THE EIGENVALUE BEHAVIOUR FOR THE BOUNDARY VALUE PROBLEMS RELATED TO SELF-SIMILAR MEASURES ON $\mathbb{R}^d$

K. Naimark and M. Solomyak

1. Introduction

Suppose that a compactly supported probability measure μ on $\mathbb{R}^d$ is given. In a domain $X \subset \mathbb{R}^d$ such that $\text{Supp}\,\mu \subset \overline{X}$, we consider spectral problems whose "weak" description is as follows: let $\mathcal{V} = \mathring{L}_2^1(X)$ or $\mathcal{V} = \{v \in L_2^1(X) : \int_{\overline{X}} v d\mu = 0\}$. The spaces $\mathring{L}_2^1(X)$ and $L_2^1(X)$ are closely related to the Sobolev spaces $H_0^1(X)$ and $H^1(X)$ respectively and coincide with them if X is, say, a bounded domain with the sufficiently smooth boundary. Definitions and some details about these spaces are given in Section 3.

We seek the eigenvalues λ and eigenfunctions $u \in \mathcal{V}$ of the equation

$$(1) \qquad \lambda \int_X \nabla u \cdot \nabla \overline{v} dx = \int_{\overline{X}} u\overline{v} d\mu, \quad \text{any } v \in \mathcal{V};$$

see Section 4 for an accurate statement of the problem. If μ is absolutely continuous, $d\mu = V dx$, then (1) corresponds to the classical boundary value problem

$$(2) \qquad -\lambda \Delta u = V u,$$

with the Dirichlet or Neumann conditions on ∂X respectively.

Suppose in addition that the operator associated with the problem (1) is compact. Then its nonzero spectrum consists of a sequence of positive eigenvalues λ_k, $\lambda_k \to 0$. Let $n(t)$ be the corresponding distribution function:

$$n(t) = \#\{k : \lambda_k > t\}, \quad t > 0.$$

We are interested in the behaviour of $n(t)$ when $t \to 0$. This behaviour is well known for absolutely continuous measures, when we are dealing with the equation (2). Namely, under some, rather mild assumptions on V (and, for the Neumann boundary value problem, also on ∂X), one has

$$(3) \qquad n(t) = O(t^{-\frac{d}{2}}).$$

Moreover, $n(t)$ has a regular asymptotic behaviour at 0:

$$(4) \qquad \lim_{t \to 0} t^{\frac{d}{2}} n(t) = \frac{(4\pi)^{-\frac{d}{2}}}{\Gamma(\frac{d}{2} + 1)} \int_X V^{\frac{d}{2}} dx;$$

see [BS] for proofs and a detailed discussion of (3) and (4).

Much less is known for singular measures μ. In this case (1) can not in general be rewritten as a classical boundary value problem. Notice only that (1) implies $\Delta u(x) = 0$ outside Supp μ. In this paper we consider the case of *self-similar measures* μ which constitute an important subclass of general singular measures. For such a μ, we show that $n(t)$ has a powerlike growth at $t = 0$:

$$(5) \qquad ct^{-\delta} \leq n(t) \leq Ct^{-\delta}, \quad t \leq t_0.$$

We give an expression for δ in (5) and discuss the relations between δ and the Hausdorff dimensions of the measure μ and of its support.

Spectral problems related to fractals have been rather popular during the last several years; e.g., see the papers by Lapidus [La] and by Levitin and Vassiliev [LeVa], which include, in particular, a survey of the topic. The reference lists in these two papers together cover most of the literature devoted to this class of problems.

However, the problems investigated in all these works are essentially different from the one considered here. The only exception known to us is the paper [Fu] by Fujita, where an estimate equivalent to (5) was obtained for the one-dimensional case. We are thankful to M. L. Lapidus who informed us about [Fu].

In fact, for $d = 1$ a much more detailed analysis is possible. This was done in [SVe], where it was shown that depending on arithmetic properties of parameters defining μ, $t^{\delta} n(t)$ behaves at $t = 0$ either as a constant or as a function, periodic in $\log t$. Such a detailed result was obtained in [SVe] on the base of the "renewal equation", well known in Probability and Dynamical Systems; e.g., see [Fe], [Wo] and [LeVa]. Here we also use the

renewal equation when analyzing an auxiliary combinatorial problem. For this problem we get detailed results of the same type as was obtained in [SVe] for $n(t)$. The eigenvalue distribution function we are dealing with, can be estimated from above and from below through the solution of this combinatorial problem. As a result, for $d \geq 2$ we were able to get only the two-sided estimate (5).

Earlier the renewal equation was applied to various spectral problems involving self-similar sets by Kigami and Lapidus [KLa], Lapidus [La], and Levitin and Vassiliev [LeVa]. For the usage of the renewal theory in other problems related to fractals see [LauW], [St2].

The main purpose of this paper is a detailed exposition of the results announced in [NS]. We give also an analysis of the spectral problem (1) corresponding to the Neumann-type boundary condition. Besides, here we consider a much wider class of domains X than in [NS]; in particular, we treat the case $X = \mathbb{R}^d$.

2. Self-similar measures

The notions of self-similar set and self-similar measure were introduced by Hutchinson [Hu]. Let $\mathcal{S} = \{S_1, ..., S_m\}$ be a set of contractive similitudes on $\mathbb{R}^d$, $h_1, ..., h_m$ their coefficients of contraction. Also, let a system of positive numbers ("weights") $\mathbf{p} = \{p_1, ..., p_m\}$ be given, such that $p_1 + ... + p_m = 1$. Then there exists a unique nonempty compact set $\mathcal{C} = \mathcal{C}(\mathcal{S}) \subset \mathbb{R}^d$ such that $\mathcal{C} = \bigcup_{k=1}^m S_k \mathcal{C}$, and a unique boundedly supported probability Borelian measure $\mu = \mu(\mathcal{S}, \mathbf{p})$ which satisfies the self-similarity property

$$(6) \qquad \mu = \sum_{k=1}^m p_k \mu \circ S_k^{-1}.$$

Furthermore, $\mathcal{C}(\mathcal{S}) = \text{Supp } \mu(\mathcal{S}, \mathbf{p})$. Measures $\mu(\mathcal{S}, \mathbf{p})$, corresponding to the same set $\mathcal{S}$ but to different families $\mathbf{p}$, are mutually singular.

