2, \mathbb{R})$ (see also [KY] for another treatment and [G3] for a recent global result). In a different direction, Katok-Spatzier[KS] and M. Kanai[K2] independently obtained the smooth local rigidity of the projective action of irreducible lattices in higher rank noncompact semisimple Lie groups on the maximal boundaries.

To explain our approach, we first recall the notion of contact Anosov flows. Let X be a C^∞ vector field on a C^∞ closed manifold M equipped with a Riemannian metric $\|\cdot\|$. The flow φ_t generated by X is called contact if φ_t preserves a smooth 1-form α such that $\alpha \wedge (d\alpha)^{n-1}$ is everywhere non-degenerate $(2n-1 = \dim M)$. φ_t is said to be Anosov if there exists a flow-invariant decomposition of the tangent bundle of $M : TM = E^+ \oplus \mathbb{R}X \oplus E^-$ and constants $A > 0$, $a > 0$ such that

(i) $\forall\, Y \in E^+$, $\forall t \ge 0$, $\|D\varphi_{-t}(Y)\| \le A\|Y\|e^{-at}$,
(ii) $\forall\, Y \in E^-$, $\forall t \ge 0$, $\|D\varphi_t(Y)\| \le A\|Y\|e^{-at}$.

The distributions E^+, E^-, $E^+ \oplus \mathbb{R}X$, $E^- \oplus \mathbb{R}X$ of an Anosov flow are all integrable with C^∞ leaves. The corresponding foliations W^{su}, W^{ss}, W^u, W^s are called strong unstable, strong stable, unstable, stable foliations of φ_t. These Anosov foliations are in general only Hölder continuous. Benoist, Foulon and Labourie, generalizing previous works of Ghys [G1], Hurder-Katok [HK], Kanai [K1], Feres-Katok[FK] and Feres [F], obtained the following result.

Theorem. [BFL] *Let* M *be a closed manifold of dimension* $2n - 1 > 3$*. Let* φ_t *be a contact Anosov flow on* M *with* C^∞ *Anosov foliations. Then after a proper reparameterization,* φ_t *is* C^∞*-time preservingly conjugate to the geodesic flow of a locally symmetric space of negative curvature.*

The strategy we adopt in the proof of the main theorem involves the construction of a new Anosov flow from the perturbed action (this is carried out in §1, §2) and then prove that the new flow is contact with C^∞-Anosov foliations (this is carried out in §3). Theorem 1 is proved in §4.

1. Suspension and holonomy

(1.1) We first recall the notion of a foliated bundle and a related suspension construction. A good reference is [HT]. Let $\tau = (P, E, M)$ be a C^1 bundle [HT] with bundle projection $P : E \to M$ and fibres V_x, $x \in M$, a foliation of τ is a foliation $\mathcal{F}$ of E where leaves are transverse to the fibres and of complementary dimension. Assume that the fibres are compact. For each $u \in E$ the leaf $\mathcal{F}_u$ through u has a unique topology making $P\big|_{\mathcal{F}_u} : \mathcal{F}_u \to M$ a covering space. For any $x, y \in M$ if $\lambda : [0,1] \to M$ is a path from x to y, a homeomorphism $h(\lambda) : V_x \to V_y$ is defined by $h^{-1}(\lambda) : w \to \lambda_w(1)$ where $\lambda_w : [0,1] \to \mathcal{F}_w$ is the unique path in $\mathcal{F}_w$, starting at $w \in V_y$, which covers λ. Holonomy must be defined backwards, as above, so that $h(\lambda \# \mu) = h(\lambda) \circ h(\mu)$ where $\lambda \# \mu$ means the sum of paths. Setting $x = y$ we obtain a homomorphism $\pi_1(M, x) \xrightarrow{h} \mathrm{Homeo}(V_x)$. Fix the base point x and identify V_x with the fibre V of the bundle $\tau = (P, E, M)$; we call $\pi_1(M) \xrightarrow{h} \mathrm{Homeo}(V)$ the holonomy homomorphism of the foliated bundle $(\tau, \mathcal{F})$.

Next we consider an inverse construction which is called the suspension construction. Let V be any space and M a space with universal cover $\widetilde{M}$. Let $h : \pi(M) \to \mathrm{Homeo}(V)$ be a homomorphism. Then $\pi_1(M)$ acts on $\widetilde{M} \times V$ by $\gamma(b, v) = (\gamma b, \gamma v)$. Let $E = (\widetilde{M} \times \widetilde{V})/\pi_1(M)$ be the quotient space and let $P{:}(\widetilde{M} \times V)/\pi_1(M) \to \widetilde{M}/\pi(M) = M$ be the natural map of orbits. Then (P, E, M) is a foliated bundle. The leaves of the foliation $\mathcal{F}$ are the images of the set $\widetilde{M} \times v$, $v \in V$. The holonomy homomorphism of this foliated bundle is the homomorphism h we start with. Conversely, if $h : \pi_1(M) \to \mathrm{Homeo}(V)$ is the holonomy homomorphism of a foliated bundle $(\tau, \mathcal{F})$ with compact fibre V, then the foliated bundle obtained by the suspension construction starting from h is naturally isomorphic to $(\tau, \mathcal{F})$.

