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MORE CONSTRAINTS ON SYMPLECTIC FORMS FROM SEIBERG-WITTEN INVARIANTS

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Recently, Seiberg and Witten (see [SW1], [SW2], [W]) introduced a remarkable new equation which gives differential-topological invariants for a compact, oriented 4-manifold with a distinguished integral cohomology class which reduces mod(2) to the 2nd Steiffel-Whitney class of the manifold. A brief mathematical description of these new invariants is given in the recent preprint [KM1].

Using the Seiberg-Witten equations, I proved in [T] the following:

Theorem 1. *Let X be a compact, oriented, 4 dimensional manifold with $b^{2+} \geq 2$. Let ω be a symplectic form on X with $\omega \wedge \omega$ giving the orientation. Then the first Chern class of the canonical bundle of a compatible, almost complex structure on X has Seiberg-Witten invariant equal to ± 1.*

(A corollary of this theorem is the assertion that connect sums of non-negative definite compact, oriented 4-manifolds do not admit symplectic forms which are compatible with the orientation.)

Subsequently, I have found that a slight modification of the proof of Theorem 1 gives further results about symplectic 4-manifolds. The purpose of this note is to report on these additional results.

The first result below constrains the *other* cohomology classes on X which have non-zero Seiberg-Witten invariant. In the theorem below, $[\omega]$ denotes the cohomology class of the symplectic form ω, and $K \to X$ is the canonical bundle for any almost complex structure on X which is compatible with ω. Also, the symbol $\bullet$ denotes the bilinear pairing on cohomology as given by cup product and evaluation on the fundamental class of X.

Theorem 2. *Let X be a compact, oriented symplectic manifold with $b^{2+} \geq 2$ and with symplectic form ω which is compatible with the given orientation. Let $c \in H^2(X;\mathbb{Z})$ have non-zero Seiberg-Witten invariant. Then*

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$|c \bullet [\omega]| \leq c_1(K) \bullet [\omega]$ *and if equality holds then either* $\pm c$ *is equal to* $c_1(K)$. *In particular, if* X *is to admit a symplectic form, then* $c_1(K) \bullet [\omega] \geq 0$.

Theorem 2 is proved in [W] for Kähler manifolds with $b^2 \geq 3$. (The assertion that $c_1(K) \bullet [\omega] \geq 0$ for Kähler manifolds with $b^2 \geq 2$ follows from the fact that such manifolds have $(b^{2+} - 1)/2$ holomorphic sections of K.)

There is another proof of Theorem 2 which uses a result recently announced by Donaldson concerning the existence of symplectic submanifolds of a symplectic manifold. Using Donaldson's existence assertion, Theorem 2 follows with a proof of an adjunction type formula for the line bundles with non-zero Seiberg-Witten classes. (Donaldson has also noted the other proof.) The aforementioned adjunction formula for the Seiberg-Witten classes is the analog of a formula proved by Kronheimer and Mrowka [KM2] for their basic class description of Donaldson's polynomial. A version of the Seiberg-Witten adjunction formula is proved in [KM1] and the general version will be discussed in a separate paper with other authors.

Another variant of the proof of Theorem 1 yields

Theorem 3. *The manifold* $\mathbb{CP}^2$ *has no symplectic form* ω *for which* $c_1(K) \bullet [\omega] > 0$. (*The standard Kähler structure on* $\mathbb{CP}^2$ *has* $c_1(K) \bullet [\omega] < 0$.)

Note that the inequalities for $c_1(K) \bullet [\omega]$ in Theorems 2 and 3 go in opposite ways. But, there is no contradiction here because $\mathbb{CP}^2$ has $b^{2+} = 1$.

