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THETA FUNCTIONS FOR SL(n) VERSUS GL(n)

Ron Donagi and Loring W. Tu

1. Introduction

Over a smooth complex projective curve C of genus g one may consider two types of moduli spaces of vector bundles, $\mathcal{M} := \mathcal{M}(n, d)$, the moduli space of semistable bundles of rank n and degree d on C, and $\mathcal{S}M := \mathcal{S}M(n, L)$, the moduli space of those bundles whose determinant is isomorphic to a fixed line bundle L on C. We call the former a *full moduli space* and the latter a *fixed-determinant moduli space*. Since the spaces $\mathcal{S}M(n, L)$ are all isomorphic as L varies in $\mathrm{Pic}^d(C)$, we also write $\mathcal{S}M(n, d)$ to denote any one of them.

On both moduli spaces there are well-defined *theta bundles*, as we recall in Section 2. While the theta bundle θ on $\mathcal{S}M$ is uniquely defined, the theta bundles θ_F on $\mathcal{M}$ depend on the choice of complementary vector bundles F of minimal rank over C. For any positive integer k, sections of θ_F^k generalize the classical theta functions of level k on the Jacobian of a curve, and so we call sections of θ^k over $\mathcal{S}M$ and θ_F^k over $\mathcal{M}$ *theta functions of level k for $SL(n)$ and $GL(n)$* respectively.

Our goal is to study the relationship between these two spaces of theta functions. We prove a simple formula relating their dimensions, and then formulate a conjectural duality.

Theorem 1. *If $h = \gcd(n, d)$ is the greatest common divisor of n and d, and $L \in \mathrm{Pic}^d(C)$, then*

$$\dim H^0(\mathcal{S}M(n, L), \theta^k) \cdot k^g = \dim H^0(\mathcal{M}(n, d), \theta_F^k) \cdot h^g.$$

In the case $k = 1, d = 0$, both sides are computed explicitly in [BNR]. This case of our result then shows that the two computations are really equivalent to each other. More generally, the celebrated Verlinde formulas ([BL], [BS], [Bo], [F], [S], [V]) evaluate the left-hand side, so our theorem gives the dimension of the space of $GL(n)$ theta functions. A similar argument should relate the Verlinde number for any reductive group to that of its semisimple part. [*Added in proof*: This has now been done by T. Pantev, cf. [P].]

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In [Li, p. 547] there is a conjecture on the dimension of the space of first-order $GL(n)$ theta functions. Our Theorem 1, together with the rank 2, degree 1, and level $k = 1$ case of the Verlinde formula ([Be], [La], [S]), disproves this conjecture.

Theorem 1 is consistent with and therefore lends credence to another, so far conjectural, relationship between these two types of theta functions. To explain this, start with integers $\bar{n}, \bar{d}, h, k$ such that $\bar{n}, h, k$ are positive and $\gcd(\bar{n}, \bar{d}) = 1$. Let $F \in \mathcal{M}(\bar{n}, \bar{d})$ and write

$$\mathcal{S}M_1 = \mathcal{S}M(h\bar{n}, (\det F)^h) \quad \text{and} \quad \mathcal{M}_2 = \mathcal{M}(k\bar{n}, k(\bar{n}(g-1) - \bar{d})).$$

The tensor product map τ sends $\mathcal{S}M_1 \times \mathcal{M}_2$ to $\mathcal{M}(hk\bar{n}^2, hk\bar{n}^2(g-1))$.

Conjecture 2. *The tensor product map induces a natural duality between* $H^0(\mathcal{S}M_1, \theta^k)$ *and* $H^0(\mathcal{M}_2, \theta_F^h)$.

This conjecture may be yet another instance where the physics has gone well ahead of the mathematics. At least in the special case of degree zero (that is, $\bar{n} = 1$, $\bar{d} = 0$), some variants of this conjecture seem to be folklore in Conformal Field Theory, cf. [NS] for a physics discussion, and the references listed there for some representation-theoretic results. (We thank A. Beauville for this reference.) We have not been able to find any references to the general situation, nor a mathematical proof even in the special case. It is possible that the physics results could be translated into a mathematical argument, as has been done successfully for the Verlinde conjectures ([BL], [F]), but we have not attempted this. We content ourselves in Section 6 with the precise statement of the general conjecture, followed by a list of the available algebro-geometric evidence for it.