If M and V are C^k manifolds and if $\Gamma = h(\pi_1(M))$ is contained in $\mathrm{Diff}^k(V)$ then the suspension construction gives a C^k-foliated bundle with C^k foliation $\mathcal{F}$. Conversely, the holonomy group Γ of a C^k foliated bundle with a C^k foliation is contained in $\mathrm{Diff}^k(V)$. Assume $h_1 : \pi_1(M) \to \mathrm{Diff}^k(V)$ is another homomorphism and M is compact. Then $\pi_1(M)$ is finitely generated. The foliation $\mathcal{F}_1$ corresponding to the suspension of h_1 is C^1-closed to $\mathcal{F}$ (in terms of their tangent distribution) if and only if the holonomy h_1 is C^1-closed to h among a set of finitely many generators. It is also easy to see that the existence of a C^k-diffeomorphism H which conjugates the leaves of the two foliation is equivalent to the existence of a C^k-diffeomorphism h on V which conjugates the actions of h and h_1.

(1.2) If M is a closed Riemannian manifold of negative curvature, then the unit tangent bundle SM carries two foliated bundle structures corresponding to the stable foliation W^s and unstable foliation W^u of the geodesic flow. To describe the holonomy map of these foliations, let us fix a point x in the universal cover $\widetilde{M}$. For any vector $v \in S\widetilde{M}$, denote by $v(t)$ the geodesic with initial velocity $v : \dot{v}(0) = v$. Denote by $v(\infty)$ the point in the ideal boundary $\partial\widetilde{M}$ represented by the geodesic ray $v(t)$, $t \geq 0$. The map $P_x : S_x\widetilde{M} \to \partial\widetilde{M} : v \mapsto v(\infty)$ induces a homeomorphism. The holonomy $h^s : \Gamma = \pi_1(M) \to \mathrm{Homeo}(S_x\widetilde{M})$ of the stable foliation W^s is given by $h^s(\gamma)(v) = P_x^{-1} \circ \rho_0(\gamma) \circ P_x(v)$ for $v \in S_x\widetilde{M}$ where $\rho_0 : \Gamma \to \mathrm{Homeo}(\partial\widetilde{M})$ represents the canonical action of Γ on $\partial\widetilde{M}$. On the other hand, there exists a geodesic reflection $\sigma_x : \partial\widetilde{M} \to \partial\widetilde{M}$ which is defined by $\sigma_x(P_x(v)) = P_x(-v), \forall\, v \in S_x\widetilde{M}$. The holonomy $h^u : \Gamma \to \mathrm{Homeo}(S_x\widetilde{M})$ of the unstable foliation W^u is given by $h^u(\gamma)(v) = P_x^{-1} \circ \sigma_x \circ \rho_0(\gamma) \circ \sigma_x \circ P_x(v)$. In summary, after identifying $\partial\widetilde{M}$ with S_xM via P_x, the holonomy of W^s is isomorphic to $\rho_0(\Gamma)$ and the holonomy of W^u is isomorphic to $\sigma_x \circ \rho_0 \circ \sigma_x$. In this model, the space of geodesics in $S\widetilde{M}$ is canonically identified with $\partial\widetilde{M} \times \partial\widetilde{M} \setminus D$ where $D = \{(\xi, \sigma_x(\xi)) | \xi \in \partial\widetilde{M}\}$. The group Γ acts on this space via $\gamma(\xi, \eta) = (\rho_0(\gamma)\xi,\ \sigma_x \circ \rho_0(\gamma) \circ \sigma_x\eta)$.

(1.3) From now on, we assume $\widetilde{M}$ is a symmetric space of negative curvature (so $\widetilde{M} = H^n_{\mathbb{R}}$, $H^n_{\mathbb{C}}$, $H^n_{\mathbb{H}}$, the real, complex, quaternionic, hyperbolic space, or $H^2_{\mathbb{O}}$, the hyperbolic Cayley plane). Then $\partial\widetilde{M}$ carries a C^∞ structure such that the holonomy h^s, h^u are contained in $\mathrm{Diff}^\infty(\partial\widetilde{M})$.

