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MACDONALD'S POLYNOMIALS AND REPRESENTATIONS OF QUANTUM GROUPS

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Introduction

Recently I. Macdonald defined a family of systems of orthogonal symmetric polynomials depending on two parameters q, k which interpolate between Schur's symmetric functions and certain spherical functions on $SL(n)$ over the real and p-adic fields [M]. These polynomials are labeled by dominant integral weights of $SL(n)$, and (as was shown by I. Macdonald) are uniquely defined by two conditions: 1) they are orthogonal with respect to a certain weight function, and 2) the matrix transforming them to Schur's symmetric functions is strictly upper triangular with respect to the standard partial ordering on weights ("strictly" means that the diagonal entries of this matrix are equal to 1). Another definition of Macdonald's polynomials is that they are (properly normalized) common eigenfunctions of a commutative set of n self-adjoint partial difference operators $M_1, ..., M_n$ (Macdonald's operators) in the space of symmetric polynomials.

In this paper we present a formula for Macdonald's polynomials which arises from the representation theory of the quantum group $U_q(\mathfrak{gl}_n)$. This formula expresses Macdonald's polynomials as vector-valued characters—(weighted) traces of intertwining operators between certain modules over $U_q(\mathfrak{gl}_n)$. This result was announced in [EK]. It is an interesting problem to relate this construction to a recent paper of Noumi ([No]) which gives an interpretation of Macdonald's polynomials for special values of k as zonal spherical functions on a homogeneous space for a quantum group.

The paper is organized as follows. In Section 1, we define Macdonald's inner product, orthogonal polynomials, and commuting difference operators, and compute the eigenvalues of these operators. In Section 2, we review some facts about representations of quantum groups that will be needed in the following sections. In Section 3 we introduce weighted traces of intertwiners (vector-valued characters) and prove an analogue of the

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Weyl orthogonality theorem for them. In Section 4 we formulate the main result—the explicit formula for Macdonald's polynomials for positive integer values of k–and give a complete proof of this formula. In Section 5, we generalize the result of Section 4 to the case of an arbitrary k. In Section 6, we construct Macdonald's operators from the generators of the center of $U_q(\mathfrak{gl}_n)$, and derive an explicit formula for generic (non-symmetric) eigenfunctions of Macdonald's operators using this construction.

1. Macdonald's polynomials

Here we give the definition and main properties of Macdonald's polynomials for the root system of type A_{n-1}, following [M].

Let $R \subset \mathbb{C}^n$ be the root system of type A_{n-1}: $R = \{\alpha_{ij}\}_{i\neq j}$, $\alpha_{ij} = \varepsilon_i - \varepsilon_j$, $\varepsilon_i = (0,\dots,0,1,0,\dots,0) \in \mathbb{Z}^n$ (1 in the i^{th} place). Also, let $R^+ = \{\alpha_{ij}\}_{i<j}$ be the set of positive roots, and $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ be the simple roots. Let Q be the lattice spanned by the roots, and let $Q^+ \subset Q$ be the semigroup spanned by the positive roots.

A sequence $\lambda = (\lambda_1 \dots \lambda_n) \in (\mathbb{Z}_+)^n$ is called a partition if $\lambda_i \geq \lambda_{i+1}$. We define a partial order on partitions: $\lambda > \mu$ if $\sum \lambda_i = \sum \mu_i$ and $\lambda_1 = \mu_1, \dots, \lambda_k = \mu_k$, $\lambda_{k+1} > \mu_{k+1}$ for some $k < n$.

Let us consider polynomials of n variables $x_1 \dots x_n$: $\mathcal{A} = \mathbb{C}[x_1,\dots,x_n]$. For any $\lambda \in \mathbb{Z}^n$, let $x^\lambda = x_1^{\lambda_1} \dots x_n^{\lambda_n}$. We have an obvious action of the Weyl group S_n on $\mathcal{A}$. We can take a basis of $\mathcal{A}^{S_n}$ formed by the orbitsums $m_\lambda = \sum_{\mu \in S_n\lambda} x^\mu$, where λ runs through the set of all partitions. These functions are orthogonal with respect to the inner product given by $\langle f, g\rangle_0 = [f\bar{g}]_0$, where $\bar{g}(x_1,\dots,x_n) = g(x_1^{-1},\dots,x_n^{-1})$, and $[\ ]_0 \colon \mathbb{C}[x_1^{\pm1},\dots,x_n^{\pm1}] \to \mathbb{C}$ is the constant term.

The main object of our study are Macdonald's polynomials, defined in [M]. This is a family of polynomials depending on two independent variables q, t and defined by the following theorem:

Theorem. (Macdonald) *There exists a unique family of polynomials $P_\lambda(x;q,t) \in \mathbb{C}(q,t)[x]$ $(x = (x_1,\dots,x_n))$, where λ is a partition and $\mathbb{C}(q,t)$ is the field of rational functions in q, t, satisfying the following properties:*

(1) *$P_\lambda(x;q,t)$ is symmetric under the action of S_n on the x's.*
(2) *$P_\lambda(x;q,t) = m_\lambda(x) + \sum_{\mu<\lambda} c_{\lambda\mu} m_\mu(x)$.*
(3) *For fixed q, t the polynomials $P_\lambda(x;q,t)$ are orthogonal with respect to the inner product given by $\langle f, g\rangle_{q,t} = [f\bar{g}\Delta_{q,t}]_0$, where*

$$\Delta_{q,t}(x) = \prod_{i\neq j}\prod_{m=0}^{\infty} \frac{1 - q^{2m}x_i x_j^{-1}}{1 - q^{2m}t^2 x_i x_j^{-1}} = \prod_{\alpha\in R}\prod_{m=0}^{\infty} \frac{1 - q^{2m}x^\alpha}{1 - q^{2m}t^2 x^\alpha}. \tag{1.1}$$

These polynomials are called Macdonald's polynomials (our notation differs slightly from that of Macdonald: what we denote by $P_\lambda(x;q,t)$ in the notations of [M] would be $P_\lambda(q^2,t^2)$).

It is often convenient to consider Macdonald's polynomials for $t=q^k$, $k\in Z_+$; for example, for $k=0$ these polynomials reduce to the orbitsums m_λ, and for $k=1$ to Schur's symmetric functions. However, most of the properties of Macdonald's polynomials obtained for $t=q^k$ can be generalized to the case when q,t are independent variables.

The proof of the theorem is based on the use of the following family of operators in the space $\mathbb{C}(q,t)[x_1,\dots,x_n]^{S_n}$:

$$(1.2)\qquad M_r=t^{r(r-n)}\sum_{i_1<i_2<\dots<i_r}\Bigg(\prod_{\substack{j\notin\{i_1\dots i_r\}\\ l=1\dots r}}\frac{t^2x_{i_l}-x_j}{x_{i_l}-x_j}\Bigg)T_{q^2,x_{i_1}}\dots T_{q^2,x_{i_r}}$$

where $(T_{q^2,x_i}f)(x_1,\dots,x_n)=f(x_1,\dots,q^2x_i,\dots,x_n)$, and $r=1,\dots,n$ (cf. [Ch]).

Proposition 1.1. (Macdonald)

(1) $[M_i,M_j]=0$

(2) *M_r is self-adjoint with respect to the inner product $\langle\cdot,\cdot\rangle_{q,t}$.*

(3) $M_rP_\lambda(x;q,t)=c^r_\lambda P_\lambda(x;q,t)$, *where* $c^r_\lambda=\sum_{i_1<\dots<i_r}\prod_{l=1}^r q^{2\lambda_{i_l}}t^{(n+1-2i_l)}$.

