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MODULI OF COMPACT LEGENDRE SUBMANIFOLDS OF COMPLEX CONTACT MANIFOLDS

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0. Introduction

Let X be a compact complex submanifold of a complex manifold Y with normal bundle N such that $H^1(X,N)=0$. In 1962 Kodaira [K] proved that such a submanifold X belongs to the complete analytic family $\{ X_t \mid t \in M \}$ of complex submanifolds X_t of Y with the moduli space M being a $\big(\dim_{\mathbb{C}} H^0(X,N)\big)$-dimensional complex manifold. Moreover, there is a canonical isomorphism $k_t : T_tM \longrightarrow H^0(X_t,N_t)$ which associates a global section of the normal bundle N_t of $X_t \hookrightarrow Y$ to any tangent vector at the corresponding point $t \in M$.

In this paper, we study complete analytic families of compact complex Legendre submanifolds of complex contact manifolds. There are two main results about these families which we discuss: the first result gives an analogue of the above mentioned Kodaira's existence theorem, while the second result unveils rich geometric structures — families of torsion-free affine connections — canonically induced on moduli spaces of compact Legendre submanifolds. The latter phenomenon of inducing geometric structures on moduli spaces of compact complex submanifolds is very much in the spirit of Penrose's twistor theory which has already demonstrated in a spectacular way how a number of non-linear differential equations in the form of constraints on curvature tensors of torsion-free connections can be treated with the help of complex analysis (see e.g. the books [B-B, B-E, Be, M, P-R, WW] and references cited therein).

Surprisingly enough, no generality is lost when one considers Legendre relative deformation problems instead of the Kodaira ones: the class of moduli spaces studied by Kodaira can be canonically realized as a *proper* subclass of the class of moduli spaces of compact Legendre submanifolds. Our interest in Legendre moduli spaces stems from the fact that when one tries to find a twistor interpretation of a torsion-free affine connection with irreducibly acting holonomy group one normally ends up in the Legendre moduli space rather than in the Kodaira one [Br, L1]. Moreover, the main motivation for the present work is a conjecture that *any torsion-free*

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connection with irreducibly acting holonomy group can be constructed by twistor methods as sketched in sections 5 and 6 of this paper.

1. Complex contact manifolds

Let Y be a complex $(2n+1)$-dimensional manifold. A complex contact structure on Y is a rank $2n$ holomorphic subbundle $D \subset TY$ of the holomorphic tangent bundle to Y such that the Frobenius form

$$\begin{aligned} \Phi : D \times D &\longrightarrow TY/D \\ (v,w) &\longrightarrow [v,w] \bmod D \end{aligned}$$

is non-degenerate. A complex n-dimensional submanifold $X \hookrightarrow Y$ is called a Legendre submanifold if $TX \subset D$. The normal bundle of any Legendre submanifold $X \hookrightarrow Y$ is isomorphic to $J^1 L_X$ [L2], where L_X is the restriction to X of the contact line bundle $L \equiv TY/D$, and, therefore, fits into the exact sequence

$$0 \longrightarrow \Omega^1 X \otimes L_X \longrightarrow N \stackrel{pr}{\longrightarrow} L_X \longrightarrow 0.$$

2. Existence theorem

Let Y be a complex contact manifold. An analytic family $\{X_t \mid t \in M\}$ of compact submanifolds of Y [K] is called an *analytic family of Legendre submanifolds* if, for any point $t \in M$, the corresponding subset $X_t \hookrightarrow Y$ is a Legendre submanifold. According to Kodaira [K], there is a natural linear map

$$k_t : T_t M \longrightarrow H^0(X_t, N_t).$$

We say that the analytic family $\{X_t \hookrightarrow Y : t \in M\}$ of compact Legendre submanifolds is *complete at a point* $t \in M$ if the composition

$$s_t : T_t M \stackrel{k_t}{\longrightarrow} H^0(X_t, N_t) \stackrel{pr}{\longrightarrow} H^0(X_t, L_{X_t})$$

provides an isomorphism between the tangent space to M at the point t and the vector space of global sections of the contact line bundle over X_t. The analytic family $\{X_t \hookrightarrow Y \mid t \in M\}$ is called *complete* if it is complete at each point of the moduli space M.

