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TOPOLOGICAL DEFORMATION RIGIDITY OF HIGHER RANK LATTICE ACTIONS

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Abstract. We prove that the linear actions of irreducible higher rank lattices on tori or nilmanifolds are topologically deformation rigid provided that the actions do not have compact part.

1. Introduction and Statement of Results

The rigidity of lattice group actions with hyperbolic behavior was investigated by a number of authors ([H1,2], [KL1,2], [KLZ], [Q1,2]). They assume that the actions under their consideration are Anosov actions (i.e., there exists an Anosov element in the group) and this plays an important role in their works.

We are interested in the partially hyperbolic group actions (i.e., there exists a partially hyperbolic element in the groups). A relatively easy case to study is the deformation rigidity of the higher rank lattice group actions on tori or nilmanifolds. In [H1], S. Hurder proved that the Anosov actions of irreducible higher rank lattice Γ on a compact manifold with dense periodic points are topologically deformation rigid. As a corollary, the linear Anosov actions of such lattices on tori (see definition before Corollary 1.2) are always topologically deformation rigid since the periodic points are all the rational points and hence are dense.

Let G be a connected semisimple Lie group with finite center. Let $G = KAN$ be an Iwasawa decomposition, where K is a maximal compact subgroup of G, A is isomorphic to the additive group $\mathbb{R}^s$, and N is a simply connected nilpotent Lie group. Then $\mathbb{R}$-rank$(G) = s$. Let Γ be a lattice in G. We say that Γ is a *higher rank lattice* if $\mathbb{R}$-rank$(G) \geq 2$. Let Γ be an irreducible lattice in G (i.e., for every normal subgroup of positive dimension G_1 of G, the projection of Γ to G/G_1 is dense in G/G_1). In the rest this paper, we will assume that Γ is a higher rank lattice in a connected semisimple Lie group with finite center and without compact factor.

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It is well-known that a lattice in a connected Lie group is finitely generated (see [R], 6.18). Fix (once and for all) a set of generators $\lambda_1, \dots, \lambda_k$ of Γ. Let M be a compact manifold. Denote by $R(\Gamma, \mathrm{Diff}^1(M))$ the set of all homomorphisms from Γ to $\mathrm{Diff}^1(M)$ with the topology of pointwise convergence. The topology can also be described as follows (see [R], 6.2). Identify $R(\Gamma, \mathrm{Diff}^1(M))$ with a closed subset of $(\mathrm{Diff}^1(M))^k$ via $\rho \mapsto (\rho(\lambda_1), \dots, \rho(\lambda_k))$; then the topology on $R(\Gamma, \mathrm{Diff}^1(M))$ is simply the subspace topology inherited from $(\mathrm{Diff}^1(M))^k$. By an *ϵ-deformation* of ρ_0 we mean a continuous path ρ_t ($t \in [0,1]$) in the space $R(\Gamma, \mathrm{Diff}^1(M))$ such that $d_{C^1}(\rho_t(\lambda_i), \rho_0(\lambda_i)) < \epsilon$ for all $i = 1, \dots, k$ and $t \in [0,1]$. It is clear that if $\epsilon_0 > \epsilon_1$, then an ϵ_1-deformation is an ϵ_0-deformation. We say the action ρ_0 is *topologically deformation rigid* if there exists $\epsilon > 0$ such that for any ϵ-deformation ρ_t of ρ_0, there exists a continuous path $\phi_t \in \mathrm{Homeo}(M)$ such that $\rho_t(\gamma) = \phi_t^{-1}\rho_0(\gamma)\phi_t$ for all $\gamma \in \Gamma$, $t \in [0,1]$.

Recall that a continuous foliation L with C^1-leaves is a C^1-*lamination* if the tangent bundle TL is continuous (§7, p. 115 of [HPS]); a C^1-diffeomorphism $f\colon M \to M$ is r-normally hyperbolic to a C^1-lamination L ($r = 0, 1$) iff f preserves L and Tf is r-normally hyperbolic over TL (i.e., there exist a tangent bundle splitting $TM = N^u \oplus TL \oplus N^s$, a tangent map splitting $Tf = N^u f \oplus Lf \oplus N^s f$, and a riemannian metric $\langle \cdot, \cdot \rangle$, such that $\inf_{p\in M} m(N^u f) > 1$, $\sup_{p\in M} \|N^s_p f\| < 1$, $\inf_{p\in M} m(N^u f)\|L_p f\|^{-r} > 1$, $\sup_{p\in M} \|N^s_p f\| m(L_p f)^{-r} < 1$; where for a linear map A, $m(A) = \inf\{\, \|A(v)\|;\ \|v\| = 1 \,\}$. See §7, p. 116 of [HPS]). To state our result, we need the concept of "plaque expansiveness" (see §7, p. 116 of [HPS]) for a lamination. We will give the definition before Lemma 2.7. We point out a fact that will be used in Corollary 1.2: if f is a C^1 diffeomorphism of M which is 0-normally hyperbolic at the C^1 foliation C then f is plaque expansive (Theorem 7.2 of [HPS]).

Recall that a point p is a *periodic point* for an action ρ if $\rho(\Gamma)p$ is a finite set (or, equivalently, if there exists a normal subgroup Γ_p of finite index such that $\rho(\Gamma_p)p = p$, see Lemma 2.1). Now we are able to state the following result.

Theorem 1.1. *Let Γ be an irreducible higher rank lattice subgroup in a connected semisimple Lie group with finite center and without compact factor. Let ρ_0 be a C^1 action of Γ on a compact smooth manifold M. Assume that there exist finitely many elements $\gamma_1, \gamma_2, \dots, \gamma_s \in \Gamma$ such that*

(1) *for each $i = 1, \dots, s$, there exists a $\rho_0(\gamma_i)$-invariant, plaque expansive C^1-lamination $\mathcal{C}(\rho_0(\gamma_i))$ such that $\rho_0(\gamma_i)$ is 1-normally hyperbolic at $\mathcal{C}(\rho_0(\gamma_i))$ in the sense of Hirsch, Pugh and Shub;*
(2) *$\cap_{i=0}^{s} T_x\mathcal{C}(\rho_0(\gamma_i)) = \{0\}$ for all $x \in M$;*

(3) *the set of periodic points is dense in M.*

Then ρ_0 is topologically deformation rigid.

