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THE CENTER OF A QUANTUM AFFINE ALGEBRA AT THE CRITICAL LEVEL

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Dedicated to the memory of Ansgar Schnizer, our friend and colleague who died tragically in Japan.

Abstract. We construct central elements in a completion of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ at the critical level $c = -g$ from the universal R-matrix (g being the dual Coxeter number of the simple Lie algebra $\mathfrak{g}$), using the method of Reshetikhin and Semenov-Tian-Shansky [RS]. This construction defines an action of the Grothendieck algebra of the category of finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$ on any $U_q(\hat{\mathfrak{g}})$-module from category $\mathcal{O}$ with $c = -g$. We explain the connection between the central elements from [RS] and transfer matrices in statistical mechanics. In the quasiclassical approximation this connection was explained in [FFR], and it was mentioned that one could generalize it to the quantum case to get Bethe vectors for transfer matrices. Using this connection, we prove that the central elements from [RS] (for all finite dimensional representations) applied to the highest weight vector of a generic Verma module at the critical level generate the whole space of singular vectors in this module. We also compute the first term of the quasiclassical expansion of the central elements near $q = 1$, and show that it always gives the Sugawara current with a certain coefficient.

1. Central elements and singular vectors

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ of rank r. Let $\mathfrak{h}$ be the Cartan subalgebra in $\mathfrak{g}$. Let $\alpha_1, \ldots, \alpha_r$ be the simple roots of $\mathfrak{g}$.

Let $U_q(\hat{\mathfrak{g}})$ be the quantum affine algebra corresponding to $\mathfrak{g}$, and let $U_q(\tilde{\mathfrak{g}})$ be its extension by the scaling element d (see [Dr1]; however, our notations will be as in [FR]). We assume that q is a formal parameter but sometimes we will use the specialization $q = 1$. Let $U_q(\tilde{\mathfrak{n}}^{\pm})$ be subalgebras of $U_q(\tilde{\mathfrak{g}})$ generated by the positive and negative root elements, respectively. Let $\{a_i, i \geq 0\}$ be a homogeneous basis of $U_q(\tilde{\mathfrak{n}}^{+})$ $(a_0 = 1)$, and let $\{a^i\}$

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be the dual basis of $U_q(\tilde{\mathfrak{n}}^-)$ (with respect to the Drinfeld pairing, [Dr1]). Then the universal R-matrix of $U_q(\tilde{\mathfrak{g}})$ is (see [Dr1]):

$$\tilde{\mathcal{R}} = q^{c\otimes d+d\otimes c}\mathcal{R} = q^{c\otimes d+d\otimes c+\Sigma_{j=1}^{r} X_j\otimes X_j}(1+\sum_{i>0} a_i\otimes a^i), \tag{1.1}$$

where c is the central element, and X_j is an orthonormal basis of the Cartan subalgebra in $\mathfrak{g}$ with respect to the invariant form $\langle\, ,\rangle$ normalized by $\langle\theta,\theta\rangle = 2$ (θ is the maximal root of $\mathfrak{g}$).

Let V be a finite-dimensional representation of $U_q(\hat{\mathfrak{g}})$. Let $\pi_V : U_q(\hat{\mathfrak{g}}) \to \mathrm{End}(V)$ be the corresponding homomorphism. For $z\in\mathbb{C}^*$, let $V(z)$ be the representation of $U_q(\hat{\mathfrak{g}})$ defined by $\pi_{V(z)}(a) = \pi_V(z^d a z^{-d})$.

Consider the following quantum currents (cf. [FRT],[RS],[FR],[DF]):

$$L_V^+(z) = (\mathrm{Id}\otimes\pi_{V(z)})(\mathcal{R}'), \qquad L_V^-(z) = (\mathrm{Id}\otimes\pi_{V(z)})(\mathcal{R}^{-1}), \tag{1.2}$$

where $\mathcal{R}'$ is obtained from $\mathcal{R}$ by permutation of factors. They are series in z with coefficients in some completion of $U_q(\hat{\mathfrak{g}})\otimes\mathrm{End}(V)$.