We benefited greatly from conversations with Rob Lazarsfeld on the subject of Theorem 1, with Alexis Kouvidakis on the proof of Proposition 5, and with Raoul Bott and Andras Szenes on special cases of the theorem and the conjecture. We thank them, as well as Ludmil Katzarkov and Tony Pantev, for their help.

Notation and Conventions.

$h^0(\ )$	$\dim H^0(\ )$
J_d	$\operatorname{Pic}^d(C)$, i.e. the set of isomorphism classes of line bundles of degree d on C
J or J_0	$\operatorname{Pic}^0(C)$
$L_1 \boxtimes L_2$	$\pi_1^* L_1 \otimes \pi_2^* L_2$ if L_i is a line bundle on X_i and $\pi_i : X_1 \times X_2 \to X_i$ is the i-th projection
$S^h C$	the hth symmetric product of C
T_n	the group of n-torsion bundles on C

2. Theta bundles

We recall here the definitions of the theta bundles on a fixed-determinant moduli space and on a full moduli space. Our definitions are slightly different from but equivalent to those in [DN].

For $L \in \mathrm{Pic}^d(C)$, the Picard group of $\mathcal{S}M := \mathcal{S}M(n, L)$ is $\mathbb{Z}$ and the theta bundle θ on $\mathcal{S}M$ is the positive generator of $\mathrm{Pic}(\mathcal{S}M)$.

When n and d are such that $\chi(E) = 0$ for $E \in \mathcal{M}(n, d)$, i.e., when $d = (g-1)n$, there is a natural divisor $\Theta \subset \mathcal{M}(n, n(g-1))$:

$$\Theta = \text{closure of } \{E \text{ stable in } \mathcal{M}(n, n(g-1)) \mid h^0(E) \neq 0\}.$$

The theta bundle θ over $\mathcal{M}(n, n(g-1))$ is the line bundle corresponding to this divisor.

We say that a semistable bundle F is *complementary* to another bundle E if $\chi(E \otimes F) = 0$. We also say that F is *complementary* to $\mathcal{M}(n, d)$ if $\chi(E \otimes F) = 0$ for any $E \in \mathcal{M}(n, d)$. It follows easily from the Riemann-Roch theorem that if $E \in \mathcal{M}(n, d)$, $h = \gcd(n, d)$, $n = h\bar{n}$, and $d = h\bar{d}$, then F has rank n_F and degree d_F, where

$$n_F = k\bar{n} \quad \text{and} \quad d_F = k(\bar{n}(g-1) - \bar{d})$$

for some positive integer k.

If F is complementary to $\mathcal{M}(n, d)$, let

$$\tau_F : \mathcal{M}(n, d) \to \mathcal{M}(nn_F, nn_F(g-1))$$

be the tensor product map

$$E \mapsto E \otimes F.$$

Pulling back the theta bundle θ from $\mathcal{M}(nn_F, nn_F(g-1))$ via τ_F gives a line bundle $\theta_F := \tau_F^* \theta$ over $\mathcal{M}(n, d)$. (This bundle may or may not correspond to a divisor in $\mathcal{M}(n, d)$.) Let $\det : \mathcal{M}(n, d) \to J_d(C)$ be the determinant map. When $\mathrm{rk}\, F$ is the minimal possible: $\mathrm{rk}\, F = \bar{n} = n/h$, then θ_F is called a *theta bundle over* $\mathcal{M}(n, d)$; otherwise, it is a multiple of a theta bundle. Indeed, we extract from [DN] the formula:

Proposition 3. *Let F and F_0 be two bundles complementary to $\mathcal{M}(n, d)$. If $\mathrm{rk}\, F = a \,\mathrm{rk}\, F_0$, then*

$$\theta_F \simeq \theta_{F_0}^{\otimes a} \otimes \det{}^*(\det F \otimes (\det F_0)^{-a}),$$

where we employ the usual identification of $\mathrm{Pic}^0(C)$ with $\mathrm{Pic}^0(J_0)$.

