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A FACTORIZATION THEOREM WITH APPLICATIONS TO INVARIANT SUBSPACES AND THE REFLEXIVITY OF ISOMETRIES

Hari Bercovici

Abstract. We prove a factorization result for spaces of vector-valued square integrable functions, and give two applications. The first one involves factorization results related to invariant subspaces of the Hardy space of the unit ball in $\mathbb{C}^d$. The second application is a proof of the fact that arbitrary commutative families of isometries on a Hilbert space generate reflexive algebras.

1. Factorization results

Let (Ω, Σ, μ) be a measure space, and let $\mathcal{F}$ be a complex Hilbert space. We denote by $L^2(\mu, \mathcal{F})$ the Hilbert space of all Bochner measurable, square integrable (classes of) functions $f : \Omega \to \mathcal{F}$. For $x, y \in L^2(\mu, \mathcal{F})$ we denote by $x \cdot y \in L^1(\mu)$ the function defined by the pointwise scalar product:

$$(x \cdot y)(\omega) = (x(\omega), y(\omega)), \quad \omega \in \Omega.$$

A problem of interest in operator theory is that of factoring a given function $f \in L^1(\mu)$ as $f = x \cdot y$, with at least one of the vectors x, y belonging to some prescribed closed subspace $\mathcal{H}$ of $L^2(\mu, \mathcal{F})$; of course factorization is always possible with $x, y \in L^2(\mu, \mathcal{F})$ unless $\mathcal{F} = \{0\}$. Conditions were given in [1] which imply the possibility of approximate factorization. For the purposes of this paper it will be convenient to say that a subspace $\mathcal{H}$ of $L^2(\mu, \mathcal{F})$ has the *approximate factorization property* if for every nonnegative function $h \in L^1(\mu)$ and every $\varepsilon > 0$ there exists a vector $x \in \mathcal{H}$ such that

$$\|h - x \cdot x\|_1 < \varepsilon.$$

The main result of this section is that the function x in an approximate factorization of h can be chosen so that $\|x(\omega)\|^2 \geq h(\omega)$ almost everywhere.

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We conclude the section by applying the factorization results to invariant subspaces in the Hardy space of the unit ball in $\mathbb{C}^d$. Another application is given in the second section of the paper, where it is shown that any commuting family of isometries on a Hilbert space generates a reflexive algebra. This result was proved by Deddens [3] for a single isometry, and by Li and M^{c}Carthy [6] for finite families of isometries.

We state now the main result.

1.1. Theorem. *Assume that $\mathcal{H} \subset L^2(\mu, \mathcal{F})$ is a closed subspace with the approximate factorization property. Then for every nonnegative function $h \in L^1(\mu)$ and every $\varepsilon > 0$ there exists a vector $x \in \mathcal{H}$ such that $\|x(\omega)\|^2 \geq h(\omega)$ almost everywhere, and $\|x\|^2 < \|h\|_1 + \varepsilon$.*

Proof. We start by observing that the result is invariant under a change of measure. More precisely, assume that $\varphi \colon \Omega \to (0, +\infty)$ is a measurable function, and a measure μ' is defined by $d\mu' = \varphi\, d\mu$. There is a unitary operator $U \colon L^2(\mu, \mathcal{F}) \to L^2(\mu', \mathcal{F})$ given by $Uf = \varphi^{-1/2} f$, $f \in L^2(\mu, \mathcal{F})$, as well as an isometry $V \colon L^1(\mu) \to L^1(\mu')$ given by $Vh = h/\varphi$. The space $\mathcal{H}' = U\mathcal{H}$ is closed in $L^2(\mu', \mathcal{F})$, and for every $x, y \in L^2(\mu, \mathcal{F})$ we have $V(x \cdot y) = (Ux) \cdot (Uy)$. Thus both the hypothesis and the conclusion of the theorem are unaffected by substituting $\mathcal{H}'$ for $\mathcal{H}$.