2. The quantum group $U_q\mathfrak{gl}_n$ and its representations

Let q be a formal variable. By definition ([D1, J]), the quantum group $U_q\mathfrak{gl}_n$ is a Hopf algebra with unit over the field $\mathbb{C}(q)$ of rational functions in q with generators e_i,f_i, $i=1\dots n-1$, q^{h_i}, $i=1\dots n$, in which multiplication, comultiplication, counit and antipode are defined by the relations

$$(2.1)\qquad \left\{\begin{matrix}[h_i,h_j]=0 & [h_i,e_i]=e_i\\ [h_i,f_i]=-f_i & [h_{i+1},e_i]=-e_i\\ [h_{i+1},f_i]=f_i & [h_i,e_j]=[h_i,f_j]=0,\quad j\neq i,i+1\end{matrix}\right\};$$

$$[e_i,f_j]=\delta_{ij}\frac{q^{h_i-h_{i+1}}-q^{h_{i+1}-h_i}}{q-q^{-1}};$$

$$\left\{\begin{matrix}e_i^2e_j-(q+q^{-1})e_ie_je_i+e_je_i^2=0\\ f_i^2f_j-(q+q^{-1})f_if_jf_i+f_jf_i^2=0\end{matrix}\right\},\qquad i=j\pm1;$$

$$[f_i,f_j]=0=[e_i,e_j],\qquad |i-j|>1;$$

$$\Delta e_i = e_i \otimes q^{(h_{i+1}-h_i)/2} + q^{(h_i-h_{i+1})/2} \otimes e_i;$$
$$\Delta f_i = f_i \otimes q^{(h_{i+1}-h_i)/2} + q^{(h_i-h_{i+1})/2} \otimes f_i;$$
$$\Delta h_i = h_i \otimes 1 + 1 \otimes h_i;$$
$$\epsilon(q^{h_i}) = 1, \qquad \epsilon(e_i) = \epsilon(f_i) = 0,$$
$$S(e_i) = -q^{-1}e_i, \qquad S(f_i) = -qf_i, \qquad S(h_i) = -h_i.$$

In the limit $q \to 1$, $U_q\mathfrak{gl}_n$ becomes the universal enveloping algebra of the Lie algebra $\mathfrak{gl}_n$: one can identify the generators with the matrix units as follows: $e_i = E_{ii+1}$, $f_i = E_{i+1i}$, $h_i = E_{ii}$.

Like its classical analogue, $U_q\mathfrak{gl}_n$ admits the following polarization:

$$U_q\mathfrak{gl}_n = U^- \cdot U^0 \cdot U^+,$$

where $U^{\pm}$ is the subalgebra generated by e_i (respectively, f_i), and U^0 is the algebra generated by q^{h_i}. $U_q\mathfrak{gl}_n$ also admits an algebra automorphism ω (Cartan involution), which transposes U^+ and U^-: $\omega e_i = -f_i$, $\omega f_i = -e_i$, $\omega h_i = -h_i$. Note that ω is a coalgebra antiautomorphism.

The representation theory of $U_q\mathfrak{gl}_n$ is quite parallel to the classical case. Unless otherwise stated, we consider only finite-dimensional representations. Define the Cartan subalgebra $\mathfrak{h}$ to be the linear span of h_i; then every $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n$ can be considered as a weight, i.e. an element of $\mathfrak{h}^*$ by $\lambda(h_i) = \lambda_i$. We have a bilinear form on the weights given by $\langle \lambda, \mu \rangle = \sum \lambda_i \mu_i$, which allows us to identify $\mathfrak{h} \simeq \mathfrak{h}^*$. We keep the notations R, R^+, Q, Q^+ from Section 1.

Define the set of integral weights $P = \{\lambda | \lambda_i - \lambda_j \in \mathbb{Z}\}$ and the set of dominant weights $P_+ = \{\lambda | \lambda_i - \lambda_{i+1} \in \mathbb{Z}_+\}$. Note that $\lambda \in P_+$ iff $\lambda + a(1, \dots, 1)$ is a partition for some $a \in \mathbb{C}$. We have a natural order on P which is defined precisely in the same way as in Section 1. It is also convenient to introduce fundamental weights $\omega_i = (1, \dots, 1, 0 \dots, 0)$ (i ones), $i = 1 \dots, n-1$. Then $\lambda \in P_+$ iff $\lambda = a(1, \dots, 1) + \sum n_i \omega_i$, $n_i \in \mathbb{Z}_+$. As usual, let $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha = (\frac{n-1}{2}, \frac{n-3}{2}, \dots, \frac{1-n}{2})$.

For every $\lambda \in P_+$ we denote by V_λ the finite-dimensional irreducible representation of $U_q\mathfrak{gl}_n$ with highest weight λ.

It is known that for any V, W the representations $V \otimes W$ and $W \otimes V$ are isomorphic, but the isomorphism is non-trivial. More precisely (see [D1]), there exists a universal R-matrix $\mathcal{R} \in U_q\mathfrak{gl}_n \hat{\otimes} U_q\mathfrak{gl}_n$ ($\hat{\otimes}$ should be understood as a completed tensor product) such that

$$\check{R}_{V,W} = P \circ \pi_V \otimes \pi_W(\mathcal{R}) \colon V \otimes W \to W \otimes V \tag{2.2}$$

is an isomorphism of representations. Here P is the transposition: $Pv\otimes w = w\otimes v$. Also, it is known that $\mathcal{R}$ has the following form:

$$\mathcal{R} = q^{-\sum h_i\otimes h_i}\mathcal{R}^*, \quad \mathcal{R}^*\in U^+\otimes U^- \\ (\epsilon\otimes 1)(\mathcal{R}^*) = (1\otimes\epsilon)(\mathcal{R}^*) = 1\otimes 1. \tag{2.3}$$

We will also need the notion of dual representation. Namely, if V is a representation of $U_q\mathfrak{gl}_n$ then by definition V^* is the representation of $U_q\mathfrak{gl}_n$ in the dual space to V given by $\langle xv^*, v\rangle = \langle v^*, S(x)v\rangle$. One easily checks that in this case the canonical pairing $V^*\otimes V\to\mathbb{C}$ and embedding $\mathbb{C}\to V\otimes V^*$ are homomorphisms of representations (the order of the factors is important here). Also, one has canonical isomorphisms: $\mathrm{Hom}_{U_q\mathfrak{gl}_n}(V,W) = \mathrm{Hom}_{U_q\mathfrak{gl}_n}(\mathbb{C}, V^*\otimes W) = (V^*\otimes W)^{U_q\mathfrak{gl}_n}$. For an irreducible representation V_λ, $(V_\lambda)^* \simeq V_{\lambda^*}$, where $(\lambda_1\dots\lambda_n)^* = (-\lambda_n,\dots,-\lambda_1)$. If we take $(V^*)^*$, we get another action of $U_q\mathfrak{gl}_n$ on the same space V. These two actions are isomorphic: $q^{-2\rho}: V\to V^{**}$ is an isomorphism of $U_q\mathfrak{gl}_n$-modules. This is due to the fact that $S^2(x) = q^{-2\rho}xq^{2\rho}$.