In particular, θ_F depends only on $\mathrm{rk}\, F$ and $\det F$. If θ_F is a theta bundle on $\mathcal{M}(n, d)$, then for any $L \in \mathrm{Pic}^d(C)$, θ_F restricts to the theta bundle on $\mathcal{S}M(n, L)$.

3. A Galois covering

Let $\tau : Y \to X$ be a covering of varieties, by which we mean a finite étale morphism. A *deck transformation* of the covering is an automorphism $\phi : Y \to Y$ that commutes with τ. The covering is said to be *Galois* if the group of deck transformations acts transitively (hence simply transitively) on a general fiber of the covering.

Denote by $J = \mathrm{Pic}^0(C)$ the group of isomorphism classes of line bundles of degree 0 on the curve C, and $G = T_n$ the subgroup of torsion points of order n. Fix $L \in \mathrm{Pic}^d(C)$ and let $\mathcal{S}M = \mathcal{S}M(n, L)$ and $\mathcal{M} = \mathcal{M}(n, d)$. Recall that the tensor product map

$$\begin{aligned} \tau : \mathcal{S}M \times J &\to \mathcal{M} \\ (E, M) &\mapsto E \otimes M \end{aligned}$$

gives an n^{2g}-sheeted étale map ([TT], Prop. 8). The group $G = T_n$ acts on $\mathcal{S}M \times J$ by

$$N.(E, M) = (E \otimes N^{-1}, N \otimes M).$$

It is easy to see that G is the group of deck transformations of the covering τ and that it acts transitively on every fiber. Therefore, $\tau : \mathcal{S}M \times J \to \mathcal{M}$ is a Galois covering.

Proposition 4. *If $\tau : Y \to X$ is a Galois covering with finite abelian Galois group G, then $\tau_*\mathcal{O}_Y$ is a vector bundle on X which decomposes into a direct sum of line bundles indexed by the characters of G:*

$$\tau_*\mathcal{O}_Y = \sum_{\lambda \in \hat{G}} L_\lambda,$$

where $\hat{G}$ is the character group of G.

Proof. Write $\mathcal{O} = \mathcal{O}_Y$. The fiber of $\tau_*\mathcal{O}$ at a point $x \in X$ is naturally a complex vector space with basis $\tau^{-1}(x)$. Hence, $\tau_*\mathcal{O}$ is a vector bundle over X. The action of G on $\tau^{-1}(x)$ induces a representation of G on $(\tau_*\mathcal{O})(x)$ equivalent to the regular representation. Because G is a finite abelian group, this representation of G decomposes into a direct sum of one-dimensional representations indexed by the characters of G:

$$(\tau_*\mathcal{O})(x) = \sum_{\lambda \in \hat{G}} L_\lambda(x).$$

Thus, for every $\lambda \in \hat{G}$, we obtain a line bundle L_λ on X such such $\tau_*\mathcal{O} = \sum_\lambda L_\lambda$. $\square$

4. Pullbacks

We consider the tensor product map

$$\begin{array}{rcl} \tau : \mathcal{S}M(n_1, L_1) \times \mathcal{M}(n_2, d_2) & \to & \mathcal{M}(n_1 n_2, n_1 d_2 + n_2 d_1) \\ (E_1, E_2) & \mapsto & E_1 \otimes E_2, \end{array}$$

where $d_1 = \deg L_1$. For simplicity, in this section we write $\mathcal{S}M_1 = \mathcal{S}M(n_1, L_1)$, $\mathcal{M}_2 = \mathcal{M}(n_2, d_2)$, and $\mathcal{M}_{12} = \mathcal{M}(n_1 n_2, n_1 d_2 + n_2 d_1)$.