This being said, fix a nonnegative function $h \in L^1(\mu)$, and observe that the conclusion of the theorem is true with $x = 0$ if $\|h\|_1 = 0$. We assume therefore that $\|h\|_1 \neq 0$. Upon replacing h by $h/\|h\|_1$ we may actually restrict ourselves to the case when $\|h\|_1 = 1$. Denote $\sigma = \{\, \omega : h(\omega) \neq 0 \,\}$, and set $\varphi = h + \chi_{\Omega \setminus \sigma}$. The function $Vh = h/\varphi$ coincides with χ_σ, and $\mu'(\sigma) = \|h\|_1 = 1$. We conclude that there is no loss of generality in assuming that $h = \chi_\sigma$ and $\mu(\sigma) = 1$ to begin with. Fix a number $\alpha \in (0,1)$, and set $\delta = \alpha^2/4$. Since $\mathcal{H}$ has the approximate factorization property, we can find x_1 in $\mathcal{H}$ such that $\|(1+\alpha)\chi_\sigma - x_1 \cdot x_1\|_1 < \delta$. If we set

$$\sigma_1 = \{\, \omega \in \sigma : \|x_1(\omega)\|^2 \leq 1 + \alpha/2 \,\}$$

then $\mu(\sigma_1) \leq 2\delta/\alpha = \alpha/2$. Observe also that we have

$$\|x_1\|^2 \leq \|h\|_1 + \alpha + \delta \leq \|h\|_1 + 2\alpha = 1 + 2\alpha.$$

We will construct by induction vectors x_n such that

(a) $\mu(\sigma_n) \leq \alpha/2^n$, where $\sigma_n = \{\, \omega \in \sigma : \|x_n(\omega)\|^2 \leq 1 + \alpha/2^n \,\}$, and
(b) $\|x_{n+1} - x_n\| \leq (\alpha/2^{n-4})^{1/2}$.

Assume that x_n has been constructed, and define $g_n \in L^1(\mu)$ by $g_n = 9\chi_{\sigma_n}$. By (a), we have $\|g_n\|_1 \leq 9\alpha/2^n$. Let δ_n be a small positive number, subject to certain conditions to be specified shortly (in fact $\delta_n = \alpha^3/10^n$ will satisfy all the requirements). The approximate factorization property shows that there exists $y_n \in \mathcal{H}$ such that $\|g_n - y_n \cdot y_n\|_1 < \delta_n$. Observe that

$$\|y_n\|^2 \le \|g_n\|_1 + \delta_n \le 9\alpha/2^n + \delta_n,$$

so that $\|y_n\| \le (\alpha/2^{n-4})^{1/2}$ if δ_n is chosen sufficiently small. Define $x_{n+1} = x_n + y_n$, and note that condition (b) is satisfied. To complete the inductive process we must show that (a) is satisfied with $n+1$ in place of n, provided that δ_n is chosen sufficiently small. Consider a point $\omega \in \sigma$ such that $\left|g_n(\omega) - \|y_n(\omega)\|^2\right| < (\alpha/2^{n+3})^2$. If $\omega \notin \sigma_n$ this means that $\|y_n(\omega)\| < \alpha/2^{n+3}$. If $\|x_n(\omega)\| > 2$ then certainly $\|x_{n+1}(\omega)\| \ge 3/2$ and $\|x_{n+1}(\omega)\|^2 \ge 1 + \alpha/2^{n+1}$. If $\|x_n(\omega)\| \le 2$ then

$$\begin{aligned} \|x_{n+1}(\omega)\|^2 &\ge (\|x_n(\omega)\| - \|y_n(\omega)\|)^2 \\ &\ge \|x_n(\omega)\|^2 - 2\|x_n(\omega)\|\,\|y_n(\omega)\| \\ &\ge 1 + \frac{\alpha}{2^n} - 4\frac{\alpha}{2^{n+3}} = 1 + \frac{\alpha}{2^{n+1}}. \end{aligned}$$

On the other hand, if $\omega \in \sigma_n$, then $\|y_n(\omega)\|^2 \ge 9 - (\alpha/2^{n+3})^2 \ge 8$ and $\|x_n(\omega)\|^2 \le 2$. Therefore

$$\begin{aligned} \|x_{n+1}(\omega)\|^2 &\ge (\|y_n(\omega)\| - \|x_n(\omega)\|)^2 \\ &\ge (2\sqrt{2} - \sqrt{2})^2 = 2 \ge 1 + \frac{\alpha}{2^{n+1}}. \end{aligned}$$

We conclude that

$$\sigma_{n+1} \subset \left\{ \omega : \left|g_n(\omega) - \|y_n(\omega)\|^2\right| \ge \left(\frac{\alpha}{2^{n+3}}\right)^2 \right\},$$

and therefore $\mu(\sigma_{n+1}) \le \delta_n(2^{n+3}/\alpha)^2$. It is easy to choose now δ_n in order to satisfy (a).