Lemma 1. (Compare [S]) *Let $\rho : \Gamma \to \mathrm{Diff}^1(\partial\widetilde{M})$ be a C^1-action of Γ on $\partial\widetilde{M}$ which is sufficiently close to ρ_0 in the C^1-topology. Then there exists a Hölder continuous homeomorphism $h : \partial\widetilde{M} \to \partial\widetilde{M}$ (unique in the transversal direction) which is close to the identity and conjugates the two actions: $\rho = h^{-1} \circ \rho_0 \circ h$.*

Proof. By the suspension construction, the new action ρ gives rise to a C^1 foliation V^s on SM which is C^1-close to the foliation W^s. The unstable foliation W^u is transversal to W^s. W^u must also be transversal to V^s if the perturbation ρ is sufficiently close to ρ_0 and consequently, the 1-dimensional foliation $W^u \cap V^s$ is C^1 close to the 1-dimensional foliation $W^u \cap W^s$ (which is the orbit foliation of the geodesic flow). By the structural stability theorem of Anosov flows, there exists a homeomorphism $\overline{H} : SM \to SM$(unique in the transversal direction) which is Hölder , close to identity, and conjugates the two 1-dimensional foliations and in consequence, conjugates the foliations W^s and V^s. This gives rise to a Hölder homeomorphism $h : \partial\widetilde{M} \to \partial\widetilde{M}$ which conjugates the two actions ρ_0 and ρ. $\square$

(1.4) In what follows we continue to use the notations in the last three sections. The perturbation ρ induces an action of Γ on $\partial\widetilde{M} \times \partial\widetilde{M} \setminus D$ given by $\widehat{\gamma}(\xi, \eta) = (\rho(\gamma)\xi,\ \sigma_x \circ \rho(\gamma) \circ \sigma_x\eta)$. For the sake of simplicity we denote this action by $\widehat{\Gamma}$ and denote the canonical action $\gamma(\xi, \eta) = (\rho_0(\gamma)\xi,\ \sigma_x \circ \rho_0(\gamma) \circ \sigma_x\eta)$ by Γ. The two actions are conjugate via the homeomorphism $H \in \mathrm{Homeo}(\partial\widetilde{M} \times \partial\widetilde{M} \setminus D) : (\xi, \eta) \to (h\xi, \sigma_x \circ h \circ \sigma_x\eta)$. So we have $\widehat{\Gamma} = H^{-1} \circ \Gamma \circ H$. We should remind the reader that both actions are equivariant under the flip map $\sigma : \sigma(\xi, \eta) = (\sigma_x\eta, \sigma_x\xi)$.

(1.5) In the proof of lemma 1, we constructed an Anosov flow using the foliations W^u and V^s. Next we describe a more symmetric construction. Namely, let $\rho : \Gamma \to \mathrm{Diff}^1(\partial\widetilde{M})$ be a C^1-perturbation of ρ_0. Consider the representation $\rho' : \Gamma \to \mathrm{Diff}^1(\partial\widetilde{M})$, $\rho' = \sigma_x \circ \rho \circ \sigma_x$. By the suspension construction, ρ' gives rise to a C^1 foliation V^u on SM which is C^1-close to the unstable foliation W^u. Since W^s is transversal to W^u, V^s must also be transversal to V^u if the perturbation ρ of ρ_0 is small enough. The intersection $V^s \cap V^u$ is a 1-dimensional foliation C^1-close to the orbit foliation of the geodesic flow $W^s \cap W^u$. Any smooth parameterization of the 1-dimensional foliation $V^s \cap V^u$ gives rise to an Anosov flow φ_t which is orbit equivalent to the geodesic flow g_t via a Hölder homeomorphism $\overline{H} : SM \to SM$. The space of orbits of φ_t is canonically identified with $\partial\widetilde{M} \times \partial\widetilde{M} \setminus D$. The fundamental group Γ acts on this space via $\widehat{\gamma}\ (\xi, \eta) = (\rho(\gamma)\xi,\ \sigma_x \circ \rho(\gamma) \circ \sigma_x\eta)$. The homeomorphism $\overline{H}$ induces the homeomorphism $H \in \mathrm{Homeo}(\partial\widetilde{M} \times \partial\widetilde{M} \setminus D)$ which conjugates the action $\widehat{\Gamma}$ and Γ (see §1.4).