One corollary of the theorem is the topological deformation rigidity of linear actions. By a *linear action* ρ_0 of Γ on torus $\mathbb{T}^n$ or nilmanifold $\mathbb{N}$ with dimension n, we mean an action induced by a homomorphism $\pi_0 \colon \Gamma \to SL(n, \mathbb{Z})$ (see [Q1] for a discussion of linear actions on nilmanifolds). Recall a homomorphism $\pi_0 \colon \Gamma \to SL(n, \mathbb{Z})$ determines a representation of Γ on $\mathbb{R}^n$. We say that π_0 has *compact part* if there exists an invariant vector subspace $V \subset \mathbb{R}^n$, such that the spectrum of γ $\mathrm{Spect}(\pi_0(\gamma)|_V) \subset S^1$ for all $\gamma \in \Gamma$. The following result asserts that if the action is linear with hyperbolicity in every direction in the tangent bundle (i.e., every nonzero vector in the tangent bundle is stretched by some element in Γ), then the action is rigid.

Corollary 1.2. *Let Γ be as in the Theorem* 1.1. *Assume that ρ_0 is an action of Γ on a torus or a nilmanifold induced by a homomorphism $\pi_0 \colon \Gamma \to SL(n, \mathbb{Z})$ without compact part. Then ρ_0 is topologically deformation rigid.*

Proof. Let $\gamma^{(0)} \in \Gamma$ such that $\mathrm{Spect}(\pi_0(\gamma^{(0)})) \not\subset S^1$. For any $\gamma \in \Gamma$ let $E_1(\gamma) = \{\, v \in \mathbb{R}^n : v = 0 \text{ or } \lim_{k\to\pm\infty} \frac{1}{k} \ln(\|\pi_0(\gamma^k)v\|) = 0 \,\}$. Let $V_0 = \cap_{\gamma\in\Gamma} E_1(\gamma^{-1}\gamma^{(0)}\gamma)$. It is easy to show that V_0 is π_0 invariant.

Indeed, if $0 \neq v \in V_0$, $\lim_{k\to\pm\infty} \frac{1}{k} \ln(\|\pi_0(\gamma^{-1}(\gamma^{(0)})^k\gamma)v\|) = 0$. This implies that $\lim_{k\to\pm\infty} \frac{1}{k} \ln(\|\pi_0((\gamma^{(0)})^k)(\pi_0(\gamma)v)\|) = 0$. Or, $\pi_0(\gamma)v \in E_1(\gamma^{(0)})$ for all $\gamma \in \Gamma$. A similar argument shows that $\pi_0(\gamma)v \in E_1(\delta^{-1}\gamma^{(0)}\delta)$ for all $\gamma, \delta \in \Gamma$. Hence V_0 is π_0 invariant.

Since π_0 has no compact part, either $V_0 = \{0\}$ or there exists $\gamma^{(1)} \in \Gamma$ such that $\mathrm{Spect}(\pi_0(\gamma^{(1)})|_{V_0}) \not\subset S^1$. In the latter case, let $V_1 = V_0 \cap (\cap_{\gamma\in\Gamma} E_1(\gamma^{-1}\gamma^{(1)}\gamma))$. Then either $V_1 = \{0\}$ or there exists $\gamma^{(2)} \in \Gamma$ such that $\mathrm{Spect}(\pi_0(\gamma^{(1)})|_{V_1}) \not\subset S^1$. In the latter case, let $V_2 = V_1 \cap (\cap_{\gamma\in\Gamma} E_1(\gamma^{-1}\gamma^{(2)}\gamma))$. We repeat the argument and notice that V_i is a strictly decreasing sequence, we know that there exists positive integer h, such that $V_h = \{0\}$, or equivalently, $\cap_{i=1}^{h}(\cap_{\gamma\in\Gamma} E_1(\gamma^{-1}\gamma^{(i)}\gamma)) = \{0\}$. Now it is easy to see that we may find finitely many elements $\gamma_1, \ldots, \gamma_s \in \Gamma$, such that $\cap_{i=1}^{s} E_1(\gamma_i) = \{0\}$.

For each i, $E_1(\gamma_i)$ corresponds to a $\rho_0(\gamma_i)$-invariant distribution (again denoted by $E_1(\gamma_i)$) that is characterized by that $v \in E_1(\gamma_i)$ iff v has 0 Lyapunov exponent. We claim that $E_1(\gamma_i)$ is integrable. Indeed, let X, Y be two vector fields in $E_1(\gamma_i)$, then $(\rho_0(\gamma_i))_*[X, Y] = [(\rho_0(\gamma_i))_*X, (\rho_0(\gamma_i))_*Y]$. It is then not hard to see that if $[X, Y] \neq 0$, then $[X, Y]$ has 0 Lyapunov exponent. Hence $[X, Y] \in E_1(\gamma_i)$, and $E_1(\gamma_i)$ is integrable. Now the resulting smooth foliation $\mathcal{C}(\rho_0(\gamma_i))$ is easily seen to be a plaque expansive

C^1-lamination at which $\rho_0(\gamma_i)$ is 1-normally hyperbolic, (1) in Theorem 1.1 is satisfied. (2) is clearly satisfied. Notice that all the rational points in the torus (or the nilmanifold) are periodic points, so the set of periodic points is dense in the manifold. Thus (3) is also satisfied. Therefore, ρ_0 is topologically deformation rigid. □

For some special higher rank lattice groups, for instance any subgroup of finite index Γ in $SL(n,\mathbb{Z})$ $(n \geq 3)$ or $Sp(2n,\mathbb{Z})$ $(n \geq 2)$, Corollary 1.2 has the following simple form.

Corollary 1.3. *Let Γ be a subgroup of finite index in $SL(n,\mathbb{Z})$ $(n \geq 3)$ or $Sp(2n,\mathbb{Z})$ $(n \geq 2)$ and let ρ_0 be an action of Γ on a torus or a nilmanifold induced by a finite dimensional continuous representation of $SL(n,\mathbb{R})$ $(n \geq 3)$ or $Sp(2n,\mathbb{R})$ $(n \geq 2)$ that does not contain the trivial representation. Then ρ_0 is topologically deformation rigid.* □

The main idea of the proof of Theorem 1.1 is to show that if the deformation is small enough, the periodic points persist for $t \in [0,1]$, and that the correspondence between periodic points extends to a continuous conjugacy. The proof will be completed after Lemma 3.7. We point out that our proof is a refinement of the argument in [H1].

2. Analysis of periodic points

The proof of the theorem relies on the fact that periodic points for ρ_0 are dense and persist under small perturbations. In the rest of the paper, we assume that all the conditions in Theorem 1.1 are satisfied. We first give an equivalent description for the periodic points.

Lemma 2.1. *For any action ρ of Γ on M, $p \in M$ is a periodic point for ρ iff there exists a normal subgroup of finite index $\Gamma_p \subset \Gamma$ such that p is a fixed point for $\rho(\Gamma_p)$.*

Proof. Let $Orb(p)$ be the orbit of p under ρ. The action of Γ on $Orb(p)$ defines a homomorphism of Γ into the permutation group on the finite set $Orb(p)$. The kernel is therefore a normal subgroup of finite index I and fixes p. It is easy to see that $I \leq |\mathrm{Perm}(Orb(p))| = (\mathrm{Card}(Orb(p)))!$.

