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ISOSPECTRAL CONVEX DOMAINS IN EUCLIDEAN SPACE

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0. Introduction

Mark Kac's question "Can one hear the shape of a drum?" [K] asks whether the eigenvalue spectrum of the Laplace operator on a domain in the Euclidean plane determines the domain up to congruence. Urakawa [U] constructed examples of isospectral domains in $\mathbf{R}^n$ for $n \geq 4$. Kac's original question was answered negatively in [GWW]; see [BCDS] for many other examples. Thus far, all examples of noncongruent isospectral domains in Euclidean space are nonconvex. Thus the following question has remained open: *are there pairs of convex domains in Euclidean space which are isospectral but not congruent?* In this note, we show that one can modify slightly Urakawa's construction to obtain a pair of convex domains in Euclidean n-space for $n \geq 4$ which are noncongruent yet isospectral for both Dirichlet and Neumann boundary conditions. The proof of Neumann isospectrality uses very recent results of Hubert Pesce; we wish to thank him for discussing his work with us, for reading and commenting on a preliminary version, and for furnishing references.

1. Urakawa's isospectral domains in $\mathbf{R}^n$

Consider a root system in a Euclidean space $\mathbf{R}^{n+1}$, and let C be a closed Weyl chamber. In [BB] (see also [B1]), P. Bérard and G. Besson determined the spectra of spherical domains of the form $C \cap \mathbf{S}^n$ in terms of the Weyl group exponents. Urakawa [U] noticed that if one chooses two root systems R_1 and R_2 whose sets of Weyl group exponents coincide, then the corresponding closed Weyl chambers C_1, C_2 give rise to isospectral domains $D_1 = C_1 \cap \mathbf{S}^n$, $D_2 = C_2 \cap \mathbf{S}^n$ in the sphere; moreover, he showed that for $0 < \epsilon < 1$, the truncated cones $\Omega_1(\epsilon) = \{t\mathbf{p} \in \mathbf{R}^{n+1} : \mathbf{p} \in D_1, \epsilon < t < 1\}$ and $\Omega_2(\epsilon) = \{t\mathbf{p} \in \mathbf{R}^{n+1} : \mathbf{p} \in D_2, \epsilon < t < 1\}$ are isospectral domains in

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$\mathbf{R}^{n+1}$. (His argument breaks down for $\epsilon = 0$ due to the presence of the radial variable t in the denominator.) In what follows, we show:

Theorem. *The convex domains $\Omega_1 = \Omega_1(0)$ and $\Omega_2 = \Omega_2(0)$ in $\mathbf{R}^{n+1}$ are isospectral for both Dirichlet and Neumann boundary conditions.*

It is not difficult to see that the domains Ω_1 and Ω_2 are Dirichlet isospectral, using the mini-max characterization of eigenvalues: the kth Dirichlet eigenvalue of $\Omega_i(\epsilon)$ is given by

$$\lambda_k^i(\epsilon) = \inf_V \sup_{f \in V} \frac{\int |df|^2}{\int |f|^2}$$

where the infimum is taken over all k-dimensional subspaces V of the space $C_0^\infty(\Omega_i(\epsilon))$ of smooth functions compactly supported in the interior of $\Omega_i(\epsilon)$. Since $\Omega_i(\epsilon) \supseteq \Omega_i(\epsilon')$ for $\epsilon > \epsilon'$, and since the domains $\Omega_i(\epsilon)$ exhaust Ω_i, one can easily argue that $\lambda_k^i(0) = \lim_{\epsilon \to 0^+} \lambda_k^i(\epsilon)$. Since $\lambda_k^1(\epsilon) = \lambda_k^2(\epsilon)$ for each $\epsilon > 0$, it follows that $\lambda_k^1(0) = \lambda_k^2(0)$, so Ω_1 and Ω_2 are Dirichlet isospectral.

The Neumann isospectrality is more interesting; the mini-max characterization of Neumann eigenvalues involves the space C^∞ rather than C_0^∞, so functions on subdomains can no longer be extended by zero, and hence questions of convergence of Neumann eigenvalues as the domain varies are more delicate. Instead, we note that the Neumann isospectrality of Ω_1 and Ω_2 is equivalent to their isospectrality as orbifolds, as in [GWW]; we then use an interesting partial converse to Sunada's Theorem recently discovered by H. Pesce [P2].