Lemma 1. *If an analytic family $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact Legendre submanifolds is complete at a point $t_0 \in M$, then there is an open neighbourhood $U \subseteq M$ of the point t_0 such that the family $\{X_t \hookrightarrow Y \mid t \in M\}$ is complete at all points $t \in U$.*

We say that an analytic family $\{X_t \hookrightarrow Y \mid t \in M\}$ of compact Legendre submanifolds is *maximal at a point* $t_0 \in M$, if, for any other analytic family $\{X_{\tilde{t}} \hookrightarrow Y \mid \tilde{t} \in \tilde{M}\}$ of compact Legendre submanifolds such that $X_{t_0} = X_{\tilde{t}_0}$ for a point $\tilde{t}_0 \in \tilde{M}$, there exists a neighbourhood $\tilde{U} \subset \tilde{M}$ of $\tilde{t}_0$ and a holomorphic map $f : \tilde{U} \longrightarrow M$ such that $f(\tilde{t}_0) = t_0$ and $X_{f(\tilde{t}')} = X_{\tilde{t}'}$ for each $\tilde{t}' \in \tilde{U}$. The family $\{X_t \hookrightarrow Y \mid t \in M\}$ is called *maximal* if it is maximal at each point t in the moduli space M.

Lemma 2. *If an analytic family of compact Legendre submanifolds*

$$\{ X_t \hookrightarrow Y \mid t \in M \}$$

is complete at a point $t_0 \in M$*, then it is maximal at the point* t_0*.*

The map $s_t : T_t M \longrightarrow H^0(X_t, L_{X_t})$ studied by the previous two Lemmas will also play a fundamental role in our study of the rich geometric structures induced canonically on moduli spaces of complete analytic families of compact Legendre submanifolds described by the following theorem.

Theorem 3. *Let* X *be a compact complex Legendre submanifold of a complex contact manifold* (Y, L)*. If* $H^1(X, L_X) = 0$*, then there exists a complete analytic family* $\{X_t \hookrightarrow Y \mid t \in M\}$ *of compact Legendre submanifolds containing* X*. This family is maximal and* $\dim M = \dim_{\mathbb{C}} H^0(X, L_X)$*.*

This theorem is proved by working in local coordinates adapted to the contact structure and expanding the defining functions of nearby compact Legendre submanifolds in terms of local coordinates on the moduli space M. This is much in the spirit of the original proof of Kodaira's theorem of the existence and completeness of compact submanifolds of complex manifolds. The essential difference from the Kodaira case is that the infinite sequence of obstructions to agreements on overlaps of formal power series is situated now in $H^1(X, L_X)$ rather than in $H^1(X, N)$ [Me].

3. Interconnections between Kodaira and Legendre moduli spaces

If $X \hookrightarrow Y$ is a complex submanifold, there is an exact sequence of vector bundles

$$0 \longrightarrow N^* \longrightarrow \Omega^1 Y\big|_X \longrightarrow \Omega^1 X \longrightarrow 0,$$

which induces a natural embedding $\mathbb{P}(N^*) \hookrightarrow \mathbb{P}(\Omega^1 Y)$ of total spaces of the associated projectivised bundles. The manifold $\hat{Y} = \mathbb{P}(\Omega^1 Y)$ carries a natural contact structure such that the constructed embedding $\hat{X} =$

$\mathbb{P}(N^*) \hookrightarrow \hat{Y}$ is a Legendre one [Ar][1]. Indeed, the contact distribution $D \subset T\hat{Y}$ at each point $\hat{y} \in \hat{Y}$ consists of those tangent vectors $V_{\hat{y}} \in T_{\hat{y}}\hat{Y}$ which satisfy the equation $< \hat{y}, \tau_*(V_{\hat{y}}) >= 0$, where $\tau : \hat{Y} \longrightarrow Y$ is a natural projection and the angular brackets denote the pairing of 1-forms and vectors at $\tau(\hat{y}) \in Y$. Since the submanifold $\hat{X} \subset \hat{Y}$ consists precisely of those projective classes of 1-forms in $\Omega^1 Y\big|_X$ which vanish when restricted on TX, we conclude that $T\hat{X} \subset D|_{\hat{X}}$.

