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A VANISHING THEOREM FOR SEIBERG-WITTEN INVARIANTS

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Abstract. It is shown that the quotients of Kähler surfaces under free anti-holomorphic involutions have vanishing Seiberg-Witten invariants.

Various vanishing theorems have played important roles in gauge theory. The first among them, due to S. K. Donaldson [2], states that the Donaldson invariants vanish for any smooth closed oriented 4-manifold X which decomposes to a connected sum $X_1 \# X_2$ with $b_2^+(X_1) > 0$, $b_2^+(X_2) > 0$. (Here $b_2^+(X_i)$ is the dimension of a maximal subspace of $H_2(X_i,\mathbf{Z})$ on which the intersection pairing is positive definite.) E. Witten [12] has shown more recently that such a connected-sum manifold has also vanishing Seiberg-Witten invariants. Even though its proof is simple, Witten's vanishing theorem is equally useful as Donaldson's vanishing theorem; for example, it implies that any symplectic 4-manifold cannot be decomposed into the above connected sum by combining with Taubes' theorem in [9].

In this note we show that Seiberg-Witten invariants vanish for another class of 4-manifolds. These manifolds are obtained in connection with real algebraic geometry, and the construction was originally proposed in Donaldson [3]. See Remark 4 below. To state our theorem, recall that a map σ between two almost complex manifolds is called anti-holomorphic if $\sigma_* J_1 = -J_2 \sigma_*$ on the tangent bundles, where J_i are the almost complex structures of the manifolds. In the following, K denotes the canonical bundle of an almost complex manifold (underlying a Kähler manifold).

Theorem 1. *Let $\widetilde{X}$ be a Kähler surface with $K_{\widetilde{X}}^2 > 0$ and $b_2^+(\widetilde{X}) > 3$. Suppose that $\sigma : \widetilde{X} \to \widetilde{X}$ is an anti-holomorphic involution without fixed*

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points. Then the quotient manifold $X = \widetilde{X}/\sigma$ has vanishing Seiberg-Witten invariants.

In view of Witten's vanishing theorem, it is natural to examine whether the quotient X can be decomposed into a connected sum:

Proposition 2. *If $\widetilde{X}$ is a simply-connected Kähler surface or more generally symplectic 4-manifold with $b_2^+(\widetilde{X}) > 1$ and $\sigma : \widetilde{X} \to \widetilde{X}$ is a free involution, then the quotient manifold $X = \widetilde{X}/\sigma$ is not diffeomorphic to any connected sum $X_1 \# X_2$ with both $b_2^+(X_i) > 0$.*

Proof. Suppose on the contrary, that there is such a decomposition $X = X_1 \# X_2$. Since $\pi_1(X) = \mathbf{Z}_2$, we can assume $\pi_1(X_1) = \mathbf{Z}_2$ and $\pi_1(X_2) = 1$. A universal (double) cover $\widetilde{X}_1$ of X_1 yields a universal cover $\widetilde{X}_1 \# X_2 \# X_2$ of X, which should then be diffeomorphic to $\widetilde{X}$. This is however impossible as $\widetilde{X}$, being symplectic, cannot be decomposed into such a connected sum with $b_2^+(\widetilde{X}_1 \# X_2) > 0$ and $b_2^+(X_2) > 0$ [9]. (This kind of covering trick was initially used in [7] in a different context.) □

Proposition 2 therefore indicates that at least in the case where $\widetilde{X}$ is simply-connected, the situation in Theorem 1 is not covered by Witten's vanishing theorem. To the author's knowledge, the quotient manifold X given here, with $b_1(X) + b_2^+(X) = b_2^+(X)$ odd, is the first kind of example which is known to satisfy the conclusions in Theorem 1 and Proposition 2.

Before proving Theorem 1, we recall briefly the definition of Seiberg-Witten invariants from [8] and [12], for example. Let Y be a smooth oriented Riemannian 4-manifold. In this case, a spin^c structure λ on Y consists of a pair of $U(2)$ bundles $W^\pm$ over Y, whose sum $W^+ \oplus W^-$ is the usual spinor bundle. Let L denote the determinant (complex line) bundle of W^+. The perturbed Seiberg-Witten equations are the following pair of equations for a unitary connection A on L and a section ϕ of W^+:

$$\begin{aligned} D_A\phi &= 0 \\ \rho(F_A^+ + i\delta) &= i\theta(\phi,\phi), \end{aligned} \qquad (*)$$