Denote by x the limit of the sequence $\{x_n\}_{n=1}^{\infty}$. Since $\sum_n \mu(\sigma_n) < \infty$, it follows that $\|x_n(\omega)\|^2 \ge h(\omega)$ almost everywhere. Moreover,

$$\|x\| \le \|x_1\| + \sum_{n=1}^{\infty} \|x_{n+1} - x_n\| \le (\|h\|_1 + 2\alpha)^{1/2} + \sum_{n=1}^{\infty} \left(\frac{\alpha}{2^{n-4}}\right)^{1/2},$$

and therefore $\|x\|^2 < \|h\|_1 + \varepsilon$ for sufficiently small α. The theorem follows. $\square$

Assume that $x \in \mathcal{H}$ is such that $\|x(\omega)\|^2 \ge h(\omega)$ almost everywhere, and $g \in L^1(\mu)$ satisfies an inequality of the form $|g(\omega)| \le kh(\omega)$ almost everywhere, with k a constant. Then the function $y \in L^2(\mu, \mathcal{F})$ defined by $y(\omega) = g(\omega)x(\omega)/\|x(\omega)\|^2$ if $x(\omega) \ne 0$, $y(\omega) = 0$ if $x(\omega) = 0$, satisfies the equality $x \cdot y = g$. This observation immediately implies the following factorization result.

1.2. Corollary. *Assume that $\mathcal{H} \subset L^2(\mu, \mathcal{F})$ has the approximate factorization property.*

(1) *For every $f \in L^1(\mu)$ and every $\varepsilon > 0$ there exist $x \in \mathcal{H}$ and $y \in L^2(\mu, \mathcal{F})$ such that $x \cdot y = f$ and $\|x\| \cdot \|y\| < \|f\|_1 + \varepsilon$.*
(2) *For every sequence $\{f_n\}_{n=1}^{\infty} \subset L^1(\mu)$ there exist vectors $x \in \mathcal{H}$ and $\{y_n\}_{n=1}^{\infty} \subset L^2(\mu, \mathcal{F})$ such that $x \cdot y_n = f_n$ for all $n \geq 1$.*

In order to see how this result can be applied, we recall a result proved in [1] and [2].

1.3. Theorem. *Assume that $\mathcal{H} \subset L^2(\mu, \mathcal{F})$ is a separable subspace such that for every set $\sigma \in \Sigma$ with $\mu(\sigma) > 0$ there exists a sequence $x_n \in \mathcal{H}$ such that*

(*i*) *$\|x_n\| = 1$ and x_n tends to zero weakly as $n \to \infty$, and*
(*ii*) *$\|\chi_{\Omega \setminus \sigma} x_n\| \to 0$ as $n \to \infty$.*

Then $\mathcal{H}$ has the approximate factorization property.

Fix now an integer $d \geq 2$, denote by B^d the Euclidean unit ball in $\mathbb{C}^d$, and let μ be normalized area measure on ∂B^d. We denote, as usual, by $H^2(B^d)$ the closure in $L^2(\mu)$ of all polynomials. The space $H^\infty(B^d)$ of all bounded holomorphic functions defined on B^d can be viewed as an algebra of multiplication operators on $L^2(\mu)$ which leaves the space $H^2(B^d)$ invariant. A closed subspace $\mathcal{H} \subset H^2(B^d)$ will be said to be invariant if $ux \in \mathcal{H}$ for every $u \in H^\infty(B^d)$ and $x \in \mathcal{H}$.