Lemma 2. *If h is absolutely continuous, then φ_t preserves an absolutely continuous invariant measure.*

Proof. If h is absolutely continuous, then the conjugacy H(see §1.4) is also absolutely continuous. Since the geodesic glow g_t preserves the Liouville measure ν, the measure $\overline{H}_*(\nu)$ is an absolutely continuous measure invariant under a time change of φ_t. $\square$

2. Synchronization of the new flow

(2.1) From now on, we assume that ρ is a C^1 perturbation of ρ_0 and

$\rho(\Gamma) \subset \mathrm{Diff}^\infty(\partial\widetilde{M})$. Starting from the perturbed action, we performed a pair of suspension constructions in §1.5. The result is a new Anosov flow φ_t. The flow φ_t has C^∞ weak stable and weak unstable foliations V^s and V^u. However, the smoothness of the strong stable and strong unstable foliations V^{ss} and V^{su} obviously depend on time parameterization. We will prove in this section that if h is absolutely continuous, then there exists a unique parameterization such that the strong stable and strong unstable foliations are smooth simultaneously.

(2.2) Recall that any smooth codimension-q foliation $\mathcal{F}$ is uniquely determined by a locally decomposable q-form w (up to the multiplication by a scalar function). The tangent bundle of $\mathcal{F}$ is exactly the kernel of w. The q-form w satisfies the Frobenius condition $dw = \alpha \wedge w$, where α is a 1-form. Let $n = \dim M$. Then V^u is a codimension-$(n-1)$ smooth foliation. Let w be a locally decomposable C^∞ $(n-1)$-form defining V^u. Since $\varphi_t^* w$ is also locally decomposable and also defines V^u, we have $\varphi_t^* w = f_t w$, where f_t is a C^∞ positive function. It is easy to see that $f_t \to 0$ (as $t \to \infty$). Now consider

$$w_T = \int_0^T \varphi_t^* w \; dt = \left(\int_0^T f_t dt \right) w.$$

We have $L_Y w_T = (f_T - 1)w = \frac{f_T - 1}{\int_0^T f_t dt} w_T$, where Y is the vector field generating φ_t. Let $X = \frac{1 - f_T}{\int_0^T f_t dt} Y$ for some large number T such that $1 - f_T$ is positive. It follows that $L_X w_T = -w_T$.

Lemma 3. *Under the new parameterization X, the strong unstable foliation V^{su} is C^∞.*

Proof. For the sake of simplicity, we denote w_T by w^u and denote the flow of X by φ_t. Let α be a C^∞ 1-form such that $dw^u = \alpha \wedge w^u$. Then $-w^u = L_X w^u = i_X dw^u + d i_X w^u = i_X(\alpha \wedge w^u) = \alpha(X) w^u (\because i_X w^u \equiv 0)$. Hence $\alpha(X) \equiv -1$.

Since $L_X w^u = -w^u$, we also have $\varphi_t^* w^u = e^{-t} w^u$. Taking exterior derivative, we get

$$d(\varphi_t^* w^u) = \varphi_t^*(d\,w^u) = \varphi_t^*(\alpha \wedge w^u) = \varphi_t^* \alpha \wedge \varphi_t^* w^u = \varphi_t^* \alpha \wedge e^{-t} w^u,$$
$$d(\varphi_t^* w^u) = d(e^{-t} w^u) = e^{-t} dw^u = e^{-t} \alpha \wedge w^u.$$

It follows that $(\varphi_t^* \alpha - \alpha) \wedge w^u = 0$. Thus, for any vector $Y \in TV^{su}$, we have $[\alpha(D\varphi_t(Y)) - \alpha(Y)] w^u = i_Y[(\varphi_t^*\alpha - \alpha) \wedge w^u] = 0$. Therefore, $\alpha(Y) = \alpha(D\varphi_t(Y)) \to 0 (ast \to -\infty)$. Consequently, we have $V^{su} = \ker(\alpha) \cap \ker(w^u)$ (recall that the kernel of a differential form σ is defined by $ker(\sigma) = \{Y | i_Y(\sigma) = 0\}$). Hence V^{su} is smooth. $\square$

Lemma 4. *If h is absolutely continuous, then under the parameterization X, the strong stable foliation V^{ss} is also C^∞.*

Proof. If h is absolutely continuous, by lemma 2, φ_t preserves an absolutely continuous measure which is represented by a volume form Ω. There exists a unique $(n-1)$ form w^s defining V^{ss} which satisfies $w^s \wedge w^u = i_X\Omega$. Suppose that $dw^s = \beta \wedge w^s$. Then we have

$$\begin{aligned} 0 &= L_X\Omega = i_X(d\Omega) + d(i_X\Omega) \\ &= d(w^s \wedge w^u) = (-1)^{n-1} w^s \wedge dw^u + dw^s \wedge w^u \\ &= (\alpha + \beta) \wedge w^s \wedge w^u \end{aligned}$$

