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AVERAGE KISSING NUMBERS FOR NON-CONGRUENT SPHERE PACKINGS

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1. Introduction

Let P be a packing of n (round) balls in $\mathbb{R}^3$. (A packing of round balls, also known as a sphere packing, is a collection of round balls with disjoint interiors.) The balls may have different radii. The average kissing number of P is defined as $k(P) = 2m/n$, where m is the number of tangencies between balls in the packing. Let

$$k = \sup\{\, k(P) \mid P \text{ is a finite packing of balls in } \mathbb{R}^3 \,\}.$$

Theorem 1.

$$12.566 \approx 666/53 \le k < 8 + 4\sqrt{3} \approx 14.928.$$

(The appearance of the number of the beast in the lower bound is purely coincidental.)

The supremal average kissing number k is defined in any dimension, as are k_c, the supremal average kissing number for congruent ball packing, and k_s, the maximal kissing number for a single ball surrounded by congruent balls with disjoint interiors. (Clearly, $k_c \le k$ and $k_c \le k_s$.) It is interesting that k is always finite, because a large ball can be surrounded by many small balls in a non-congruent ball packing. Nevertheless, a simple argument presented below shows that $k \le 2k_s$ in every dimension, and clearly k_s is always finite. In two dimensions, an Euler characteristic argument shows that $k \le 6$, but it is also well-known that $k_s = k_c = 6$. One might therefore conjecture that $k = k_c$ always, or at least in dimensions such as 2, 3, 8, and 24 (and conjecturally several others) in which $k_s = k_c$ [1]. Surprisingly, in three dimensions, $k > 12$ even though $k_s = k_c = 12$.

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Remark 1. No packing P achieves the supremum $k = k(P)$, because if P' is a translate of P that meets P in only one point, then $k(P \cup P') > k(P)$.

Let $P = (P_v,\ v \in V)$ be a packing, where V is some indexing set. The *nerve* of P is a combinatorial object that encodes the combinatorics of the packing. It is the (abstract) graph $G = (V, E)$ on V, where an edge $\{u, w\}$ appears in E precisely when P_u and P_w intersect. If P is a packing of round disks in the plane, then it is easy to see that G is a planar graph. Conversely, the circle packing theorem [3] states that every finite planar graph is the nerve of some disk packing in the plane. This non-trivial theorem has received much attention lately, mostly because of its surprising relation with complex analysis. (Compare references [7], [5], and [8].)

Since the nerves of planar disk packings are understood, it is natural to ask for a description of all graphs that are nerves of ball packings in $\mathbb{R}^3$. In lieu of a complete characterization, which is probably intractable, Theorem 1 gives a necessary condition on such graphs: $2|E| < (8 + 4\sqrt{3})|V|$.

We wish to thank Gil Kalai for a discussion which led to the question of estimating k.

2. The upper bound

Theorem 2. *If P is a finite ball packing in $\mathbb{R}^3$, then $k(P) < 8 + 4\sqrt{3}$.*

As a warm-up, we will show that $k(P) \leq 24$. Let E be the set of unordered pairs of balls in P that kiss. Let $r(B)$ be the radius of a ball $B \in P$. By a famous result ([6], [4]), it is impossible for more than 12 unit balls with disjoint interiors to kiss a unit ball B. If C kisses B and $r(C) > 1 = r(B)$, then C contains a (unique) unit ball that kisses B. Thus, in a packing, B cannot kiss more than 12 balls at least as large as B. Consider a function $f : E \to P$ that assigns to $\{B, C\} \in E$ the smaller of the balls B and C, or either if they are the same size. Since f is at most 12 to 1, $|E| \leq 12|P|$. Consequently, $k(P) = 2|E|/|P| \leq 24$.

The proof of Theorem 2 is a refinement of this argument.

Proof. In addition to the above notation, we let $E(B)$ denote the set of $C \in P$ such that $\{B, C\} \in E$.