Proposition 4. *If $\{X_t \hookrightarrow Y \ \mid t \in M\}$ is a complete analytic family of compact submanifolds, then the associated family*

$$\left\{\hat{X}_t = \mathbb{P}(N_t^*) \hookrightarrow \mathbb{P}(\Omega^1 Y) \ \mid t \in M\right\}$$

of projectivized conormal bundles is a complete analytic family of compact Legendre submanifolds.

4. Canonical subspaces of Legendre moduli spaces

Let $\{X_t \hookrightarrow Y \mid t \in M\}$ be a complete family of compact Legendre submanifolds. For any point $y \in Y' \equiv \cup_{t\in M} X_t$, there is an associated subset M consisting of all Legendre submanifolds which pass through y. It is easy to show that such a subset is always an analytic subspace of M. Moreover, if the natural "derivation and then evaluation" map

$$H^0(X_t, L_{X_t}) \otimes \mathcal{O}_{X_t} \longrightarrow J^1 L_{X_t}$$

is an epimorphism for all t, then this analytic subspace has no singularities, i.e. it is a submanifold. In general, we denote the set of its regular points by α_y and call it an *alpha subspace* of M (cf. [P]).

5. Induced connections on Legendre moduli spaces

Let $\{X_t \hookrightarrow Y \mid t \in M\}$ be a complete analytic family of compact Legendre submanifolds. Consider $F = \{(y,t) \in Y \times M \mid y \in X_t\}$ and denote by $\mu : F \longrightarrow Y$ and $\nu : F \longrightarrow M$ two natural projections; F is a submanifold in $Y \times M$. If $L_F = \pi^*(L)\big|_F$, π being the natural projection $Y \times M \longrightarrow Y$, then, for any point $t \in M$, we have an isomorphism $L_F|_{\nu^{-1}(t)} \simeq L_{X_t}$, and, by the definition of completeness, there is an isomorphism of sheaves $s : TM \longrightarrow \nu_*(L_F)$ on M. If N_F is the normal bundle of $F \hookrightarrow Y \times M$, then, for any point $t \in M$, we have $N_F|_{\nu^{-1}(t)} \simeq N_t$, where N_t is the normal bundle of the submanifold X_t. The Kodaira map $k : TM \longrightarrow \nu_*(N_F)$ is not isomorphic in general.

[1]That Arnold's construction could provide a link between Kodaira and Legendre moduli spaces was pointed out to the author by C. LeBrun.

There exists a natural morphism of sheaves of $\mathcal{O}_M$-modules

$$\phi : \nu_* \left(L_F \otimes S^2(N_F^*) \right) \longrightarrow TM \otimes S^2(\Omega^1 M)$$

where S^2 stands for the symmetric square. Indeed, if χ is a germ of $\nu_* \left(L_F \otimes S^2(N_F^*) \right)$ at a point $t \in M$, then the action

$$\phi(\chi) : S^2(TM) \longrightarrow TM$$

of the corresponding germ $\phi(\chi) \in TM \otimes S^2(\Omega^1 M)$ on $S^2(TM)$ may be described as follows. First we note that there is a natural morphism

$$\begin{array}{rcl} \lambda : S^2_{\nu^{-1}(\mathcal{O}_M)} \left(\nu_*(N_F) \right) & \longrightarrow & \nu_* \left(S^2_{\mathcal{O}_F}(N_F) \right), \\ \sum_i \sigma_i \otimes_{\nu^{-1}(\mathcal{O}_M)} \zeta_i & \longrightarrow & \sum_i \sigma_i \otimes_{\mathcal{O}_F} \zeta_i, \end{array}$$

defined by the pointwise symmetric tensor product of germs of global sections of N_F over the germ of the submanifold $\nu^{-1}(t) \subset F$, $t \in M$. Combining this map with the Kodaira map k, we obtain a natural composition