1.4. Theorem. *Let $\mathcal{H} \subset H^2(B^d)$ be a nonzero invariant subspace.*

(a) *For every $f \in L^1(\mu)$, and every $\varepsilon > 0$, there exists $x \in \mathcal{H}$ and $y \in L^2(\mu)$ such that $x \cdot y = f$ and $\|x\| \cdot \|y\| < \|f\|_1 + \varepsilon$.*
(b) *For every sequence $\{f_n\}_{n=1}^{\infty} \subset L^1(\mu)$ there exist vectors $x \in \mathcal{H}$ and $\{y_n\}_{n=1}^{\infty} \subset L^2(\mu)$ such that $x \cdot y_n = f_n$ for $n \geq 1$.*

Proof. By Theorem 1.1 it suffices to show that $\mathcal{H}$ has the approximate factorization property. Hence it suffices to show that $\mathcal{H}$ satisfies the hypotheses of Theorem 1.3. Let indeed σ be an arbitrary subset with positive measure of ∂B^d, and let $z \in \mathcal{H}$ be a nonzero function. Of course, z is almost nowhere equal to zero. Choose open subsets $G_n \supset \sigma$ such that $\mu(G_n) \to \mu(\sigma)$ as $n \to \infty$. By virtue of Theorem 4.1 in [9], there exist functions $u_n \in H^\infty(B^d)$ such that $|u_n| = \chi_{G_n} + 1/n$ almost everywhere. Then the unit vectors $y_n = u_n z / \|u_n z\| \in \mathcal{H}$ satisfy the condition $\|\chi_{B^d \setminus \sigma} y_n\| \to 0$ as $n \to \infty$. Choose now a nonconstant inner function $v \in H^\infty(B^d)$ (i.e., $|v| = 1$ a.e.; see again Theorem 4.1 in [9]), and observe that $v^N y$ tends to zero weakly as $N \to \infty$ for every $y \in L^2(\mu)$. We conclude that a sequence of the form $x_n = v^{N_n} y_n$ will satisfy conditions (i) and (ii) of Theorem 1.3. $\square$

2. Commuting families of isometries

Let A be a collection of operators on a Hilbert space $\mathcal{H}$. Recall that AlgLat(A) denotes the algebra of all operators on $\mathcal{H}$ which leave invariant all the invariant subspaces of A. The set A is said to be reflexive if AlgLat(A) coincides with the weakly closed unital algebra generated by A. In this section we prove that any collection A of commuting isometries is reflexive; in fact any subset of the weakly closed algebra generated by A is reflexive. In the case of a single isometry the result was proved by Deddens [3]. Earlier, Sarason [10] proved that any collection of analytic Toeplitz operators, in particular the unilateral shift, is reflexive. For families of two or more isometries there were several partial results [8], [7]. Most recently, a proof for a finite number of isometries was given by McCarthy and Li. For arbitrary families of isometries, Horák and Müller [5] proved recently that AlgLat(A) is contained in a certain commutative algebra. Since our proof depends on this result, we would like to formulate it in more detail.

Fix a commuting set A of isometries on a Hilbert space $\mathcal{H}$, and denote by S the multiplicative semigroup generated by A, i.e., the set of all finite products of elements in A. Clearly, A is reflexive if and only if S is reflexive since $S \subset \mathrm{AlgLat}(A) = \mathrm{AlgLat}(S)$. The simultaneous unitary extension of the isometries in S will be considered next. We will recall briefly how this extension is obtained; this construction is somewhat different from the one given in [11] and [5]. Define a relation ρ on $S \times \mathcal{H}$ by setting $(V,h)\ \rho\ (W,k)$ if $Vk = Wh$. It is easy to see that ρ is an equivalence relation. Denote by $[V,h]$ the equivalence class of (V,h), and observe that $S \times \mathcal{H}/\rho$ becomes a pre-Hilbert space with the operations $[V,h] + [W,k] = [VW, Wh + Vk]$, $\lambda[V,h] = [V,\lambda h]$, and the norm $\|[V,h]\| = \|h\|$. Let $\mathcal{K}$ be the completion of $S \times \mathcal{H}/\rho$, and note that $\mathcal{H}$ can be embedded isometrically in $\mathcal{K}$ if we identify $h \in \mathcal{H}$ with $[V, Vh] \in \mathcal{K}$. Each isometry $W \in S$ can be extended to a unitary $\widetilde{W}$ on $\mathcal{K}$ satisfying