Let $\rho > 1$ be a constant to be determined below. For each ball $B \in P$, let $S(B)$ be the concentric spherical shell with radius $\rho r(B)$. For each $B, C \in P$, define

$$a(B, C) = \frac{\operatorname{area}(C \cap S(B))}{\operatorname{area}(S(B))}. \tag{1}$$

Since the interiors of the balls in P are disjoint, for any B,

$$1 \geq \sum_{C \in P} a(B,C) \geq \sum_{C \in E(B)} a(B,C). \tag{2}$$

Summing over B,

$$|P| \geq \sum_{\{B,C\} \in E} \left(a(B,C) + a(C,B)\right). \tag{3}$$

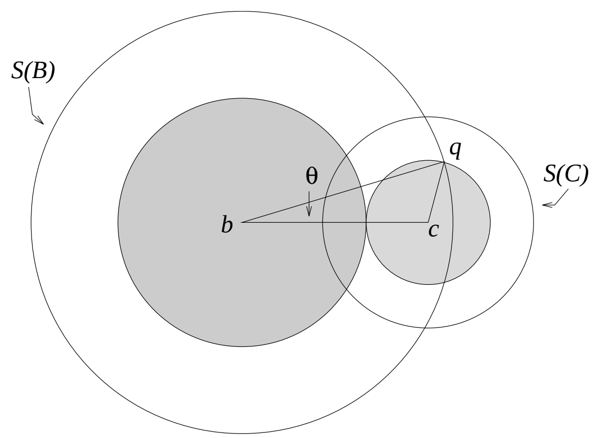

FIGURE 1. The intersection of B and C with a plane passing through their centers.

We will obtain a lower bound on $a(B,C)+a(C,B)$ for two kissing balls B and C. Suppose that B intersects $S(C)$ and C intersects $S(B)$, as shown in Figure 1. Let b and c be the centers of B and C. Let q be a point on the relative boundary in $S(B)$ of the spherical disk $C \cap S(B)$. Clearly,

$$\begin{aligned} d(b,c) &= r(B) + r(C) \\ d(b,q) &= \rho r(B) \\ d(c,q) &= r(C), \end{aligned}$$

where $d(x,y)$ is the distance from x to y. Let $\theta = \angle cbq$ be the angular radius of $C \cap S(B)$. By the law of cosines,

$$\cos\theta = \frac{(r(B)+r(C))^2 + (\rho r(B))^2 - r(C)^2}{2(r(B)+r(C))\rho r(B)} = \frac{r(B) + \rho^2 r(B) + 2r(C)}{2\rho(r(B)+r(C))}. \tag{4}$$

Also,

$$\operatorname{area}(C \cap S(B)) = \frac{1-\cos\theta}{2} \operatorname{area}(S(B)). \tag{5}$$

Combining equations (1), (4) and (5),

$$a(B,C) = \frac{1}{2} - \frac{r(B) + \rho^2 r(B) + 2r(C)}{4\rho(r(B) + r(C))}. \tag{6}$$

Switching B and C and adding,

$$a(B,C) + a(C,B) = 1 - \frac{3+\rho^2}{4\rho}. \tag{7}$$

Isn't it remarkable that $a(B,C) + a(C,B)$ does not depend on $r(B)$ and $r(C)$? We now choose $\rho = \sqrt{3}$ to maximize the right side of equation (7). Then $a(B,C) + a(C,B) = 1 - \frac{\sqrt{3}}{2}$, under the assumption that $S(B) \cap C$ and $S(C) \cap B$ are non-empty. If $S(B) \cap C = \emptyset$, $a(B,C) = 0$, which is greater than the negative value at the right side of equation (6). As a result, $a(B,C) + a(C,B) \geq 1 - \frac{\sqrt{3}}{2}$ in the general case. Applying this inequality to inequality (3) yields $|P| \geq |E| \left(1 - \frac{\sqrt{3}}{2}\right)$, which gives

$$k(P) = 2|E|/|P| \leq 8 + 4\sqrt{3}.$$

In conclusion, $k \leq 8 + 4\sqrt{3}$. By Remark 1, $k(P) < k$, establishing Theorem 2. □

Remark 2. In fact, $k < 8 + 4\sqrt{3}$. Let $B \in P$. Since each ball $C \in E(B)$ that intersects $S(B)$ must have $r(C) \geq (\rho - 1)r(B)/2$, there is a finite bound for the number of balls $C \in E(B)$ such that $a(B,C) > 0$. Therefore there is some $\alpha < 1$ (depending on ρ but not P) such that

$$\sum_{C \in E(B)} a(B,C) \leq \alpha.$$

Using this inequality in place of inequality (2) in the above proof would multiply the upper bound by a factor of α. A good estimate for α would consequently strengthen Theorem 2.