Now, following [RS], introduce the L-matrix

$$L_V(z) = (q^{-cd}\otimes 1)L_V^+(z)(q^{cd}\otimes 1)L_V^-(z)^{-1}, \tag{1.3}$$

and consider the current

$$l_V(z) = \mathrm{Tr}\,|_V\big((1\otimes q^{2\rho})L_V(z)\big), \tag{1.4}$$

where $\rho\in\mathfrak{h}^*$ is the half-sum of the positive roots of $\mathfrak{g}$. This is a formal series in z infinite in both directions, and its components belong to a completion of the quantum affine algebra. However:

Lemma 1.1. *Let U be any highest weight module over $U_q(\hat{\mathfrak{g}})$. Then for any $u\in U$, $l_V(z)u\in U((z))$ (i.e. it is a series in z finite in the negative direction and its coefficients belong to U).*

This lemma follows from the definition of the universal R-matrix.

It turns out that at the critical level $l_V(z)$ becomes central. This construction of central elements is due to Reshetikhin and Semenov-Tian-Shansky [RS], and is analogous to the construction of the center for $U_q(\mathfrak{g})$ due to Drinfeld [Dr2] and Reshetikhin [R].

Theorem 1.2. ([RS]) *Let U be any highest weight module over $U_q(\hat{\mathfrak{g}})$ with central charge $c=-g$, where g is the dual Coxeter number of $\mathfrak{g}$. Then $l_V(z)a = a l_V(z)$ on U for any $a\in U_q(\hat{\mathfrak{g}})$.*

Proof. It follows from the definition of $L^\pm$ that $PL_V^-(z)^{-1}$ is an intertwining operator between completions of $U\otimes V(z)$ and $V(q^{-k}z)\otimes U$, and

$L_V^+(q^{-k}z)P$ is an intertwining operator between completions of $V(q^{-k}z)\otimes U$ and $U\otimes V(q^{-2k}z)$, where k is the central charge of U, and P is permutation of factors. Thus, $L_V(z)$ is an intertwiner between completions of $U\otimes V(z)$ and $U\otimes V(q^{-2k}z)$. In particular, if $k=-g$, crossing symmetry (cf.[FR]) implies that $(1\otimes q^{2\rho})L_V(z)$ is a homomorphism between completions of $U\otimes V(z)$ and $U\otimes V(z)^{**}$. Therefore, the theorem follows from the fact that whenever $\Phi: U\otimes X\to U\otimes X^{**}$ is an intertwiner, $\mathrm{Tr}\,|_X(\Phi): U\to U$ is also an intertwiner. □

Let l_V^0 be the central element of $U_q(\mathfrak{g})$ defined by $l_V^0=\mathrm{Tr}\,|_V((\mathrm{Id}\otimes\pi_V)(1\otimes q^{2\rho})(R_{21}R))$, where R is the universal R-matrix for $U_q(\mathfrak{g})$ ([Dr2],[R]). If U is a $U_q(\mathfrak{g})$-module of highest weight λ then l_V^0 is a scalar in U: $l_V^0|_U=\chi_V(q^{2(\lambda+\rho)})$, where χ_V is the character of V as a $U_q(\mathfrak{g})$-module.

Proposition 1.3. (*Properties of* l_V) *Let* U *be as in Theorem* 1.2. *Then in* U*:*

(*i*) $l_V(z)$ *is regular at* $z=0$ *and* $l_V(0)=l_V^0$;

(*ii*) *for any exact sequence* $0\to V_1\to V_2\to V_3\to 0$ *of finite-dimensional representations of* $U_q(\hat{\mathfrak{g}})$ *one has* $l_{V_2}(z)=l_{V_1}(z)+l_{V_3}(z)$;

(*iii*)

$$l_{V(z_1)}(z_2)=l_V(z_1z_2); \tag{1.5}$$

(*iv*)

$$l_{V_1}(z_1)l_{V_2}(z_2)=l_{V_1(z_1/z_2)\otimes V_2}(z_2); \tag{1.6}$$

Proof. (i) Let $u_0\in U$ be the highest weight vector. Let $l_V[n]$ denote the coefficient to z^n in $l_V(z)$. Then $l_V(z)u_0=l_V^0u_0+\sum_{n>0}z^nl_V[n]u_0$. Let $u\in U$. Pick $a\in U_q(\tilde{\mathfrak{n}}^-)$ such that $u=au_0$. Then by Theorem 1.2 $l_V(z)u=l_V^0u+\sum_{n>0}z^nl_V[n]u$.