$$S^2(TM) \xrightarrow{S^2(k)} S^2 \left(\nu_*(N_F) \right) \xrightarrow{\lambda} \nu_* \left(S^2(N_F) \right) \xrightarrow{\chi} \nu_* L_F \xrightarrow{s^{-1}} TM$$

which explains the action of $\phi(\chi)$ on $S^2(TM)$. In $TM \otimes S^2(\Omega^1 M)$, the image of $\nu_* \left(L_F \otimes S^2(N_F^*) \right)$ is a subsheaf of $\mathcal{O}_M$-modules which we denote by Λ. We define a Λ-*connection* on M as a section of the quotient of the $TM \otimes S^2(\Omega^1 M)$-bundle of affine torsion-free connections modulo the action of Λ. On an open covering $\{U_i \mid i \in I\}$ of M a Λ-connection is represented by a collection of ordinary torsion-free affine connections $\{\nabla_i \mid i \in I\}$ which, on the overlaps $U_{ij} = U_i \cap U_j$, have their differences in $H^0(U_{ij}, \Lambda)$. Therefore locally, a Λ-connection is the same thing as an equivalence class of torsion-free connections under the relation $\nabla_1 \sim \nabla_2$ if $\nabla_1 - \nabla_2$ is a local section of $\Lambda \subset TM \otimes S^2(\Omega^1 M)$, but globally they are different — the obstruction for the existence of a Λ-connection on M lies in $H^1 \left(M, TM \otimes S^2(\Omega^1 M)/\Lambda \right)$, while the obstruction for the existence of the above mentioned equivalence class of torsion-free affine connections is an element of $H^1 \left(M, TM \otimes S^2(\Omega^1 M) \right)$. A submanifold of M is said to be totally geodesic relative to a Λ-connection if it is totally geodesic relative to *each* of its local representatives ∇_i.

Theorem 5. *Let $\{X_t \hookrightarrow Y \mid t \in M\}$ be a complete family of compact Legendre submanifolds. If $H^1 \left(X_t, L_{X_t} \otimes S^2(N_t^*) \right) = 0$ for each $t \in M$, then M comes equipped canonically with an induced Λ-connection such that, for every $y \in Y'$, the associated alpha subspace α_y is totally geodesic.*

Let $(X, \mathcal{O}_X)$ be an analytic subspace of a complex manifold $(Y, \mathcal{O}_Y)$ defined by a sheaf of ideals $J \subset \mathcal{O}_Y$. The mth-order infinitesimal neighbourhood of X in Y is the ringed space $X^{(m)} = (X, \mathcal{O}_X^{(m)})$ with the structure sheaf $\mathcal{O}_X^{(m)} = \mathcal{O}_Y / J^{m+1}$. With the $(m+1)$th-order infinitesimal neighbourhood of X in Y, there is naturally associated an *mth-order conormal sheaf* of $\mathcal{O}_X^{(m)}$-modules $\mathcal{O}_X^{(m)}(N^*) = J/J^{m+2}$ (cf. [E-L]). This sheaf is an ideal subsheaf of $\mathcal{O}_X^{(m+1)}$. By construction, $\mathcal{O}_X^{(0)}(N^*)$ is the usual conormal sheaf N^*, while $\mathcal{O}_X^{(1)}(N^*)$ fits into the exact sequence of $\mathcal{O}_X^{(1)}$-modules

$$0 \longrightarrow S^2(N^*) \longrightarrow \mathcal{O}_X^{(1)}(N^*) \longrightarrow N^* \longrightarrow 0.$$

When X is a point in Y, then $\left(\mathcal{O}_X^{(0)}(N^*)\right)^*$ is identical to the tangent space $T_X Y$ at $X \in Y$, while $\left(\mathcal{O}_X^{(1)}(N^*)\right)^*$ is usually denoted by $T_X^{[2]} Y$ and called the *second-order tangent space* at $X \in Y$.