3. The lower bound

Theorem 3. *There exists a sequence of finite packings $\{P_n\}$ with*

$$\lim_{n \to \infty} k(P_n) = 666/53.$$

Observe that all questions about nerves of ball packings and average kissing numbers are invariant under sphere-preserving transformations such as stereographic projection from the 3-sphere S^3 to $\mathbb{R}^3$ and inversion in a sphere.

There exists a packing D in S^3 of 120 congruent spherical balls such that each ball kisses exactly 12 others [2], or 720 kissing points in total. The existence of D already implies that $k(P) > 12$ for some packing P, because by Remark 1, $k > k(D) = 12$.

The proof of Theorem 3 is a refinement of this construction.

Proof. We give an explicit description of D. Let S^3 be the unit 3-sphere in $\mathbb{R}^4$ and let $\tau = \frac{1+\sqrt{5}}{2}$ be the golden ratio. Choose the centers of the balls of D to be the points in the orbits of $\frac{1}{2}(\tau, 1, 1/\tau, 0)$, $\frac{1}{2}(1, 1, 1, 1)$, and $(1, 0, 0, 0)$ under change of sign of any coordinate and even permutations of coordinates. The radius of each ball is 18°. We will need the following four properties of D, which can be verified using the explicit description or by other means: The 12 balls that kiss a given ball have an icosahedral arrangement with 30 mutual kissing points, the centers of two kissing balls of D are 36° apart, the centers of two next-nearest balls of D are 60° apart, and D is self-antipodal. (If X is a point, set of points, or set of set of points in S^3, the antipode of X is given by negating all coordinates in $\mathbb{R}^4$ and is denoted $-X$.)

Let $B_0 \in D$ be a ball with center b and let $P_0 = D \setminus \{B_0, -B_0\}$. The packing P_0 has $720 - 24 = 696$ kissing points and 118 balls. Let R be the set of 12 balls in D that kiss B_0, and let S be the unique sphere centered at b which contains the 30 kissing points between the balls in R. Let $I_S : S^3 \to S^3$ be inversion in the sphere S. Observe that S meets the boundary of each $B \in R$ orthogonally in a circle (because, by symmetry, it is orthogonal to the boundary at each kissing point), and therefore each $B \in R$ is invariant under I_S. Let $\sigma : S^3 \mapsto S^3$ be the map $\sigma(p) = I_S(-p)$. This map σ contracts $S^3 \setminus \{-b\}$ towards b, sends $-S$ to S, and preserves spheres. Because I_S leaves each $B \in R$ invariant, σ sends $-R$ to R. For each $n > 0$, let

$$P_n = P_{n-1} \cup \sigma^n(P_0).$$

We claim that the sphere S does not intersect any ball in $P_0 \setminus R$. Assuming this claim, the packing $Q = P_0 \setminus (R \cup -R)$ lies between $-S$ and S, and $\sigma^n(Q)$ is separated from $\sigma^{n+1}(Q)$ by $\sigma^n(S)$. Therefore each P_n consists of an alternation of layers

$$-R, Q, \sigma(-R) = R, \sigma(Q), \sigma^2(-R), \sigma^2(Q), \dots, \sigma^n(-R)$$

such that each layer only intersects the two neighboring layers and intersects only in kissing points. In particular, each P_n is a packing. Moreover,

P_{n+1} has $118-12=106$ more balls and $696-30=666$ more kissing points than P_n does. Therefore

$$\lim_{n\to\infty} k(P_n) = 2\frac{666}{106} = \frac{666}{53}.$$

It remains to check the claim. Let B_1, B_2 be two kissing balls in R. Let b_1 and b_2 be their centers and let p be their kissing point. Evidently the angular radius of S is $\angle b0p$. Using the inclusion $S^3 \subset \mathbb{R}^4$ and the notation of vector calculus,

$$b_1 \cdot b_2 = b \cdot b_1 = b \cdot b_2 = \tau/2,$$
$$b \cdot b = b_1 \cdot b_1 = b_2 \cdot b_2 = 1,$$
$$p = \frac{b_1+b_2}{|b_1+b_2|},$$
$$\angle b0p = \cos^{-1}\left(\frac{b\cdot(b_1+b_2)}{|b_1+b_2|}\right) = \cos^{-1}\left(\sqrt{\frac{2+\tau}{5}}\right) \approx 31.717^\circ.$$

On the other hand, the center of a ball in P_0 which is not in R is at least 60° away from b, and therefore the closest point of any such ball is at least 42° away from b. Thus, S does not intersect any such ball. □

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