$$\widetilde{W}[V,h] = [V, Wh].$$

We denote $\widetilde{S} = \{\, \widetilde{W} : W \in S \,\}$. More generally, every operator T commuting with S, i.e., $T \in S'$, has a unique extension $\widetilde{T}$ on $\mathcal{K}$ commuting with $\widetilde{S}$, such that

$$\widetilde{T}[V,h] = [V, Th], \qquad V \in S,\ h \in \mathcal{H}.$$

The map $T \to \widetilde{T}$ is an isometry of S' onto the collection of those operators in $\widetilde{S}'$ which leave $\mathcal{H}$ invariant.

We have now the necessary notation to state the main results of [5].

2.1. Theorem. *Every operator $T \in \operatorname{AlgLat}(S)$ belongs to S'. Moreover, the operator $\widetilde{T}$ (which is defined since $T \in S'$) is in the double commutant $\widetilde{S}''$ of $\widetilde{S}$.*

In order to explain our approach to proving reflexivity, recall that a linear space $\mathcal{B}$ of linear operators is said to be *elementary* if for every weak operator continuous functional φ on $\mathcal{B}$ there exist $x, y \in \mathcal{H}$ such that $\varphi(T) = (Tx, y)$ for all $T \in \mathcal{B}$. The following result was proved in [4].

2.2. Lemma. *Let A be a set of operators on a Hilbert space. If $\operatorname{AlgLat}(A)$ is contained in an elementary linear space $\mathcal{B}$ then A is reflexive.*

Returning now to the setting of a semigroup S of commuting isometries, let us denote by $\mathcal{B}$ the algebra of all operators $T \in S'$ for which $\widetilde{T}$ belongs to $\widetilde{S}''$. Our main result is as follows.

2.3. Theorem.

(1) *For every weak* continuous functional φ on $\mathcal{B}$, and every $\varepsilon > 0$, there exist $x, y \in \mathcal{H}$ such that $\varphi(T) = (Tx, y)$ for all $T \in \mathcal{B}$, and $\|x\| \cdot \|y\| < \|\varphi\| + \varepsilon$.*

(2) *For every sequence $\{\varphi_n\}_{n=1}^{\infty}$ of weak* continuous functionals on $\mathcal{B}$ there exist vectors $x, y_n \in \mathcal{H}$ such that $\varphi_n(T) = (Tx, y_n)$ for all $T \in \mathcal{B}$ and all $n \geq 1$.*

An immediate consequence of the preceding results is as follows.

2.4. Theorem. *Every commuting set A of isometries is reflexive. Moreover, every subset of $\operatorname{AlgLat}(A)$ is reflexive.*

Proof. We would like to restrict ourselves to the case in which $\widetilde{S}$ has a countable *-cyclic set contained in $\mathcal{H}$, i.e., there is a countable subset $C \subset \mathcal{H}$ such that the linear span of $\{\, \widetilde{V}^* \widetilde{W} x : V, W \in S, x \in C \,\}$ is dense in $\mathcal{H}$. To show that this is possible, let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of weak* continuous functionals on $\mathcal{B}$. Each φ_n can be written as $\varphi_n(T) = \sum_{j=0}^{\infty} (Tx_{jn}, y_{jn})$ for $T \in \mathcal{B}$. Let C denote the collection of all the vectors x_{jn}, y_{jn}, and denote by $\mathcal{K}_0$ the *-cyclic space of $\widetilde{S}$ generated by C. Denote by $\mathcal{H}_0$ the intersection of $\mathcal{K}_0$ with $\mathcal{H}$. Further, set $S_0 = \{\, V|\mathcal{H}_0 : V \in S \,\}$ and $\widetilde{S}_0 = \{\, \widetilde{V}|\mathcal{K}_0 : \widetilde{V} \in \widetilde{S} \,\}$. Finally, denote by $\mathcal{B}_0$ the algebra of all operators $T_0 \in S_0'$ for which there exists $T' \in \widetilde{S}_0''$ such that $T'|\mathcal{H}_0 = T_0$. Observe that the subspace $\mathcal{K}_0$ is reducing for $\widetilde{S}$ and therefore it reduces $\widetilde{S}''$ as well. It follows that every operator $T \in \mathcal{B}$ leaves $\mathcal{H}_0$ invariant, and $T|\mathcal{H}_0 \in \mathcal{B}_0$.