The other direction is clear. □

The second condition in Theorem 1.1 may be viewed as a transversality condition. It turns out that it is an open condition (see Lemma 2.4). Let $\epsilon_0 > 0$ be small and ρ_t an ϵ_0-deformation. For each element γ_j $(j = 1, \ldots, s)$ as in the theorem, we know by Theorem 7.1 of [HPS] that there exists a C^1-lamination $\mathcal{C}(\rho_t(\gamma_i))$ such that $(\rho_0(\gamma_i), \mathcal{C}(\rho_0(\gamma_i)))$ is leaf conjugate to $(\rho_t(\gamma_i), \mathcal{C}(\rho_t(\gamma_i)))$; i.e., there exists a homeomorphism $h_t^{(i)}$ mapping the leaf L in $\mathcal{C}(\rho_0(\gamma_i))$ to the leaf $L_t^{(i)}$ in $\mathcal{C}(\rho_t(\gamma_i))$, such that $h_t^{(i)}\rho_0(\gamma_i)(L) = \rho_t(\gamma_i)h_t^{(i)}(L)$. The homeomorphism $h_t^{(i)}$ obtained in [HPS] depends on the choice of a smooth normal vector bundle η_i complement to $T\mathcal{C}(\rho_0(\gamma_i))$. From now on we fix one such normal bundle, and then it is easy to see that $h_t^{(i)}$ depends continuously on t. Moreover, $h_t^{(i)} \to \mathrm{Id}_M$ uniformly; i.e., for any $K > 0$, there exists $\epsilon(K) > 0$, such that if $\epsilon < \epsilon(K)$ and ρ_t is any ϵ-deformation, then $d(h_t^{(i)}(x), x) < 1/K$ for all $x \in M$. In the rest of the paper, we assume that ρ_t is an ϵ_0-deformation. We also adopt the notation introduced above.

In the next lemma, we show that set of the non-periodic points of each of the diffeomorphisms $\rho_0(\gamma_i)$ in Theorem 1.1 is dense. The density of non-periodic points is assumed in the development of the Mather Theory that is described later.

Lemma 2.2. *Let the diffeomorphism $f\colon M \to M$ be 0-normally hyperbolic at a C^1-lamination $\mathcal{C}$ with C^1 leaves in the sense of Hirsh, Pugh and Shub. Then the set of the non-periodic points of f is dense.*

Proof. Let P_n denote the set of points $x \in M$ such that $f^n(x) = x$. Since P_n is a closed set, by the Baire Category Theorem it is enough to show that P_n has no interior points (since $\cup_n P_n$ is of the first category). Suppose the contrary and let p_0 be an interior point of P_n, and U be an open set such that $x \in U \subset P_n$. Since $f^n|_U = \mathrm{Id}_U$, we have $Df^n_{p_0} = \mathrm{Id}\colon T_{p_0} \to T_{p_0}$, contrary to the normal hyperbolicity of f. $\square$

For a C^1 diffeomorphism $f\colon M \to M$ on a compact smooth manifold with Riemannian metric d given by inner product $\langle\cdot,\cdot\rangle$, we may define an operator f_* on the space $\mathrm{Vec}^0(TM)$ of C^0 vector fields by the formula $f_*v(x) = Dfv(f^{-1}(x))$. According to Mather ([Mat], [P]), the operator L obtained by complexification of f_* possesses a spectrum consisting of all the points between full circles, provided that the non-periodic points of f are dense in M. Each of the connected components is thus a $[\lambda_i, \mu_i]$-ring (λ_i, μ_i are the inner radius, outer radius respectively, of the ring), and corresponding to each $[\lambda_i, \mu_i]$-ring there is a continuous subbundle E_i in TM such that for each $\delta > 0$, $q_i(\lambda_i - \delta)^n\|v_i\| \le \|Df^n v_i\| \le q_i'(\mu_i + \delta)^n\|v_i\|$ for some $q_i, q_i' > 0$, all $v_i \in E_i(x)$, $x \in M$, $n > 0$ ([P], [BP]). It is easy to see that if f is 0-normally hyperbolic at a continuous foliation $\mathcal{C}$ with C^1 leaves then it is partially hyperbolic in the sense that the spectrum consists of at least two nontrivial rings. It is also easy to see that the tangent bundle $T\mathcal{C}$ is the union of some $E_i's$. In the case that f is 1-normally hyperbolic at $\mathcal{C}$ in the sense of Hirsch, Pugh and Shub, the foliation $\mathcal{C}$ persists (let $\mathcal{C}'$ denote the new foliation for the perturbation f'), and the distance between the tangent bundles $d(T\mathcal{C}, T\mathcal{C}') \to 0$ if $f' \to f$ (Theorem 1, [P]).

We summarize the above discussion together with Theorem 7.1 of [HPS] as following.

Lemma 2.3. *Let f be a diffeomorphism 1-normally hyperbolic at $\mathcal{C}$ in the sense of Hirsch, Pugh and Shub, where $\mathcal{C}$ is an f-invariant, plaque expansive C^1-lamination. Then for any C^1 nearby diffeomorphism f' there exists an f'-invariant, plaque expansive C^1-lamination $\mathcal{C}(f')$, such that f' is 1-normally hyperbolic at $\mathcal{C}(f')$ and $T\mathcal{C}(f') \to T\mathcal{C}$ when f' C^1 approaches to f.* $\square$

We remark that Theorem 6.8 of [HPS] also gives that $T\mathcal{C}(f') \to T\mathcal{C}$ when f' C^1 approaches to f. But we feel that Lemma 2.2 (that is needed to apply the Mather Theory) has its own interest although it is simple.