(ii) The matrix $(1\otimes q^{2\rho})L_V(z)$ is block-triangular, and its trace is the sum of the traces of its diagonal blocks.

(iii) Straightforward.

(iv) Using Theorem 1.2 and property (iii), we get

$$\begin{aligned} l_{V_1}(z_1)l_{V_2}(z_2) &= \mathrm{Tr}\,|_{V_2}((1\otimes q^{2\rho})L^+_{V_2}(q^gz)l_{V_1(z_1/z_2)}(z_2)L^-_{V_2}(z)^{-1})\\ &= \mathrm{Tr}\,|_{V_1(z_1/z_2)}\mathrm{Tr}\,|_{V_2}\big((1\otimes q^{2\rho}\otimes q^{2\rho})L^+_{V_2}(q^gz)L^+_{V_1(z_1/z_2)}(q^gz)\\ &\qquad\cdot L^-_{V_1(z_1/z_2)}(z)^{-1}L^-_{V_2}(z)^{-1}\big)\\ &= \mathrm{Tr}\,|_{V_1(z_1/z_2)\otimes V_2}((1\otimes q^{2\rho})L^+_{V_1(z_1/z_2)\otimes V_2}(q^gz)L^-_{V_1(z_1/z_2)\otimes V_2}(z)^{-1})\\ &= l_{V_1(z_1/z_2)\otimes V_2}(z_2). \quad \square \end{aligned} \tag{1.7}$$

Remark. If U is a module from category $\mathcal{O}$ but not necessarily highest weight then property (i) no longer holds, and all Fourier components of $l_V(z)$ could be nontrivial operators.

Properties (ii)–(iv) imply:

Corollary 1.4. *The map $V \to l_V(1)$ defines an action of the Grothendieck algebra Gr of finite dimensional representations of the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ on any completed highest weight module U over this algebra with $c = -g$.*

Remark. By definition, finite-dimensional representations with the same components of the Jordan-Hölder series correspond to the same element in the Grothendieck algebra.

Let now U be as in Theorem 2, and let u_0 be the highest weight vector in U. Let $l_V[n]$ denote the coefficient to z^n in $l_V(z)$. Then for every V and n, $l_V[n]u_0$ is a singular vector in U. A natural question is: do such vectors span the space of singular vectors in U if the highest weight is generic? We will prove that answer is positive.

More precisely, let $\omega_1, \dots, \omega_r$ be the fundamental weights of $\mathfrak{g}$. Let $V_1, \dots, V_r$ be deformations of the fundamental representations, i.e. irreducible finite-dimensional representations of $U_q(\hat{\mathfrak{g}})$ such that the restrictions of V_j to $U_q(\mathfrak{g})$ are $V_j = L_{\omega_j} \oplus \sum_{\omega < \omega_j} c_\omega L_\omega$, where L_ω is the irreducible $U_q(\mathfrak{g})$ module with highest weight ω (cf. [Dr3],[Dr4],[CP]). These representations are the simplest in the case $\mathfrak{g} = \mathfrak{sl}_{r+1}$: in this case $V_i = \Lambda_q^i \mathbb{C}^{r+1}$ are just the quantum exterior powers pulled back from $U_q(\mathfrak{g})$ to $U_q(\hat{\mathfrak{g}})$ by the evaluation homomorphism $p : U_q(\hat{\mathfrak{g}}) \to U_q(\mathfrak{g})$ defined in [J].

Let $y_{jn} = l_{V_j}[n]$. Let A be the free polynomial algebra in y_{jn}, $j = 1, \dots, r$, $n > 0$. Let U_0 be the space of singular vectors in U. Then U_0 is an A-module, via $(y, u) \to yu$.

Let λ be generic. This means that we regard $S = q^{2(\lambda+\rho)}$ as an indeterminate taking values in the Cartan subgroup of the Lie group corresponding to $\mathfrak{g}$. Let U be the Verma module with highest weight λ and central charge $-g$.