It is $\mu(\mathcal{S}, \mathbf{p})$ that we are interested in. For the rest of the paper $\mu = \mu(\mathcal{S}, \mathbf{p})$.

We always suppose that $\mathcal{S}$ meets the "open set condition" (see [Hu]). This means that there exists a bounded open set $\Omega \subset \mathbb{R}^d$, such that

$$(7) \qquad 1. \ \bigcup_{k=1}^m S_k(\Omega) \subset \Omega, \qquad 2. \ S_k(\Omega) \bigcap S_l(\Omega) = \emptyset \ \text{ for } \ k \neq l.$$

Evidently, (7) implies that $\mathcal{C}(\mathcal{S}) \subseteq \overline{\Omega}$. It follows also that $\sum_{k=1}^{m} h_k^d \leq 1$. In fact, we will assume that Ω is a domain (a connected open set). Another assumption is

$$(8) \qquad \qquad \mu(\partial\Omega) = 0.$$

It is known that otherwise $\mu(\Omega) = 0$ (see [LauW]). Assumption (8) is of a rather technical character and can sometimes be withdrawn.

The simplest particular case of a self-similar measure is δ_a (the delta-measure supported at a point a). We obtain it taking $m = 1$.

3. Embedding theorems

Let $X \subseteq \mathbb{R}^d$ be an arbitrary domain. Since the open set condition (7) does not imply any regularity restriction on the open set, the problem (1) is intimately connected not with the usual Sobolev space H^1 but with its less familiar counterpart, L_2^1. We recall the definitions of function classes $\mathring{L}_2^1(X)$ and $L_2^1(X)$ (all the information needed can be found in [M]).

$\mathring{L}_2^1(X)$ is defined as the closure of $C_0^\infty(X)$ in the metric of Dirichlet integral $\int_X |\nabla u|^2 dx$. If $d \geq 3$, or if $d = 2$ and $\mathbb{R}^2 \setminus X$ is not a set of zero inner $(2,1)$-capacity, or if $d = 1$ and $X \neq \mathbb{R}^1$, this is actually a space of functions ([M], §11.2). $\mathring{L}_2^1(X)$ coincides with the Sobolev space $H_0^1(X)$ if, for example, X is bounded.

The definition of $L_2^1(X)$ is as follows:

$$L_2^1(X) = \{u \in H_{loc}^1(X) : \int_X |\nabla u|^2 dx < \infty\}.$$

If $u \in L_2^1(X)$ implies $u \in L_2(X)$, then clearly $L_2^1(X)$ coincides with $H^1(X)$ algebraically and topologically. In particular, this is true if X is a bounded domain with the Lipschitz boundary.

$L_2^1(X)$ can be considered as a Hilbert space with respect to any metric form

$$(9) \qquad \qquad \int_X |\nabla u|^2 dx + \left| \int_X \psi u dx \right|^2 ,$$

where $\psi \in C_0^\infty(X)$ is such that $\int_X \psi dx \neq 0$. The choice of ψ is arbitrary, and the corresponding norms are equivalent.

The space
$$L_2(\mu) := L_2(\mathbb{R}^d, \mu)$$
can be identified in a standard way with $L_2(\mathcal{C}(S), \mu)$. Let $X \subseteq \mathbb{R}^d$ be a domain such that $\Omega \subseteq X$. Our goal here is to study the embeddings of $\mathring{L}_2^1(X)$ and $L_2^1(X)$ into $L_2(\mu)$. This problem does not arise in the case $d = 1$, because then $L_2^1(X) \subset C(\overline{X})$ and hence the embedding $L_2^1(X) \subset L_2(\mu)$ is bounded and compact for any finite compactly supported μ.

If $d \geq 2$, the problem is more complex. We start with the case $X = \mathbb{R}^d$. Denote

$$(10) \qquad A_k = h_k^{2-d} p_k, \quad k = 1, ..., m, \qquad \overline{A}(\mathcal{S}, \mathbf{p}) = \max_k A_k.$$

Proposition 1. *Suppose that (7) (the open set condition) is satisfied. Then*

$$(11) \qquad\qquad L_2^1(\mathbb{R}^d) \subset L_2(\mu) \quad compactly$$

if and only if

$$(12) \qquad\qquad \overline{A}(\mathcal{S}, \mathbf{p}) < 1.$$

Proof. If $d = 1$, then (12) is satisfied automatically, and, as was noted above, (11) also holds. So below we discuss only the case $d \geq 2$.

Necessity. Assume that (11) is valid. Then first of all μ can not have a nonzero delta-component. Indeed, for $d \geq 2$ the Sobolev space H^1 contains unbounded functions, so the above embedding is not even bounded.

Now suppose that $h_1^{2-d} p_1 \geq 1$. Let $a \in \text{Supp } \mu$. We may assume that $a \neq S_1(a)$. Indeed, m contractions S_k have finite number (not greater than m) of fixed points, while $\text{Supp } \mu$ must be infinite; otherwise μ would be a linear combination of delta-measures, which is impossible.

Fix a function $U_0 \in C_0^\infty(\mathbb{R}^d)$ such that $U_0(a) > 0$. Define

$$(13) \qquad\qquad U_n(x) = p_1^{-\frac{n}{2}} U_0(S_1^{-n}(x)),$$

then, in view of the self-similarity of μ, $\int |U_n|^2 d\mu = \int |U_0|^2 d\mu$. Clearly, $U_n \in H^1(\mathbb{R}^d) \subset L_2^1(\mathbb{R}^d)$. Moreover, $\{U_n\}$ is bounded in $L_2^1(\mathbb{R}^d)$:
$$(14)$$
$$\|U_n\|_{L_2^1(\mathbb{R}^d)}^2$$
$$= \int_{\mathbb{R}^d} |\nabla U_n|^2 dx + \left| \int_{\mathbb{R}^d} \psi U_n dx \right|^2 \leq \int_{\mathbb{R}^d} |\nabla U_n|^2 dx + C \int_{\mathbb{R}^d} |U_n|^2 dx$$
$$= (p_1 h_1^{2-d})^{-n} \left(\int_{\mathbb{R}^d} |\nabla U_0|^2 dx + C h_1^{2n} \int_{\mathbb{R}^d} |U_0|^2 dx \right) \leq C' \|U_0\|_{H^1(\mathbb{R}^d)}^2.$$

Now recall that $S_1(a) \neq a$. If b is the fixed point of S_1, then $|S_1^n(a) - b| = h_1^n |a - b|$ for any n. Therefore, U_0 may be chosen so that Supp U_n are mutually disjoint. Thus

$$(15) \qquad \int |U_n - U_m|^2 d\mu = \int |U_n|^2 d\mu + \int |U_m|^2 d\mu = 2 \int |U_0|^2 d\mu > 0,$$

which, together with (14), contradicts the compactness of the embedding.