Hence $[\alpha(X)+\beta(X)]w^s \wedge w^u = i_X[(\alpha+\beta)\wedge w^s \wedge w^u] = 0$. Therefore $\beta(X) \equiv 1$. It follows that $L_X w^s = i_X dw^s + d\, i_X w^s = i_X(\beta \wedge w^s) = \beta(X)w^s = w^s$. From the argument for lemma 3, we have that $V^{ss} = \ker(\beta) \cap \ker(w^s)$; hence, it is C^∞. □

3. The new flow is contact

(3.1) In what follows we assume that ρ is a sufficiently small C^1 perturbation of ρ_0 with $\rho(\Gamma) \subset \mathrm{Diff}^\infty(\partial\widetilde{M})$ and moreover, the map h (see lemma 1) is absolutely continuous. We keep the notations of section 2. In particular, by lemma 4, there exists a time parameterization φ_t of the perturbed 1-dimensional foliation such that the strong stable and strong unstable foliations V^{ss} and V^{su} of φ_t are C^∞ foliations. Hence the 1-form σ defined by $\sigma(X) = 1$, $\sigma\big|_{E^{ss}\oplus E^{su}} = 0$ is a C^∞ form invariant under the flow φ_t and $d\sigma$ is a C^∞ flow-invariant 2-form.

Lemma 5.

(1) *For local vector fields Y, $Z \in E^{ss} \oplus E^{su}$, we have*

$$d\sigma(Y, Z) = -\sigma([Y, Z]);$$

(2) $X \subset \ker(d\sigma)$*;*
(3) $\sigma \wedge d\sigma \not\equiv 0$*;*
(4) $d\sigma\big|_{E^s} = d\sigma\big|_{E^u} = 0$.

Proof. (1) Let $Y, Z \in E^{ss} \oplus E^{su}$ be local C^∞ vector fields Y, Z. Then we have

$$d\sigma(Y, Z) = Y(\sigma(Z)) - Z(\sigma(Y)) - \sigma([Y, Z]) = -\sigma([Y, Z]).$$

(2) For any local vector field $Y \subset E^{ss}$ or E^{su}, we have $d\sigma(X, Y) = d\sigma(Dg_t(X), Dg_t(Y)) = d\sigma(X, Dg_t(Y)) \to 0$ (as $t \to \infty$ or $t \to -\infty$).

(3) If $\sigma \wedge d\sigma \equiv 0$, then $E^{ss} \oplus E^{uu}$ is integrable. This is impossible (see [P], theorem 3.1).
(4) This follows from the same argument as in (2).

Consider the distribution $F \overset{\text{def}}{=} \{Y \in E^{ss} \oplus E^{su} \big| i_Y(d\sigma) = 0\}$. Denote $F^s = F \cap E^{ss}$ and $F^u = F \cap E^{su}$. Then we have $F = F^s \oplus F^u$. □

Definition. For each point $v \in SM$, we define the index of v by $I(v) = \dim F$. The stable (resp. unstable) index of v is defined to be $I^s(v) = \dim F^s$ (resp. $I^u(v) = \dim F^u$). We also define the rank $r(v)$ of v to be the smallest number $p > 0$ such that $(d\sigma)^p \neq 0$ but $(d\sigma)^{p+1} = 0$ at v.

Lemma 6. *There exists a dense open subset $U \subset SM$ such that for all $v \in U$, $I(v) = n^s + n^u$, $I^s(v) = n^s$, $I^u(v) = n^u$, $r(v) = n_0$ for some integers $0 \leq n^s$, n^u, $n_0 \leq n - 1$.*

Proof. Clearly, the functions $I(v)$, $I^s(v)$, $I^u(v)$ are lower semi-continuous and the function $r(v)$ is upper semi-continuous. Hence the sets

$$\begin{aligned} U_1 &= \{v \in SM \big| I(v) \text{ is minimal}\}, \\ U_2 &= \{v \in SM \big| I^s(v) \text{ is minimal}\}, \\ U_3 &= \{v \in SM \big| I^u(v) \text{ is minimal}\}, \\ U_4 &= \{v \in SM \big| r(v) \text{ is maximal}\} \end{aligned}$$

are nonempty and open. Since the functions $I(v), I^s(v), I^u(v)$ and $r(v)$ are all φ_t-invariant, the sets U_1, U_2, U_3, U_4 are also φ_t-invariant. By the ergodicity of φ_t with respect to the Lebesgue measure, each U_i must also be dense in SM. Hence $U \overset{\text{def}}{=} U_1 \cap U_2 \cap U_3 \cap U_4$ is open, dense and φ_t-invariant. □

Lemma 7.