Theorem 1.5. (*Main result*) *For $q \neq 1$ the space U_0 is a free A-module of rank 1 generated by u_0.*

The proof of this theorem will be given in Section 3.

Remark. This result is a quantum analogue of the theorem of Feigin and Frenkel [FF] who showed that certain central elements in the completion of $U(\hat{\mathfrak{g}})/(c+g)$ applied to a highest weight vector in a generic Verma module generate the space of all singular vectors in this module.

Corollary 1.6. *The space U_0 is a cyclic Gr-module generated by u_0.*

2. Central elements, Bethe vectors, and transfer matrices

In Section 1 we defined a representation of the Grothendieck algebra Gr of finite-dimensional $U_q(\hat{\mathfrak{g}})$-modules on a critical level module. Representations of this algebra also occur in statistical mechanics as transfer matrices. In this section we will connect these two representations.

Let W be a $U_q(\hat{\mathfrak{g}})$-module with central charge 0. Fix $\beta \in \mathfrak{h}^*$. Define *transfer matrices* to be the following operators in W with coefficients in the ring of formal power series in z:

$$T_V(z) = \operatorname{Tr}|_V (R^{VW}(z)(q^\beta \otimes 1)), \tag{2.1}$$

where $R^{VW}(z)$ is the projection of the universal R-matrix $\mathcal{R}$ to $V(z) \otimes W$, and V is a finite-dimensional representation of $U_q(\hat{\mathfrak{g}})$. Obviously, transfer matrices preserve weight, i.e. commute with the Cartan subalgebra in $U_q(\mathfrak{g})$. So we can restrict their action to a weight subspace in W, say $W[\mu]$, $\mu \in \mathfrak{h}^*$.

It follows from the definition of the R-matrix that the transfer matrices are pairwise commutative and satisfy properties (ii)–(iv) of Proposition 1.3. Hence, similarly to $l_V(z)$, they define a representation of Gr. Therefore, it is natural to ask if there is any relation between l_V and T_V. It turns out there is a close connection between them. To explain this connection, we need to introduce intertwiners.

Let $\lambda = (\beta + \mu - 2\rho)/2$, $\nu = \lambda - \mu$. Let $M_{\lambda,-g}$ denote the Verma module over $U_q(\hat{\mathfrak{g}})$ with highest weight λ and central charge $-g$. Consider intertwining operators

$$\Phi : M_{\lambda,-g} \to M^c_{\nu,-g} \otimes W, \tag{2.2}$$

where M^c denotes the (complete) dual module to M twisted by the Cartan involution (the contragredient module). Such operators were defined and studied in [FR]. The following fact is known about them:

Lemma 2.1. *The space of intertwiners* (2.2) *is isomorphic to the weight space* $W[\lambda - \nu]$; *the isomorphism is given by* $\Phi \to \langle \Phi \rangle$, *where* $\langle \Phi \rangle = \langle v_\nu, \Phi v_\lambda \rangle \in W$ *(here* $\langle \, , \rangle$ *is the Shapovalov form, and* v_λ, v_ν *are the vacuum vectors of* $M_{\lambda,-g}$, $M^c_{\nu,-g}$*).*

Lemma 2.1 allows us to connect the elements l_V and T_V.

Proposition 2.2.

$$\langle \Phi l_V(z) \rangle = T_V(z) \langle \Phi \rangle . \tag{2.3}$$

Proof. For the proof we need the commutation relation

$$\Phi L^+_V(zq^{-g}) = R^{VW}(z) L^+_V(zq^{-g})\Phi, \tag{2.4}$$

which is a special case of (4.47) in [FR]. Applying this relation, we get

$$\Phi L_V(z) = R^{VW}(z) L_V^+(zq^{-g})\Phi L_V^-(z)^{-1},$$

or

$$\begin{aligned}\langle \Phi L_V(z)\rangle &= R^{VW}(z)\left\langle L_V^+(zq^{-g})\Phi L_V^-(z)^{-1}\right\rangle \\ &= R^{VW}(z)\left\langle q^{\sum X_i\otimes X_i}\Phi q^{\sum X_i\otimes X_i}\right\rangle \\ &= R^{VW}(z)(q^{\lambda+\nu}\otimes 1)\langle\Phi\rangle \\ &= R^{VW}(z)(q^{\beta-2\rho}\otimes 1)\langle\Phi\rangle . \qquad (2.5)\end{aligned}$$

Now, multiplying the V-components of both sides of (2.5) by $q^{2\rho}$ and taking the trace, and using the definitions of l_V and T_V, we get (2.3). □

In statistical mechanics one is interested in finding Bethe vectors—common eigenvectors of transfer matrices. Let us characterize Bethe vectors in the language of intertwiners.