Sufficiency. Let $h_k^{2-d} p_k < 1$, $k = 1, ..., m$. Then for ε positive and sufficiently small we have $\left(h_k^{2-d} \right)^{1+\varepsilon} p_k < 1$, any k. Choose q such that $2 < q < 2 + 2\varepsilon$. We will prove first that under our assumptions the embedding $H^1(\mathbb{R}^d) \subset L_q(\mu)$ is compact. For a finite measure μ, this yields the compactness of $H^1(\mathbb{R}^d) \subset L_2(\mu)$. (11) follows from here because μ is compactly supported.

We will need the following result (see [M], Theorems 3 and 4 in §8.8):

The embedding $H^1(\mathbb{R}^d) \subset L_q(\mu)$ is compact if and only if

$$(16) \qquad \lim_{\delta \to +0} \sup_{x \in \mathbb{R}^d, \rho \in (0,\delta)} \rho^{1 - \frac{d}{2}} \left[\mu(B(x, \rho)) \right]^{\frac{1}{q}} = 0 \quad \text{for } d > 2,$$

$$(17) \qquad \lim_{\delta \to +0} \sup_{x \in \mathbb{R}^d, \rho \in (0,\delta)} |\log \rho|^{\frac{1}{2}} \left[\mu(B(x, \rho)) \right]^{\frac{1}{q}} = 0 \quad \text{for } d = 2,$$

where $B(x, \rho)$ denotes the ball of radius ρ centered at x.

Now we concentrate on the proof of (16) and (17).

For a multiindex $\mathbf{j} = (j_1, ..., j_n) \in \{1, ..., m\}^n$ we denote

$$(18) \qquad \begin{aligned} S(\mathbf{j}) &= S_{j_n} ... S_{j_1}, \\ \Omega(\mathbf{j}) &= S(\mathbf{j})\Omega, \\ p(\mathbf{j}) &= p_{j_1} ... p_{j_n}, \\ h(\mathbf{j}) &= h_{j_1} ... h_{j_n}, \\ A(\mathbf{j}) &= A_{j_1} ... A_{j_n}. \end{aligned}$$

Also, denote

$$(19) \qquad p_{max} = \max_{k=1,...,m} p_k, \qquad h_{min} = \min_{k=1,...,m} h_k.$$

Here we apply the following result (see [Fa], Lemma 8.5 and the proof of Theorem 8.6):

Let the open set condition (7) be satisfied, and $0 < \rho < 1$. Then there exist constants c, l and a family of multiindices $\mathbf{j}_1, ..., \mathbf{j}_r$ satisfying

$$\Omega(\mathbf{j}_\alpha) \cap \Omega(\mathbf{j}_\beta) = \emptyset \quad for \ \alpha \neq \beta$$

and

$$c\rho \leq h(\mathbf{j}_\alpha) \leq \rho, \qquad any \ \alpha,$$

such that any ball B_ρ of radius ρ intersects no more than l of the sets $\Omega(\mathbf{j}_\alpha)$.

Therefore, for $d > 2$ we get:

$$\rho^{1-\frac{d}{2}} \left[\mu(B(x,\rho))\right]^{\frac{1}{q}} \leq \rho^{1-\frac{d}{2}} \left[l \max_\alpha \mu(\overline{\Omega(\mathbf{j}_\alpha)})\right]^{\frac{1}{q}}$$

$$\leq l^{\frac{1}{q}} \rho^{1-\frac{d}{2}} \left[\max_{h(\mathbf{j}_\alpha) \leq \rho} p(\mathbf{j}_\alpha)\right]^{\frac{1}{q}} \leq l^{\frac{1}{q}} \rho^{1-\frac{d}{2}} \rho^{(d-2)(1+\varepsilon)\frac{1}{q}} = l^{\frac{1}{q}} \rho^{(d-2)\left(\frac{1+\varepsilon}{q} - \frac{1}{2}\right)}.$$

Since $q < 2(1+\varepsilon)$, we have $\frac{1+\varepsilon}{q} - \frac{1}{2} > 0$. Hence

$$\lim_{\delta \to +0} \sup_{x \in \mathbb{R}^d, \rho \in (0,\delta)} \rho^{1-\frac{d}{2}} \left[\mu(B(x,\rho))\right]^{\frac{1}{q}} \leq l^{\frac{1}{q}} \lim_{\delta \to +0} \delta^{(d-2)\left(\frac{1+\varepsilon}{q} - \frac{1}{2}\right)} = 0.$$

For $d = 2$, we get in a similar way:

$$|\log \rho|^{\frac{1}{2}} \left[\mu(B(x,\rho))\right]^{\frac{1}{q}} \leq l^{\frac{1}{q}} |\log \rho|^{\frac{1}{2}} \left[\max_{h(\mathbf{j}_\alpha) \leq \rho} p(\mathbf{j}_\alpha)\right]^{\frac{1}{q}}.$$

$h(\mathbf{j}_\alpha) \leq \rho$ implies $(h_{min})^{|\mathbf{j}_\alpha|} \leq h(\mathbf{j}_\alpha) \leq \rho$, so $|\mathbf{j}_\alpha| \geq (\log h_{min})^{-1} \log \rho$. Hence

$$\max_{h(\mathbf{j}_\alpha) \leq \rho} p(\mathbf{j}_\alpha) \leq (p_{max})^{|\mathbf{j}_\alpha|} \leq \exp\left(\frac{\log p_{max}}{\log h_{min}} \log \rho\right) = \rho^{\frac{\log p_{max}}{\log h_{min}}},$$

and we obtain

$$\lim_{\delta \to +0} \sup_{x \in \mathbb{R}^d, \rho \in (0,\delta)} |\log \rho|^{\frac{1}{2}} \left[\mu(B(x,\rho))\right]^{\frac{1}{q}} \leq l^{\frac{1}{q}} \lim_{\delta \to +0} |\log \delta|^{\frac{1}{2}} \delta^{\frac{\log p_{max}}{q \log h_{min}}} = 0.$$

So we have proved that for $d \geq 2$ the embedding $H^1(\mathbb{R}^d) \subset L_q(\mu)$ is compact for some $q > 2$, which implies (11). $\square$

We could not obtain a sufficient condition for the embedding $L_2^1(\mathbb{R}^d) \subset L_2(\mu)$ to be bounded. Clearly, $\overline{A}(\mathcal{S}, \mathbf{p}) \leq 1$ is necessary, but it is not sufficient even if $d = 2$. Indeed, if $p_1 h_1^{d-2} = p_1 = 1$, then we have only one contraction S_1, and μ is a delta-measure. So if $d = 2$, the boundedness and the compactness of the embedding are equivalent. It is unknown to us whether this is still true for $d > 2$.