A sketch of the proof of Theorem 5. For each $t \in M$, there is a natural epimorphism $\mathcal{O}_X^{(1)}(N_t^*) \longrightarrow TX_t \otimes L_{X_t}^*$, whose kernel, denoted by $\mathcal{L}_{X_t}^{(1)}(N_t^*)$, fits into an exact sequence of $\mathcal{O}_X^{(1)}$-modules

$$0 \longrightarrow S^2(N^*) \longrightarrow \mathcal{L}_X^{(1)}(N^*) \longrightarrow L_X^* \longrightarrow 0.$$

Next consider the restriction of the dual of the contact line bundle L to the first order infinitesimal neighbourhood of X_t in Y, $L_{X_t}^{*(1)} \equiv L^*|_{X_t^{(1)}}$, which, as an $\mathcal{O}_{X_t}^{(1)}$-module, has the following extension structure

$$0 \longrightarrow N_t^* \otimes L_{X_t}^* \xrightarrow{i_1} L_{X_t}^{*(1)} \longrightarrow L_{X_t}^* \longrightarrow 0.$$

Let E be the canonical global section of $N \otimes N^* \simeq \mathrm{End}(N)$ whose value at each point of X is the multiplicative identity in the corresponding stalk of endomorphism groups. Then the global section $\tilde{E} \equiv pr \otimes id\,(E) \in H^0(X_t, L_{X_t} \otimes N_t^*)$, where $pr : N_t \longrightarrow L_{X_t}$, defines a canonical monomorphism of $\mathcal{O}_{X_t}^{(1)}$-modules

$$\begin{array}{cccccc} i_2 : & N_t^* \otimes L_{X_t}^* & \longrightarrow & \left(N_t^* \otimes L_{X_t}^*\right) \otimes \left(L_{X_t} \otimes N_t^*\right) & \cong & N_t^* \otimes N_t^* \\ & f & \longrightarrow & f \otimes \tilde{E}. & & \end{array}$$

Consider the composition

$$i_3 : N_t^* \otimes L_{X_t}^* \xrightarrow{j} N_t^* \otimes L_{X_t}^* \ \oplus \ N_t^* \otimes L_{X_t}^* \xrightarrow{i_1 \oplus i_2} L_{X_t}^{*(1)} \ \oplus \ N_t^* \otimes N_t^*,$$

where

$$\begin{array}{rcl} j : N_t^* \otimes L_{X_t}^* & \longrightarrow & N_t^* \otimes L_{X_t}^* \ \oplus \ N_t^* \otimes L_{X_t}^* \\ f & \longrightarrow & f \oplus (-f). \end{array}$$

The quotient sheaf of $\mathcal{O}_{X_t}^{(1)}$-modules,

$$\tilde{L}_{X_t}^{*\,(1)} \equiv \left(L_{X_t}^{*\,(1)} \ \oplus \ N_t^* \otimes N_t^* \right) / i_3 \left(N_t^* \otimes L_{X_t}^* \right)$$

fits into the commutative diagram

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & N_t^* \otimes L_{X_t}^* & \xrightarrow{i_1} & L_{X_t}^{*\,(1)} & \longrightarrow & L_{X_t}^* & \longrightarrow & 0 \\ & & i_2\downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & N_t^* \otimes N_t^* & \xrightarrow{i_4} & \tilde{L}_{X_t}^{*\,(1)} & \longrightarrow & L_{X_t}^* & \longrightarrow & 0. \end{array}$$

Then the sheaf of $\mathcal{O}_{X_t}^{(1)}$-modules, $\mathcal{L}_X^{(1)}(L^*) \equiv \tilde{L}_X^{*\,(1)} / i_4 \left(\wedge^2 N^* \right)$, fits into the exact sequence of $\mathcal{O}_{X_t}^{(1)}$-modules

$$0 \longrightarrow S^2(N_t^*) \longrightarrow \mathcal{L}_X^{(1)}(L^*) \longrightarrow L_X^* \longrightarrow 0.$$