Here $D_A : \Gamma(W^+) \to \Gamma(W^-)$ is the Dirac operator, F_A^+ is the self-dual part of the curvature of A, ρ is the Clifford multiplication, and θ is a pairing defined by matrix multiplication. (See [8] for details.) For a generic perturbation $\delta \in \Omega^+(X)$, the moduli space $M_{\lambda,\delta}$ of irreducible solutions (A,ϕ) is either empty or a smooth manifold of dimension

$$d_\lambda = \frac{1}{4}[c_1(L)^2 - (2e_Y + 3s_Y)],$$

where e_Y, s_Y are respectively the Euler characteristic and signature of Y. (The pair (A, ϕ) is called irreducible if $\phi \neq 0$.) The moduli space $M_{\lambda,\delta}$ will be compact if the metric on Y is chosen so that $(*)$ admit no reducible solutions, which can be achieved in a path-connected subset of metrics if $b_2^+(Y) > 1$. In this situation and if $d_\lambda \geq 0$ also, the Seiberg-Witten invariant $\mathrm{SW}(Y, \lambda)$ of Y with respect to λ is then defined to be the integral of the maximal power of the Chern class of the circle bundle $M^0_{\lambda,\delta} \to M_{\lambda,\delta}$, where $M^0_{\lambda,\delta}$ is the framed moduli space. (The integral makes sense only if $b_1(Y)+b_2^+(Y)$ is odd; if $b_1(Y)+b_2^+(Y)$ is even, the Seiberg-Witten invariant is defined to be zero.)

The full details of the definition of Seiberg-Witten invariants will not be needed in this paper. In fact, for the purpose of proving Theorem 1, it is enough to note that the perturbation δ will be generic and the Seiberg-Witten invariant will be zero if there are no reducible and irreducible solutions to $(*)$. The issue of reducible solutions is dealt with in the following simple observation.

Lemma 3. *Set $\delta = 0$ in $(*)$. If the smooth 4-manifold Y satisfies $2e_Y + 3s_Y > 0$, then there is no reducible solution to $(*)$ for any metric and any spin^c structure λ on Y with $d_\lambda \geq 0$.*

Proof. Since $d_\lambda = \frac{1}{4}[c_1(L)^2 - (2e_Y + 3s_Y)]$, its nonnegativeness yields $c_1(L)^2 > 0$. If there is a reducible Seiberg-Witten solution for some metric, that is, a connection A on L with $\frac{i}{2\pi}F_A$ being anti-self dual, then $c_1(L)^2 = \int_Y (\frac{i}{2\pi}F_A)^2 \leq 0$. This is a contradiction. □

It is interesting to observe that the situation in Lemma 3 is different from the Donaldson theory. There the inequality $c_1(L)^2 \leq 0$ (from a reducible anti-self dual connection on an $\mathrm{SU}(2)$ bundle E) does not contradict the nonnegativity of the (virtual) dimension $d_E = 8c_2(E) - 3(e_Y + s_Y)/2$ of the moduli space of ASD connections, as $c_2(E) = -c_1(L)^2 \geq 0$. Thus a condition such as $e_Y + s_Y > 0$ is not helpful to rule out the existence of reducible ASD connections.

Proof of Theorem 1. Let h be the Kähler metric on $\widetilde{X}$ and ω the associated Kähler form. As σ is anti-holomorphic, one sees easily that $\widetilde{g} = h + \sigma^* h$ is an equivariant Kähler metric on $\widetilde{X}$ with Kähler form $\widetilde{\omega} = \omega - \sigma^*\omega$. Through the double cover $p : \widetilde{X} \to X$, $\widetilde{g}$ pushes down to a metric g on X. Both $\widetilde{g}$ and g will be fixed for the rest of the proof. The standard Euler characteristic and signature formulae applied to p also yield $b_2^+(X) = \frac{1}{2}[b_2^+(\widetilde{X}) - 1] > 1$.

Consider an arbitrary spin^c structure λ on X. It pulls back to a spin^c structure $\widetilde{\lambda}$ on $\widetilde{X}$ through the double covering p. Without much difficulty one verifies that the associated bundles $W^\pm$ and L of λ pull back to the corresponding associated bundles $\widetilde{W}^{\pm}$ and $\widetilde{L}$ of $\widetilde{\lambda}$. Assume the dimension $d_\lambda \geq 0$ from now on.