(1) *The distributions F, F^s, F^u are C^∞ on U.*
(2) *$n^s = n^u = (n-1) - n_0$.*

Proof. (1) This is because $F + \mathbb{R}X$ is the kernel of the C^∞ 2-form $d\alpha$ which has constant rank on U.
(2) By the Darboux theorem, around each point in U, there exists local coordinates $(x_1, x_2, \ldots, x_{n-1}, y_1, y_2, \cdots, y_{n-1}, s)$ such that $d\sigma = dx_1 \wedge dy_1 + \cdots + dx_{n_0} \wedge dy_{n_0}$ and moreover, (i) if $\sigma \wedge (d\sigma)^{n_0} \equiv 0$ in the local coordinate chart, then $\sigma = x_1 dy_1 + \cdots + x_{n_0} dy_{n_0}$; (ii) if $\sigma \wedge (d\sigma)^{n_0} \neq 0$, then $\sigma = x_1 dy_1 + \cdots + x_{n_0} dy_{n_0} + ds$. We claim that the first case (i) is impossible. This is because if $\sigma \wedge (d\sigma)^{n_0} \equiv 0$ in a small neighborhood in U, then $i_X(\sigma \wedge (d\sigma)^{n_0}) = (d\sigma)^{n_0} \equiv 0$ which is a contradiction. □

Lemma 8. *The distributions F^s, F^u and $F \oplus \mathbb{R}X$ are integrable on U.*

Proof. For arbitrary vector fields Y_1, Y_2, Z such that $Y_1, Y_2 \subset \ker(d\sigma)$, we have, by the Jacobi identity,

$$\begin{aligned} 0 = d^2\sigma(Y_1, Y_2, Z) &= Y_1(d\sigma(Y_2, Z)) + Y_2(d\sigma(Z, Y_1)) + Z(d\sigma(Y_1, Y_2)) \\ &\quad - d\sigma([Y_1, Y_2], Z) - d\sigma([Y_2, Z], Y_1) - d\sigma([Z, Y_1], Y_2) \\ &= -d\sigma([Y_1, Y_2], Z). \end{aligned}$$

Hence $[Y_1, Y_2] \subset \ker(\sigma)$. It follows from the Frobenius theorem that $F \oplus \mathbb{R}X = \ker(d\sigma)$ is integrable. Since $F^s = (F \oplus \mathbb{R}x) \cap E^{ss}$, $F^u = (F \oplus \mathbb{R}X) \cap E^{su}$ and both E^{ss} and E^{su} are integrable, F^s, F^u must also be integrable. $\square$

Consider the set $\mathcal{R} = \{v \in U \big|$ There exist two sequences $T_k \to \infty$ and $t_k \to -\infty$ such that $\lim_{k\to\infty} \varphi_{t_k}(v) = \lim_{k\to\infty} \varphi_{T_k}(v) = v\}$. Clearly $\mathcal{R}$ contains all periodic points in U. By the closing lemma ([Ma]), the set of periodic points in U is dense. Hence $\mathcal{R}$ is dense in U.

Lemma 9. *For each $v \in \mathcal{R}$, we have $V^{ss}(v) \subset U$, $V^{su}(v) \subset U$.*

Proof. This follows directly from the definition of $\mathcal{R}$ and the contracting property of V^{ss} (as $t \to \infty$) and V^{su} (as $t \to -\infty$). $\square$

For each $v \in \mathcal{R}$, the restriction of F^u to $V^{su}(v)$ generates a C^∞-foliation $\mathcal{F}^u(v)$ along $V^{su}(v)$. Next, we lift everything to the universal covering $\widetilde{SM}$ and keep the same notations. Recall that there exists a canonical identification between the orbit space $\widetilde{SM}/V^u$ (resp. $\widetilde{SM}/V^s$) and $\partial\widetilde{M}$ via the suspension construction: $\widetilde{SM}/V^u \xrightarrow{P^u} \partial\widetilde{M}$ (resp. $\widetilde{SM}/V^s \xrightarrow{P^s} \partial\widetilde{M}$). The map P^u (resp. P^s) induces a projection $\pi^u : \widetilde{SM} \to \partial\widetilde{M}$ (resp. $\pi^s : \widetilde{SM} \to \partial\widetilde{M}$) defined by $\pi^u(v) = P^u(V^u(v))$ (resp. $\pi^s(v) = P^s(V^s(v))$). The restriction of π^u to $V^{su}(v)$ defines a diffeomorphism $\pi^u_v : V^{su}(v) \to \partial\widetilde{M} \setminus \{\sigma_x(\pi^s(v))\}$ (see §1.2). The foliation $\mathcal{F}^u(v)$ projects to a foliation $L^u(v)$ on $\partial\widetilde{M} \setminus \{\sigma_x(\pi^s(v))\}$ under the diffeomorphism π^u_v. Given any other point $v_1 \in \mathcal{R}$, $v_1 \neq v$, the stable foliation V^s induces a canonical holonomy map $f^s : V^{su}(v) \to V^{su}(v_1)$ such that for all $w \in V^{su}(v)$, $f^s(w) = V^{su}(v_1) \cap V^s(w)$. Thus there exists $T_0 \in \mathbb{R}$ such that $\lim_{t\to\infty} d(\varphi_{t+T_0}(w), \varphi_t(f^s(w))) = 0$ for any Riemannian metric d on $\widetilde{SM}$ lifted from SM. By lemma 8, the distribution $F^s \oplus F^u \oplus \mathbb{R}X$ is integrable on U. Hence the foliation $\mathcal{F}^{su}$ is invariant under the local holonomy along V^s (by 'local' we mean that the holonomy between nearby leaves in U such that the holonomy process stays in U). It follows that there exists $t_0 > 0$ such that the foliation $\mathcal{F}^u(\varphi_{t_0+T_0}(w))$ is mapped to