Finally, if V is an irreducible representation of $U_q\mathfrak{gl}_n$ let us consider the action of $U_q\mathfrak{gl}_n$ in V given by $\pi_{V^\omega}(x) = \pi(\omega x)$, where ω is the Cartan involution defined above. We denote V endowed with this action by V^ω. One can easily check that $V\simeq (V^\omega)^*$ (which is, of course, equivalent to saying that $V^\omega\simeq V^*$); that is, there exists non-degenerate pairing $V\otimes V^\omega\to\mathbb{C}$ which commutes with the action of $U_q\mathfrak{gl}_n$. In other words, there exists a non-degenerate bilinear pairing (Shapovalov form) $(\cdot,\cdot)_V\colon V\otimes V\to\mathbb{C}$ such that $(xv,v')_V = (v,\omega S(x)v')_V$. This form is symmetric (which relies on $\omega S\omega = S^{-1}$).

Note also that $(V\otimes W)^\omega = W^\omega\otimes V^\omega$ and that if $\Phi\colon V\to W$ is an intertwiner then Φ is also an intertwiner considered as a map $V^\omega\to W^\omega$.

3. Traces of intertwiners and the generalized Weyl orthogonality theorem

Let V, U be finite-dimensional representations of $U_q\mathfrak{gl}_n$, and $\Phi\colon V\to V\otimes U$ be a non-zero intertwining operator for $U_q\mathfrak{gl}_n$.

Definition. *A vector-valued character is the following function of* $x = (x_1,\dots,x_n)$:

$$\chi_\Phi(x_1,\dots,x_n) = \mathrm{Tr}\,|_V(\Phi x_1^{h_1}\dots x_n^{h_n}). \tag{3.1}$$

From the definition it is clear that χ_Φ is a linear combination of monomials x^μ where μ runs over the set of weights of V. Thus, we can consider

χ as an element of $\mathcal{A}\otimes U$, where $\mathcal{A}$ is the group algebra of the weight group:

$$\mathcal{A} = \mathbb{C}(q)[P] \simeq \Big\{ \sum_{\lambda\in P} a_\lambda x^\lambda \Big| a_\lambda \in \mathbb{C}(q), \text{ almost all } a_\lambda = 0 \Big\}.$$

We will sometimes call elements of $\mathcal{A}$ generalized Laurent polynomials in x_i; also, we will write x^h instead of $x_1^{h_1}\dots x_n^{h_n}$. Note that the elements of $\mathcal{A}$ can also be interpreted as functions on $\mathfrak{h}$ by letting $x_i(\sum z_j h_j) = e^{z_i}$. This is the same as considering the function on $\mathfrak{h}$ given by $\chi(h) = \operatorname{Tr}|_V(\Phi e^h)$.

In particular, for $V = V_\lambda$, $\chi_\Phi \in x^\lambda \mathbb{C}(q)[\frac{x_2}{x_1},\dots,\frac{x_n}{x_{n-1}}]\otimes U$; the highest term of χ is ux^λ and the lowest term is $u'x^{-\lambda^*}$ for some $u, u' \in U$. Note that in the contrast with the classical case, χ_Φ is not S_n-symmetric if U is not a trivial representation.

Using the notion of dual representation, we can rewrite χ_Φ as follows: we can identify Φ with an intertwiner $\Phi\colon V^*\otimes V \to U$; then $\chi_\Phi(x) = \Phi(1\otimes x^h)\sum v^i\otimes v_i$, where v_i, v^i are the dual bases in V, V^*. Note that $\sum v^i\otimes v_i = (1\otimes q^{-2\rho})\mathbf{1}$, where $\mathbf{1} = i(1)$, $i\colon \mathbb{C}\to V^*\otimes V$ being an embedding of $U_q\mathfrak{gl}_n$-modules. In particular, this implies that $\chi_\Phi(q^{2\rho}) = 0$ if U is a non-trivial irreducible representation.

Generalized Weyl Orthogonality Theorem. *Let $\Phi_\lambda : V_\lambda \to V_\lambda\otimes U$, $\Phi_\mu\colon V_\mu \to V_\mu\otimes U$ be intertwiners, and $\lambda\neq\mu$. Then the characters $\chi_1 = \chi_{\Phi_\lambda}$, $\chi_2 = \chi_{\Phi_\mu}$ are orthogonal with respect to the following inner product: $\langle f, g\rangle_1 = [(f,\bar g)_U\Delta]_0$, where $\Delta = \prod_{\alpha\in R}(1-x^\alpha)$, $(\cdot,\cdot)_U$ is the Shapovalov form and all the other notations are as in Section 1.*

Proof. As was explained above, we can as well consider Φ_μ as an intertwiner $V_\mu^\omega \to U^\omega\otimes V_\mu^\omega$. Thus, $(\chi_1(x),\chi_2(x^{-1}))_U = \operatorname{Tr}|_{V_\lambda\otimes V_\mu^\omega}(\Psi x^h\otimes x^h)$ (note the change of sign of h in the second factor!), where the intertwiner $\Psi\colon V_\lambda\otimes V_\mu^\omega \to V_\lambda\otimes V_\mu^\omega$ is defined as the following composition

$$V_\lambda\otimes V_\mu^\omega \xrightarrow{\Phi_\lambda\otimes\Phi_\mu^\omega} V_\lambda\otimes U\otimes U^\omega\otimes V_\mu^\omega \xrightarrow{\mathrm{Id}\otimes(\cdot,\cdot)_U\otimes\mathrm{Id}} V_\lambda\otimes V_\mu^\omega.$$

Since $V_\lambda\otimes V_\mu^\omega = \bigoplus N_\nu V_\nu$, we see that $(\chi_1(x),\chi_2(x^{-1}))_U$ is a linear combination of usual characters $\chi_\nu(x)$. But since these characters are the same as for $\mathfrak{gl}_n$, we know that $[\chi_\nu(x)\Delta]_0 = 0$ unless $\nu = 0$. On the other hand, it is known that if $\lambda\neq\mu$ then the decomposition of $V_\lambda\otimes V_\mu^\omega$ does not contain the trivial representation (i.e. $N_0 = 0$); thus, in this case χ_1 and χ_2 are orthogonal. □

4. The main theorem

Through this section, we assume $k\in\mathbb{N}$ and show how one gets Macdonald's polynomials $P_\lambda(x;q,q^k)$ as vector-valued characters. Let U be the finite-dimensional representation of $U_q\mathfrak{gl}_n$ with the highest weight $(k-1)n\omega_1-(k-1)(1,\dots,1)=(k-1)(n-1,-1,\dots,-1)$; as a $U_q\mathfrak{sl}_n$-module, this is a q-analogue of the representation $S^{(k-1)n}\mathbb{C}^n$. Note that all the weight subspaces in U are one-dimensional; this property will be very useful to us.

Lemma. *A non-zero $U_q\mathfrak{gl}_n$-homomorphism $\Phi\colon V_\lambda\to V_\lambda\otimes U$ exists iff $\lambda-(k-1)\rho\in P_+$; if it exists, it is unique up to a factor.*

As we discussed before, it suffices to prove this lemma for $\mathfrak{gl}_n$, which is a standard exercise.