a) Proof of Theorem 2

The reader should first become familiar with the proof of Theorem 1 in [T], for the proof of Theorem 2 will proceed almost verbatim as that of Theorem 1 modulo some minor changes in notation. To begin, one should fix a metric on X for which the symplectic form ω is self dual. Then, the $\text{spin}_{\mathbb{C}}$ bundle for K^{-1} splits as $S_+ = \mathbb{I} \oplus K^{-1}$ where the form ω acts (by Clifford multiplication) on the $\mathbb{I}$ summand as multiplication by $-i$ and on the other summand as multiplication by i. Remember that there is a unique connection A_0 on K^{-1} which is such that the $\text{spin}_{\mathbb{C}}$ covariant derivative $\widehat{\nabla}_{A_0}$ induces the trivial covariant derivative d on the $\mathbb{I}$ summand. (This induced covariant derivative is $2^{-1} \cdot (1 + i\omega) \cdot \widehat{\nabla}_{A_0} \cdot 2^{-1} \cdot (1 + i\omega)$.)

To prove Theorem 2, assume that there is a line bundle L over X whose first Chern class has non-zero Seiberg-Witten invariant which violates the conditions in the theorem. Such an assumption will be seen to lead directly to a contradiction. In deriving this contradiction, it is necessary to remark first that if $c_1(L)$ has non-zero Seiberg-Witten invariant, then so does $-c_1(L)$ (see [W]). Thus, if Theorem 2's conditions are violated, they are violated by an L with

$$c_1(L) \bullet [\omega] + c_1(K) \bullet [\omega] \leq 0. \tag{1}$$

The line bundle L can be written as $K^{-1} \otimes E^2$, where $E \to X$ is another complex line bundle. With this understood, the spin$_{\mathbb{C}}$ spinors for L decompose as $S_{L+} = E \oplus (E \otimes K^{-1})$ and a spinor ψ will be written as $(\alpha \cdot u_0, \beta)$ where α is a section of E and β one of $E \otimes K^{-1}$. Here u_0 is (as in [T]) the unit length, A_0–covariantly constant section of the summand $\mathbb{I}$ in $\mathbb{I} \oplus K^{-1}$. A choice of connection A on the line bundle L gives a spin$_{\mathbb{C}}$ covariant derivative $\widehat{\nabla}_A$ on S_{L+}. This $\widehat{\nabla}_A$ induces covariant derivatives on the two summands of S_{L+}. These induced covariant derivatives are written as:

$$\begin{aligned} &1)\quad 2^{-1} \cdot (1 + i\omega) \cdot \widehat{\nabla}_A(\alpha u_0) \equiv (\nabla_a \alpha) \cdot u_0, \\ &2)\quad 2^{-1} \cdot (1 - i\omega) \cdot \widehat{\nabla}_A \beta \equiv \nabla'_A \beta. \end{aligned} \tag{2}$$

Here, ∇_a is a covariant derivative on E and ∇'_A is one on $E \otimes K^{-1}$.

With these preliminaries out of the way, consider now the perturbed Seiberg-Witten equation in (6) of [T] with the parameter $r \in [0, \infty)$ as an equation for a connection A on L and a section $\psi = (\alpha u_0, \beta)$ of S_{L+}:

(3)

$$\begin{aligned} D_A \psi &= 0, \\ P_+ F_A &= P_+ F_{A_0} + i \cdot (|\alpha|^2 - |\beta|^2 - 1) \cdot \omega - i \cdot (\alpha^* \beta + \alpha \beta^*) \\ &\quad - i \cdot 4 \cdot r \cdot (1 + r \cdot |\alpha|^2)^{-1} \cdot (\alpha^* \langle b, \nabla_a \alpha \rangle - \alpha \cdot \langle b, \nabla_a \alpha \rangle^*). \end{aligned}$$

Here, b is the section of $K^{-1} \otimes T^* X_{\mathbb{C}}$ which is defined (as in (1) in [T]) by the equation $\widehat{\nabla}_{A_0} u_0 = b$. (In (3), D_A denotes the spin$_{\mathbb{C}}$ Dirac operator on S_{L+} as defined by the Riemannian metric and A.)