2. Sunada's Theorem; Pesce's strengthening and partial converse

In [S], T. Sunada introduced a systematic method for constructing isospectral manifolds. Many interesting examples have been constructed using this method, and some examples constructed prior to [S] are in retrospect best understood via Sunada's theorem. Recently, Pesce [P2] strengthened Sunada's theorem and gave a partial converse. In order to state Pesce's version of Sunada's theorem, we fix some notation and recall some preliminaries, following [P1].

Given a compact Lie group G, let $Irr(G)$ denote the set of isomorphism classes of irreducible unitary representations of G. Denote by 1_G the trivial one-dimensional representation of G. Let $R(G)$ denote the Grothendieck group of G, so $R(G)$ is a free abelian group with basis $Irr(G)$. Let $\Gamma < G$ be a subgroup. Given a unitary representation W of Γ, let $\mathrm{W}{\uparrow}_\Gamma^G$ denote the induced representation of G from Γ; for $V \in R(G)$, let $V{\downarrow}_\Gamma^G$ denote the restriction of V to Γ. The group $R(G)$ has a natural symmetric bilinear form $\langle , \rangle_G$ relative to which the basis $Irr(G)$ is orthonormal. If $\sigma \in Irr(G)$ and V is a representation of G, then $\langle \sigma, V\rangle_G$ is just the multiplicity of the irreducible representation σ as a constituent of V; for any $V \in R(G)$, we have $V = \sum_{\sigma \in Irr(G)} \langle \sigma, V\rangle_G \sigma$. Recall also that Frobenius reciprocity asserts that induction and restriction are adjoint: if $W \in R(\Gamma)$ and $V \in R(G)$, then $\langle W{\uparrow}_\Gamma^G, V\rangle_G = \langle W, V{\downarrow}_\Gamma^G\rangle_\Gamma$.

Now let K be a closed subgroup of the Lie group G, and let V be a unitary representation of G. Let V^K denote the subspace of K-invariant elements of V (i.e., the vectors fixed by every element of K), and let V_K denote the smallest closed G-invariant subspace of V containing V^K.

Remark 2.1. Suppose that the direct sum decomposition of V into irreducibles is given by $V = \sum_{\sigma \in Irr(G)} n_\sigma \sigma$ (here $n_\sigma = \langle \sigma, V\rangle_G$ is the multiplicity of σ as a constituent of V). Then the irreducible decomposition of V_K is given by $V_K = \sum_{\sigma \in Irr_K(G)} n_\sigma \sigma$, where $Irr_K(G)$ denotes the subset of $Irr(G)$ consisting of irreducible unitary representations σ of G such that the trivial representation of K occurs as a constituent of $V{\downarrow}_K^G$, i.e., such that $\langle 1_K, V{\downarrow}_K^G\rangle_K \neq 0$. Indeed V_K is generated as a G-space by the irreducible subrepresentations σ containing nontrivial K-fixed vectors, so this is clear.

Pesce declares two representations V_1, V_2 of G to be K-*equivalent* if $(V_1)_K$ and $(V_2)_K$ are unitarily equivalent representations of G; by the remark above, this means that all irreducible representations of G which contain K-fixed vectors occur with the same multiplicities in V_1 and V_2.

Now let M be a connected smooth manifold with a smooth proper action by a Lie group G. Then (see [Bo]) there is a compact subgroup K which is subconjugate to all isotropy subgroups and is actually conjugate to most:

(1) For any $x \in M$, there is a $g \in G$ such that $G_x \supseteq gKg^{-1}$;
(2) There is a dense open subset U of M such that for all $x \in U$, K is conjugate to G_x.

The subgroup K is called the *generic stabilizer* or the *principal isotropy subgroup* of the G-action.

Theorem 2.2 (Pesce [P2])**.** *Let (M, g) be a complete Riemannian manifold, $G < Iso(M, g)$ a Lie subgroup of the isometry group, Γ_1, Γ_2 discrete subgroups of G such that the orbit spaces $\Gamma_1\backslash M$ and $\Gamma_2\backslash M$ are compact manifolds. Let K be the generic stabilizer of the G-action, and suppose* *that the induced representations* $(1_{\Gamma_1})\uparrow_{\Gamma_1}^{G}$ *and* $(1_{\Gamma_2})\uparrow_{\Gamma_2}^{G}$ *are* K*-equivalent. Then* $\Gamma_1\backslash M$ *and* $\Gamma_2\backslash M$ *are isospectral.*

Remarks.