Let us consider the (non-zero) intertwiners

$$\Phi_\lambda : V_{\lambda+(k-1)\rho}\to V_{\lambda+(k-1)\rho}\otimes U,\qquad \lambda\in P_+, \tag{4.1}$$

and the corresponding traces

$$\varphi_\lambda(x)=\chi_{\Phi_\lambda}(x)=\operatorname{Tr}|_{V_{\lambda+(k-1)\rho}}(\Phi_\lambda x^h). \tag{4.2}$$

As we discussed before, $\varphi_\lambda(x)$ has the form $\varphi_\lambda(x)=x^{\lambda+(k-1)\rho}p(x)$, $p(x)\in\mathbb{C}(q)[\frac{x_2}{x_1},\dots,\frac{x_n}{x_{n-1}}]\otimes U$. It takes values in the zero-weight subspace $U[0]$, which is one-dimensional; therefore, we can regard it as a complex-valued function. We choose the normalization of Φ and the identification $U[0]\simeq\mathbb{C}$ in such a way that the coefficient at the highest term is one: $\varphi_\lambda(x)=x^{\lambda+(k-1)\rho}+\cdots$.

Proposition 4.1.

$$\varphi_0(x)=\prod_{i=1}^{k-1}\prod_{\alpha\in R^+}(x^{\alpha/2}-q^{2i}x^{-\alpha/2}). \tag{4.3}$$

Proof. First, we prove the following statement:

Lemma 1. *$\varphi_\lambda(x)$ is divisible by $(1-q^{2j}x^{-\alpha})$ for any positive root α and $1\le j\le k-1$.*

The proof is done in several steps. Let us introduce $F_i=f_iq^{(h_{i+1}-h_i)/2}$; then

$$\Delta(F_i)=F_i\otimes q^{h_{i+1}-h_i}+1\otimes F_i \tag{4.4}$$

Let F be a (non-commutative) polynomial in $F_1,\dots,F_{n-1}$ of weight $-\alpha$, $\alpha\in Q^+$. Let $\varphi_\lambda^F=\operatorname{Tr}|_{V_\lambda}(\Phi_\lambda Fx^h)$. Also, let us fix a basis in U: $U[\alpha]=\mathbb{C}w_\alpha$, $\alpha\in Q$. Then

Lemma 2. *There exists a polynomial $P_F \in \mathbb{C}(q)[P]$ such that*

$$\varphi_\lambda^F(x) = \frac{P_F(x)\varphi_\lambda(x)}{\prod_{\beta\le\alpha,\,\beta\in Q^+}(1-q^{\langle\beta,\beta\rangle}x^{-\beta})} w_{-\alpha}. \tag{4.5}$$

Proof is by induction in $\alpha \in Q^+$. For $\alpha = 0$ the statement is obvious. Now, let $\alpha = \sum m_i\alpha_i$, $\sum m_i = m$ and assume that the statement is proved for all $\alpha' < \alpha$. Take $F = F_{j_1}\dots F_{j_m}$. Then $\Delta F = \Delta(F_{j_1})\dots\Delta(F_{j_m}) = \sum_i q^{\sigma_i}F(\gamma_i)\otimes\tilde{F}(\alpha-\gamma_i)q^{-h_{\gamma_i}} + F\otimes q^{-h_\alpha}$, where $\gamma_i \in Q^+,\ \gamma_i<\alpha,\ \sigma_i\in\mathbb{Z}$, and $F(\gamma)$ has weight $-\gamma$. Therefore, using the intertwining property of Φ_λ and the cyclic property of the trace, we get

$$\varphi_\lambda^F(x) = \operatorname{Tr}(\Delta(F)\Phi_\lambda x^h) = q^{-h_\alpha}\operatorname{Tr}(F\Phi_\lambda x^h) + A = q^{\langle\alpha,\alpha\rangle}x^{-\alpha}\varphi_\lambda^F(x) + A,$$

where $A = \sum q^{\sigma_i}\tilde{F}(\alpha-\gamma_i)q^{-h_{\gamma_i}}\operatorname{Tr}(F(\gamma_i)\Phi_\lambda x^h)$. Thus,

$$\varphi_\lambda^F(x) = \frac{1}{1-q^{\langle\alpha,\alpha\rangle}x^{-\alpha}}A.$$

On the other hand, it follows from the induction assumption that A is an expression of the form (4.5) containing only the factors $1-q^{\langle\beta,\beta\rangle}x^{-\beta}$ with $\beta<\alpha$ in the denominator. This completes the proof of Lemma 2.

Lemma 3. *Let $\alpha\in R^+$, and let F_α be a (non-commutative) polynomial in F_i which in the limit $q=1$ becomes a root element of $\mathfrak{sl}_n$. Then $P_{F_\alpha^{k-1}}$ is a non-zero polynomial relatively prime to $\prod_{j=1}^{k-1}(1-q^{2j}x^{-\alpha})$.*

It suffices to prove this lemma for $q=1$. But for $q=1$, $\Delta(F_\alpha) = F_\alpha\otimes 1 + 1\otimes F_\alpha$, and therefore

$$\begin{aligned}\varphi_\lambda^{F_\alpha^{k-1}} &= \operatorname{Tr}(\Phi_\lambda F_\alpha^{k-1}x^h) = F_\alpha\operatorname{Tr}(\Phi_\lambda F_\alpha^{k-2}x^h) + \operatorname{Tr}(\Phi F_\alpha^{k-2}x^hF_\alpha)\\ &= F_\alpha\varphi_\lambda^{F_\alpha^{k-2}} + x^{-\alpha}\varphi_\lambda^{F_\alpha^{k-1}},\end{aligned}$$

so

$$\varphi_\lambda^{F_\alpha^{k-1}} = (1-x^{-\alpha})^{-1}F_\alpha\varphi_\lambda^{F_\alpha^{k-2}}(x) = \dots = (1-x^{-\alpha})^{1-k}\varphi_\lambda(x)F_\alpha^{k-1}w_0.$$

Since $F_\alpha^{k-1}w_0 = c_\alpha w_{(1-k)\alpha}$ for some $c_\alpha\ne 0$, we see that

$$\begin{aligned}P_{F_\alpha^{k-1}} &= c_\alpha\frac{\prod_{\beta\le(k-1)\alpha}(1-x^{-\beta})}{(1-x^{-\alpha})^{k-1}}\\ &= c_\alpha\prod_{\substack{\beta<(k-1)\alpha\\ \beta\ne s\alpha}}(1-x^{-\beta})\prod_{s=1}^{k-1}(1+x^{-\alpha}+\dots+x^{(1-s)\alpha}).\end{aligned}$$

One can easily see that this polynomial is relatively prime to $1 - x^{-\alpha}$. Thus, we have proved Lemma 3.

Now, let us return to the proof of Lemma 1. Let us write

$$\mathrm{Tr}(\Phi_\lambda F_\alpha^{k-1} x^h) = \frac{P_{F_\alpha^{k-1}} \varphi_\lambda(x)}{\prod_{j=1}^{k-1}(1 - q^{2j}x^{-\alpha})}.$$

Since the left-hand side is a non-zero Laurent polynomial in x_i, and $P_{F_\alpha^{k-1}}$ is relatively prime to $\prod_{j=1}^{k-1}(1-q^{2j}x^{-\alpha})$, we see that $\varphi_\lambda(x)$ must be divisible by $\prod_{j=1}^{k-1}(1 - q^{2j}x^{-\alpha})$. So Lemma 1 is proved.