Lemma 2.4. *There exists $\epsilon_1 > 0$ $(\epsilon_1 < \epsilon_0)$, such that for any ϵ_1-deformation ρ_t, $\cap_{i=0}^s T\mathcal{C}(\rho_t(\gamma_i)) = 0$.*

Proof. Otherwise, there exist a sequence $0 < \epsilon_n \to 0$, for each n an ϵ_n-deformation ρ_t^n, $t_n \in [0,1]$, $p_n \in M$ and $v_n \in \cap_{i=0}^k T_{p_n}\mathcal{C}(\rho_{t_n}^n(\gamma_i))$ with $\|v_n\| = 1$. Without loss of generality, we assume that $p_n \to p_0$ and $v_n \to v_0 \in T_{p_0}M$. Now we apply Lemma 2.3 to obtain that $v_0 \in T_{p_0}\mathcal{C}(\rho_0(\gamma_i))$ for all $i = 1, 2, \ldots, s$. This violates condition (2) in Theorem 1.1. $\square$

Lemma 2.5. *There exists $\epsilon_1' > 0$ $(\epsilon_1' < \epsilon_0)$, such that for any ϵ_1'-deformation ρ_t and any normal subgroup of finite index $\Gamma_* \subset \Gamma$, $\rho_t(\Gamma_*)$ has finitely many fixed points.*

Proof. Suppose that the statement is not true. Then there exist a sequence $0 < \epsilon_n \to 0$, for each n an ϵ_n-deformation ρ_t^n, $t_n \in [0,1]$ and a normal subgroup $\Gamma_n \subset \Gamma$ of finite index, such that $\rho_{t_n}^n(\Gamma_n)$ has infinitely many fixed points. Fix a positive integer n and let $n_i > 0$ be positive integers such that $\gamma_i^{n_i} \in \Gamma_n$, $i = 1, \ldots, s$. Choose a sequence $\{ p_k : k = 1, 2, \ldots \}$ of fixed points for $\rho_{t_n}^n(\Gamma_n)$, hence common fixed points for $f_j \overset{def}{=} \rho_{t_n}^n(\gamma_j^{n_j})$. Without loss of generality we assume that $p_k \to p_0 \overset{def}{=} p_0(n)$ and it is clear that p_0 is also a common fixed point for f_j. Let $m = \dim(M)$. With the help of a coordinate chart, we may assume further that $f_j \colon \mathbb{R}^m \to \mathbb{R}^m$, $p_0 = 0$. Near the origin, $f_j(x) = A_j x + o(\|x\|)$, where A_j is an $m \times m$-matrix. Since $f_j(p_k) = p_k$, we have $A_j p_k + o(\|p_k\|) = p_k$ and hence $A_j \frac{p_k}{\|p_k\|} + \frac{o(\|p_k\|)}{\|p_k\|} = \frac{p_k}{\|p_k\|}$. Without loss of generality we may assume that $\frac{p_k}{\|p_k\|} \to v_n$, and hence we have $A_j v_n = v_n$. In other words, we obtain a vector $v_n \in T_{p_0(n)}M$ fixed by all Tf_j. Therefore v_n is a unit vector in the center distribution for every f_j and hence in the center distribution for every $\rho_0^n(\gamma_j)$, $j = 1, 2, \ldots, s$, contrary to Lemma 2.4. $\square$

We introduce the following notation. We denote by $\mathcal{C}(p, \rho_t(\gamma_i))$ the leaf of $\mathcal{C}(\rho_t(\gamma_i))$ passing through p, by $\mathcal{C}(p, \delta, \rho_t(\gamma_i))$ the closed δ-ball in $\mathcal{C}(p, \rho_t(\gamma_i))$ centered at p (using the submanifold metric $d_{i,t}$).

Lemma 2.6. *There exist an $\epsilon_2 > 0$ ($\epsilon_2 < \min\{\epsilon_1, \epsilon_1'\}$, ϵ_1, ϵ_1' as in Lemmas* 2.4, 2.5)*, and a $\delta_0 > 0$, such that for any ϵ_2-deformation ρ_t, any $p \in M$, $\cap_{i=1}^s \mathcal{C}(p, \delta_0, \rho_t(\gamma_i)) = \{p\}$.*

Proof. Otherwise there exist a sequence $\delta_n \to 0$, a sequence of $1/n$-deformations ρ_t^n, $p_n, q_n \in M$, $t_n \in [0,1]$, such that $p_n \neq q_n$, and $q_n \in \cap_{i=1}^s \mathcal{C}(p_n, \delta_n, \rho_{t_n}^n(\gamma_i))$. Without loss of generality, we assume that $p_n \to p_0$, $q_n \to q_0$. Since $d_{i,t}(p_n, q_n) \le \delta_n$, $p_0 = q_0$. Let $m = \dim(M)$. With the help of a coordinate chart at p_0, we may assume also that a neighborhood of p_0 is an open set in $\mathbb{R}^m$ and $p_0 = 0$. Since $T\mathcal{C}(\rho_{t_n}^n(\gamma_i))$ is a continuous bundle, converges uniformly to $T\mathcal{C}(\rho_0(\gamma_i))$ (Lemma 2.3), we may assume that (locally) $\mathcal{C}(\rho_{t_n}^n(\gamma_i))$ is the graphs of the 3-parameter family of maps $z = f(x, y, t, n)\colon \mathbb{R}^{m_1} \to \mathbb{R}^m$, $x \to f(x, y, t, n)$, $y \in \mathbb{R}^{m_2}$, $t \in [0,1]$, $m_1 + m_2 = m$ and $\mathcal{C}(\rho_0(\gamma_i))$ is the graphs of the 1-parameter family of maps $z = f_0(x, y)$, where m_1 is the dimension of foliation $\mathcal{C}(\rho_{t_n}^n(\gamma_i))$. The continuity of the tangent bundles of the foliations and the continuous dependence on the parameters permit us to assume that $D_x f(x, y, t, n)$ is continuous in (x, y, t), and when $n \to \infty$, $D_x f(x, y, t, n) \to D_x f_0(x, y)$ uniformly in t. Let $p_n = f(x_n, y_n, t_n, n)$, $q_n = f(x_n', y_n', t_n, n)$. Since p_n, q_n are in the same (local) leaf, we have $y_n = y_n'$. Observe that $q_n - p_n = f(x_n, y_n, t_n, n) - f(x_n', y_n, t_n, n) = \int_0^1 \frac{d}{ds} f(x_n' + s(x_n - x_n'), y_n, t_n, n) ds = (\int_0^1 D_x f(x_n' + s(x_n - x_n'), y_n, t_n, n) ds)(x_n' - x_n)$, we obtain $\frac{q_n - p_n}{\|q_n - p_n\|} = (\int_0^1 D_x f(x_n' + s(x_n - x_n'), y_n, t_n, n) ds) \frac{(x_n' - x_n)}{\|q_n - p_n\|}$. Without loss of generality, we assume that $\frac{q_n - p_n}{\|q_n - p_n\|} \to v$, $\frac{(x_n' - x_n)}{\|q_n - p_n\|} \to w$, $x_n, x_n' \to x_0$, $y_n \to y_0$. Now let $n \to \infty$, we have $0 \neq v = (\int_0^1 D_x f_0(x_0, y_0) ds) w$. In other words, $0 \neq v \in T\mathcal{C}(p_0, \rho_0(\gamma_i))$ for all $i = 1, \dots, s$, contrary to Lemma 2.4. □

