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SYMPLECTIC REDUCTION AND RIEMANN-ROCH FOR CIRCLE ACTIONS

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1. Introduction

Let M be a $2n$ dimensional compact symplectic manifold, $L \longrightarrow M$ a complex Hermitian line bundle and ∇ a Hermitian connection on L whose curvature form, ω, is the symplectic form. By equipping M with an almost-complex structure, J, which is compatible with ω and positive-definite, one gets a Riemannian metric, $g(v,w) = \omega(Jv,w)$, and a splitting of the complexified tangent bundle of M into the $+i$ and $-i$ eigenspaces of J:

$$TM \otimes \mathbb{C} = T^{1,0}M \oplus T^{0,1}M. \tag{1.1}$$

Let

$$\wedge T^*M \otimes \mathbb{C} = \oplus_{i,j} \wedge^{i,j} T^*M$$

denote the associated bigrading of the bundle of forms, and

$$\Omega^{i,j}(M,L) = C^\infty(M, \wedge^{i,j} T^*M \otimes L)$$

the space of L-valued forms of type (i,j). Given a Hermitian connection ∇_K on the canonical line bundle $K = \wedge^{0,n} T^*M$, one can construct a Spin^c-Dirac operator [1]

$$\partial\!\!\!/_{\mathbb{C}} : \Omega^{0,\mathrm{even}}(M,L) \longrightarrow \Omega^{0,\mathrm{odd}}(M,L).$$

(If M is Kähler, L holomorphic, and ∇, ∇_K the canonical connections on L and K, then $\partial\!\!\!/_{\mathbb{C}}$ is equal to the Dirac operator for the twisted Dolbeault complex.) Let $Q(M)$ be the virtual vector space

$$Q(M) = \ker \partial\!\!\!/_{\mathbb{C}} - \operatorname{coker} \partial\!\!\!/_{\mathbb{C}}. \tag{1.2}$$

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[1] For more details on the construction, see e.g. Lawson-Michelsohn [LM], or Duistermaat [D].

Since any two ω-compatible J's, and any two Hermitian connections on L, K can be smoothly deformed into each other, the dimension of $Q(M)$ is a symplectic invariant of M.

Suppose now that a compact Lie group, G, acts equivariantly on the pair (M, L) and preserves ∇. The infinitesimal action of G on sections of L is given by the formula of Kostant:

$$D_v s = \nabla_v s - i\langle \phi, v \rangle s \tag{1.3}$$

(c.f. [K], page 169) where $v \in \mathfrak{g}$ and $\phi : M \longrightarrow \mathfrak{g}^*$ is the moment map. In particular, letting $Z = \phi^{-1}(0)$ and letting $\iota : Z \longrightarrow M$ be the inclusion map, the induced action of G on sections of $\iota^* L$ is given by

$$D_v s = \nabla_v s \tag{1.4}$$

i.e., sections of $\iota^* L$ are G-invariant iff they are autoparallel along orbits of G. If in addition G acts freely on Z, zero is a regular value of ϕ ([GS3], page 185), and hence, Z is a compact submanifold of M. Moreover, the fact that G acts freely on Z implies that the orbit space

$$M_G = Z/G$$

is a compact manifold and the projection map, $\pi : \; Z \longrightarrow M_G$, is a principal G-fibration. Since G also acts freely on $\iota^* L$ (if it acts freely on Z) the orbit space

$$L_G = \iota^* L/G$$

is a line bundle over M_G, and from (1.4) it is easy to see that there is a unique connection, ∇_G, on L_G with the property

$$\pi^* \nabla_G = \iota^* \nabla.$$

In particular, the curvature form, ω_G, of ∇_G satisfies

$$\pi^* \omega_G = \iota^* \omega$$

and hence, by Marsden-Weinstein, is the reduced symplectic form on M_G. Coming back to (1.1), if J is G-invariant there is a natural representation of G on $Q(M)$ which, up to isomorphism, is independent of the choice of J. In particular, denoting by $Q(M)^G$ the (virtual) vector space of G-fixed vectors in $Q(M)$; the dimension of $Q(M)^G$ is a symplectic invariant of the action of G on M.