$\mathcal{F}^u(\varphi_{t_0}(f^s(w)))$ by the holonomy $f^s : V^{su}(\varphi_{t_0+T_0}(w)) \to V^{su}(\varphi_{t_0}(f^s(w)))$. Since the foliation $\mathcal{F}^u$ and the holonomy f^s is flow-invariant, it follows that $f^s : V^{su}(v) \to V^{su}(v_1)$ preserves the foliation $\mathcal{F}^u$. Hence we obtain the following lemma.

Lemma 10. *For any two points v, $w \in \mathcal{R}$, we have $f^s(\mathcal{F}^u(v)) = \mathcal{F}^u(w)$.*

Proposition 11. *The flow φ_t is contact.*

Proof. By lemma 10, the holonomy invariant foliations $\mathcal{F}^u(v)$, $v \in \mathcal{R}$ project to a C^∞-foliation L^u on $\partial\widetilde{M}$ which is clearly invariant under the action $\rho(\Gamma)$ on $\partial\widetilde{M}$. Such a foliation is necessarily trivial ([F]). Hence $n_s = 0$ or $n-1$. If $n^s = n-1$, then $d\sigma \equiv 0$ on U which is obviously a contradiction (see [P]). Thus $n^s = 0$ and $(d\sigma)^{n-1} \neq 0$ everywhere on U. From the argument for lemma 7, we see that the zero set of $\sigma \wedge (d\sigma)^{n-1}$ is nowhere dense in U. Hence $\sigma \wedge (d\sigma)^{n-1} \neq 0$ everywhere on an open dense subset $U_1 \subset U$. But then $\sigma \wedge (d\sigma)^{n-1}$ defines an invariant volume for φ_t. It follows from the Livshitz theorem that $\sigma \wedge (d\sigma)^{n-1} \neq 0$ everywhere on SM. Consequently, σ is a φ_t-invariant contact form on SM. $\square$

4. Proof of theorem 1 and other comments

(4.1) We finish the proof of theorem 1. If $\dim(\partial\widetilde{M}) = 1$, then theorem 1 follows from Ghys [G2]. This is because by [G2], there exists a Fuchsian group $\Gamma_1 \subset PSL(2,\mathbb{R})$ and a C^∞ map $h_1 \in \mathrm{Diff}^\infty(S^1)$ such that $h_1^{-1} \circ \rho(\Gamma) \circ h_1 = \rho_0(\Gamma_1)$, where $\rho_0 : \Gamma_1 \to \mathrm{Proj}\ (S^1)$ is the canonical projective representation. Hence, we have $\rho_0(\Gamma_1) = (h \circ h_1)^{-1} \circ \rho_0(\Gamma) \circ (h \circ h_1)$. Since the map $h \circ h_1$ is absolutely continuous with respect to the Lebesgue measure, by Mostow's rigidity theorem [Mo, pp. 178], $h \circ h_1 \in PSL(2,\mathbb{R})$, it follows that $h \in \mathrm{Diff}^\infty(S^1)$.

If $\dim(\partial\widetilde{M}) > 1$, by Benoist, Foulon and Labourie's theorem, there exists a locally symmetric space N, such that after a proper reparameterization, φ_t is C^∞-time preservingly conjugate to the geodesic flow $\overline{g}_t$ of N via a diffeomorphism $H_1 : SM \to SN$. The map $H_1 \circ \overline{H}$ (see §1.5 for the map $\overline{H}$) defines a C^∞ orbit equivalence between the geodesic flows $\overline{g}_t$ on N and g_t on M. By the Mostow rigidity theorem, $H_1 \circ \overline{H}$ must be induced by the tangent map of an isometry between M, N up to a time shift. It follows that $\overline{H} \in C^\infty$ and consequently, $h \in C^\infty$.