Now we give the definition of "plaque expansiveness" and other related concepts following [HPS]. We say that P is a C^1-*plaque* in an n_1-dimensional immersed submanifold $X \subset M$ if P is the image of a C^1 embedding τ from a unit ball $B^{n_1} \subset \mathbb{R}^{n_1}$ to X. We say that a family of such pairs $\mathcal{P} = \{(P_\alpha, \tau_\alpha)\}_{\alpha \in A}$ *plaquates* an n_1-dimensional C^1-lamination $\mathcal{C}$ if P_α is a plaque in a leaf C_α of $\mathcal{C}$ for each α, $M = \cup_\alpha \tau_\alpha(Int(B^{n_1}))$ and $\{\tau_\alpha\}_{\alpha \in A}$ is precompact in $\mathrm{Emb}^1(B^{n_1}, M)$. For each C^1-lamination $\mathcal{C}$ of M, there exists a family $\mathcal{P} = \{(P_\alpha, \tau_\alpha)\}_{\alpha \in A}$ that plaquates $\mathcal{C}$, and the diameters of the plaques can be chosen to be arbitrarily small (Theorem 6.2 of [HPS] and its proof). By a β-*pseudo orbit* of $f\colon M \to M$ we mean a bi-infinite sequence $\{p_i\}$ such that $d(f(p_n), p_{n+1}) \leq \beta$ for all $n \in \mathbb{Z}$. If $f\colon M \to M$ preserves $\mathcal{C}$, then we say that a pseudo orbit $\{p_n\}$ *respects* $\mathcal{P}$ if $f(p_n)$, p_{n+1} lie in a common plaque of $\mathcal{P}$. We say that f is *plaque expansive* if there exists a $\beta > 0$ with the following property: there exists a family $\mathcal{P}$ of plaques, such that if $\{p_n\}$, $\{q_n\}$ are β-pseudo orbits which respect $\mathcal{P}$ and if $d(p_n, q_n) \leq \beta$ for all $n \in \mathbb{Z}$ then for each n, p_n and q_n lie in a common plaque.

The definition of "plaque expansiveness" is independent of d and $\mathcal{P}$ with small plaques (Remark 1, p.116 of [HPS]). From now on, we fix β as above, choose a plaquation $\mathcal{P}$, such that for each plaque $P \in \mathcal{P}$ and each $i \in \{1, \dots, s\}$, there exists p_i such that $P \subset \mathcal{C}(p_i, \delta_0/2, \rho_0(\gamma_i))$.

Lemma 2.7. *Let ϵ_1 be as in Lemma 2.4, δ be as in Lemma 2.6, β be as above. There exits a β_0 with the following properties.*

Let $i_0 \in \{1, \dots, s\}$ and p_0 be a periodic point of $\rho_0(\gamma_{i_0})$ with orbit $\{p^{(0)} = p_0, p^{(1)}, \dots, p^{(r-1)}\}$ with $\rho_0(\gamma_{i_0})(p^{(i)}) = p^{(i+1)} \mod (r)$. Assume that ρ_t is an ϵ_1-deformation and $p_t^{(i)}$ is a continuous path of periodic points for $\rho_t(\gamma_{i_0})$ such that $p_0^{(i)} = p^{(i)}$ and $\rho_t(\gamma_{i_0})p_t^{(i)} = p_t^{(i+1)} \mod (r)$ $(i = 0, \dots, r-1)$. If $d(p_0^{(i)}, p_t^{(i)}) \leq \beta_0$ for all $i = 0, \dots, r-1$, then $p_t^{(i)} \in h_t^{(i_0)}\mathcal{C}(p_0^{(i)}, \delta_0/2, \rho_0(\gamma_{i_0}))$ for $i = 0, \dots, r-1$. Therefore in this case, we have $h_t^{(i_0)}\mathcal{C}(p_0^{(i)}, \rho_0(\gamma_{i_0})) \subset \mathcal{C}(p_t^{(i)}, \rho_t(\gamma_{i_0}))$.

Proof. This follows from the construction of the leaf-conjugacy in [HPS] (see Theorem 6.8 and the comment immediately before Theorem 7.1 of [HPS]). For instance, the $\rho_t(\gamma_{i_0})$-orbit of $p_t = p_t^{(0)}$ is β-shadowed by the $\rho_0(\gamma_{i_0})$-orbit of p_0, so the leaf conjugacy is forced to carry a local leaf of $\mathcal{C}(\rho_0(\gamma_{i_0}))$ containing p_0 to a local leaf of $\mathcal{C}(\rho_t(\gamma_{i_0}))$ containing p_t provided that $p_t^{(0)}$ is in the image of $\exp\colon (\eta_{i_0})_{p'}(\beta) \to M$ for some p' (p' and p_0 in the same local leaf), where $(\eta_{i_0})_{p'}(\beta)$ is the β-ball of $(\eta_{i_0})_{p'}$ centered at 0. □

Lemma 2.8. *Let δ_0 be as in Lemma 2.6, β_0 be as in Lemma 2.7. For any $a > 0$, there exists $\epsilon_3 > 0$ ($\epsilon_3 < \epsilon_2$, ϵ_2 as in Lemma 2.6) such that if ρ_t is an ϵ_3-deformation and $h_t^{(i)}\mathcal{C}(x, \delta_0/2, \rho_0(\gamma_i)) \subset \mathcal{C}(p, \delta_0, \rho_t(\gamma_i))$ for all $i = 1, \dots, s$, then $d(x, p) < a$. In particular, we may choose ϵ_3 to be such that $d(x, p) < \delta_0/8$, $\beta_0/2$.*

Proof. Otherwise, there exist $a_0 > 0$, $1/n$-deformation ρ_t^n, $t_n \in [0, 1]$, and $p_n, x_n \in M$, such that $h_{t_n}^{(i)}\mathcal{C}(x_n, \delta_0/2, \rho_0(\gamma_i)) \subset \mathcal{C}(p_n, \delta_0, \rho_{t_n}^n(\gamma_i))$ for all $i = 1, \dots, s$, and $d(p_n, x_n) \geq a_0$. Without loss of generality, we assume that $p_n \to p$ and $x_n \to x$. Since $\rho_t^n \to \rho_0$ and $h_{t_n}^{(i)} \to \mathrm{Id}_M$ when $n \to \infty$, we have $\mathcal{C}(p_n, \delta_0, \rho_{t_n}^n(\gamma_i)) \to \mathcal{C}(p, \delta_0, \rho_0(\gamma_i))$ and $\mathcal{C}(x, \delta_0/2, \rho_0(\gamma_i)) \subset \mathcal{C}(p, \delta_0, \rho_0(\gamma_i))$

for all $i = 1, \dots, s$. Since $p \neq x$, we have that the set $\cap_i \mathcal{C}(p, \delta_0, \rho_0(\gamma_i))$ has more than 2 points, contrary to Lemma 2.6. The last statement in the lemma is clear. □