Proposition 2. *Suppose that (7) is satisfied and $d \geq 2$. Let $X \subseteq \mathbb{R}^d$ be a domain such that $\Omega \subseteq X$. If $d = 2$, suppose also that $\mathbb{R}^2 \setminus X$ has nonzero inner (2,1)-capacity. If (12) is satisfied, that is, $\overline{A}(\mathcal{S}, \mathbf{p}) < 1$, then*

$$\mathring{L}_2^1(X) \subset L_2(\mu) \quad compactly.$$

If, in addition, $\mu(\partial\Omega) = 0$, then (12) is also necessary for the compactness of $\mathring{L}_2^1(X) \subset L_2(\mu)$.

Proof. The proof is easily reduced to the one of Proposition 1.

Necessity. Since $\mu(\partial\Omega) = 0$, we can choose $a \in \text{Supp } \mu$ so that $a \in \Omega$. Moreover, Supp $\mu \setminus \partial\Omega$ is infinite (otherwise μ includes a delta-measure). Therefore a again can be taken in such a way that $a \neq S_1(a)$.

Suppose that $h_1^{2-d}p_1 \geq 1$. Define U_n as in (13). Clearly, we may assume Supp $U_n \subset \Omega$. So if $u_n := U_n|_X$, then $u_n \in \mathring{L}_2^1(X)$ and

$$\|u_n\|^2_{\mathring{L}_2^1(X)} = \int_{\mathbb{R}^d} |\nabla U_n|^2 dx \leq \|U_0\|^2_{H^1(\mathbb{R}^d)}.$$

Therefore, $\{u_n\}$ is bounded in $\mathring{L}_2^1(X)$. The remaining part of the proof is the same.

Sufficiency. Follows evidently from Proposition 1.   □

Before going over to $L_2^1(X)$, let us discuss in more detail the sense of the embedding considered. Suppose first that $\mu(\partial\Omega) = 0$. Then the integral

$$(20) \qquad\qquad Q_\mu[u] = \int |u|^2 d\mu$$

is well defined on the set $L_2^1(X) \cap C^\infty(X)$, which is dense in $L_2^1(X)$. If Q_μ is bounded in $L_2^1(X)$, that is, if for any $u \in L_2^1(X) \cap C^\infty(X)$ we have

$$(21) \qquad\qquad Q_\mu[u] \leq C\left(\int_X |\nabla u|^2 + \left|\int_X \psi u dx\right|^2\right),$$

then we extend Q_μ to the whole of $L_2^1(X)$ by continuity. For $L_2^1(\mathbb{R}^d)$ and $\mathring{L}_2^1(X)$ the sense of the embeddings considered is the same, with $C_0^\infty(\mathbb{R}^d)$ or $C_0^\infty(X)$ as the initial domain of Q_μ.

The situation changes if $\mu(\partial\Omega) = 1$, because then we can not start with $u \in C^\infty(X)$. In order that $Q_\mu[u]$ be well defined on $L_2^1(X)$, we need an additional assumption:

$$(22) \qquad L_2^1(X) \cap C^\infty(X \cup \operatorname{Supp} \mu) \quad \text{is dense in} \quad L_2^1(X).$$

Now the same scheme as above works; if (21) is valid for any $u \in L_2^1(X) \cap C^\infty(X \cup \operatorname{Supp} \mu)$, then we extend Q_μ to $L_2^1(X)$ by continuity.

The condition (22) withdraws some pathological cases. Say, if $d = 2$, X is the unit disc with a cut along a radius and μ is supported on this radius, then $Q_\mu[u]$ makes no sense on $L_2^1(X)$.

Another notion we need below is the *extension property* for L_2^1.

We say that a domain $X \subset \mathbb{R}^d$ admits the extension property for L_2^1 and write

$$X \in \mathcal{E}(L_2^1),$$

if there exists a continuous linear operator $\Pi : L_2^1(X) \to L_2^1(\mathbb{R}^d)$ such that $\Pi f|_X = f$ for any $f \in L_2^1(X)$. The extension property for H^1 and the class $\mathcal{E}(H^1)$ are defined in the same way.

The following statement (due to V. Maz'ya) shows that for a bounded X these two properties are equivalent. (If X is unbounded, then this is not the case.)

Lemma 1. *Let $X \subset \mathbb{R}^d$ be a bounded domain. Then*

$$X \in \mathcal{E}(L_2^1) \iff X \in \mathcal{E}(H^1).$$

Proof. The implication "$\Longrightarrow$" is evident. Conversely, if $X \in \mathcal{E}(H^1)$, then $H^1(X) \subset L_q(X)$ for some $q > 2$. So for any $u \in H^1(X)$

$$(23) \qquad \left(\int_X |u|^q dx \right)^{\frac{2}{q}} \le C \left(\int_X |\nabla u|^2 dx + \int_{X_0} |u|^2 dx + \int_{X'} |u|^2 dx \right),$$

where $X_0 \subset X$ is a domain with the Lipschitz boundary, and $X' = X \setminus X_0$. Estimating the last term on the right-hand side of (23) by the Hölder inequality, we get
(24)

$$\left(\int_X |u|^q dx \right)^{\frac{2}{q}} \le$$

$$C \left(\int_X |\nabla u|^2 dx + \int_{X_0} |u|^2 dx \right) + C(\operatorname{meas} X')^{1-\frac{2}{q}} \left(\int_{X'} |u|^q dx \right)^{\frac{2}{q}}.$$

Choosing X_0 in such a way that $C(\text{meas}X')^{1-\frac{2}{q}} < \frac{1}{2}$, we obtain

$$(25) \qquad \|u\|_{L_2(X)} \le C_1 \|u\|_{L_q(X)} \le C_2 \left(\int_X |\nabla u|^2 dx + \int_{X_0} |u|^2 dx \right)^{\frac{1}{2}}.$$

Since the right-hand side of (25) is an equivalent norm in $L_2^1(X)$ and $H^1(X)$ is dense in $L_2^1(X)$, we are done. $\square$

It follows from this Lemma that for bounded $X \in \mathcal{E}(H^1)$ we have $L_2^1(X) = H^1(X)$. Recall that any bounded domain with the Lipschitz boundary belongs to $\mathcal{E}(H^1)$; e.g., see [Ste]. For the rest of the paper we denote $\mathcal{E} = \mathcal{E}(L_2^1)$.