Let β be generic. Then any singular vector in $M_{\lambda,-g}$ is of weight λ with respect to $\mathfrak{h}$, so the submodule generated by this vector is isomorphic to the module $M_{\lambda,-g}$. This makes legitimate the following definition:

Definition. We say that an intertwiner Φ is a Bethe operator if its restriction to every Verma submodule in $M_{\lambda,-g}$ is proportional to Φ.

Then we have an obvious proposition.

Proposition 2.3. *If Φ is a Bethe operator then $\langle\Phi\rangle$ is a Bethe vector.*

3. Proof of Theorem 1.5

We will work with the Drinfeld (loop) realization of quantum affine algebras. In this realization the algebra $U_q(\hat{\mathfrak{g}})$ is described as an algebra generated by elements x_{ij}, $\xi_{ij}^{\pm}$, and c (central element), where $1\le i\le r$, and $j\in\mathbb{Z}$, satisfying the relations listed in [Dr4],[KhT]. These elements are quantum analogues of $h_i\otimes t^j$, $e_i\otimes t^j$, $f_i\otimes t^j$, c, in the affine Lie algebra $\hat{\mathfrak{g}}$. We need to use only the quotient of $U_q(\hat{\mathfrak{g}})$ by the relation $c=0$, which we denote by $U_q(L\mathfrak{g})$ (quantum loop algebra). In this algebra, we define three subalgebras: U^+, H, U^-, generated by ξ_{ij}^+, x_{ij}, ξ_{ij}^-, respectively. About these algebras we only need to know that H is abelian, and $[H,U^+]\subset U^+$, $[H,U^-]\subset U^-$.

By a Drinfeld weight we mean an infinite set of numbers $D=\{d_{ij}\}$, $i=1,\dots,r$, $j\in\mathbb{Z}$. We will only use weights with $d_{ij}=0$, $j<0$.

To every Drinfeld weight D one can associate a one-dimensional module over HU^+ in which x_{ij} acts by d_{ij} and U^+ acts trivially. We will denote this module also by D. Set $W(D) = \mathrm{Ind}_{HU^+}^{U_q(L\mathfrak{g})} D$. The module W has the following property: its weights with respect to the Cartan subalgebra $\mathfrak{h}$ are $D_0 = (d_{10}, \ldots, d_{r0})$ and lower, and the D_0-weight subspace is one-dimensional. Let w_0 be the generator of this subspace. Then for any $\lambda, \nu \in \mathfrak{h}^*$ such that $\lambda = \nu + D_0$ there exists a unique intertwining operator Φ of the form (2.2) with $\langle\Phi\rangle = w_0$, where $W = W(D)$.

Since the transfer matrices defined in Section 3 preserve weight, we have $T_V(z)w_0 = t_V(z,D)w_0$, where $t_V(z,D)$ is some series with scalar coefficients.

Lemma 3.1. *Let ϕ_{ij}, $i = 1, \ldots, r$, $j \geq 1$, be arbitrary numbers. Let $\phi_i(z) = \sum_{n=1}^{\infty} \phi_{in} z^n$. Then the Drinfeld highest weight D can be chosen in such a way that*

(i) $\beta + D_0 = 2(\lambda + \rho)$,
(ii) $t_{V_i}(z, D) = t_{V_i}(0, D) + \phi_i(z)$, $i = 1, \ldots, r$, *and*
(iii) the components of D are rational functions in the variables q and $S = q^{2(\lambda+\rho)}$.