Lemma 2.9. *For any $a > 0$, there exists $\epsilon_4 > 0$ ($\epsilon_4 < \epsilon_3$, ϵ_3 as in Lemma 2.7), such that if ρ_t is an ϵ_4-deformation and p_t, $t \in [0, c]$ is a continuous path of periodic points for ρ_t, then $d(p_t, p_0) < a$. In particular, we may choose ϵ_4 such that $d(p_t, p_0) < \beta_0$ (β_0 as in Lemma 2.7).*

Proof. To avoid unnecessary complication, we assume that p_0 is a fixed point. We take ϵ_4 to be less than ϵ_3 and also be small enough that $d(h_t^{(i)}(x), x) < \delta_0/8$ for all $i = 1, \dots, s$, $t \in [0, 1]$ and $x \in M$. By Lemma 2.7, it is easy to see that there exists $0 < t_0 \leq 1$ such that for all $t \in [0, t_0]$, $h_t^{(i)} \mathcal{C}(p_0, \delta_0/2, \rho_0(\gamma_i)) \subset \mathcal{C}(p_t, \delta_0, \rho_t(\gamma_i))$. Let $T = \sup\{\, t_0 : h_t^{(i)} \mathcal{C}(p_0, \delta_0/2, \rho_0(\gamma_i)) \subset \mathcal{C}(p_t, \delta_0, \rho_t(\gamma_i)),\ t \in [0, t_0],\ i = 1, \dots, s \,\}$. By Lemma 2.8, $d(p_0, p_t) < \beta_0/2, \delta_0/8$ for all $t \in [0, T)$, and hence $d(p_0, p_T) \leq \beta_0/2, \delta_0/8$. By Lemma 2.7, $p_T \in h_T^{(i)} \mathcal{C}(p_0, \delta_0/2, \rho_0(\gamma_i))$ for $i = 1, \dots, s$. Observe that

$$\begin{aligned} d(h_T^{(i)}(x), p_T) &\leq d(h_T^{(i)}(x), x) + d(x, p_0) + d(p_0, p_T) \\ &\leq d(h_T^{(i)}(x), x) + d_{i,0}(x, p_0) + d(p_0, p_T) \\ &\leq \delta_0/8 + \delta_0/2 + \delta_0/8 = 3\delta_0/4 < \delta_0, \end{aligned}$$

so $h_T^{(i)} \mathcal{C}(p_0, \delta_0/2, \rho_0(\gamma_i))$ is in the interior of $\mathcal{C}(p_T, \delta_0, \rho_T(\gamma_i))$. We claim that $T = c$. Otherwise, there exists $b > 0$ such that $d(p_t, p_0) < \beta_0$ (since $d(p_T, p_0) \leq \beta_0/2$) for $t \in [0, T + b]$. Hence $p_t \in h_t^{(i)} \mathcal{C}(p_0, \delta_0/2, \rho_0(\gamma_i))$ for $t \in [0, T + b]$ and $i = 1, \dots, s$ by Lemma 2.7. Notice that $d(p_t, p_0) < \delta_0/8$ by Lemma 2.8, we obtain that $d(h_t^{(i)}(x), p_t) \leq d(h_t^{(i)}(x), x) + d(x, p_0) + d(p_0, p_t) \leq d(h_t^{(i)}(x), x) + d_{i,0}(x, p_0) + d(p_0, p_t) \leq \delta_0/8 + \delta_0/2 + \delta_0/8 = 3\delta_0/4 < \delta_0$ for all $t \in [0, T + b]$, so $h_t^{(i)} \mathcal{C}(p_0, \delta_0/2, \rho_0(\gamma_i))$ is in the interior of $\mathcal{C}(p_t, \delta_0, \rho_t(\gamma_i))$ for all $t \in [0, T + b]$. Contrary to the maximality of T. Hence $d(p_t, p_0) \leq a, \beta_0/2$ by Lemma 2.8. Last statement is obvious. □

We remark that this lemma has the following corollary. That is, when $\epsilon \to 0$, any continuous path p_t of any ϵ-deformation ρ_t shrinks to a point p_0.

Lemma 2.10. *There exists $\epsilon_5 > 0$ ($\epsilon_5 < \epsilon_4$, ϵ_4 as in Lemma 2.9), such that if ρ_t is an ϵ_5-deformation and p_t, $t \in [0, c]$ is a continuous path of periodic points for ρ_t, then $p_t = \cap_i h_t^{(i)} \mathcal{C}(p_0, \delta_0/2, \rho_0(\gamma_i))$ (δ_0 as in Lemma 2.6).*

Therefore, such continuous path of periodic points for ρ_t starting from p_0 is unique.

Proof. It suffices to show that there exists ϵ_5, such that for any ϵ_5-deformation ρ_t and a continuous path of periodic points p_t for ρ_t ($t \in [0, c]$), $h_t^{(i)}\mathcal{C}(p_0, \delta_0/2, \rho_0(\gamma_i)) \subset \mathcal{C}(p_t, \delta_0, \rho_t(\gamma_i))$ and $p_t \in h_t^{(i)}\mathcal{C}(p_0, \delta_0/2, \rho_0(\gamma_i))$ for all $i = 1, \dots, s$.

The fact that $p_t \in h_t^{(i)}\mathcal{C}(p_0, \delta_0/2, \rho_0(\gamma_i))$ follows from Lemmas 2.7, 2.9.

To show that there exists ϵ_5 such that

$$h_t^{(i)}\mathcal{C}(p_0, \delta_0/2, \rho_0(\gamma_i)) \subset \mathcal{C}(p_t, \delta_0, \rho_t(\gamma_i)),$$

we assume otherwise. Then there exist a $\frac{1}{n}$-deformation ρ_t^n, $i_n \in \{1, \dots, s\}$, a continuous path of periodic points p_t^n for ρ_t^n ($t \in [0, c_n]$), a sequence $t_n \in [0, c_n]$, and $q_n \in \mathcal{C}(p_0^n, \delta_0/2, \rho_0(\gamma_{i_n}))$, such that

$$h_{t_n}^{(i)}(q_n) \notin \mathcal{C}(p_{t_n}^n, \delta_0, \rho_{t_n}^n(\gamma_{i_n})).$$

Without loss of generality, we assume that $i_n = 1$, $p_{t_n}^n \to p, p_0^n \to p', q_n \to q_0$. Lemma 2.9 asserts that $p = p'$. Then we obtain after taking limit $q_0 \notin \mathcal{C}(p, \delta_0, \rho_0(\gamma_1))$. At the same time we also have $q_n \in \mathcal{C}(p_0^n, \delta_0/2, \rho_0(\gamma_1))$. After taking the limit we obtain $q_0 \in \mathcal{C}(p', \delta_0/2, \rho_0(\gamma_1)) = \mathcal{C}(p, \delta_0/2, \rho_0(\gamma_1))$, a contradiction. □

In the next lemma, we prove that there exists a sequence of normal subgroups $\Gamma_j \subset \Gamma$ of finite index, such that $\rho_0(\Gamma_j)$ has finitely many fixed points, $\Gamma_j \subset \Gamma_{j-1}$, and each periodic point of Γ is a fixed point for Γ_{j_0} for some j_0 (and hence for all $j \geq j_0$).