Notice that if $X \in \mathcal{E}$, then (22) is automatically satisfied, because in this case $L_2^1(X) \cap C^\infty(\overline{X})$ is dense in $L_2^1(X)$.

Proposition 3. *Suppose that (7) is satisfied. Let $d \ge 2$ and $X \subseteq \mathbb{R}^d$ be a domain such that $\Omega \subseteq X$. If $\mu(\partial\Omega) = 1$, then let also (22) be satisfied. If*

$$L_2^1(X) \subset L_2(\mu) \quad \text{compactly,}$$

then (12) is satisfied, that is, $\overline{A}(\mathcal{S}, \mathbf{p}) < 1$.

If, in addition, there exists a domain $Y \in \mathcal{E}$ such that $\Omega \subseteq Y \subseteq X$, then (12) is also sufficient for compactness of $L_2^1(X) \subset L_2(\mu)$.

Proof. The proof is quite similar to the one of Proposition 1; the slight changes needed are clear. $\square$

4. The operators $T_{\mu,X}^D$ and $T_{\mu,X}^N$. Formulation of the main results

Here we give an accurate statement of the problems to be considered below.

If the conditions of the sufficiency part of Proposition 2 are satisfied, then $Q_\mu[u]$ generates in $\mathring{L}_2^1(X)$ a compact, self-adjoint and nonnegative operator, which we denote by $T_{\mu,X}^D$. The equality

$$T_{\mu,X}^D u = f, \qquad u, f \in \mathring{L}_2^1(X)$$

means that

$$\int_X \nabla f \cdot \nabla \overline{v} dx = \int_X u\overline{v} d\mu, \qquad \text{any } v \in \mathring{L}_2^1(X).$$

In particular, every eigenpair (λ, u) meets the equation (1), with $\mathcal{V} = \mathring{L}_2^1(X)$.

To define $T_{\mu,X}^N$, we need to introduce a special metric form on $L_2^1(X)$.

Assume that we are in the conditions of the sufficiency part of Proposition 3, so that the embedding $L_2^1(X) \subset L_2(\mu)$ is compact. Along with the quadratic functional $Q_\mu[u]$ introduce a linear functional

$$(26) \qquad \phi_\mu(u) = \int u d\mu.$$

This makes sense for $u \in C^\infty(X)$ if $\mu(\partial\Omega) = 0$, or for $u \in C^\infty(X \cup \mathrm{Supp}\ \mu)$ if $\mu(\partial\Omega) = 1$. Clearly,

$$|\phi_\mu(u)|^2 \leq Q_\mu[u],$$

so ϕ_μ is bounded on $L_2^1(X)$. As usual, we extend ϕ_μ to the whole of $L_2^1(X)$ by continuity. Notice that according to our definitions, we have for any $u \in L_2^1(X)$

$$(27) \qquad Q_\mu[u] = Q_\mu[u|_\Omega] \qquad \text{and} \qquad \phi_\mu(u) = \phi_\mu(u|_\Omega).$$

Consider now the metric form on $L_2^1(X)$:

$$(28) \qquad D_\mu^2[u] = \int_X |\nabla u|^2 dx + |\phi_\mu(u)|^2.$$

Lemma 2. *Suppose that the assumptions of Proposition 3 are satisfied. Then $L_2^1(X)$ is complete with respect to D_μ.*

Proof. Our arguments are quite standard. Let $\{u_k\}$ be a Cauchy sequence in the metric D_μ, then $\{u_k|_Y\}$ is a Cauchy sequence in $H^1(Y)$. So there exists $v \in H^1(Y)$ such that $\|u_k|_Y - v\|_{H^1(Y)} \to 0$. Besides, it follows from the completeness of $L_2^1(X)$ factorized by constants ([M], §1.1.13), that there exists a function $u \in L_2^1(X)$ such that $\int_X |\nabla(u_k - u)|^2 dx \to 0$, and therefore $\int_Y |\nabla(u_k - u)|^2 dx \to 0$. It follows that $u - v = C = \mathrm{const}$ on Y. Now, $u' := u - C \in L_2^1(X)$ and $u_k \to u'$ in the metric D_μ. $\square$

Suppose now that the conditions of the sufficiency part of Proposition 3 are satisfied. Then by $T_{\mu,X}^N$ we denote the compact, self-adjoint and nonnegative operator generated by $Q_\mu[u]$ in $L_2^1(X)$ with D_μ as the metric form. The equality

$$T_{\mu,X}^N u = \lambda u, \qquad u \in L_2^1(X)$$

means that
(29)
$$\lambda \left(\int_X \nabla u \cdot \nabla \overline{v} dx + \int_{\overline{X}} u d\mu \cdot \int_{\overline{X}} \overline{v} d\mu \right) = \int_{\overline{X}} u \overline{v} d\mu, \qquad \text{any } v \in L_2^1(X).$$

The equation (29) has the trivial eigenvalue $\lambda = 1$, with the corresponding eigenfunction $u_0 \equiv 1$. For any eigenfunction orthogonal to u_0, (29) is equivalent to (1), with $\mathcal{V} = \{v \in L_2^1(X) : \int_{\overline{X}} v d\mu = 0\}$.