With these observations, remark that one can define functionals φ_{0n} on $\mathcal{B}_0$ by setting $\varphi_{0n}(T_0) = \sum_{j=0}^{\infty} (T_0 x_{jn}, y_{jn})$ for $T_0 \in \mathcal{B}_0$. If the conclusion of Theorem 2.3 were true for the algebra $\mathcal{B}_0$, it would follow at once from the preceding remarks that the conclusion of Theorem 2.3 would hold for the original functionals φ_n.

Observe also that $\bigcup_{V\in S}\widetilde{V}^*\mathcal{H}_0$ is dense in $\mathcal{K}_0$. This shows that there is no loss of generality in assuming that $\mathcal{H}_0=\mathcal{H}$ and $\mathcal{K}_0=\mathcal{K}$ to begin with. Under this additional assumption, spectral theory implies the existence of a probability space (Ω,Σ,μ) (Ω can be taken to be $\mathbb{T}^S$), of measurable unimodular functions f_V, $V\in S$, on Ω, of measurable sets $\sigma_1=\Omega\supset\sigma_2\supset\cdots$ and of a unitary operator $U\colon\mathcal{K}\to\bigoplus_{j\geq 1}L^2(\mu|\sigma_j)$ such that $U\widetilde{V}U^*$ is the operator of multiplication by f_V for every $V\in S$. In order to simplify notation, we will assume that $\mathcal{K}=\bigoplus_{j\geq 1}L^2(\mu|\sigma_j)$ so that U is just the identity operator. Observe that $\mathcal{K}$ can be identified as a subspace of $L^2(\mu,\mathcal{F})$ (with a separable space $\mathcal{F}$). We claim that the subspace $\mathcal{H}$ has the approximate factorization property. Assume indeed that $h\in L^1(\mu)$ is a nonnegative function. Then there exists $y\in\mathcal{K}$ such that $h=y\cdot y$; indeed y can be chosen in the first component of the direct sum decomposition of $\mathcal{K}$. Clearly we have $h=\widetilde{V}y\cdot\widetilde{V}y$ for every $V\in S$. Now, there are vectors of the form $\widetilde{V}y$ which are as close as we want to $\mathcal{H}$, and therefore if we set $x=P_{\mathcal{H}}\widetilde{V}y$, then $x\cdot x$ will be as close as we want to h.

We are now ready to prove the first assertion of Theorem 2.3. Fix a weak*-continuous functional φ on the algebra $\mathcal{B}$ and a number $\varepsilon>0$. φ can be written as $\varphi(T)=\sum_{n=0}^{\infty}(Tx_n,y_n)$, $T\in\mathcal{B}$, where the vectors $x_n,y_n\in\mathcal{H}$ satisfy $\sum_{n=0}^{\infty}\|x_n\|\cdot\|y_n\|<\|\varphi\|+\varepsilon/2$. Denote $f=\sum_{n=0}^{\infty}x_n\cdot y_n$, and observe that $\|f\|_1<\|\varphi\|+\varepsilon/2$. By Corollary 1.2, we can choose vectors $x\in\mathcal{H}$ and $z\in\mathcal{K}$ such that $x\cdot z=f$ and $\|x\|\cdot\|z\|<\|f\|_1+\varepsilon/2<\|\varphi\|+\varepsilon$. If we denote $y=P_{\mathcal{H}}z$ then we have $(Tx,y)=(Tx,z)$ for every $T\in\mathcal{B}$. Since every operator T in $\mathcal{B}$ is the restriction to $\mathcal{H}$ of a multiplication operator by some function u in $L^\infty(\mu)$ we deduce that

$$\varphi(T)=\sum_{n=0}^{\infty}(ux_n,y_n)=\int_\Omega uf\,d\mu=(ux,z)=(Tx,z),$$

we conclude that $\varphi(T)=(Tx,y)$, as desired. The second part of the statement of the theorem follows in an analogous manner. □

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Mathematics Department, Indiana University, Bloomington, IN 47405
E-mail address: bercovic@indiana.edu