(1) If M is a manifold with boundary, then either Dirichlet or Neumann boundary conditions can be imposed.
(2) In Sunada's original formulation, G was a finite group, and the induced representations were required to be equivalent. The theorem was strengthened to permit G to be a Lie group in [DG]. Pesce's insight [P1] that the induced representations need only be K-equivalent affords much greater flexibility, and permits an understanding of isospectral manifolds such as those of Ikeda [I] which formerly did not appear to fit into Sunada's framework.
(3) One can relax the requirement that Γ_1 and Γ_2 act freely. In this case, the conclusion is that $\Gamma_1\backslash M$ and $\Gamma_2\backslash M$ are isospectral as orbifolds, as in [GWW]. A representation-theoretic proof of Sunada's theorem permitting the extension to orbifolds was first given by Bérard [B2]; his proof furnishes an explicit combinatorial "transplantation" of eigenfunctions on $\Gamma_1\backslash M$ to eigenfunctions on $\Gamma_2\backslash M$.

We now turn to a special case of Pesce's converse to Sunada's theorem.

Theorem 2.3 (Pesce [P2])**.** *Let* $G = O(n+1)$ *and* $K = O(n)$*, so* $M = G/K$ *is the* n*-sphere* $\mathbf{S}^n$*. Let* $\Gamma_1, \Gamma_2 < G$ *be discrete subgroups. Suppose that* $\Gamma_1\backslash M$ *and* $\Gamma_2\backslash M$ *are isospectral orbifolds. Then* $(1_{\Gamma_1})\uparrow_{\Gamma_1}^{G}$ *and* $(1_{\Gamma_2})\uparrow_{\Gamma_2}^{G}$ *are* K*-equivalent.*

For the reader's convenience, we briefly record the proof. Let Γ denote Γ_1 or Γ_2. Viewing functions on $\Gamma\backslash M$ as Γ-invariant functions on M, it is clear that $\mathit{spec}(M) \supseteq \mathit{spec}(\Gamma\backslash M)$. Now G acts on the space $\text{Ł}^2(M, \mathbf{C})$ of complex-valued square-integrable functions on M, and for each $\lambda \in \mathit{spec}(M)$, the λ-eigenspace $E_\lambda(M)$ is G-invariant. The multiplicity of $\lambda \in \mathit{spec}(M)$ as an eigenvalue of $\Gamma\backslash M$ is just the dimension $\dim(E_\lambda(\Gamma\backslash M)) = \dim(E_\lambda(M)^\Gamma)$ of the λ-eigenspace of $\Gamma\backslash M$, i.e., the multiplicity of the trivial representation of Γ in $E_\lambda(M)\downarrow_\Gamma^G$. Thus

$$\begin{aligned}\dim(E_\lambda(\Gamma\backslash M)) = \langle 1_\Gamma, E_\lambda(M)\downarrow_\Gamma^G\rangle_\Gamma &= \langle 1_\Gamma, \Big(\sum_{\sigma\in Irr(G)} \langle\sigma, E_\lambda(M)\rangle_G\,\sigma\Big)\downarrow_\Gamma^G\rangle_\Gamma \\ &= \sum_{\sigma\in Irr(G)} \langle\sigma, E_\lambda(M)\rangle_G \langle 1_\Gamma, \sigma\downarrow_\Gamma^G\rangle_\Gamma \\ &= \sum_{\sigma\in Irr(G)} \langle\sigma, E_\lambda(M)\rangle_G \langle (1_\Gamma)\uparrow_\Gamma^G, \sigma\rangle_G,\end{aligned}$$

by Frobenius reciprocity. Since $\Gamma_1\backslash M$ and $\Gamma_1\backslash M$ are isospectral, it follows that

$$\text{(2.4)}\quad \begin{aligned}&\text{For all } \lambda \in \mathit{spec}(M),\\ &\sum_{\sigma\in Irr(G)} \langle \sigma, E_\lambda(M)\rangle_G \big\{\langle (1_{\Gamma_1})\uparrow^G_{\Gamma_1}, \sigma\rangle_G - \langle (1_{\Gamma_2})\uparrow^G_{\Gamma_2}, \sigma\rangle_G\big\} = 0.\end{aligned}$$

It is a classical fact [BGM] that the eigenspaces $E_\lambda(M)$ of the sphere are all absolutely irreducible, so exactly one of the $\langle \sigma, E_\lambda(M)\rangle_G$ is 1, while all others are 0, and hence the condition (2.4) becomes:

$$\text{(2.4}'\text{)}\quad \begin{aligned}&\text{For all irreducible representations } \sigma \text{ of } G \text{ occurring in } L^2(M, \mathbf{C}),\\ &\text{the multiplicities of } \sigma \text{ in } (1_{\Gamma_1})\uparrow^G_{\Gamma_1} \text{ and in } (1_{\Gamma_2})\uparrow^G_{\Gamma_2} \text{ coincide.}\end{aligned}$$