Now it is easy to prove Proposition 4.1: Lemma 1 implies that $\varphi_0 = f(x)\prod_{j=1}^{k-1}\prod_{\alpha\in R^+}(1-q^{2j}x^{-\alpha})$ for some Laurent polynomial $f(x)$; comparing the highest and the lowest terms on both sides we see that $f = 1$. This completes the proof of Proposition 4.1 □

Now we can formulate our main theorem:

Theorem 1. *If λ is a partition then $\varphi_\lambda(x)$ is divisible by $\varphi_0(x)$, and the ratio $\varphi_\lambda(x)/\varphi_0(x)$ is the Macdonald's polynomial $P_\lambda(x; q, q^k)$.*

Proof. Let us first prove that $\varphi_\lambda(x)$ is divisible by $\varphi_0(x)$, and the ratio is a symmetric generalized Laurent polynomial in x_i with highest term x^λ. Consider the tensor product $V = V_\lambda \otimes V_{(k-1)\rho}$. It decomposes as follows: $V = V_{\lambda+(k-1)\rho} + \sum_{\mu<\lambda} N_{\lambda\mu} V_{\mu+(k-1)\rho}$. Consider the intertwiner $\Phi = \mathrm{Id}_{V_\lambda} \otimes \Phi_0 : V \to V \otimes U$. On one hand, it follows from the definition that $\mathrm{Tr}(\Phi x^h) = \chi_{V_\lambda}\varphi_0$, where χ_{V_λ} is the (usual) character of V_λ. On the other hand, the decomposition of V implies that $\mathrm{Tr}(\Phi x^h) = \varphi_\lambda + \sum_{\mu<\lambda} a_{\lambda\mu}\varphi_\mu$, and thus $\varphi_\lambda/\varphi_0 = \chi_{V_\lambda} + \sum_{\mu<\lambda} a_{\lambda\mu}\varphi_\mu/\varphi_0$. Since χ_{V_λ} is a symmetric Laurent polynomial in x_i, it follows by induction in λ that $\varphi_\lambda/\varphi_0$ is also a symmetric Laurent polynomial.

Using S_n-symmetry and the fact that λ is a partition, it is easy to show that in fact $\varphi_\lambda(x)/\varphi_0(x)$ is a polynomial, i.e. belongs to $\mathbb{C}[x_1, \ldots, x_n]^{S_n}$. Thus, it suffices to prove that these ratios are orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_{q,t}$. This immediately follows from the generalized Weyl orthogonality theorem and Proposition 4.1. Indeed, we know from the generalized Weyl orthogonality theorem that $[\varphi_\lambda\bar{\varphi}_\mu\Delta]_0 = 0$ if $\lambda \neq \mu$. Therefore, $[(\varphi_\lambda/\varphi_0)\overline{(\varphi_\mu/\varphi_0)}\varphi_0\bar{\varphi}_0\Delta]_0 = 0$. Due to Proposition 4.1, $\varphi_0\bar{\varphi}_0\Delta = \prod_{\alpha\in R}\prod_{i=0}^{k-1}(1-q^{2i}x^\alpha) = \Delta_{q,t}(x)$, which proves the orthogonality of $\{\varphi_\lambda(x)/\varphi_0(x)\}$ with respect to the inner product $\langle\cdot,\cdot\rangle_{q,t}$. □

5. The case of generic k

In this section we show how to get Macdonald's polynomials for the case when q and t are independent variables. However, it will be convenient to introduce formal variable k such that $t = q^k$; thus, q and q^k are algebraically independent variables. One can check that all the formulas can be rewritten in such a way that k appears only in the expression q^k and thus we could avoid using k, writing everything entirely in terms of q, t; however, this would make our construction less transparent. Also, we must consider the algebra $U_q\mathfrak{gl}_n$, as well as the representations, over the field $\mathbb{C}(q,t)$ rather than $\mathbb{C}(q)$.

Let M_μ be a Verma module with highest weight μ over $U_q\mathfrak{gl}_n$, and v_μ be the corresponding highest weight vector. We choose a homogeneous basis a_i in U^-; then the basis in M_μ is given by $a_i v_\mu$. In particular, this applies to the module $M_{\lambda+(k-1)\rho}$, which is a natural generalization of the module considered in the previous section. Note that if k is a formal variable then this module is irreducible.

We can also introduce the analogue of the module U. Indeed, let

$$W_k = \{(x_1 \dots x_n)^{k-1} p(x), p(x) \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}], \deg p = 0\}$$

with the action of $U_q\mathfrak{gl}_n$ given by

(5.1)
$$h_i \mapsto x_i \frac{\partial}{\partial x_i} - (k-1), \; e_i \mapsto x_i D_{i+1}, \; f_i \mapsto x_{i+1} D_i$$
$$(D_i f)(x_1, \dots, x_n) = \frac{f(x_1, \dots, q x_i, \dots, x_n) - f(x_1, \dots, q^{-1} x_i, \dots, x_n)}{(q - q^{-1}) x_i}$$

W_k is an irreducible infinite-dimensional module over $U_q\mathfrak{gl}_n$. The set of weights of W_k is the root lattice Q, and every weight subspace is one-dimensional: $W_k[\lambda] = \mathbb{C} w_\lambda, \; w_\lambda = (x_1 \dots x_n)^{k-1} x^\lambda$.

If we replace the formal variable k in the formulas above by a positive integer k then W_k has a finite-dimensional submodule $U_k = W_k \cap \mathbb{C}[x_1, \dots, x_n]$; it coincides with the module U defined in Section 4.

Lemma 5.1. *For every $\lambda \in P_+$ there exists a unique up to a constant factor intertwiner*

$$\tilde{\Phi}^k_\lambda \colon M_{\lambda+(k-1)\rho} \to M_{\lambda+(k-1)\rho} \otimes W_k. \tag{5.2}$$

We use the notation $\tilde{\Phi}$ to distinguish these intertwiners from those for finite-dimensional modules introduced in Section 4; the same convention applies to all other notations.

The proof is based on the general fact: if the Verma module M_μ is irreducible then the space of intertwiners $M_\mu \to M_\mu \otimes W$ is in one-to-one correspondence with the zero-weight subspace $W[0]$.

Let us fix the normalization of $\tilde{\Phi}^k_\lambda$ by choosing a highest-weight vector $v_{\lambda+(k-1)\rho} \in M_{\lambda+(k-1)\rho}[\lambda + (k-1)\rho]$ and requiring that $\tilde{\Phi}^k_\lambda v_{\lambda+(k-1)\rho} = v_{\lambda+(k-1)\rho} \otimes w_0 + \cdots$. Then one can find explicit formulas for matrix elements of $\tilde{\Phi}$ as follows: write

$$\tilde{\Phi}^k_\lambda(a_i v_{\lambda+(k-1)\rho}) = \sum \tilde{R}^{ijl}_\lambda a_j v_{\lambda+(k-1)\rho} \otimes w_l. \tag{5.3}$$

Then the condition for $\tilde{\Phi}$ to be an intertwiner can be rewritten as a system of linear equations on $\tilde{R}^{ijl}_\lambda$. Due to Lemma 5.1, this system has a unique solution. From this approach one can easily see that $\tilde{R}^{ijl}_\lambda$ is a rational function in q, q^k.

Similarly to Section 4, define the trace: $\tilde{\varphi}^k_\lambda(x) = \operatorname{Tr}|_{M_{\lambda+(k-1)\rho}}(\tilde{\Phi}^k_\lambda x^h)$.

Again, $\tilde{\varphi}^k_\lambda(x)$ takes values in $W_k[0]$, which is one-dimensional; so we consider $\tilde{\varphi}$ as a scalar-valued function, identifying $W_k[0] \simeq \mathbb{C}$ so that $w_0 \mapsto 1$. Then

$$\tilde{\varphi}^k_\lambda(x) = x^{\lambda+(k-1)\rho}\Big(1 + \sum_{\mu \in Q_+} \tilde{R}_{\lambda\mu}(q, q^k) x^{-\mu}\Big), \tag{5.4}$$

and $\tilde{R}_{\lambda\mu}$ are rational functions of q, q^k.