The upshot of all these constructions is that the $\mathcal{O}_{X_t}^{(1)}$-module

$$\left[\mathcal{L}_{X_t}^{(1)}(N_t^*) - \mathcal{L}_{X_t}^{(1)}(L^*) \right] \in \operatorname{Ext}^1_{\mathcal{O}_{X_t}^{(1)}} \left(L_{X_t}^*, S^2(N_t^*) \right)$$

is actually a locally free $\mathcal{O}_X$-module. Moreover, denoting its dual by $\Delta_{X_t}^{[2]}$, one obtains the following commutative diagram of vector spaces

$$\begin{array}{ccccccccc} 0 & \rightarrow & T_t M & \rightarrow & T_t^{[2]} M & \rightarrow & S^2(T_t M) & \rightarrow & 0 \\ & & \Big\downarrow & & \Big\downarrow & & \Big\downarrow & & \\ 0 & \rightarrow & H^0(X_t, L_{X_t}) & \rightarrow & H^0 \left(X_t, \Delta_{X_t}^{[2]} \right) & \rightarrow & H^0 \left(X_t, S^2(N_{X_t}) \right) & \rightarrow & 0 \end{array}$$

which extends the canonical isomorphism $T_t M \rightarrow H^0(X_t, L_{X_t})$ to *second* order infinitesimal neighbourhoods of $t \hookrightarrow M$ and $X_t \hookrightarrow Y$. Since $H^1 \left(X_t, L_{X_t} \otimes S^2(N_t^*) \right) = 0$, the exact sequence of locally free sheaves

$$0 \longrightarrow L_{X_t} \xrightarrow{\alpha} \Delta_{X_t}^{[2]} \longrightarrow S^2(N_t) \longrightarrow 0$$

splits, and any splitting, i.e. any morphism $\beta : \Delta_{X_t}^{[2]} \rightarrow L_{X_t}$ such that $\beta \circ \alpha = id$, induces via the above commutative diagram an associated splitting of the exact sequence

$$0 \longrightarrow T_t M \longrightarrow T_t^{[2]} M \longrightarrow S^2(T_t M) \longrightarrow 0$$

which is equivalent to a torsion-free affine connection at $t \in M$. The set of all splittings of the extension for $\Delta_{X_t}^{[2]}$ is a principle homogeneous space for the group $H^0\left(X_t, L_{X_t} \otimes S^2(N_t^*)\right)$. Therefore, the set of all induced torsion-free affine connections is a principle homogeneous space for the group $\Lambda_t \subset T_t M \otimes S^2(\Omega_t^1 M)$. $\square$

Example 1. Let (Y, L) be a $(2n+1)$-dimensional complex contact manifold and $X \hookrightarrow Y$ a Legendre submanifold which is isomorphic to a non-degenerate quadratic hypersurface (quadric) in $\mathbb{CP}^{n+1}$. Suppose also that $L_X \simeq \mathcal{O}_{\mathbb{CP}^{n+1}}(1)|_{i(X)}$, where $i : X \hookrightarrow \mathbb{CP}^{n+1}$ is the standard embedding. According to [L2], the $(n+2)$-dimensional moduli space M of the associated complete family of Legendre submanifolds comes equipped canonically with a holomorphic conformal structure $[g_{\alpha\beta}]$. On the other hand, $H^1\left(X_t, L_{X_t} \otimes S^2(N_t^*)\right) = 0$, and so, by Theorem 5, the moduli space M has a distinguished Λ-connection. How are these canonical structures related to each other? The conormal bundle of $X \hookrightarrow Y$ fits into the exact sequence [L1, M]

$$0 \longrightarrow N^* \longrightarrow \Omega^1 M \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow L_X[-2] \longrightarrow 0,$$

where $\Omega^1 M$ denotes the cotangent vector space to the point in M corresponding to X, $[1]$ denotes a line in $S^2(\Omega^1 M)$ spanned by the conformal metric, $[m] = [1]^{\otimes m}$ for $m > 0$, and $[m] = ([1]^*)^{\otimes |m|}$ for $m < 0$. Then