But $L^2(M, \mathbf{C}) = (1_K)\uparrow^G_K$, so $\langle \sigma, L^2(M, \mathbf{C})\rangle_G = \langle \sigma\downarrow^G_K, 1_K\rangle_K$; thus the σ occurring in $L^2(M, \mathbf{C})$ are precisely those containing nontrivial K-fixed vectors, i.e., those $\sigma \in Irr_K(G)$. By Remark 2.1, this asserts precisely that $(1_{\Gamma_1})\uparrow^G_{\Gamma_1}$ and $(1_{\Gamma_2})\uparrow^G_{\Gamma_2}$ are K-equivalent.

Remark. The above argument can be reversed to yield a proof of Theorem 2.2 in the special case of M a homogeneous space.

We now turn to the proof that the convex domains Ω_1 and Ω_2 of §1 are Neumann isospectral. The group $G = O(n+1)$ acts on $\mathbf{R}^{n+1}$. Let Γ_1, Γ_2 be the two Coxeter groups associated with the root systems R_1 and R_2 in $\mathbf{R}^{n+1}$. Then by [U], the spherical domains $D_1 = C_1 \cap \mathbf{S}^n$ and $D_2 = C_2 \cap \mathbf{S}^n$ are Neumann isospectral, hence isospectral as orbifold quotients of $\mathbf{S}^n$, as in [GWW], since a Γ_i-invariant eigenfunction on $\mathbf{S}^n$ is one which extends by reflection across walls of the Weyl chamber C_i and hence has zero normal derivative on the boundary of D_i. By Theorem 2.3, $(1_{\Gamma_1})\uparrow^G_{\Gamma_1}$ and $(1_{\Gamma_2})\uparrow^G_{\Gamma_2}$ are K-equivalent, where $K = O(n)$.

Now consider the action of G on the unit ball $\mathbf{B}^{n+1}$ in $\mathbf{R}^{n+1}$. The generic stabilizer is $K = O(n)$ (indeed, this is the isotropy group at every point except the origin). By the above, $(1_{\Gamma_1})\uparrow^G_{\Gamma_1}$ and $(1_{\Gamma_2})\uparrow^G_{\Gamma_2}$ are K-equivalent, so by Theorem 2.2, $\Gamma_1\backslash\mathbf{B}^{n+1}$ and $\Gamma_2\backslash\mathbf{B}^{n+1}$ are isospectral orbifolds with boundary: for each domain, the singular locus is the union of the walls of the Weyl chambers, and the boundary is the intersection of the domain with $\mathbf{S}^n$ (here we impose Neumann boundary conditions on the orbifold boundary, that is, on $\mathbf{S}^n$). But the orbifold isospectrality of $\Gamma_1\backslash\mathbf{B}^{n+1}$ and $\Gamma_2\backslash\mathbf{B}^{n+1}$ is precisely the Neumann isospectrality of the underlying domains Ω_1 and Ω_2. This completes the proof.

Remark. Urakawa's examples of isospectral domains occur in all Euclidean spaces of dimension at least 4; thus there exist pairs of isospectral convex domains in $\mathbf{R}^n$ for each $n \geq 4$. One of the simplest examples of [U] is given by the root systems $A_3 \times A_1$ and $I_2(3) \times I_2(4)$. The domains are described explicitly as follows. Let $e_1, \ldots, e_4$ denote the standard basis of $\mathbf{R}^4$. Let $u_1 = e_3$, $u_2 = e_1 - e_2 + e_3$, $u_3 = e_1 + e_2 + e_3$, $u_4 = e_4$, $v_1 = e_1$, $v_2 = e_1 + \sqrt{3}e_2$, $v_3 = e_3$, $v_4 = e_3 + e_4$. Then a Weyl chamber for $A_3 \times A_1$ is given by $C_1 = \left\{\sum_{i=1}^{4} a_i u_i : a_i \geq 0, i = 1, \ldots, 4\right\}$, while a chamber for $I_2(3) \times I_2(4)$ is given by $C_2 = \left\{\sum_{i=1}^{4} a_i v_i : a_i \geq 0, i = 1, \ldots, 4\right\}$.

Added in proof. Lizhen Ji has recently shown that the Neumann isospectrality can be established by passage to the limit, as in the Dirichlet argument above; Hubert Pesce has recently given a representation-theoretic argument for the Dirichlet isospectrality, in the spirit of the Neumann argument above.

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