We will be concerned in this paper with a conjecture of Guillemin-Sternberg which asserts that

$$\dim Q(M_G) = \dim Q(M)^G. \tag{1.5}$$

(See [GS1].) Several months ago the abelian version of this conjecture was proved, independently, by M. Vergne [Ve] and E. Meinrenken [Me1]. [2] A proof for the general case was recently given in [Me3]. [3]

The purpose of this note is to give a short, simple proof of their result for the group S^1. This proof is based on the "symplectic cutting" construction of E. Lerman, a description of which can be found in [Le]. In this article Lerman proves that if $\phi : M \longrightarrow \mathbb{R}$ is the moment map associated with the action of S^1 on M, the disjoint unions

$$M_+ = M_{S^1} \cup \phi^{-1}(\mathbb{R}^+)$$

and

$$M_- = M_{S^1} \cup \phi^{-1}(\mathbb{R}^-)$$

are compact symplectic manifolds, and the actions of S^1 on $\phi^{-1}(\mathbb{R}^+)$ and $\phi^{-1}(\mathbb{R}^-)$ can be extended to Hamiltonian actions of S^1 on M_+ and M_- by letting S^1 act trivially on M_{S^1}. In §3 below we will "pre-quantize" this result by showing that the restrictions of L and ∇ to $\phi^{-1}(\mathbb{R}^+)$ can be extended to a line bundle and connection, L_+ and ∇_+, on M_+ which are preserved by S^1. Thus, in particular, M_+ can be "quantized" by attaching to it the virtual vector space, $Q(M_+)$, and the representation of S^1 on this space. We will prove (1.5) by computing the character of this representation, using the equivariant index theorem of Atiyah-Segal-Singer, and verifying directly that the multiplicity with which the zero weight of S^1 occurs in $Q(M_+)$ is the dimension of $Q(M_{S^1})$. We will then compare the characters of the representations of S^1 on $Q(M)$ and $Q(M_+)$ and verify the following.

Theorem 1. *If m is a nonnegative integer, the multiplicity with which m occurs as a weight of S^1 in $Q(M_+)$ is equal to the multiplicity with which it occurs as a weight in $Q(M)$; and if m is negative, this multiplicity is zero.*

It follows that, for $m = 0$, the multiplicity with which m occurs as a weight in $Q(M)$ is equal to the dimension of $Q(M_{S^1})$.

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[2] They actually proved that for (1.5) to be true it suffices that zero be a regular value of ϕ (in which case M_G is an orbifold and L_G an orbifold line bundle).

[3] More elementary proofs are available for rank one groups (see Jeffrey-Kirwan [JK] and Meinrenken [Me2]), for L replaced with some high tensor power L^N (see Guillemin-Sternberg [GS2] and Meinrenken [Me1]), or if one assumes that a sufficiently large neighborhood of zero is contained in the set of regular values of ϕ (see Martin-Weitsman [MW]).

Comments

1. The proof of (1.5) which we have just sketched can be extended, with the appropriate adaptions, to orbifolds.
2. Let H be a compact Lie group. Given a Hamiltonian action of H on M which commutes with the action of S^1, one gets representations of H on $Q(M_{S^1})$ and $Q(M)^{S^1}$; and our proof of (1.5) will show that these representations are isomorphic.
3. From the orbifold version of (1.5) for S^1, one can deduce the orbifold version of (1.5) for $S^1 \times \cdots \times S^1$ by "reduction in stages."

2. The equivariant index formula

Let χ be the character of the representation of S^1 on $Q(M)$. The equivariant index formula [ASS] for the Spin^c Dirac operator asserts that if x is a generic element of $\mathbb{R}/2\pi\mathbb{Z}$, $\chi(e^{ix})$ can be written as a sum of local contributions over the connected components F of the fixed point set of S^1

$$\sum_F \chi_F(e^{ix}) \tag{2.1}$$

where

$$\chi_F(e^{ix}) = \int_F \frac{\mathrm{Ch}(L,x)\ \mathrm{Td}(F)}{D(F,x)} \tag{2.2}$$

the terms $\mathrm{Ch}(L,x),\dots$ in the integrand being defined as follows: Recall that the definition of $\not\partial_{\mathbb{C}}$ involved the choice of an S^1-invariant almost-complex structure on M. From this one gets complex structures on the tangent bundle and normal bundle of F. $\mathrm{Td}(F)$ is the Todd class of the tangent bundle (with respect to this complex structure), and $\mathrm{Ch}(L,x)$ is the equivariant Chern character of L.