$$0 \longrightarrow L_X \otimes S^2(N^*) \longrightarrow L_X \otimes_{\mathbb{C}} S^2(\Omega^1 M) \longrightarrow \Omega^1 M \otimes_{\mathbb{C}} L_X^{\otimes 2}[-2] \longrightarrow 0,$$

and hence $H^0\left(X, L_X \otimes S^2(N^*)\right) =$

$$\begin{aligned} \text{ker: } S^2(\Omega^1 M) \otimes TM \ &\longrightarrow \ \Omega^1 M \otimes S_0^2(\Omega^1 M)[2] \\ f_{\alpha\beta}^{\gamma} \ &\longrightarrow \ f_{\alpha\beta}^{\delta}\, g_{\gamma\delta} + f_{\alpha\gamma}^{\delta}\, g_{\beta\delta} - \frac{2\, g_{\beta\gamma}}{n+2} f_{\alpha\delta}^{\delta} \end{aligned}$$

where S_0^2 denotes trace-free symmetric tensor product. Thus two local affine connections, $\Gamma_{\beta\gamma}^{\alpha}$ and $\tilde{\Gamma}_{\beta\gamma}^{\alpha}$, on M represent one and the same Λ-connection if and only

$$\tilde{\Gamma}_{\alpha\beta}^{\gamma} - \Gamma_{\alpha\beta}^{\gamma} = p_\alpha\, \delta_\beta^\gamma + p_\beta\, \delta_\alpha^\gamma - g_{\alpha\beta}\, p^\gamma$$

for some 1-form $p = p_\alpha\, dt^\alpha$. Since null geodesics of $[g_{\alpha\beta}]$ are totally geodesic relative to the induced Λ-connection, we conclude that the latter is locally the set of all torsion-free affine connections on M which preserve the conformal structure (such connections are called Weyl connections).

Theorem 5 implies

Corollary 6. *Let $\{X_t \hookrightarrow Y \mid t \in M\}$ be a complete family of compact Legendre submanifolds. If*

$$H^0\left(X_t, L_{X_t} \otimes S^2(N_t^*)\right) = H^1\left(X_t, L_{X_t} \otimes S^2(N_t^*)\right) = 0$$

for each $t \in M$, then M comes equipped canonically with an induced torsion-free connection such that, for every $y \in Y'$, the associated alpha subspace α_y is totally geodesic.

Example 2. Let (Y, L) be a 3-dimensional complex contact manifold and $X = \mathbb{CP}^1$ its Legendre submanifold with $L_X = \mathcal{O}(k)$, $k \in \mathbb{N}^+$. Then the normal bundle N is isomorphic to $\mathbb{C}^2 \otimes \mathcal{O}(k-1)$. It is easy to check that both cohomology groups $H^0\left(X, L_X \otimes S^2(N^*)\right)$ and $H^1\left(X, L_X \otimes S^2(N^*)\right)$ vanish if and only if $k = 3$. By Theorem 3, the Legendre moduli space generated by $X \hookrightarrow Y$ is a 4-dimensional complex manifold M. By Corollary 6, M comes equipped with a distinguished torsion-free affine connection. This connection has been first found and studied by Bryant [Br] with the help of different methods.

Torsion-free affine connections generated on Legendre moduli spaces by Theorem 5 or Corollary 6 are called *induced connections*.

7. Holonomy of induced connections

One can use the fact that the normal bundle N of a Legendre submanifold $X \hookrightarrow Y$ is of the form $J^1 L_X$ to show that the vector space $H^0(X, L_X \otimes N^*)$ has a canonical Lie algebra structure. There is a natural representation of this finite-dimensional Lie algebra on the vector space $H^0(X, L_X)$: any vector $s \in H^0(X, L_X \otimes N^*)$ defines a map

$$\begin{array}{rccc} \psi_s: & H^0(X, L_X) & \longrightarrow & H^0(X, L_X) \\ & \alpha & \longrightarrow & < s, j^1(\alpha) >, \end{array}$$

where the angular brackets stand for the natural pointwise L_X-valued pairing of a global section of $L_X \otimes N^*$ with a global section of N. We have, for any $s, t \in H^0(X, L_X \otimes N^*)$,

$$\psi_s \circ \psi_t - \psi_t \circ \psi_s = \psi_{[s,t]}.$$

Theorem 7. *Let ∇ be an induced connection on a Legendre moduli space M. If the function*

$$f : t \longrightarrow \dim H^0(X_t, L_{X_t} \otimes N_t^*)$$

is constant on M, then the holonomy algebra of ∇ is a subalgebra of the Lie algebra $H^0(X, L_X \otimes N^*)$.