Lemma 2.11. *Let P_j ($1 \leq j \in \mathbb{Z}$) be the set of periodic points with period j, i.e., $\#(\rho_0(\Gamma)(p)) = j$ iff $p \in P_j$. Let Λ_j be the union of $P_1, \dots, P_j$. Let $\Gamma_j = \{\, \gamma \in \Gamma : \rho_0(\gamma)x = x$ for all $x \in \Lambda_j \,\}$. Then Γ_j is a normal subgroup of Γ of finite index ($j = 1, 2, \dots$). Each periodic point of Γ is a fixed point for Γ_{j_0} for some j_0 (and hence for all $j \geq j_0$).*

Proof. We first show that P_j is finite for all $j = 1, 2, \dots$. Otherwise, there exists a sequence $q_n \in P_j$ such that $q_j \neq q_k$ for $j \neq k$ and $q_n \to q_0$ for a $q_0 \in M$. Since for any $p \in P_j$, and any $\gamma \in \Gamma$, the orbit of p under γ $\mathrm{Orb}(p, \gamma) = \{\, \rho_0(\gamma^l)p : l \in \mathbb{Z} \,\}$ is a finite set with $h \leq j$ elements, there exists $r \leq h$ such that $\rho_0(\gamma^r)p = p$. Therefore, $\rho_0(\gamma^{j!})p = p$ for any $p \in P_j$, $\gamma \in \Gamma$. Let f_i denote $\rho_0(\gamma_i^{j!})$; we have $f_i(p_n) = p_n$ for all $n \in \mathbb{Z}$. Let $n \to \infty$; we obtain $f_i(p_0) = p_0$. Without loss of generality, we assume that an open set containing q_0 is an open set in $\mathbb{R}^n$ and $q_0 = 0$.

Observe that $f_i(x) = Ax + o(\|x\|)$, we have $q_n = Aq_n + o(\|q_n\|)$, and hence $\frac{q_n}{\|q_n\|} = A\frac{q_n}{\|q_n\|} + o(1)$. Let a subsequence of $\frac{q_n}{\|q_n\|}$ converge to v. Then it is easy to see that v is in the center distribution $T\mathcal{C}(\rho_o(\gamma_i))$, contrary to the condition (2) of Theorem 1.1.

Now we established that the set Λ_j is a finite set, and clearly it is $\rho_0(\Gamma)$-invariant. Notice that the action of Γ on Λ_j defines a representation $\Gamma \to \mathrm{Perm}(\Lambda_j)$ into the permutation group on the set Λ_j; we denote the kernel of it by Γ_j. So Γ_j is normal and has finite index less than $|\mathrm{Perm}(\Lambda_j)| = (\mathrm{Card}(\Lambda_j))!$.

It is clear that every periodic point is in Λ_{j_0} for some integer $j_0 > 0$. Therefore it is a fixed point for Γ_{j_0} (and hence for all $j \geq j_0$). $\square$

3. Construction of the conjugacy

To construct the conjugacy, we first prove the persistence of the periodic points. Then we construct a conjugacy between periodic points. Finally we will show that it extends to a homeomorphism and thus gives a topological conjugacy. We need the results by D. Stowe (Theorem A, [S]) and by G. A. Margulis (a special case of Theorem 3′(iii), Introduction [Mar]).

Proposition 3.1 (Stowe). *Let Γ_0 be a finitely-generated discrete group acting C^1 on a smooth manifolds M and let p be a fixed point of the action. Assume that the group cohomology $H^1(\Gamma_0, T_pM)$ of Γ_0 with coefficients in the isotropy linear representation on T_pM vanishes. Then p is stable under perturbations of the action.*

Proposition 3.2 (Margulis). *Let $\Gamma_0 \subset G$ be an irreducible lattice in a connected semisimple Lie group with finite center, and no non-trivial compact factor groups. Then $H^1(\Gamma_0, \pi)$ vanishes for every representation π of the group Γ_0 on a finitely-dimensional real vector space.*

Since for each $j \geq 1$, Γ_j as in Lemma 2.11 is a subgroup of finite index of irreducible lattice Γ, Γ_j is itself a irreducible lattice in G. The above results assert that for any fixed point p_0 of $\rho_0(\Gamma_j)$, it persists under small perturbation ρ. I.e., for a small perturbation ρ, there exists a fixed point p in a neighborhood of p_0 and p depends continuously on the perturbation. For a deformation, we actually have more. From now on, we assume that ρ_t is an ϵ_5-deformation (ϵ_5 as in Lemma 2.10).

Lemma 3.3. *Fix a positive integer j. Let p be a fixed point for $\rho_0(\Gamma_j)$. Then there exists a continuous path p_t of fixed points for $\rho_t(\Gamma_j)$, $t \in [0, 1]$, with $p_0 = p$.*

Proof. Using the remark above we conclude that there exists $t_0 > 0$ such that ρ_t has a continuous path of fixed points p_t, $t \in [0, t_0)$, $p_0 = p$.

Let $T = \sup\{ t_1 :$ there exists a continuous path p_t with $t \in [0, t_1)$, $p_0 = p \}$. Then $p_t = \cap_i h_t^{(i)} \mathcal{C}(p_0, \delta_0/2, \rho_0(\gamma_i))$ by Lemma 2.10. It is clear that $\lim_{t \to T} p_t$ exists and is a fixed point for $\rho_T(\Gamma_i)$ (denoted by p_T).

We claim that $T = 1$. Otherwise, by the remark above for the fixed point p_T for $\rho_T(\Gamma_j)$, we may extend further the continuous path of fixed points p_t and violate the maximality of T. □

Now we construct a conjugacy between the set Λ_0 of periodic points of $\rho_0(\Gamma)$ and the set Λ_t of periodic points of $\rho_t(\Gamma)$. For any $p \in \Lambda_0$, p is a fixed point for some Γ_j by Lemma 2.11. We define a map $h_t \colon \Lambda_0 \to \Lambda_t$ by $p \mapsto p_t$. We need to verify that this map is well defined, that it can be extended to a continuous map $h_t \colon M \to M$, one-to-one, onto and that it is actually a conjugacy.

