Mathematical Research Letters **2**, 75–77 (1995)

ON THE NON-EXISTENCE OF COCOMPACT LATTICES FOR $SL(n)/SL(m)$

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In this note we continue the investigation of manifolds locally modelled on non-Riemannian homogeneous spaces, along the lines developed in [3], [4]. If H/J is a homogeneous space of a Lie group H, a natural but special class of compact forms of H/J (i.e., compact manifolds locally modelled on H/J) are those compact manifolds of the form $\Gamma\backslash H/J$ where $\Gamma \subset H$ is a cocompact lattice for H/J, i.e., a discrete subgroup acting freely and properly discontinuously on H/J. It remains an open problem to classify those H/J that admit such a discrete subgroup or those admitting a compact form. This problem is not completely solved even for the basic test case of $H = SL(n,\mathbb{R})$, $J = SL(m,\mathbb{R})$, $2 \leq m < n$, and where the embedding $J \subset H$ is the standard one. (See [2] for a survey on the general problem.) In this note we prove:

Theorem 1. *If $m \geq 2$ and $n \geq m+3$, then $SL(n,\mathbb{R})/SL(m,\mathbb{R})$ does not admit a cocompact lattice.*

Remarks.

(1) It is natural to believe that there are no compact forms not only no cocompact lattices.

(2) The results of [3] establish that for $n \geq 2m$, $m \geq 2$, $n \geq 5$, there are no compact forms.

(3) For $2n/3 < m$, the assertion in Theorem 1 has been proven by T. Kobayashi [1] by completely different methods.

Proof. Suppose such a group Γ existed. Set $H = SL(n,\mathbb{R})$, $J = SL(m,\mathbb{R})$, and $G = SL(n-m,\mathbb{R})$, with the natural embedding in the centralizer of J in H. Since G centralizes J we have a natural action of G on the compact manifold $M = \Gamma\backslash H/J$ given by $g \cdot (\Gamma hJ) = \Gamma hg^{-1}J$, and this action preserves a smooth volume form on M [4]. The projection $\Gamma\backslash H \to \Gamma\backslash H/J$ is a principal bundle with structure group J on which G acts by principal bundle automorphisms. Following the arguments of [3], [4], since

Received December 21, 1994.

Research of second author partially supported by NSF grant.

$\mathbb{R}$-rank$(G) \geq 2$ from the assumption $n \geq m+3$, we can apply superrigidity for actions on principal bundles. We deduce that there are

(1) a homomorphism $\pi : G \to J$,
(2) a compact subgroup $K \subset Z_J(\pi(G))$, and
(3) a measurable section $s : A \to \Gamma\backslash H$, where $A \subset \Gamma\backslash H/J$ is a G-invariant set of positive measure, such that

$$s(gm) = gs(m)\pi(g)^{-1}c(g,m),$$

where $c(g,m) \in K$.

We can, of course, view π as a representation $\pi : G \to SL(m,\mathbb{R})$ and consider separately the following cases:

(1) π is irreducible;
(2) π is reducible.

In case (i), it follows that the centralizer $Z_J(\pi(G))$ is itself compact. One can then apply the main result of [3] to deduce that this is impossible using the existence of an $\mathbb{R}$-split one parameter subgroup in $Z_H(JG)$. (See [3] for details.) It thus suffices to consider the case in which π is reducible. We shall need the following simple algebraic fact.

Lemma 2. *Let $SL(2,\mathbb{R}) \subset SL(3,\mathbb{R})$ be the standard embedding and $A \subset SL(2,\mathbb{R})$ the diagonal matrices. Then for any finite dimensional representation of $SL(3,\mathbb{R})$, there is a non-zero vector fixed by A.*

We postpone the proof of Lemma 2 and proceed with the proof of Theorem 1.

If we let μ be the finite G-invariant volume on M restricted to A, then $\int_{c\in K} c_*(s_*\mu)d\mu_K(c)$ is a finite $gr(\pi)$-invariant measure on $\Gamma\backslash H$ where μ_K is the Haar measure of K and $gr(\pi)$ is the graph of π. (Cf. [3], proof of Theorem 2.2.) It follows that for any subgroup $L \subset gr(\pi)$ and any conjugate hLh^{-1} in H, there is a hLh^{-1}-invariant probability measure on $\Gamma\backslash H$. Since Γ acts properly on H/J, J acts properly on $\Gamma\backslash H$, and hence, any closed subgroup of J that preserves a probability measure on $\Gamma\backslash H$ must be compact. (Cf. [5], proof of Lemma 2.7.) Therefore, to show the existence of Γ is impossible, it suffices to find a non-compact closed $L \subset gr(\pi)$ and an element $h \in H$ such that $hLh^{-1} \subset J$.

Consider $SL(2,\mathbb{R}) \subset G = SL(n-m,\mathbb{R}) \subset H$ via the standard embedding. Thus, this copy of $SL(2,\mathbb{R})$ acts on the standard basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ by fixing all e_i except e_{m+1} and e_{m+2}. Since π is reducible, by lemma 2 there are two linearly independent vectors in $\mathbb{R}^m$ that are invariant under $\pi \mid A$, where $A \subset SL(2,\mathbb{R})$ is the split Cartan subgroup. Conjugating π by an element $j \in J = SL(m,\mathbb{R})$, we can assume these vectors are the first two standard basis vectors. Let $L = gr(\pi \mid A)$. Then it is clear that if w is the element of the Weyl group of $SL(n, \mathbb{R})$ with respect to the standard basis that interchanges e_k and e_{k+m} for $k = 1, 2$ and fixes all the others, then $wjLj^{-1}w^{-1} \subset J$. As we have already observed, this suffices to complete the proof of Theorem 1. $\square$

Proof of Lemma 2. We work with the Lie algebras $\mathfrak{a} \subset \mathfrak{sl}(2,\mathbb{R}) \subset \mathfrak{sl}(3,\mathbb{R})$. Let (π, V) be a representation of $\mathfrak{sl}(3,\mathbb{R})$, which we may assume is non-trivial. Write $\pi \mid \mathfrak{sl}(2,\mathbb{R}) = (\pi_{\text{even}}, V_{\text{even}}) \oplus (\pi_{\text{odd}}, V_{\text{odd}})$ where π_{even} is the sum of those irreducible components of $\pi \mid \mathfrak{sl}(2,\mathbb{R})$ with even highest weight and π_{odd} the corresponding sum for odd highest weight. By the standard theory of representations for $\mathfrak{sl}(2,\mathbb{R})$, it suffices to see that π_{even} is non-zero. However, there is $X \in \mathfrak{sl}(3,\mathbb{R})$ (namely the matrix E_{13}) that is an eigenvector of weight 1 under representation $ad_{\mathfrak{sl}(3,\mathbb{R})} \mid \mathfrak{a}$. Thus, $\pi(X)(V_{\text{odd}}) \subset V_{\text{even}}$. If $V_{\text{even}} = (0)$, then $\pi(X) = 0$ which is impossible by the simplicity of $\mathfrak{sl}(3,\mathbb{R})$. This completes the proof. $\square$

Acknowledgement

This work was completed while the second author was a visitor at l'Université de Paris-sud, Orsay, and he wishes to express his appreciation for the hospitality shown him.

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