$D(F,x)$ is an equivariant class associated with the normal bundle of F. Using the "splitting principle", one can assume without loss of generality that the normal bundle of F splits equivariantly into a sum of complex line bundles, E_j, $j = 1,\dots,r_F$ (where $r_F = \operatorname{codim} F/2$). Since F is connected, S^1 acts on E_j by multiplication by a fixed character, e^{ia_jx}, $a_j \in \mathbb{Z}$; and, by definition

$$D(F,x) = \prod \left(1 - e^{ia_jx-\alpha_j}\right) \tag{2.3}$$

where $\alpha_j \in H^2(F,\mathbb{R})$ is the Chern class of E_j. (If the normal bundle doesn't split equivariantly into line bundles, the individual terms in this product aren't well-defined, but the product is.)

Noting that $\mathrm{Ch}(L, x)$ is represented by the equivariant characteristic form, $e^{i\phi x+\omega}$, we can rewrite $\chi_F(e^{ix})$ as

$$\chi_F(e^{ix}) = e^{i\phi_F x} \int_F \frac{e^{\omega}\,\mathrm{Td}(F)}{\prod(1 - e^{ia_j x - \alpha_j})} \tag{2.4}$$

where $e^{i\phi_F\, x}$ is the character of the representation of S^1 on the fiber of L at points of F. This extends to a meromorphic function, $\chi_F(z)$, on the Riemann sphere, $\mathbb{C} \cup \{\infty\}$

$$\chi_F(z) = z^{\phi_F} \int_F \frac{e^{\omega}\,\mathrm{Td}(F)}{\prod(1 - z^{a_j} e^{-\alpha_j})}. \tag{2.5}$$

If $k = \phi_F$ is the maximum value of ϕ, all the weights a_j are negative and hence

$$\chi_F(z) = z^k \int_F e^{\omega}\,\mathrm{Td}(F) + O(z^{k-1}) \tag{2.6}$$

as $z \longrightarrow \infty$. Otherwise, at least one a_j is positive and

$$\chi_F(z) = O(z^{k-1}) \tag{2.7}$$

One can also express $\chi(z)$ in terms of the multiplicities $N(m)$ with which m occurs as a weight of the representation of S^1 on $Q(M)$, as the finite sum

$$\chi(z) = \sum N(m) z^m. \tag{2.8}$$

By comparing (2.8) with (2.6) – (2.7), one deduces that

$$N(k) = 0 \tag{2.9}$$

if $k > \max\,\phi$, and

$$N(k) = \int_F e^{\omega}\,\mathrm{Td}(F) = \dim Q(F) \tag{2.10}$$

if $k = \phi_F = \max\,\phi$. Similarly, by looking at the limit $z \longrightarrow 0$, one deduces that

$$N(k) = 0 \tag{2.11}$$

if $k < \min \phi$ and

$$N(k) = \dim Q(F) \tag{2.12}$$

if $k = \phi_F = \min \phi$.

3. Symplectic cutting

The action of the circle group on the complex plane $\mathbb{C}$ by multiplication by $e^{i\theta}$ preserves the symplectic form $-i\,dz \wedge d\bar{z}$, and is a Hamiltonian action with moment map $-|z|^2$. Therefore, the diagonal action of S^1 on the product $\mathbb{C} \times M$ is Hamiltonian with moment map

$$\psi(z,p) = -|z|^2 + \phi(p). \tag{3.1}$$

Moreover, S^1 acts freely on the zero level set of ψ. (Proof: If z is not equal to zero, this is obvious. If $z = 0$, $0 = \psi(z,p) = \phi(p)$, so p is on the zero level set of ϕ.) Therefore, the reduced space