Our interest lies in the behavior of the eigenvalues $\lambda_k(T_{\mu,X}^D)$ or $\lambda_k(T_{\mu,X}^N)$; we shorten the notation of both operators to T_μ when it is unambiguous. As usual, we express this behaviour in terms of the corresponding distribution function
$$n(t, T_\mu) = \#\{k : \lambda_k(T_\mu) > t\}.$$

Theorem 1. *Let (7) and (12) be satisfied, and assume $\Omega \in \mathcal{E}$. Let X be a domain in $\mathbb{R}^d$, $d \geq 2$, such that $\Omega \subseteq X$. If $d = 2$, then let $\mathbb{R}^2 \setminus X$ have nonzero inner (2,1)-capacity. Define $\delta > 0$ as the unique solution of*

(30)
$$\sum_{k=1}^m A_k^\delta = \sum_{k=1}^m \left(h_k^{2-d} p_k \right)^\delta = 1.$$

Then there exists a constant $C > 0$ such that for $T_\mu = T_{\mu,X}^D$

(31)
$$n(t, T_\mu) \leq C t^{-\delta}, \quad \text{for any } t > 0.$$

If, in addition, $\mu(\partial\Omega) = 0$, then there exist positive constants c and t_0 such that for $T_\mu = T_{\mu,X}^D$

(32)
$$n(t, T_\mu) \geq c t^{-\delta}, \quad \text{for any } t \in (0, t_0).$$

Theorem 2. *Let (7) and (12) be satisfied, and assume $\Omega \in \mathcal{E}$. Let X be a domain in $\mathbb{R}^d$, $d \geq 2$, such that $\Omega \subseteq X$. If $\mu(\partial\Omega) = 1$, then let also (22) be satisfied. Define $\delta > 0$ as the unique solution of (30).*

Then the results of Theorem 1 are valid for $T_\mu = T_{\mu,X}^N$.

5. Proof of the main results: basic construction

Theorems 1 and 2 will be proved simultaneously. We use the variational approach. Self-similarity of μ allows us to present a construction giving for $n(t, T_\mu)$ estimates of order $O(t^{-\delta})$ both from above and from below.

Let a finite set of multiindices $\mathbf{j}_1, ..., \mathbf{j}_r$, whose entries are $1, ..., m$ and whose length is not fixed, be chosen in such a way that the family $\Xi = \{\Omega(\mathbf{j}_1), ..., \Omega(\mathbf{j}_r)\}$ satisfies

$$1.\ \Omega(\mathbf{j}_\alpha) \cap \Omega(\mathbf{j}_\beta) = \emptyset \text{ for } \alpha \neq \beta, \qquad 2.\ \mu\left(\cup_{\alpha=1}^r \Omega(\mathbf{j}_\alpha)\right) = 1.$$

It was supposed that $\Omega \in \mathcal{E}$ in both Theorems 1 and 2. So the conditions of Proposition 3 are satisfied for $X = \Omega$. This yields

$$(33) \qquad \int_{\overline{\Omega}} |u|^2 d\mu \leq K \left(\int_\Omega |\nabla u|^2 dx + \left| \int_{\overline{\Omega}} u d\mu \right|^2 \right), \qquad \text{any } u \in L_2^1(\Omega),$$

with $K = \|T_{\mu,\Omega}^N\|^2$.

Let $\mathcal{V} = \mathring{L}_2^1(X)$ or $\mathcal{V} = \{v \in L_2^1(X) : \int_{\overline{X}} v d\mu = 0\}$, for Theorems 1 or 2 respectively. Consider the subspace $F \subset \mathcal{V}$:

$$(34) \qquad\qquad F = \{u \in \mathcal{V} : \int_{\overline{\Omega(\mathbf{j}_\alpha)}} u d\mu = 0, \quad \alpha = 1, ..., r\}.$$

Clearly, codim $F \leq r$. For any $u \in F$ and any $\alpha = 1, ..., r$ (33) yields:

$$\int_{\overline{\Omega}} |u(S^{-1}(\mathbf{j}_\alpha)x)|^2 d\mu \leq K \int_\Omega |\nabla u(S^{-1}(\mathbf{j}_\alpha)x)|^2 dx,$$

so

$$\int_{\overline{\Omega(\mathbf{j}_p)}} |u(x)|^2 d\mu = p(\mathbf{j}_\alpha) \int_{\overline{\Omega}} |u(S^{-1}(\mathbf{j}_\alpha)x)|^2 d\mu$$

$$\leq K p(\mathbf{j}_\alpha) h(\mathbf{j}_\alpha)^{2-d} \int_{\Omega(\mathbf{j}_\alpha)} |\nabla u(x)|^2 dx = K A(\mathbf{j}_\alpha) \int_{\Omega(\mathbf{j}_\alpha)} |\nabla u(x)|^2 dx,$$

with the same constant factor K as in (33).

Denote
$$t_+ = t_+(\Xi) = \max\{A(\mathbf{j}_\alpha), \alpha = 1, ..., r\},$$
$$t_- = t_-(\Xi) = \min\{A(\mathbf{j}_\alpha), \alpha = 1, ..., r\}.$$

Then for any $u \in F$ we get:

$$(35) \quad \begin{aligned} \int_{\overline{\Omega}} |u|^2 d\mu &= \sum_{\alpha=1}^r \int_{\overline{\Omega(\mathbf{j}_\alpha)}} |u(x)|^2 d\mu \leq K \sum_{\alpha=1}^r A(\mathbf{j}_\alpha) \int_{\Omega(\mathbf{j}_\alpha)} |\nabla u(x)|^2 dx \\ &\leq K t_+ \sum_{\alpha=1}^r \int_{\Omega(\mathbf{j}_\alpha)} |\nabla u(x)|^2 dx \leq K t_+ \int_X |\nabla u(x)|^2 dx. \end{aligned}$$

So we have constructed a subspace of codimension not greater than r on which

$$(36) \qquad \int_{\overline{\Omega}} |u|^2 d\mu \leq Kt_+ \int_X |\nabla u(x)|^2 dx.$$

Therefore, in view of the variational principle, we obtain both for $T_{\mu,X}^D$ and $T_{\mu,X}^N$:

$$(37) \qquad n(Kt_+, T_\mu) \leq r.$$

Now, we turn to the estimates from below. In the proof of Theorem 2, we will deal here with $L_2^1(X)$ rather than with $\mathcal{V}$. Since the codimension of $\mathcal{V}$ in $L_2^1(X)$ is 1, this suffices for the proof of (32).