Theorem 2. *$\tilde{\varphi}^k_\lambda(x)/\tilde{\varphi}^k_0(x)$ is the Macdonald's polynomial $P_\lambda(x; q, q^k)$.*

Proof. Let k be a positive integer, $\lambda \in P_+$ and compare the traces and intertwiners defined above with φ^k_λ, Φ^k_λ defined in Section 4. First, note that U defined in Section 4 is a submodule in the module W_k defined in the beginning of this section, so we can as well consider Φ as an intertwiner $V_{\lambda+(k-1)\rho} \to V_{\lambda+(k-1)\rho} \otimes W_k$. Next, the irreducible module $V_{\lambda+(k-1)\rho}$ is a factor module of the Verma module $M_{\lambda+(k-1)\rho}$. Moreover, if

$$\mu = \sum n_i \alpha_i, \qquad n_i \in \mathbb{Z}_+, \ \sum n_i < k \tag{5.5}$$

then $\dim V_{\lambda+(k-1)\rho}[\lambda + (k-1)\rho - \mu] = \dim M_{\lambda+(k-1)\rho}[\lambda + (k-1)\rho - \mu]$. Thus, if we consider elements a_i of the basis in U^- such that $\mu = -$weight a_i satisfies condition (5.5) then the vectors $a_i v_{\lambda+(k-1)\rho}$ form a basis in the corresponding weight subspaces of $V_{\lambda+(k-1)\rho}$. Let us consider the restriction of the operator Φ to these subspaces. Then it can be written in the form $\Phi^k_\lambda(a_i v_{\lambda+(k-1)\rho}) = \sum R^{ijl;k}_\lambda a_j v_{\lambda+(k-1)\rho} \otimes w_l$, where the coefficients $R^{ijl;k}_\lambda(q)$ are rational functions of q. They can be found by solving the system of equations expressing the intertwining property of Φ. This is the same system which defined the coefficients $\tilde{R}^{ijl}_\lambda(q, q^k)$ in the expansion (5.3) of the intertwiners $\tilde{\Phi}$, but now we consider k as a positive integer, not a formal variable. Still one can check that if we restrict ourselves to considering only $R^{ijl;k}_\lambda$ such that both $-$wt a_i, $-$wt a_j satisfy (5.5) then this system has a unique solution. Thus, we have the following lemma.

Lemma. *For fixed λ, i, j, l such that $-$weight a_i, $-$weight a_j satisfy* (5.5),

$$R_\lambda^{ijl;k}(q) = \tilde{R}_\lambda^{ijl}(q, q^k) \tag{5.6}$$

for $k \in \mathbb{Z}_+$, $k \gg 0$. Here the right-hand side should be understood as a rational function of q obtained by substituting $t = q^k$, $k \in \mathbb{Z}_+$ in the rational function of two variables $\tilde{R}_\lambda^{ijl}(q,t)$.

Corollary 1. *If we write*

$$\varphi_\lambda^k(x) = x^{\lambda+(k-1)\rho}\Big(1 + \sum_{\mu\in Q^+} R_{\lambda\mu}^k(q)x^{-\mu}\Big),$$

then for fixed λ, μ and $k \in \mathbb{Z}_+, k \gg 0$ we have $R_{\lambda\mu}^k(q) = \tilde{R}_{\lambda\mu}(q, q^k)$.

Let us consider the ratios $\tilde{\varphi}_\lambda^k/\tilde{\varphi}_0^k, \varphi_\lambda^k/\varphi_0^k$. Clearly, they can be written in the form

$$\frac{\tilde{\varphi}_\lambda^k(x)}{\tilde{\varphi}_0^k(x)} = x^\lambda\Big(1+\sum_{\mu\in Q^+} \tilde{Q}_{\lambda\mu}(q, q^k)x^{-\mu}\Big), \qquad \frac{\varphi_\lambda^k(x)}{\varphi_0^k(x)} = x^\lambda\Big(1+\sum_{\mu\in Q^+} Q_{\lambda\mu}^k(q)x^{-\mu}\Big) \tag{5.7}$$

(in fact, the latter sum is finite due to Theorem 1).

Then Corollary 1 above immediately implies the following:

Corollary 2. *For fixed λ, μ, $Q_{\lambda\mu}^k(q) = \tilde{Q}_{\lambda\mu}(q, q^k)$ for $k \in \mathbb{Z}_+$, $k \gg 0$.*

On the other hand, Theorem 1 in the previous section claims that if one writes Macdonald's polynomials in the form

$$P_\lambda(x; q, t) = x^\lambda\Big(1 + \sum_{\mu\in Q^+} P_{\lambda\mu}(q,t)x^{-\mu}\Big)$$

then $Q_{\lambda\mu}^k(q) = P_{\lambda\mu}(q, q^k)$ for all $k \in \mathbb{N}$. Comparing this with Corollary 2, we see that $\tilde{Q}_{\lambda\mu}(q, q^k) = P_{\lambda\mu}(q, q^k)$ if $k \in \mathbb{Z}_+$, $k \gg 0$. Since both $P_{\lambda\mu}(q,t)$, $\tilde{Q}_{\lambda\mu}(q,t)$ are rational functions in q, t, this is possible only if $P_{\lambda\mu} = \tilde{Q}_{\lambda\mu}$. Thus, $\tilde{\varphi}_\lambda^k/\tilde{\varphi}_0^k$ equals the Macdonald's polynomial $P_\lambda(x; q, t)$. □

Using a similar argument, one can prove

Proposition 5.2.

$$\tilde{\varphi}_0^k(x) = x^{(k-1)\rho}\prod_{i=1}^{\infty}\prod_{\alpha\in R^+}\frac{1 - q^{2i}x^{-\alpha}}{1 - q^{2(i+k-1)}x^{-\alpha}}. \tag{5.8}$$

Proof. Let the rational functions $S_\mu(q,t)$ be defined by the formula

$$\prod_{i=1}^{\infty}\prod_{\alpha\in R^+}\frac{1-q^{2i}x^{-\alpha}}{1-q^{2(i+k-1)}x^{-\alpha}}=1+\sum_{\mu\in Q^+}S_\mu(q,t)x^{-\mu}. \tag{5.9}$$

Proposition 4.1, formula (5.4), and Corollary 1 in the proof of Theorem 2 imply that for a fixed μ and sufficiently large positive integer k ($k>k(\mu)$), $\tilde{R}_{0\mu}(q,q^k)=S_\mu(q,q^k)$. Since both of these functions are rational, they must coincide identically: $\tilde{R}_{0\mu}(q,t)=S_\mu(q,t)$. This implies (5.9). □

Corollary. *The traces $\tilde{\varphi}^k_\lambda$, $\lambda\in P^+$, are pairwise orthogonal with respect to the pairing $\langle,\rangle_1$.*

Proof. Proposition 5.2 implies that $\Delta(x)\tilde{\varphi}^k_0(x)\overline{\tilde{\varphi}^k_0(x)}=\Delta_{q,t}(x)$, which (by virtue of Theorem 2) means that $\langle\tilde{\varphi}^k_\lambda,\tilde{\varphi}^k_\mu\rangle_1=\langle\tilde{\varphi}^k_\lambda/\tilde{\varphi}^k_0,\tilde{\varphi}^k_\lambda/\tilde{\varphi}^k_0\rangle_{q,t}=\langle P_\lambda,P_\mu\rangle_{q,t}$, which implies that $\langle\tilde{\varphi}^k_\lambda,\tilde{\varphi}^k_\mu\rangle_1=0$ when $\lambda\neq\mu$. □

6. The center of $U_q\mathfrak{gl}_n$ and Macdonald's operators

In this section we show how one can get Macdonald's operators M_r introduced in Section 1 from the quantum group $U_q\mathfrak{gl}_n$. This construction is parallel to the one for $q=1$ (see [E]).