Proposition 5. *Let $F = F_{12}$ be a bundle on C complementary to $\mathcal{M}_{12}$. Then*

$$\tau^* \theta_F \simeq \theta^c \boxtimes \theta_{E_1 \otimes F}$$

for any $E_1 \in \mathcal{S}M(n_1, L_1)$, where

$$c := \frac{n_2 \operatorname{rk} F}{\operatorname{rk} F_1} = \frac{n_2 \operatorname{rk} F}{n_1 / \gcd(n_1, d_1)}$$

and F_1 is a minimal complementary bundle to E_1.

Proof. For $E_2 \in \mathcal{M}(n_2, d_2)$, let

$$\tau_{E_2} : \mathcal{S}M_1 \to \mathcal{M}_{12}$$

be tensorization with E_2. Then

$$(\tau^* \theta_F)|_{\mathcal{S}M_1 \times \{E_2\}} = \tau_{E_2}^* \theta_F = \tau_{E_2}^* \tau_F^* \theta = \tau_{E_2 \otimes F}^* \theta = \theta^c,$$

where by Proposition 3

$$\begin{array}{rcl} c & = & \operatorname{rk}(E_2 \otimes F) / \operatorname{rk} F_1 \\ & = & \dfrac{n_2 \operatorname{rk} F}{n_1 / \gcd(n_1, d_1)}. \end{array}$$

Similarly,

$$\begin{array}{rcl} (\tau^* \theta_F)|_{\{E_1\} \times \mathcal{M}_2} & = & \tau_{E_1}^* \theta_F = \tau_{E_1}^* \tau_F^* \theta \\ & = & \tau_{E_1 \otimes F}^* \theta = \theta_{E_1 \otimes F}. \end{array}$$

Note that the bundle $\theta_{E_1 \otimes F}$ depends only on $\operatorname{rk}(E_1 \otimes F) = n_1 \operatorname{rk} F$ and $\det(E_1 \otimes F) = L_1^{\operatorname{rk} F} \otimes (\det F)^{n_1}$. Hence, both $(\tau^* \theta_F)|_{\mathcal{S}M_1 \times \{E_2\}}$ and $(\tau^* \theta_F)|_{\{E_1\} \times \mathcal{M}_2}$ are independent of E_1 and E_2. By the seesaw theorem,

$$\tau^* \theta_F \simeq \theta^c \boxtimes \theta_{E_1 \otimes F}. \quad \square$$

Corollary 6. *Let* $L \in \mathrm{Pic}^d(C)$ *and*

$$\tau : \mathcal{S}M(n, L) \times J_0 \to \mathcal{M}(n, d)$$

be the tensor product map. Suppose F *is a minimal complementary bundle to* $\mathcal{M}(n, d)$. *Choose* $N \in \mathrm{Pic}^{g-1}(C)$ *to be a line bundle such that* $N^n = L \otimes (\det F)^h$, *where* $h = \gcd(n, d)$. *Then*

$$\tau^* \theta_F = \theta \boxtimes \theta_N^{n^2/h}.$$

Proof. Apply the Proposition with $\mathrm{rk}\, F = n/h$ and $n_1 = n, d_1 = d, n_2 = 1, d_2 = 0$. Then $c = 1$. By Proposition 3,

$$\begin{aligned} \theta_{E_1 \otimes F} &= \theta_N^{n^2/h} \otimes \det{}^*(\det(E_1 \otimes F) \otimes N^{-n^2/h}) \\ &= \theta_N^{n^2/h}. \quad \square \end{aligned}$$

5. Proof of Theorem 1

We apply the Leray spectral sequence to compute the cohomology of $\tau^* \theta_F^k$ on the total space of the covering $\tau : \mathcal{S}M \times J \to \mathcal{M}$ of Section 3. Recall that $\mathcal{S}M = \mathcal{S}M(n, d)$, $J = J_0$, and $\mathcal{M} = \mathcal{M}(n, d)$. Because the fibers of τ are 0-dimensional, the spectral sequence degenerates at the E_2-term and we have

$$H^0(\mathcal{S}M \times J, \tau^* \theta_F^k) = H^0(\mathcal{M}, \tau_* \tau^* \theta_F^k). \tag{1}$$