Proof. For the proof we need an explicit realization of the universal R-matrix (1.1). Such a realization was provided by Khoroshkin and Tolstoy [KhT]:

Proposition. ([KhT], Eq. (42)) The universal R-matrix can be represented in the form

$$\mathcal{R} = \mathcal{R}^+ \mathcal{R}^0 \mathcal{R}^- q^{\sum_{j=1}^{r} X_j \otimes X_j}, \tag{3.1}$$

where $\mathcal{R}^\pm \in U^\pm \otimes U^\mp$ are of total degree 0, and

$$\mathcal{R}^0 = \exp\Big(\sum_{n>0} \sum_{i,j=1}^{r} c_{ij}^n x_{i,n} \otimes x_{j,-n}\Big), \tag{3.2}$$

and for every n the matrix (c_{ij}^n) is inverse to the matrix (A_{ij}^n), where

$$A_{ij}^n = \frac{q^{n\langle\alpha_i,\alpha_j\rangle} - q^{-n\langle\alpha_i,\alpha_j\rangle}}{n(q^{\langle\alpha_i,\alpha_i\rangle} - q^{-\langle\alpha_i,\alpha_i\rangle})(q^{\langle\alpha_j,\alpha_j\rangle} - q^{-\langle\alpha_j,\alpha_j\rangle})}. \tag{3.3}$$

Fix $\beta \in \mathfrak{h}^*$. Set $D_0 = 2(\lambda + \rho) - \beta$. Using definition (2.1) of the transfer matrix and the above proposition, we compute $t_V(z, D)$:

$$\begin{aligned} t_V(z,D) &= \mathrm{Tr}\,|_{V(z)}(\mathcal{R}^0 q^{\sum_{j=1}^{r} X_j \otimes X_j}(q^\beta \otimes 1))|_{\mathbb{C}w_0} \\ &= \mathrm{Tr}\,|_{V(z)}\Big(\exp\Big(\sum_{n>0}\sum_{i,j=1}^{r} c_{ij}^n d_{j,-n} x_{in}\Big) S\Big). \end{aligned} \tag{3.4}$$

Note that the terms $\mathcal{R}^+$ and $\mathcal{R}^-$ drop out: $\mathcal{R}^-$ disappears since we are computing its action on the Drinfeld highest weight vector, and $\mathcal{R}^+$ disappears because of taking the trace, since U^- acts nilpotently on V.

Let $b_{in} = \sum_{j=1}^{r} c_{ij}^n d_{j,-n}$. Since $d_{j,-n}$ can be chosen arbitrarily, and the matrices c_{ij}^n are invertible by the definition, the numbers b_{in} can also be arbitrary. In terms of them, equation (3.4) takes the form

$$t_V(z,D) = \mathrm{Tr}\,|_{V(z)}\Big(\exp\Big(\sum_{n>0}\sum_{i=1}^{r} b_{in}x_{in}\Big)S\Big). \tag{3.5}$$

We must show that $t_{V_i}(z,D)$ can be made an arbitrary Taylor series. By a deformation argument, it is enough to show this when $q=1$. Thus we must consider the quasiclassical limit of V_i. This limit has the form $V_i|_{q=1} = L_{\omega_i}(1) \oplus \sum_j M_{ij}$, where M_{ij} are irreducible modules over $\hat{\mathfrak{g}}$: $M_{ij} = \otimes_m N_{ijm}(t_m)$, where N_{ijm} are irreducible $\mathfrak{g}$-modules, and $N_{ijm}(t_m)$ are the corresponding evaluation modules with some parameters t_m. Also, $\lim_{q\to 1} x_{in} = h_i \otimes t^n,\ \ h_i = h_{\alpha_i}$. This information allows us to compute the right hand side of (3.5).

Let χ_i denote the characters of L_{ω_i}. Let $\psi_i(z) = \sum_{n>0} b_{in}z^n$. Note that ψ_i can be an arbitrary Taylor series with zero free term if the weight D is suitably chosen. We have

$$t_{V_i}(z,D)|_{q=1} = \chi_i(e^{\sum_l \psi_l(z)h_l}S) + \sum_j \prod_m \mathrm{Ch}_{N_{ijm}}(e^{\sum_l \psi_l(zt_m)h_l}S), \tag{3.6}$$

where Ch_V denotes the character of V as a $\mathfrak{g}$-module.

Remark. If $\mathfrak{g} = \mathfrak{sl}_{r+1}$ then only the first term occurs on the right hand side of (3.6).