$$M_+ \overset{def.}{=} \psi^{-1}(0)/S^1 \tag{3.2}$$

is well-defined. Let $\psi^{-1}(0) = W_0 \cup W_1$, where W_0 is the subset of $\psi^{-1}(0)$ where $z = 0$, and W_1 is its complement. The action of S^1 on W_1 possesses a global cross-section, namely the set where $\operatorname{Arg}(z) = 0$, and a point (z,p) is in this set iff $\phi(p) > 0$ and $z = \sqrt{\phi(p)}$. Thus,

$$W_1 \simeq \phi^{-1}(\mathbb{R}^+) \times S^1.$$

On the other hand, $(z,p) \in W_0$ iff $z = 0$ and $\phi(p) = 0$; so $W_0 \simeq \phi^{-1}(0)$ and

$$W_0/S^1 \simeq M_{S^1}.$$

Modulo a few technical details, this proves the following theorem of Lerman:

Theorem 2. *M_{S^1} imbeds in M_+ as a symplectic submanifold of codimension 2, and its complement is symplectomorphic to the open subset $\phi^{-1}(\mathbb{R}^+)$ of M. In particular, M_+ is the disjoint union*

$$M_+ = M_{S^1} \cup \phi^{-1}(\mathbb{R}^+). \tag{3.3}$$

Next, observe that there is another Hamiltonian action of the circle on $\mathbb{C} \times M$—the product of the trivial action on $\mathbb{C}$ with the given action of S^1 on M. Since this commutes with the diagonal action, it gives rise to a Hamiltonian action of S^1 on M_+. It is easy to check that the moment map associated with this action is zero on the first summand of (3.3) and, on the second summand, is the restriction of ϕ to $\phi^{-1}(\mathbb{R}^+)$. From this, it follows that the fixed point set, $(M_+)^{S^1}$, is the union

$$\left(M^{S^1} \cap \phi^{-1}(\mathbb{R}^+)\right) \cup M_{S^1},$$

i.e., in addition to the fixed points that existed prior to cutting, there is one new connected set of fixed points created by cutting—the reduced space M_{S^1}.

A parenthetical remark. If we make a small modification in the construction we have just described—namely, take $i\,dz \wedge d\bar{z}$ rather than $-i\,dz \wedge d\bar{z}$ to be the symplectic form on $\mathbb{C}$—we get in place of M_+ a symplectic manifold, M_-, which is set-theoretically the disjoint union

$$M_{S^1} \cup \phi^{-1}(\mathbb{R}^-). \tag{3.4}$$

Lerman's construction can be "pre-quantized" by taking $L_{\mathbb{C}}$ to be the trivial line bundle over $\mathbb{C}$, with fiber metric equal to the usual metric times $\exp(-|z|^2)$ and $\nabla_{\mathbb{C}}$ to be the connection which takes sections of $L_{\mathbb{C}}$, i.e., complex-valued functions f, to one forms

$$\nabla_{\mathbb{C}} f = df - \bar{z} f\, dz. \tag{3.5}$$

The product bundle on $\mathbb{C} \times M$, $L_{\mathbb{C}} \boxtimes L$, can then be equipped with the product connection $\nabla_{\mathbb{C}} \boxtimes \nabla$, which is invariant under the product action of $S^1 \times S^1$, and hence, by the "pre-quantum" version of reduction which we described in §1, this descends to an S^1-invariant line bundle and connection on M_+.

The proof of (1.5). For z large

$$\chi_F(z) = \sum_{i=0}^{k} c_{iF} z^i + O(z^{-1}),\ k = \phi_F; \tag{3.6}$$

by (2.6) – (2.7). Moreover, by (2.7), the coefficient of the leading term in this sum is zero if $\phi_F < \max \phi$. Since zero is a regular value of ϕ, $\max \phi > 0$; so by (2.8) and (3.6),

$$N(0) = \sum_{\phi_F > 0} c_{0F}. \tag{3.7}$$

The same formula, however, gives the multiplicity with which the trivial representation of S^1 occurs in $Q(M_+)$; by (2.12), this is equal to $\dim Q(M_{S^1})$ (the moment map associated with the Hamiltonian action of S^1 on M_+ achieves its minimum at zero and achieves this minimum on M_{S^1}). □

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