Note that Theorem 7 gives also a representation of the holonomy algebra on TM. It is very easy to check using this Theorem that

(1) in the case of the complete family of quadrics considered in Example 1, the Lie algebra $H^0(X, L_X \otimes N^*)$ is precisely the conformal algebra, thus confirming once again that induced torsion-free connections are Weyl connections;
(2) the holonomy group of the induced connection on the space of Legendre rational curves discussed in Example 2 is precisely Bryant's [Br] exotic group $G_3 \subset GL(4, \mathbb{C})$ (exotic in the sense that it is missing in the Berger [B] list of admissible holonomies of torsion-free affine connections);
(3) Legendre moduli spaces associated with relative deformation problems $X = CP^{2k} \times CP^1 \hookrightarrow Y$ with $L_X \simeq \mathcal{O}(1,1)$ come equipped with complexified quaternionic connections.

In fact, almost all *known* torsion-free affine connections with irreducibly acting holonomy groups can be interpreted, at least locally, as induced connections on appropriate Legendre moduli spaces generated by relative deformations of compact complex homogeneous manifolds. This partly motivates the conjecture that *any such a connection can be obtained in this way*. If so, one gets a powerful tool — the famous Bott-Borel-Weil theorem — to attempt to fill the gaps in Berger's [B] list of irreducibly acting holonomies of torsion-free affine connections.

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References

[Ar] V.I. Arnold, V.I. , *Mathematical Methods of Classical Mechanics*, Springer-Verlag, New York, 1978 [Russian: Nauka, Moscow, 1974].

[B-B] T.N. Bailey and R.J. Baston (eds.), *Twistors in Mathematics and Physics*, Cambridge University Press, 1990.

[B-E] R.J. Baston and M.G. Eastwood, *The Penrose transform, its interaction with representation theory*, Oxford University Press, 1989.

[B] M. Berger, *Sur les groupes d'holonomie des variétés á connexion affine et des variétés Riemanniennes*, Bull. Soc. Math. France **83** (1955), 279-330.

[Be] A. Besse, *Einstein manifolds*, Springer-Verlag, 1987.

[Br] R. Bryant, *Two exotic holonomies in dimension four, path geometries, and twistor theory*, Proc. Symposia in Pure Mathematics **83** (1991), 33-88.

[E-L] M. Eastwood and C. LeBrun, *Fattening complex manifolds: curvature and Kodaira-Spencer maps*, J. Geom. Phys. **8** (1992), 123-146.

[K] K. Kodaira, *A theorem of completeness of characteristic systems for analytic families of compact submanifolds of complex manifolds*, Ann. Math. **75** (1962), 146-162.

[L1] C. R. LeBrun, *Spaces of complex null geodesics in complex-Riemannian geometry*, Trans. Amer. Math. Soc. **284** (1983), 209-321.

[L2] C. R. LeBrun, *Thickenings and conformal gravity*, Commun. Math. Phys. **139** (1991), 1-43.

[M] Yu. Manin, *Gauge Field theory and Complex Geometry*, Springer-Verlag, 1988 [Russian: Nauka, Moscow, 1984].

[Me] S. A. Merkulov, *Existence and geometry of Legendre moduli*, preprint.

[P] R. Penrose, *Non-linear gravitons and curved twistor theory*, Gen. Rel. Grav. **7** (1976), 31-52.

[P-R] R. Penrose and W. Rindler, *Spinors and space-time*, Vol. 2, Cambridge University Press, 1986.

[WW] R.S. Ward and R.O. Wells, Jr., *Twistor geometry and field theory*, Cambridge University Press, 1990.

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