Now, using (3.6), we can compute the coefficients b_{jn} recursively for any given series $t_{V_i}(z,D)$. At each step we will have to solve a system of linear equations whose matrix is

$$a_{il}^n = \mathrm{Tr}\,|_{L_{\omega_i}}(h_l S) + \sum_{j,m} t_m^n \,\mathrm{Tr}\,|_{N_{ijm}}(h_l S)\,\mathrm{Ch}_{N_{ijm}}(S)^{-1}\,\mathrm{Ch}_{M_{ij}}(S), \quad 1 \le i,l \le n. \tag{3.7}$$

So we must show that the determinant of this matrix is not identically zero for any $n>0$.

Using the fact that the highest weights of N_{ijm} are lower than ω_i, and the formula $\mathrm{Tr}\,|_V(h_i S) = \frac{\partial}{\partial h_i}\mathrm{Tr}_V(S)$, we find that only the first term in (3.7) contributes to the determinant, and therefore this determinant is equal to the Jacobian of the fundamental characters, i.e. the Weyl denominator: $\det(a_{il}^n) = \frac{\partial(\chi_1,\dots,\chi_r)}{\partial(h_1,\dots,h_r)}$, which is not identically zero. This proves the lemma. $\square$

Lemma 3.2. *For any $n > 0$, the operators $l_{V_i}[j]$ in U are algebraically independent for $i = 1, \dots, r,\ \ j = 1, \dots, n$.*

Proof. Let $t_V(z, D) = \sum_{n \geq 0} t_V(D)[n] z^n$. Then Proposition 2.2 implies that $\langle \Phi l_V[n] \rangle = t_V(D)[n] \langle \Phi \rangle$. Therefore, if there were a nontrivial polynomial relation between the elements $l_{V_i}[j]$ in $U = M_{\lambda, -g}$, say $P(\{l_{V_i}[j]\}) = 0$, the same polynomial relation would have to hold for the numbers $t_{V_i}(D)[j]$ for any D. But this is impossible since by Lemma 3.1 these numbers can be arbitrary. $\square$

Now we can finish the proof of the theorem. Lemma 3.2 implies that the action of the algebra A on the vacuum vector u_0 defines an embedding of A into U_0 as a graded vector space. On the other hand, it is known that the character of U_0 at $q = 1$ is equal to the character of A (see e.g. [FF]). This implies that for $q \neq 1$ the dimension of the homogeneous subspace in U_0 of degree d is at most that for A. This means that the embedding of A into U_0 is an isomorphism, i.e. that U_0 is a free module of rank 1 over A. The theorem is proved. $\square$

4. Quasiclassical limit

In this section we compute the first term of the quasiclassical expansion of the central elements introduced in Section 1.

Let the number C_V be defined by $\mathrm{Tr}\,|_V(ab) = C_V \langle a, b \rangle$, $a, b \in \mathfrak{g}$ (here we abuse the notation by using the same symbol V for the quasiclassical limit of V at $q \to 1$ regarded as a $\mathfrak{g}$-module).

Theorem 4.1. *In any $U_q(\hat{\mathfrak{g}})$-module U from the category $\mathcal{O}$ with $c = -g$*

$$\lim_{q \to 1} \frac{l_V(z) - \dim V}{(q - q^{-1})^2} = C_V \left(\sum_{j \in \mathbb{Z}} z^{-j} T_j + \frac{1}{2} \langle \rho, \rho \rangle \right), \tag{4.1}$$

where T_j are the Sugawara elements:

$$T_j = \frac{1}{2} \sum_{a \in B} \sum_{n \in Z} : a[n] a[j-n] :, \tag{4.2}$$

B is an orthonormal basis of $\mathfrak{g}$ with respect to $\langle\, , \rangle$, $a[n] = a \otimes t^n \in \hat{\mathfrak{g}}$, $: a[n] a[m] :$ equals $a[n] a[m]$ when $m > n$ and $a[m] a[n]$ otherwise.