By Cor. 6 and the Künneth formula, the left-hand side of (1) is

$$\begin{aligned} H^0(\mathcal{S}M \times J, \tau^* \theta_F^k)) &= H^0(\mathcal{S}M \times J, \theta^k \boxtimes \theta_N^{kn^2/h})) \\ &= H^0(\mathcal{S}M, \theta^k) \otimes H^0(J, \theta_N^{kn^2/h}). \end{aligned}$$

By the Riemann-Roch theorem for an abelian variety,

$$h^0(J, \theta_N^{kn^2/h}) = (kn^2/h)^g.$$

So the left-hand side of (1) has dimension

$$h^0(\mathcal{S}M, \theta^k) \cdot (kn^2/h)^g. \tag{2}$$

Next we look at the right-hand side of (1). By the projection formula and Prop. 4,

$$\begin{aligned}\tau_*\tau^*\theta_F^k &= \theta_F^k \otimes \tau_*\mathcal{O}\\ &= \theta_F^k \otimes \sum_{\lambda\in\hat{G}} L_\lambda\\ &= \sum_{\lambda\in\hat{G}} \theta_F^k \otimes L_\lambda.\end{aligned}$$

Our goal now is to show that for any character $\lambda \in \hat{G}$,

$$H^0(\mathcal{M}, \theta_F^k \otimes L_\lambda) \simeq H^0(\mathcal{M}, \theta_F^k). \tag{3}$$

This will follow from two lemmas.

Lemma 7. *The line bundle L_λ on $\mathcal{M}$ is the pullback under* $\det : \mathcal{M} \to J_d$ *of some line bundle N_λ of degree 0 on $J_d := \mathrm{Pic}^d(C)$.*

Lemma 8. *For F a vector bundle as above, k a positive integer, and M a line bundle of degree 0 over C,*

$$H^0(\mathcal{M}, \theta_{F\otimes M}^k) \simeq H^0(\mathcal{M}, \theta_F^k).$$

Assuming these two lemmas, let's prove (3). By Proposition 3,

$$\theta_{F\otimes M} = \theta_F \otimes \det{}^* M^{n_F};$$

hence,

$$\theta_{F\otimes M}^k = \theta_F^k \otimes \det{}^* M^{n_F k}.$$

If $L_\lambda = \det{}^* N_\lambda$, and we choose a root $M = N_\lambda^{1/(n_F k)}$, then

$$\theta_F^k \otimes L_\lambda = \theta_F^k \otimes \det{}^* N_\lambda = \theta_{F\otimes M}^k.$$

Equation (3) then follows from Lemma 8.

Proof of Lemma 7. Define $\alpha : \mathcal{S}M \times J \to J$ to be the projection onto the second factor, $\beta : \mathcal{M} \to J$ to be the composite of $\det : \mathcal{M} \to J_d$ followed by multiplication by $L^{-1} : J_d \to J$, and $\rho : J \to J$ to be the n-th tensor power map. Then there is a commutative diagram

$$\begin{array}{ccc}\mathcal{S}M \times J & \xrightarrow{\tau} & \mathcal{M}\\ \alpha\downarrow & & \downarrow\beta\\ J & \xrightarrow[\rho]{} & J.\end{array}$$

Furthermore, in the map $\alpha : \mathcal{S}M \times J \to J$ we let $G = T_n$ act on J by

$$N.M = N \otimes M, \qquad M \in J,$$

and in the map $\beta : \mathcal{M} \to J$ we let G act trivially on both $\mathcal{M}$ and J. Then all the maps in the commutative diagram above are G-morphisms.

By the push-pull formula ([H], Ch. III, Prop. 9.3, p. 255),

$$\tau_* \alpha^* \mathcal{O}_J = \beta^* \rho_* \mathcal{O}_J.$$

By Proposition 4, $\rho_* \mathcal{O}_J$ is a direct sum of line bundles V_λ on J, where $\lambda \in \hat{G}$. In fact, these V_λ are precisely the n-torsion bundles on J; in particular, their degrees are zero. If $\tau_{L^{-1}} : J_d \to J$ is multiplication by the line bundle L^{-1}, we set $N_\lambda := \tau_{L^{-1}}^* V_\lambda$. Then

$$\begin{aligned} \tau_* \mathcal{O}_{\mathcal{S}M \times J} &= \beta^* \sum_{\lambda \in \hat{G}} V_\lambda \\ &= \det{}^* \tau_{L^{-1}}^* \sum V_\lambda \\ &= \sum \det{}^* N_\lambda. \end{aligned}$$