Recall that here we suppose $\mu(\partial\Omega) = 0$. Let $a \in \operatorname{Supp} \mu$. We may assume $a \in \Omega$. Fix a nonnegative function $u_0 \in C_0^\infty(\Omega)$ such that $u_0(a) > 0$. Define

$$u_{\mathbf{j}_\alpha}(x) = \begin{cases} u_0(S^{-1}(\mathbf{j}_\alpha)x), & x \in \Omega(\mathbf{j}_\alpha); \\ 0, & x \in X \setminus \Omega(\mathbf{j}_\alpha), \end{cases}$$

then $u_{\mathbf{j}_\alpha} \in \mathring{L}_2^1(X)$. Moreover, $\operatorname{Supp} u_{\mathbf{j}_\alpha} \subset \Omega(\mathbf{j}_\alpha)$, so the supports of $u_{\mathbf{j}_\alpha}$ for different α are mutually disjoint. Hence $u_{\mathbf{j}_\alpha}$ are mutually orthogonal both in L_2^1 and in $L_2(\mu)$ scalar products. We have

$$(38) \qquad \int_X |u_{\mathbf{j}_\alpha}|^2 d\mu = \int_{\Omega(\mathbf{j}_\alpha)} |u_{\mathbf{j}_\alpha}|^2 d\mu = p(\mathbf{j}_\alpha) \int_\Omega |u_0|^2 d\mu,$$

and

$$
\begin{aligned}
\int_X |\nabla u_{\mathbf{j}_\alpha}|^2 dx + \left| \int u_{\mathbf{j}_\alpha} d\mu \right|^2 &= \int_{\Omega(\mathbf{j}_\alpha)} |\nabla u_{\mathbf{j}_\alpha}|^2 dx + \left| \int_{\Omega(\mathbf{j}_\alpha)} u_{\mathbf{j}_\alpha} d\mu \right|^2 \\
(39) \qquad &= (h(\mathbf{j}_\alpha))^{d-2} \int_\Omega |\nabla u_0|^2 dx + p(\mathbf{j}_\alpha) \left| \int_\Omega u_0 d\mu \right|^2 \\
&\leq (h(\mathbf{j}_\alpha))^{d-2} \left(\int_\Omega |\nabla u_0|^2 dx + \left| \int_\Omega u_0 d\mu \right|^2 \right).
\end{aligned}
$$

So we have constructed a subspace of dimension r on which

$$(40) \qquad \frac{\int_X |u|^2 d\mu}{\|u\|_{\mathring{L}_2^1(X)}^2} \geq \frac{\int_X |u|^2 d\mu}{\|u\|_{L_2^1(X)}^2} \geq A(\mathbf{j}_\alpha) \frac{\int_\Omega |u_0|^2 d\mu}{\int_\Omega |\nabla u_0|^2 dx + \left| \int_\Omega u_0 d\mu \right|^2} \geq Ct_-.$$

Applying again the variational principle, we get both for $T^D_{\mu,X}$ and $T^N_{\mu,X}$:

(41) $$n(Ct_-, T_\mu) \geq r.$$

Now we describe a suitable sequence of families

$$\Xi_N = \left\{ \Omega(\mathbf{j}^N_1), ..., \Omega(\mathbf{j}^N_{r(N)}) \right\}.$$

Take $\Xi_0 = \{\Omega\}$. Suppose Ξ_N is already constructed. Let $\mathbf{j}_0$ be a multiindex for which $\Omega(\mathbf{j}_0) \in \Xi_N$ and

$$A(\mathbf{j}_0) = t_+(\Xi_N).$$

Then Ξ_{N+1} is obtained from Ξ_N by the replacement of $\Omega(\mathbf{j}_0)$ by the collection $S_1\Omega(\mathbf{j}_0), ..., S_m\Omega(\mathbf{j}_0)$. (If there are several such multiindices $\mathbf{j}_0$, we do the same with all the corresponding sets $\Omega(\mathbf{j}_0)$.)

It is easy to see that for the sequence $\{\Xi_N\}^\infty_{N=0}$ we have

(42) $$t_+(\Xi_N) \leq \bar{A}(\mathcal{S}, \mathbf{p})t_-(\Xi_N).$$

Now Theorems 1 and 2 can be proved with the help of an auxiliary combinatorial problem.

6. Combinatorial problem

Consider a class of paths on $(\mathbb{Z}_+)^m$. The initial point of each path is 0 and every link is parallel to one of the coordinate axes and goes in positive direction. Let a linear function $L(x)$ in $\mathbb{R}^m$ be given:

$$L(x) = L(x_1, ..., x_m) = c_1 x_1 + ... + c_m x_m, \quad c_k > 0, \quad k = 1, ..., m.$$

Denote

$$c_{min} = \min_{k=1,...,m} c_k, \qquad c_{max} = \max_{k=1,...,m} c_k.$$

Fix $s \geq 0$. Let $\mathcal{N}(s)$ be the family of paths whose last link intersects the hyperplane $L(x) = s$. (The path is in $\mathcal{N}(s)$ if the endpoint of its last link belongs to this hyperplane, and is not in $\mathcal{N}(s)$, if the starting point does.) Denote $\#\mathcal{N}(s) = N(s)$. Our aim is to estimate $N(s)$. This we will do in an "indirect" way. Namely, let $\mathcal{W}(s)$ be the family of paths that lie in

the pyramid $(\mathbb{R}_+)^m \bigcap \{L(x) \leq s\}$ and cannot be extended inside it and let $W(s) = \#\mathcal{W}(s)$. For $s < 0$ we set $W(s) = 0$. Then

(43) $$W(s) \leq N(s) \leq W(s + c_{max}).$$

Indeed, consider a path from $\mathcal{W}(s)$. The endpoint of its last link either lies on the hyperplane $L(x) = s$—then this path belongs also to $\mathcal{N}(s)$—or it is beneath the hyperplane; in this case the path may be continued to become the one from $\mathcal{N}(s)$. So to any path of $\mathcal{W}(s)$ corresponds at least one path from $\mathcal{N}(s)$, and this correspondence is clearly injective. Hence $W(s) \leq N(s)$.