Lemmas 2, 3 in [T] have essentially verbatim analogs for this L version of the perturbed Seiberg-Witten equation. Also, because of (1), the L version of Lemma 4 in [T] also holds. Thus, Proposition 5 in [T] also has a self-evident L version. Consider now whether one can prove the L analog of Lemma 6 in [T]. Proceeding as in Step 1 of Section c of [T], one finds the L-analog of Lemma 8 by directly copying the arguments in [T]. Likewise, the L version of (30) in [T] also holds, but the L version of (31) in [T] may not. Instead, one has

$$\begin{aligned} \int (|\nabla_a \alpha|^2 + (|\alpha|^2 - 1 - |\beta|^2)(|\alpha|^2 - 1)) - \int (1 - |\alpha|^2 - |\beta|^2) \\ = \int \langle D_A(\alpha u_0), D_A \beta \rangle. \end{aligned} \tag{4}$$

In the case where $L = K^{-1}$, the second integral on the right side above is zero. In general, this integral can be identified using (3) as equal to 2π times the left hand side of (1). Thus, in the case where (1) holds, one has

$$\int (|\nabla_a \alpha|^2 + (|\alpha|^2 - 1 - |\beta|^2)(|\alpha|^2 - 1)) \leq \int \langle D_A(\alpha u_0), D_A \beta \rangle, \tag{5}$$

which implies that (32) in [T] holds for the L-version of (5).

The remaining steps in the proof of Theorem 1 of [T] can then be carried out through to the end. (These only involve integrations by parts, the triangle inequality and a standard Sobolev inequality.) These steps lead directly to a contradiction; namely that $|\alpha| \equiv 1$ and that $\nabla_a \alpha \equiv 0$. To avoid this contradiction, one is inescapably forced to conclude that (1) can't be negative; and if (1) is equal to zero, then $L = K^{-1}$.

b) Proof of Theorem 3

Suppose that ω is a symplectic form on $\mathbb{CP}^2$ and let K denote the canonical bundle for a compatible almost complex structure. The arguments in [T] for the proof of Theorem 1 show that the perturbed Seiberg-Witten equation (3) has a unique solution up to gauge (A_0, u_0) for all r sufficiently large. Lemma 4 in [T] holds, and this means that the $r = 0$ version of (3) computes a Seiberg-Witten invariant of ± 1 for K^{-1}. But, note that $\mathbb{CP}^2$ has a metric with positive scalar curvature, and this means that the Seiberg-Witten invariant for K^{-1} is zero [W] when computed by Seiberg and Witten's original equation

$$\begin{aligned} D_A \psi &= 0, \\ P_+ F_A &= i \cdot (|\alpha|^2 - |\beta|^2) \cdot \omega - i \cdot (\alpha^* \beta + \alpha \beta^*). \end{aligned} \tag{6}$$

The disagreement between the $r = 0$ version of (3) and (6) leads directly to Theorem 3. Indeed, consider the family of equations below as parameterized by $s \in [0, 1]$:

$$\begin{aligned} D_A \psi &= 0, \\ P_+ F_A &= s \cdot P_+ F_{A_0} + i \cdot (|\alpha|^2 - |\beta|^2 - s) \cdot \omega - i \cdot (\alpha^* \beta + \alpha \beta^*). \end{aligned} \tag{7}$$

The $s = 1$ version of (7) is the $r = 0$ version of (3), and the $s = 0$ version of (7) is (6). The only way (3) and (6) can disagree on the Seiberg-Witten invariant of K^{-1} is if there exists some $s \in [0, 1]$ for which (7) has a $\psi \equiv 0$ solution. (The vernacular for this is that there is a "wall crossing" for some value of the parameter s.) If a solution $(A, 0)$ solves (7) for some particular

$s \in [0,1]$, then wedge both sides of (7) with the symplectic form ω and integrate over $\mathbb{CP}^2$ to conclude that

$$c_1(K^{-1}) \bullet [\omega] = s \cdot c_1(K^{-1}) \bullet [\omega] + s \cdot [\omega] \bullet [\omega]. \tag{8}$$

This last possiblity is forbidden if $c_1(K^{-1}) \bullet [\omega] < 0$. (Note that (8) will occur for some s if $c_1(K^{-1}) \bullet [\omega] > 0$ as happens with the standard Kähler structure on $\mathbb{CP}^2$.)

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