By Prop. 4, $\tau_* \mathcal{O}_{\mathcal{S}M \times J} = \sum L_\lambda$. Since both L_λ and $\det{}^* N_\lambda$ are eigenbundles of $\tau_* \mathcal{O}_{\mathcal{S}M \times J}$ corresponding to the character $\lambda \in \hat{G}$,

$$L_\lambda = \det{}^* N_\lambda. \quad \square$$

Proof of Lemma 8. Tensoring with $M \in J_0(C)$ gives an automorphism

$$\begin{aligned} \tau_M : \mathcal{M} &\to \mathcal{M} \\ E &\mapsto E \otimes M, \end{aligned}$$

under which

$$\theta_{F \otimes M} = \tau_M^* \theta_F.$$

Hence,

$$\theta_{F \otimes M}^k = \tau_M^* (\theta_F^k)$$

and the lemma follows. $\square$

Returning now to Eq. (1), its right-hand side is

$$\begin{aligned} H^0(\mathcal{M}, \tau_* \tau^* \theta_F^k) &= \sum_{\lambda \in \hat{G}} H^0(\mathcal{M}, \theta_F^k \otimes L_\lambda) \\ &\simeq \sum_{\lambda \in \hat{G}} H^0(\mathcal{M}, \theta_F^k), \qquad \text{(by (3))} \end{aligned}$$

which has dimension

$$h^0(\mathcal{M}, \theta_F^k) \cdot n^{2g}.$$

By (2) the left-hand side of Eq. (1) has dimension

$$h^0(\mathcal{S}M, \theta^k) \cdot (kn^2/h)^g.$$

Equating these two expressions gives

$$h^0(\mathcal{M}, \theta_F^k) = h^0(\mathcal{S}M, \theta^k) \cdot (\frac{k}{h})^g.$$

This completes the proof of Theorem 1.□

6. A conjectural duality

As in the Introduction we start with integers $\bar{n}$, $\bar{d}$, h, k such that $\bar{n}$, h, k are positive and $\gcd(\bar{n}, \bar{d}) = 1$. Take

$$n_1 = h\bar{n},\ d_1 = h\bar{d},\ n_2 = k\bar{n},\ d_2 = k(\bar{n}(g-1) - \bar{d}),\ \textit{and}\ L_1 \in \mathrm{Pic}^{d_1}(C).$$

The tensor product induces a map

$$\tau : \mathcal{S}M(n_1, L_1) \times \mathcal{M}(n_2, d_2) \to \mathcal{M}(n_1 n_2, n_1 n_2(g-1)).$$

As before, write $\mathcal{S}M_1 = \mathcal{S}M(n_1, L_1)$, $\mathcal{M}_2 = \mathcal{M}(n_2, d_2)$, and $\mathcal{M}_{12} = \mathcal{M}(n_1 n_2, n_1 n_2(g-1))$. Let $F_2 = F$ and $F_{12} = \mathcal{O}$ be minimal complementary bundles to $\mathcal{M}_2$ and $\mathcal{M}_{12}$ respectively.

By the pullback formula (Proposition 5)

$$\tau^* \theta_{\mathcal{O}} = \theta^{n_2/\bar{n}} \boxtimes \theta_{E_1}.$$

But by Proposition 3,

$$\theta_{E_1} = \theta_F^h \otimes \det{}^*(L \otimes (\det F)^{-h}).$$

If $L = (\det F)^h$, then $\theta_{E_1} = \theta_F^h$ and

$$\tau^* \theta_{\mathcal{O}} = \theta^k \boxtimes \theta_F^h.$$

By the Künneth formula,

$$H^0(\mathcal{S}M_1 \times \mathcal{M}_2, \tau^* \theta_{\mathcal{O}}) = H^0(\mathcal{S}M_1, \theta^k) \otimes H^0(\mathcal{M}_2, \theta_F^h).$$