Lemma 3.4. *The map $p \mapsto p_t$ is well-defined.*

Proof. If $\Gamma_j \subset \Gamma_k$ and there exist two continuous paths p_t, p'_t such that p_t is a continuous path of fixed points p_t for $\rho_t(\Gamma_j)$ and p'_t is a continuous path of fixed points for $\rho_t(\Gamma_k)$ with $p_0 = p'_0 = p$, by the uniqueness of such continuous paths (Lemma 2.10), $p_t = p'_t$. Therefore the map is well-defined. □

Lemma 3.5. *The map $p \mapsto p_t$ can be extended to a one-to-one continuous map $h_t \colon M \to M$.*

Proof. Let $x \in M$ and p_n be a sequence of periodic points in M converging to x. Then $\mathcal{C}(p_n, \delta_0/2, \rho_0(\gamma_i)) \to \mathcal{C}(x, \delta_0/2, \rho_0(\gamma_i))$. Therefore for any limit point z of the sequence $\{(p_n)_t\}_1^\infty = \{\cap_i h_t^{(i)} \mathcal{C}(p_n, \delta_0/2, \rho_0(\gamma_i))\}_1^\infty$, $z \in h_t^{(i)} \mathcal{C}(x, \delta_0/2, \rho_0(\gamma_i))$ and hence $z \in \cap_i h_t^{(i)} \mathcal{C}(x, \delta_0/2, \rho_0(\gamma_i))$. Therefore the set of limit point(s) of the sequence $\{\cap_i h_t^{(i)} \mathcal{C}(p_n, \delta_0/2, \rho_0(\gamma_i))\}$ is non-empty and contained in $\cap_i h_t^{(i)} \mathcal{C}(x, \delta_0/2, \rho_0(\gamma_i))$. Since $h_t^{(i)} \mathcal{C}(p_n, \delta_0/2, \rho_0(\gamma_i)) \subset \mathcal{C}((p_n)_t, \delta_0, \rho_t(\gamma_i))$ (see the proof of Lemma 2.10), we have

$$h_t^{(i)} \mathcal{C}(x, \delta_0/2, \rho_0(\gamma_i)) \subset \mathcal{C}(z, \delta_0, \rho_t(\gamma_i)).$$

Therefore, $\cap_i h_t^{(i)} \mathcal{C}(x, \delta_0/2, \rho_0(\gamma_i))$ is a set of one point z (by Lemma 2.6). We define $h_t(x) = \cap_i h_t^{(i)} \mathcal{C}(x, \delta_0/2, \rho_0(\gamma_i))$. Our two assertions (one-to-one and continuous) are then clear. □

Lemma 3.6. *h_t is onto, and hence a homeomorphism.*

Proof. We will apply the degree theory for continuous maps. Without loss of generality we assume that M is an oriented manifold. Observe that for fixed t, h_t is homotopic to $h_0 = \mathrm{Id}$, therefore $\deg(h_t) = \deg(\mathrm{Id}) = 1$. On the other hand, if h_t is not onto, there exists an open set that is contained in the complement of the image of h_t (since the image of h_t is compact). So, we may C^0-approximate h_t by a smooth map h'_t such that h'_t is not onto either and $\deg(h_t) = \deg(h'_t)$. Choose a point x that is not in the image of h'_t. We know that the point x is a regular point of h'_t and hence $1 = \deg(h_t) = \deg(h'_t) = \deg_x(h'_t) = \sum_{y \in h_t'^{-1}x} \epsilon_y = 0$ where $\epsilon_y = \pm 1$ according to whether $D_y h'_t$ preserves or reverses orientation. (See Ch. 8 of [HK] for the theory involved.) This contradiction implies that the map h_t is onto. □

Lemma 3.7. *The continuous map h_t is a conjugacy between $\rho_0(\gamma)$ and $\rho_t(\gamma)$.*

Proof. For any periodic point $p \in M$ there exists an integer j such that $\rho_0(\Gamma_j)p = p$ (Lemma 2.11). For any $\gamma \in \Gamma$, the continuous path $\rho_t(\gamma)h_t(p)$ is a path of fixed points for $\rho_t(\Gamma_j)$ starting from $\rho_0(\gamma)p$. (Indeed, for any $\gamma' \in \Gamma_j$, $\gamma^{-1}\gamma'\gamma \in \Gamma_j$ and so $\rho_t(\gamma^{-1}\gamma'\gamma)h_t(p) = h_t(p)$ because $h_t(p)$ is a path of fixed points for $\rho_t(\Gamma_j)$, or equivalently, $\rho_t(\gamma')\rho_t(\gamma)h_t(p) = \rho_t(\gamma)h_t(p)$.) But $\rho_0(\gamma)p$ is a fixed point for $\rho_0(\Gamma_j)$, hence that $h_t\rho_0(\gamma)(p)$ is a continuous path of the fixed points for $\rho_t(\Gamma_j)$. By the uniqueness of such path (Lemma 2.10), we obtain $\rho_t(\gamma)h_t(p) = h_t\rho_0(\gamma)(p)$.

Since h_t is the conjugacy in a dense set, h_t is also the conjugacy between $\rho_0(\gamma)$ and $\rho_t(\gamma)$. □

Theorem 1.1 is proved.

4. Some remarks

We remark that the only place we need Γ to be a higher rank lattice is in Proposition 3.2. Hence we may replace "higher rank lattice" in Theorem 1.1 by "group with vanishing cohomology in any finite dimensional representation on $\mathbb{R}^n$," and Theorem 1.1 is still true.

We feel that any higher rank linear action on torus (or nilmanifold) is topologically deformation rigid. We will investigate this problem in the future.

The regularity of the conjugacy is another interesting problem worth investigating. Hurder [H1] obtained the smoothness of the conjugacy for a special class of irreducible higher rank lattice Anosov actions (Cartan actions), Katok and Lewis [KL1] proved that for a special class of $\mathbb{Z}^n$ Anosov actions, any topological conjugacy is actually smooth. Their arguments do not apply to our situation (at least not directly).

The proof of local rigidity using the argument of the persistence of the periodic points seems to be impossible because of the difficulty in the proof of the persistence of periodic points as observed by other authors [KL1]. Some progress has been made for the local rigidity and even global rigidity for Anosov actions (see [KL1,2], [KLZ], [Q1,2]).

In addition to the works we mentioned, we also want to point out that A. Katok and R. Spatzier [KS] proved the rigidity of a class of Anosov abelian group actions; R. Feres [F] proved the rigidity of lattice group actions that have certain hyperbolic behavior and preserve some geometric structures. All evidence indicates the rigidity of lattice group actions and other large group actions with hyperbolic behavior.

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