Suppose there is a solution pair (A, ϕ) to the Seiberg-Witten equations:

$$\begin{aligned} D_A\phi &= 0 \\ \rho(F_A^+) &= i\theta(\phi, \phi), \end{aligned} \tag{$**$}$$

where A and ϕ are respectively a connection on L and a section on W^+ . Then the pull-back $(\widetilde{A}, \widetilde{\phi})$ is a solution pair to the Seiberg-Witten equations on $\widetilde{X}$ with spin^c structure $\widetilde{\lambda}$ and Kähler metric $\widetilde{g}$. Since $2e_X + 3s_X = (2e_{\widetilde{X}} + 3s_{\widetilde{X}})/2 = K_{\widetilde{X}}^2/2 > 0$, there are no reducible Seiberg-Witten solutions for λ from Lemma 3; thus (A, ϕ) and hence $(\widetilde{A}, \widetilde{\phi})$ are both irreducible. It follows easily from the irreducibility of $(\widetilde{A}, \widetilde{\phi})$ that $\widetilde{\omega} \cdot \widetilde{L} \neq 0$ (see [12] for example). This is however not possible; $\sigma^*\widetilde{L} = \widetilde{L}$ and $\sigma^*\widetilde{\omega} = -\widetilde{\omega}$ have already forced $\widetilde{\omega} \cdot \widetilde{L} = 0$. (Note that σ preserves the orientation of $\widetilde{X}$.)

Thus the argument above shows that there is no reducible or irreducible solution to $(**)$. As noted before Lemma 3, this means that in the perturbed Seiberg-Witten equations of $(**)$, the perturbation $\delta = 0$ is generic and the Seiberg-Witten invariant of X with respect to λ is zero. Since λ is an arbitrary spin^c structure, the Seiberg-Witten invariants of X all vanish. □

Note that all minimal complex surfaces of general type satisfy the condition $K_{\widetilde{X}}^2 > 0$ in Theorem 1 [1; page 208]. (Moreover a necessary condition for $K_{\widetilde{X}}^2 > 0$ is that $\widetilde{X}$ be projective by using Grauert's ampleness criterion [1; Page 127].)

Therefore the conditions in both Theorem 1 and Proposition 2 are satisfied by all simply-connected minimal complex surfaces of general type with $b_2^+ > 3$, and these include lots of examples. For one set of examples, one can take $\widetilde{X}$ to be algebraic surfaces in $\mathbf{CP}^n$ defined by real polynomials and σ to be the complex conjugation. These include hypersurfaces $\sum_{j=1}^4 x_j^{2n} = 0$ in $\mathbf{CP}^3$, where $n > 2$. For another set of examples, consider a Kähler surface Y with anti-holomorphic involution τ. Suppose that there exits a complex curve $C \subset Y$ such that $2|[C] \in H_2(Y,\mathbf{Z})$, and that C is invariant under τ, disjoint from $\operatorname{Fix}\tau$. Then τ lifts to two anti-holomorphic involutions on the double cover $\widetilde{X}$ of Y branched over C, and one of the lifting involutions has no fixed point. As a special case, one can take $Y = \mathbf{CP}^2$, $C = \{\sum_{j=1}^3 x_j^{2n} = 0\}$ $(n > 3)$ and τ to be the complex conjugation on $\mathbf{CP}^2$.

Remark 4.

(i) Since the Seiberg-Witten invariants vanish, it follows from [9] that the quotient X in Theorem 1 does not admit any symplectic structure.

Moreover it is possible to give examples of $\widetilde{X}$ (e.g., degree $4m+2$ hypersurfaces in $\mathbf{CP}^3$), from which $b_2^+(X), b_2^-(X)$ are both even. Thus X does not even have an almost complex structure with either of its orientations in this case.

(ii) Fix a complex surface $\widetilde{X}$ but vary free anti-holomorphic involutions σ. It is an open problem whether the quotients X are diffeomorphic to each other. (It is not difficult to see that the quotients are homeomorphic to each other for most cases using [6].) Perhaps the d-complex structures introduced in [11] are a useful tool.

(iii) It is also interesting to investigate the quotient X when σ is not free. By using the generalized adjunction inequality [4], one can show easily that the Seiberg-Witten invariants of X vanish again for many cases. (Furthermore, combining with the rational blow-down formula [5] as well as a surgery formula [10], one can recover the vanishing result in Theorem 1 for hypersurfaces in $\mathbf{CP}^3$.) It remains unknown [3] whether there exits such an X which cannot be decomposed into a sum $X_1 \# X_2$ with both $b_2^+(X_i) > 0$ (other than a couple of trivial cases with $b_2^+(X) = 0$).

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