(4.2) We conclude this paper with a few comments. We would like to point out that the flow φ_t we constructed in §1.5 is more symmetric than the quasi-Fuchsian construction of Ghys [G2]. Namely, there exists a C^∞ diffeomorphism $\Sigma : SM \to SM$ which flips the orbits of $\varphi_t : \varphi_t(\Sigma(v)) = \Sigma\varphi_{-s}(v)$ (Σ might not preserve the time). Does the existence of such a flip map imply that φ_t preserves a volume? If this was true, than our assumption that h is absolutely continuous in theorem 1 is superfluous.

R. Zimmer proved that if the action of a lattice in a semi-simple Lie group with Kazhdan property T preserves a Riemannian metric, then any nearby perturbed action also preserves a Riemannian metric. In our situation, the action ρ_0 preserves a canonical conformal structure. One can easily prove that for any action $\rho : \Gamma \to \mathrm{Diff}^\infty(S^{n-1})$ which is sufficiently C^1 close to ρ_0 among a set of finitely map g generators, the following statements are equivalent (for the sake of simplicity, we consider only $\Gamma \subset SO(n,1),\ n \geq 3$):

(1) ρ is smoothly conjugate to ρ_0;
(2) ρ preserves a conformal structure;
(3) ρ is uniformly quasi-conformal.

However, although the quasi-conformal distortion of the generators of $\rho(\Gamma)$ is small, long compositions of these generators might create arbitrarily large quasi-conformal distortion. Theorem 1 says that this could not happen if h is absolutely continuous, or if the product action $\widehat{\Gamma}$ on $(S^{n-1} \times S^{n-1}) \smallsetminus D$ (see §1.4) preserves a locally finite absolutely continuous measure.

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References

[BFL] Y. Benoist, P. Foulon, F. Labourie, *Flots d'Anosov á distribution stable et instable diffé- rentiable*, J. Amer. Math. Soc. **1** (1992), 33–74.

[F] R. Feres, *Geodesic flows on manifolds of negative curvature with smooth horospheric foliations*, Ergod. Th. & Dyn. Syst. **11** (1991), 635–670.

[FK] R. Feres, A. Katok, *Anosov flows with smooth foliations and rigidity of geodesic flows in three dimensional manifolds of negative curvature*, Ergod. Th. & Dyn. Syst. **10** (1990).

[G1] E. Ghys, *Flots d'Anosov dont les feuilletages stables sont différentiables*, Ann. Scien. Ec. Norm. Sup. **20** (1987), 251–270.

[G2] E. Ghys, *Deformations de flots d'Anosov et de groupes Fuchsiens*, Ann. Inst. Fourier, Grenoble **42** (1992), 209–247.

[G3] E. Ghys, *Rigidité différentiable des groupes Fuchsiens*, Publ. Math. IHES **78** (1994), 163–185.

[HK] S. Hurder, A. Katok, *Differentiability, rigidity and Godbillon-Vey classes for Anosov flows*, Publ. Math. IHES **72** (1990), 5–61.

[HT] M. Hirsch, W. P. Thurston, *Foliated bundles, invariant measures and flat manifolds*, Ann. of Math. **101** (1975), 369–390.

[K1] M. Kanai, *Geodesic flows of negatively curved manifolds with smooth stable and unstable foliations*, Ergod. Th. & Ergod. Th. & Dyn. Syst. **8** (1988), 215–240.

[K2] M. Kanai, Talk on the "Rigidity and deformations for hyperbolic systems" conference at CIRM, Marseilles.

[KS] A. Katok, R. Spatzier, *Differential rigidity of projective lattice actions*, in preparation.

[KY] A. Kononenko, C. B. Yue, *Cohomology and rigidity of Fuchsian groups*, to appear in Isr. J. Math.

[Ma] R. Mañé, *Ergodic theory and differentiable dynamics*, Springer-Verlag, 1987.

[Mo] G. Mostow, *Strong rigidity of locally symmetric spaces*, Ann. Math. Studies 78, Princeton Univ. Press (1973).

[P] J. Plante, *Anosov flows*, Amer. Jour. of Math. **94** (1972), 729–754.

[S] D. Sullivan, *Quasiconformal homeomorphisms and dynamics, III. Structural stability implies hyperbolicity for Kleinian groups*, Acta Math. **155** (1985), 243–260.

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