For simplicity, in this section we assume that $t=q^k$, $k\in\mathbb{N}$. Consider functions f of n variables $x_1,\dots,x_n$ and introduce the ring of difference operators, acting on these functions:

$$\mathrm{DO}=\Big\{D=\sum_{\lambda\in\mathbb{Z}^n}a_\lambda T_\lambda\Big|\ \text{almost all } a_\lambda=0\Big\},$$

where $(T_\lambda f)(x_1,\dots,x_n)=f(q^{\lambda_1}x_1,\dots,q^{\lambda_n}x_n)$, and a_λ are rational functions in $x_i,q^{1/2}$ with poles only at the points where $x^\mu q^m=1$ for some $\mu\in\mathbb{Z}^n$, $m\in\frac{1}{2}\mathbb{Z}$.

As before, let us consider a non-zero intertwiner $\Phi\colon V\to V\otimes W$, where V is a highest-weight module over $U_q\mathfrak{gl}_n$ and W is an arbitrary module with finite-dimensional weight spaces (V,W need not be finite-dimensional), and define the corresponding trace $\varphi(x)=\mathrm{Tr}\,|_V(\Phi x^h)$. This function takes values in W.

Theorem 3. *For any $u\in U_q\mathfrak{gl}_n$ there exists a difference operator $D_u\in DO\otimes U_q\mathfrak{gl}_n$, independent of the choice of V, W and the intertwiner Φ such that*

$$\mathrm{Tr}\,|_V(\Phi u x^h)=D_u\,\mathrm{Tr}\,|_V(\Phi x^h). \tag{6.1}$$

D_u is defined uniquely modulo the left ideal in $U_q\mathfrak{gl}_n$ generated by $q^{h_i}-1$; thus, $D_u f$ is well defined for any function $f(x_1,\dots,x_n)$ with values in $W[0]$.

Proof. Without loss of generality we can assume that u is a monomial in the generators e_i, f_i, q^{h_i} of the form $u = u^-u^0u^+$, $u^\pm \in U^\pm$, $u^0 \in U^0$. Define sdeg $u = \deg u^+ - \deg u^-$, where $\deg e_i = -\deg\, f_i = 1$. We prove the theorem by induction in sdeg u.

If sdeg $u = 0$ then $u = u^0 = q^{\sum \lambda_i h_i}$ for some $\lambda \in \mathbb{Z}^n$. Then it follows immediately from the definition that $D_u = T_\lambda$, so the theorem holds.

Let us make the induction step. Assume that sdeg $u > 0$; then either $\deg u^+ \neq 0$ or $\deg u^- \neq 0$. We can assume that $u^- \in U[\mu]$, $\mu \neq 0$.

Since Φ is an intertwiner, $\mathrm{Tr}(\Phi u^-u^0u^+x^h) = \mathrm{Tr}(\Delta(u^-)\Phi u^0u^+x^h)$. From the definition of comultiplication one easily sees that

$$\Delta(u^-) = u^- \otimes q^{\sum \lambda_i h_i} + \sum u_j \otimes v_j$$

for some $\lambda \in \frac{1}{2}\mathbb{Z}^n$, and $u_j, v_j \in U^0U^-$ such that $\mathrm{sdeg}(u_j' u_0 u^+) < \mathrm{sdeg}\, u$. Thus,

$$\mathrm{Tr}(\Phi u x^h) = q^{\sum \lambda_i h_i}\,\mathrm{Tr}(\Phi u^0u^+x^hu^-) + \sum v_j\,\mathrm{Tr}(\Phi u^0u^+x^hu_j).$$

Since commuting with x^h does not change sdeg u_j, by the induction assumption we can write

$$\begin{aligned}\mathrm{Tr}(\Phi u x^h) &= q^{\sum \lambda_i h_i}\,\mathrm{Tr}(\Phi u^0u^+x^hu^-) + D'\,\mathrm{Tr}(\Phi x^h)\\ &= q^{\langle\lambda,\mu\rangle}x^\mu\,\mathrm{Tr}(\Phi u^0u^+u^-x^h) + D'\,\mathrm{Tr}(\Phi x^h)\\ &= q^{\langle\lambda,\mu\rangle}x^\mu\,\mathrm{Tr}(\Phi(u+[u^0u^+,u^-])x^h) + D'\,\mathrm{Tr}(\Phi x^h)\end{aligned}$$

for some $D' \in \mathrm{DO}\otimes U_q\mathfrak{gl}_n$. Since the sdeg of all terms in $[u^0u^+,u^-]$ is less than sdeg $u^-u^0u^+$, we can again apply the induction assumption and get

$$\mathrm{Tr}(\Phi u x^h) = \frac{1}{1-q^{<\lambda,\mu>}x^\mu}D''\,\mathrm{Tr}(\Phi x^h).$$

This proves the existence part of the theorem. Uniqueness follows from the following lemma:

Lemma. *Let us fix a $U_q\mathfrak{gl}_n$-module W with finite-dimensional weight spaces. If $D \in DO\otimes \mathrm{Hom}(W[0],W[\mu])$ is such that $D\varphi = 0$ for any $\varphi(x) = \mathrm{Tr}(\Phi x^h)$, $\Phi : V \to V\otimes W$, V an arbitrary highest-weight module, then $D = 0$.*

Proof of the lemma. Let us assume that $D \neq 0$. Multiplying D by a suitable polynomial of x_i we can assume that D has polynomial coefficients:

$D = \sum x^\lambda D_{(\lambda)}$, $D_{(\lambda)}$ being difference operators with constant matrix-valued coefficients. Let us take the maximal (with respect to the lexicographic ordering) λ such that $D_{(\lambda)} \neq 0$. Then if we have a trace φ as above such that $\varphi(x) = x^\mu w +$ lower order terms then, taking the highest term of $D\varphi$, we see that $D_{(\lambda)}(x^\mu w) = 0$. On the other hand, if we take μ such that $\mu + \rho \in -P_+$ then the Verma module M_μ is irreducible and thus for every $w \in W[0]$ there exists a non-zero intertwiner $\Phi: M_\mu \to M_\mu \otimes W$ such that the corresponding trace has the form $\varphi(x) = x^\mu w +$ lower order terms. Thus $D_{(\lambda)}(x^\mu w) = 0$ for all $\mu \in -P_+ - \rho$, $w \in W[0]$. Thus, if one writes $D_{(\lambda)} = \sum_\beta a_{\lambda\beta} T_\beta$ then $\sum_\beta a_{\lambda\beta} q^{\langle\beta,\mu\rangle} w = 0$ for all $w \in W[0]$, $\mu \in -P_+ - \rho$. This is possible only if all $a_{\lambda\beta} = 0$, which contradicts the assumption $D_{(\lambda)} \neq 0$. □