A similar argument shows that $N(s) \leq W(s + c_{max})$; we should only take into account that the endpoints $x^e = (x_1^e, ..., x_m^e)$ of paths from $\mathcal{N}(s)$ satisfy

$$s \leq L(x^e) = c_1 x_1^e + ... + c_m x_m^e < s + c_{max}.$$

Now notice that $W(s)$ satisfies a simple functional equation:

(44) $$W(s) = W(s - c_1) + ... + W(s - c_m) + \Delta(s),$$

where $\Delta(s) = 1$ for $s < c_{min}$ and $\Delta(s) = 0$ for $s \geq c_{min}$ (the verification is straightforward). Define δ as the unique nonnegative solution of the equation

(45) $$e^{-\delta c_1} + ... + e^{-\delta c_m} = 1.$$

Denoting $F(s) = e^{-s\delta} W(s)$, we rewrite (44) as

(46) $$F(s) = e^{-\delta c_1} F(s - c_1) + ... + e^{-\delta c_m} F(s - c_m) + e^{-\delta s} \Delta(s).$$

This is a particular case of the so called renewal equation. The solvability results for this equation can be borrowed e.g. from [Fe], Chapter XI.1. We come to the following statement:

Proposition 4. *1) If at least one of the ratios* $\frac{c_k}{c_l}$ *is irrational ("non-arithmetic case"), then there exists a positive constant* B *such that*

$$e^{-\delta s} W(s) \to B, \quad s \to \infty.$$

2) If all the ratios $\frac{c_k}{c_l}$ *are rational ("arithmetic case"), then there exists bounded, positive and bounded away from zero periodic function* $\psi(s)$ *on* $\mathbb{R}^d$ *such that*

$$e^{-\delta s} W(s) - \psi(s) \to 0, \quad s \to \infty.$$

In fact, the complete formulation of this Proposition gives explicit expressions for B and $\psi(s)$.

Corollary (of Proposition 4 and inequality (43)). *There exist positive constants c and C such that*

$$c \leq e^{-\delta s} N(s) \leq C \quad \text{for } s \geq 0.$$

7. Proof of the main results: the final step

Choose $c_k = -\ln A_k$, $k = 1, ..., m$. Then, taking (42) in account, (37) and (41) become

(47) $$n(K\bar{A}(\mathcal{S}, \mathbf{p})e^{-s}, T_\mu) \leq N(s) \leq n(Ce^{-s}, T_\mu).$$

The conclusion of Theorems 1 and 2 immediately follows from the Corollary.

8. Discussion

1) The case $d = 1$ was analyzed in [SVe]. Here $n(t, T_\mu)$ itself satisfies an equation similar to (44). As a result, more precise conclusions were obtained.

2) The Hausdorff dimension of μ equals

$$\beta = \left(\sum_{k=1}^{m} p_k \ln p_k \right) \bigg/ \left(\sum_{k=1}^{m} p_k \ln h_k \right)$$

(see [St1]). The Hausdorff dimension α of $\mathcal{C}(\mathcal{S})$ is determined by the equation $\sum_{k=1}^{m} h_k^\alpha = 1$ (see [Hu]). Always $\beta \leq \alpha$, and the equality takes place if and only if $p_k = h_k^\alpha$, $k = 1, ..., m$. These p_k's are called "natural weights".

The exponent δ meets the following conditions:

(48) $$\frac{\beta}{\beta + 1} \leq \delta \leq \frac{\alpha}{\alpha + 1} \quad \text{for } d = 1 \quad (\text{ see [SVe]}),$$

$$\delta = 1 \quad \text{for } d = 2,$$

(49) $$\frac{\alpha}{\alpha - (d - 2)} \leq \frac{\beta}{\beta - (d - 2)} \leq \delta \quad \text{for } d \geq 3$$

(the proof of (49) is similar to the proof of (48) given in [SVe]). If $d \geq 3$, the value of δ can be arbitrarily large. The equalities in (48), (49) take place only in the case of natural weights.

3) For the natural weights, $\mu(\mathcal{S}, \mathbf{p})$ is nothing but the restriction on $\mathcal{C}(\mathcal{S})$ of the Hausdorff measure of dimension α. In this case the sufficiency part of Proposition 1 can be derived from Theorem 1 of Chapter 7 of [JWal].

4) If $d = 2$, the only self-similar measures for which (12) fails, are delta-measures supported at an interior point of Ω; here $m = 1$.

5) The sign "=" in the inequality $\sum_{k=1}^{m} h_k^d \leq 1$ is rather rare. It corresponds to the case when $S_k(\Omega)$, $k = 1, ..., m$, constitute a tiling of Ω by a family of subdomains similar to Ω. If, in addition, p_k are the natural weights, $\mu(\mathcal{S}, \mathbf{p})$ is nothing but the d-dimensional Lebesgue measure on Ω. Here, clearly, $\delta = \frac{d}{2}$.

6) In the results of Section 3, the spaces $H_0^1(X)$ and $H^1(X)$ endowed with the standard metric of H^1, can be taken instead of $\overset{\circ}{L}_2^1(X)$ and $L_2^1(X)$. The minor changes needed in the assumptions are evident. These changes correspond to the passage from (1) to the equation

$$\lambda \left(\int_X (\nabla u \cdot \nabla \overline{v}) dx + \int_X u \overline{v} dx \right) = \int_{\overline{X}} u \overline{v} d\mu, \qquad \text{any } v \in \mathcal{V}.$$

The estimates (31) and (32) remain valid for this modified equation.

7) In the case $d = 1$, it was proven in [Bo] that for any finite *singular* μ one has for the problem (1)

$$(50) \qquad\qquad\qquad n(t) = o(t^{-\frac{1}{2}})$$

(instead of $O(t^{-\frac{1}{2}})$, given for $d = 1$ by (3)). [Bo] contains also the first attempt to specify "o" in (50), provided μ meets some additional restrictions. However, for self-similar μ the estimates obtained in [Bo] turn out to be not sharp. As was mentioned in Section 1, the estimate (5) for $d = 1$ was first obtained in [Fu] with the help of another approach.

8) The equation (30) for δ has some resemblance with the equation for the so called *spectral dimension* d_S of a "fractal object" (whose description includes a self-similar set endowed with a self-similar measure and with the so called *harmonic structure*; see [KLa]). The exponent d_S appears in connection with the spectral analysis of the Laplacian *on* a fractal. This Laplacian and also the spectral dimension d_S are determined by a fractal object and formally do not depend on the embedding into $\mathbb{R}^d$ (though usually the specific harmonic structure involved is implied by this embedding).

On the contrary, in our type of problems the definition of the operators T_μ depends on this embedding.

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DEPARTMENT OF THEORETICAL MATHEMATICS, THE WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 76100, ISRAEL

E-mail address: naimark@wisdom.weizmann.ac.il

DEPARTMENT OF THEORETICAL MATHEMATICS, THE WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 76100, ISRAEL

E-mail address: solom@wisdom.weizmann.ac.il