Proof. It is known (cf. [FR], Eq. (4.42)) that near the point $q = 1$ the quantum currents have the expansion

$$L_V^{\pm}(z) = 1 \otimes 1 + (q - q^{-1}) \sum_{a \in B} J_a^{\pm}(z) \otimes \pi_V(a) + O((q - q^{-1})^2), \tag{4.3}$$

where $J_a^\pm(z)$ are classical currents:

$$J_a^\pm(z) = \pm\left(\frac{1}{2}a^0 + a^\mp + \sum_{n>0} a[\mp n] z^{\pm n}\right), \qquad a = a^0 + a^+ + a^-,\ a^0 \in \mathfrak{h},\ a^\pm \in \mathfrak{n}^\pm. \tag{4.4}$$

This implies that

$$(1\otimes q^{2\rho})L_V(z) = 1\otimes 1 + (q-q^{-1})\Big(\sum_{a\in B} J_a(z)\otimes \pi_V(a) + 1\otimes\rho\Big) - (q-q^{-1})^2\Big(\sum_{a,b\in B} J_a^+(z)J_b^-(z)\otimes\pi_V(ab) + Q_V^+(z) + Q_V^-(z)\Big) + O((q-q^{-1})^3), \tag{4.5}$$

where $J_a = J_a^+ - J_a^-$, and $Q_V^\pm$ are quadratic terms.

Let us compute the expansion of $l_V(z)$ using (4.5). Since $\operatorname{Tr}|_V(a) = 0$, $a \in \mathfrak{g}$, we have (near $q=1$):

$$l_V(z) = \dim V - (q-q^{-1})^2\Big(C_V\sum_{a\in B} J_a^+(z)J_a^-(z) + K_V^+(z) + K_V^-(z)\Big), \tag{4.6}$$

where $K_V^\pm(z) = \operatorname{Tr}|_V(Q_V^\pm(z))$ are quadratic terms lying in the Borel subalgebras $U(\hat{\mathfrak{b}}^\mp)$.

From formula (4.6) we see that the limit (4.1) exists and equals $C_V\tilde{T}(z)$, where

$$\tilde{T}(z) = \sum_{a\in B} J_a^+(z)J_a^-(z) + K_V^+(z) + K_V^-(z). \tag{4.7}$$

Our purpose is to prove that $\tilde{T}(z) = T(z)$. This is the same as to show that their Fourier components are the same: $\tilde{T}[n] = T[n]$.

Combining (4.2), (4.6), we see that $T[n] - \tilde{T}[n]$ is in $U(\hat{\mathfrak{b}}^+)$ if $n>0$, in $U(\hat{\mathfrak{b}}^-)$ if $n<0$, and in both if $n=0$. On the other hand, both $T[n]$ and $\tilde{T}[n]$ are central and of degree n, which implies that so is their difference. These two facts immediately imply that this difference is zero for $n\neq 0$ and a constant independent of the module U if $n=0$.

To find this constant, let us assume that U is a Verma module with highest weight λ. Then $T_0 = \frac{1}{2}\langle\lambda,\lambda+2\rho\rangle$. Thus we get

$$\lim_{q\to 1}\frac{l_V^0 - \dim V}{(q-q^{-1})^2} = \frac{1}{2}d^2\,\mathrm{Ch}_V(x)|_{x=1}(\lambda+\rho,\lambda+\rho) = \frac{1}{2}C_V\langle\lambda+\rho,\lambda+\rho\rangle = C_V\left(T_0 + \frac{\langle\rho,\rho\rangle}{2}\right). \tag{4.8}$$

□

So far the highest weight λ has been a formal parameter. Now we specialize λ.

Corollary 4.2. *Let $\mathfrak{g} = \mathfrak{sl}_2$. Then Lemma* 3.2 *holds for arbitrary special value of λ; Theorem* 1.5 *holds for any λ such that $\langle \lambda + \rho, \alpha^\vee \rangle$ is not a positive integer for any root α of $\mathfrak{g}$ (positive or negative).*

Proof. The statement follows by deformation argument from Theorem 4.1 and the fact that T_i are algebraically independent in any Verma module. □

Remark. Probably, Corollary 4.2 is true for any $\mathfrak{g}$. However, the method we used to prove Theorem 1.5 does not seem to be powerful enough to show it: for example, Lemma 3.1 is false for $\lambda = -\rho$ even for $\mathfrak{g} = \mathfrak{sl}_2$.

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