In [BNR] it is shown that up to a constant, $\theta_{\mathcal{O}}$ has a unique section s over $\mathcal{M}_{12}$. Then $\tau^* s$ is a section of $H^0(\mathcal{S}M_1 \times \mathcal{M}_2, \tau^* \theta_{\mathcal{O}})$ and therefore induces a natural map

$$H^0(\mathcal{S}M_1, \theta^k)^\vee \to H^0(\mathcal{M}_2, \theta_F^h). \tag{4}$$

We conjecture that this natural map is an isomorphism.

Among the evidence for the duality (4), we cite the following.

i) **(Rank 1 bundles)** The results of [BNR] that

$$H^0(\mathcal{S}M(n,\mathcal{O}),\theta)^\vee \simeq H^0(\mathcal{M}(1,g-1),\theta_{\mathcal{O}}^n)$$

and

$$H^0(\mathcal{M}(n,n(g-1)),\theta_{\mathcal{O}}) = \mathbb{C},$$

are special cases of (4), for $(n_2,d_2) = (1,g-1)$ and $(n_1,d_1) = (1,0)$, respectively.

ii) **(Consistency with Theorem 1)** Given (n_1,d_1,k), a triple of integers, we define $h,\bar{n},\bar{d}$ by

$$h = \gcd(n_1,d_1), n_1 = h\bar{n}, d_1 = h\bar{d},$$

and let n_2, d_2 be as before:

$$n_2 = k\bar{n}, d_2 = k(\bar{n}(g-1)-\bar{d}).$$

Assuming n_1 and k to be positive, it is easy to check that the function

$$(n_1,d_1,k) \mapsto (n_2,d_2,h)$$

is an involution. Write

$$v(n,d,k) = h^0(\mathcal{M}(n,d),\theta_F^k) \text{ and } s(n,d,k) = h^0(\mathcal{S}M(n,d),\theta^k).$$

Then Theorem 1 assumes the form

$$v(n_1,d_1,k)\cdot h^g = s(n_1,d_1,k)\cdot k^g. \tag{5}$$

The duality (4) implies that there is an equality of dimensions

$$s(n_1,d_1,k) = v(n_2,d_2,h). \tag{6}$$

Because $(n_1,d_1,k) \mapsto (n_2,d_2,h)$ is an involution, it follows that

$$s(n_2,d_2,h) = v(n_1,d_1,k). \tag{7}$$

Putting (5), (6), and (7) together, we get

$$v(n_2,d_2,h)k^g = s(n_2,d_2,h)h^g,$$

which is Theorem 1 again.

iii) **(Degree 0 bundles)** Consider the moduli space $\mathcal{S}M(n,0)$ of rank n and degree 0 bundles. In this case,

$$n_1 = n,\ d_1 = 0,\ h = \gcd(n,0) = n,\ n_2 = k,\ d_2 = k(g-1).$$

So the conjectural duality is

$$H^0(\mathcal{S}M(n,\mathcal{O}),\theta^k)^\vee \simeq H^0(\mathcal{M}(k,k(g-1)),\theta_{\mathcal{O}}^n).$$

Because $\mathcal{M}(k, k(g-1))$ is isomorphic to $\mathcal{M}(k, 0)$ (though non-canonically), it follows that in the notation of ii)

$$s(n, 0, k) = v(k, 0, n).$$

According to R. Bott and A. Szenes, this equality follows from Verlinde's formula.

iv) **(Elliptic curves)** We keep the notation above, specialized to the case of a curve C of genus $g = 1$:

$$n_1 = h\bar{n}, d_1 = h\bar{d}, n_2 = k\bar{n}, d_2 = -k\bar{d}.$$

Set $C' := \operatorname{Pic}^{\bar{d}}(C)$. The map sending a line bundle to its dual gives an isomorphism $C' \simeq \operatorname{Pic}^{-\bar{d}}(C)$. If $L \in \operatorname{Pic}^{\bar{d}}(C)$, viewed as a line bundle on C, we let ℓ be the corresponding point in C', and $\mathcal{O}_{C'}(\ell)$ the associated line bundle of degree 1 on the curve C'. There is a natural map

$$\gamma : \operatorname{Pic}^{h\bar{d}}(C) \to \operatorname{Pic}^h(C')$$

which sends $L := L_1 \otimes \cdots \otimes L_h \in \operatorname{Pic}^{h\bar{d}}(C)$ to $L' := \mathcal{O}_{C'}(\ell_1 + \cdots + \ell_h)$, where $L_i \in \operatorname{Pic}^{\bar{d}}(C)$ corresponds to the point $\ell_i \in C'$.

From [A] and [T] we see that there are natural identifications

$$\mathcal{M}(h\bar{n}, h\bar{d}) \simeq S^h \mathcal{M}(\bar{n}, \bar{d}) \simeq S^h \operatorname{Pic}^{\bar{d}}(C) = S^h C'$$

and

$$\mathcal{M}(k\bar{n}, -k\bar{d}) \simeq S^k \mathcal{M}(\bar{n}, -\bar{d}) \simeq S^k \operatorname{Pic}^{-\bar{d}}(C) \simeq S^k C'.$$

Furthermore, there is a commutative diagram

$$\begin{array}{ccc} \mathcal{M}(h\bar{n}, h\bar{d}) & \xrightarrow{\sim} & S^h C' \\ {\scriptstyle \det}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \operatorname{Pic}^{h\bar{d}}(C) & \xrightarrow[\gamma]{} & \operatorname{Pic}^h(C'). \end{array}$$

Since the fiber of the Abel-Jacobi map $\alpha : S^h C' \to \operatorname{Pic}^h(C')$ above L' is the projective space $\mathbb{P}H^0(C', L')$, it follows that there is a natural identification

$$\mathcal{S}M(h\bar{n}, L) \simeq \mathbb{P}H^0(C', L').$$

Since the theta bundle is the positive generator of $\mathcal{S}M(h\bar{n}, L)$, it corresponds to the hyperplane bundle on $\mathbb{P}H^0(C', L')$. For $F \in \mathcal{M}(\bar{n}, \bar{d})$, let

$q \in C'$ be the point corresponding to the line bundle $Q := \det F \in \mathrm{Pic}^{\bar{d}}(C)$. Then

$$\begin{aligned} H^0(\mathcal{S}M(h\bar{n}, (\det F)^h), \theta^k) &\simeq H^0(\mathbb{P}H^0(C', \mathcal{O}_{C'}(hq)), \mathcal{O}(k)) \\ &= S^k H^0(C', \mathcal{O}_{C'}(hq))^\vee. \end{aligned}$$

Recall that each point $q \in C'$ determines a divisor X_q on the symmetric product S^kC':

$$X_q := \{q + D \mid D \in S^{k-1}C'\}.$$

The proof of Theorem 6 in [T] actually shows that if $F \in \mathcal{M}(\bar{n}, -\bar{d})$, then under the identification $\mathcal{M}(k\bar{n}, -k\bar{d}) \simeq S^kC'$, the theta bundle θ_F corresponds to the bundle associated to the divisor X_q on S^kC', where q is the point corresponding to $\det F \in \mathrm{Pic}^{\bar{d}}$. Therefore, by the calculation of the cohomology of a symmetric product in [T],

$$\begin{aligned} H^0(\mathcal{M}(k\bar{n}, -k\bar{d}), \theta_F^h) &= H^0(S^kC', \mathcal{O}(hX_q)) \\ &= S^k H^0(C', \mathcal{O}(hq)). \end{aligned}$$

So the two spaces $H^0(\mathcal{S}M(h\bar{n}, (\det F)^h), \theta^k)$ and $H^0(\mathcal{M}(k\bar{n}, -k\bar{d}), \theta_F^h)$ are naturally dual to each other.

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Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395

E-mail address: donagi@rademacher.math.upenn.edu

Department of Mathematics, Tufts University, Medford, MA 02155-7049

E-mail address: ltu@jade.tufts.edu