In general, $u \mapsto D_u$ is not an algebra homomorphism. However, if u is central: $u \in \mathcal{Z}(U_q\mathfrak{gl}_n)$ then Φu is also an intertwiner, and thus for every $v \in U_q\mathfrak{gl}_n$ we have:

$$D_{uv}\,\mathrm{Tr}(\Phi x^h) = \mathrm{Tr}(\Phi u v x^h) = D_v\,\mathrm{Tr}(\Phi u x^h) = D_v D_u\,\mathrm{Tr}(\Phi x^h).$$

This implies the following proposition:

Proposition 6.1. *$u \mapsto D_u$ is an algebra homomorphism of $\mathcal{Z}(U_q\mathfrak{gl}_n)$ to $DO \otimes U_q\mathfrak{gl}_n[0]/I$, where I is the ideal generated by $q^{h_i} - 1$.*

Proposition 6.2. *Let $c \in \mathcal{Z}(U_q\mathfrak{gl}_n)$, V be a highest-weight module over $U_q\mathfrak{gl}_n$ (not necessarily finite-dimensional), $c|_V = C \cdot \mathrm{Id}$ for some $C \in \mathbb{C}(q)$, and let $\Phi: V \to V \otimes W$ be a non-zero intertwiner. Then the trace $\varphi(x) = \mathrm{Tr}\,|_V(\Phi x^h)$ satisfies the difference equation*

$$D_c\varphi(x) = C\varphi(x). \tag{6.2}$$

This proposition is an obvious corollary of Theorem 3.

This shows that our construction allows us to construct commutative algebras of difference operators and their eigenfunctions. In general, these functions are vector-valued (they take values in the space $W[0]$); however, if we choose W as in Section 4 so that $W[0]$ is one-dimensional then we can consider the traces as scalar functions; since every central element in $U_q\mathfrak{gl}_n$ has weight zero, D_c preserves $W[0]$ and thus can be considered as a difference operator with scalar coefficients. We want to show that for appropriate choice of central elements the operators D_c are precisely Macdonald's operators (up to conjugation).

To find these central elements we will use Drinfeld's construction of central elements ([D2]), which is based on the universal R-matrix $\mathcal{R} \in U_q\mathfrak{gl}_n \hat{\otimes} U_q\mathfrak{gl}_n$ discussed in Section 2 (a similar construction was independently proposed by N. Reshetikhin [R]). Define $\mathcal{R}^{21} = P(\mathcal{R})$, $P(x \otimes y) = y \otimes x$.

Proposition 6.3. *Define* $c_r \in U_q\mathfrak{gl}_n$, $r = 1, \dots, n$ *by*

$$c_r = (\mathrm{Id} \otimes \mathrm{Tr}_{(\Lambda_q^r)^*}) \left(\mathcal{R}^{21}\mathcal{R}(1 \otimes q^{-2\rho})\right), \tag{6.3}$$

where Λ_q^r *is the q-deformation of the representation of* $\mathfrak{gl}_n$ *in the r-th exterior power of the fundamental representation* $\Lambda^r\mathbb{C}^n$. *Then*

(1) $c_r \in \mathcal{Z}(U_q\mathfrak{gl}_n)$.
(2) *If* V *is a highest-weight module with highest weight* λ, *then* $c_r|_V = \sum_I q^{2\sum_{i\in I}(\lambda+\rho)_i}\mathrm{Id}$, *where the sum is taken over all subsets* $I \subset \{1, \dots, n\}$ *of order* r.

Proof. 1. This is based on the following statement (see [D2]): if $\theta\colon U_q\mathfrak{gl}_n \to \mathbb{C}(q)$ is such that $\theta(xy) = \theta(yS^2(x))$ then the element $c_\theta = (\mathrm{Id} \otimes \theta)(\mathcal{R}^{21}\mathcal{R})$ is central. On the other hand, we know that $S^2(x) = q^{-2\rho}xq^{2\rho}$, so $\theta(x) = \mathrm{Tr}\,|_V(xq^{-2\rho})$, where V is any finite-dimensional representation of $U_q\mathfrak{gl}_n$, satisfies $\theta(xy) = \theta(yS^2(x))$. Taking $V = (\Lambda_q^r)^*$, we get statement 1 of the proposition.

2. Let v_λ be a highest-weight vector in V; let us calculate c_rv_λ. Let $w \in (\Lambda_q^r)^*[\mu]$. Then (2.3) implies

$$\mathcal{R}^{21}\mathcal{R}(v_\lambda \otimes w) = q^{-2<\lambda,\mu>}v_\lambda \otimes w + \sum v_i' \otimes w_i'$$

where wt $w_i' < \mu$. Thus, $c_rv_\lambda = (\sum_\mu(\dim\,(\Lambda_q^r)^*[\mu])q^{-2<\lambda,\mu>}q^{-2<\rho,\mu>})v_\lambda$, where the sum is taken over all the weights of $(\Lambda_q^r)^*$. Since the weights of $(\Lambda_q^r)^*$ are $\mu = (\mu_1, \dots, \mu_n)$ such that $\mu_i = 0$ or -1, $\sum \mu_i = -r$, and the multiplicity of each weight is 1, we get the desired formula. □

Remark. These central elements are closely related to those constructed in [FRT]. Essentially, the central elements constructed in [FRT] are traces of the powers of L-matrix, whereas our central elements are coefficients of the characteristic polynomial of L.

Theorem 4.

$$M_r = \varphi_0^{-1}(x) \circ D_{c_r} \circ \varphi_0(x),$$

where M_r *is Macdonald's operator introduced in Section 1,* c_r *is the central element constructed in Proposition 6.3, and* φ_0 *is the operator of multiplication by the function* φ_0 *defined by* (4.3).

Proof. This follows from the fact that M_r and $\varphi_0^{-1}(x)D_{c_r}\varphi_0(x)$ coincide on the Macdonald's polynomials $P_\lambda(x) = \varphi_\lambda(x)/\varphi_0(x)$: just compare Proposition 1.1, Theorem 1 and Proposition 6.3. Repeating the uniqueness arguments outlined in the proof of Theorem 3, but considering $\lambda \in P_+ + (k-1)\rho$

instead of $\lambda \in -P_+ - \rho$, we see that it is only possible if $M_r = \varphi_0^{-1} \circ D_{c_r} \circ \varphi_0$. □

Thus, we can use the traces of the form (4.2) to find eigenfunctions of Macdonald operators M_r. Indeed, let us consider $\lambda = (\lambda_1, \dots, \lambda_n)$ as a formal variable; then q, q^{λ_i} are algebraically independent. In this case the Verma module M_λ is irreducible, and thus there exists an intertwiner $\Phi: M_\lambda \to M_\lambda \otimes U$, where the module U is the same one we used in Section 4.

Theorem 5.

(1) *The function* $f_\lambda(x) = \mathrm{Tr}\,|_{M_\lambda}(\Phi x^h)/\varphi_0(x)$, *where* $\Phi : M_\lambda \to M_\lambda \otimes U$ *is a non-zero intertwiner and* $\varphi_0(x)$ *is defined by* (4.3), *satisfies the following system of difference equations:*

$$M_r f_\lambda(x) = \sum_I q^{2\sum_{i\in I}(\lambda+\rho)_i} f_\lambda(x). \tag{6.5}$$

(2) *The functions* $f_{\sigma(\lambda+\rho)-\rho}$, $\sigma \in S_n$ *form a basis of solutions of the system* (6.5) *in the space of generalized Laurent series*

$$\mathcal{F} = \sum_\nu x^\nu \mathbb{C}(q, q^{\lambda_i})\Big[\Big[\frac{x_2}{x_1}, \dots, \frac{x_n}{x_{n-1}}\Big]\Big].$$

Proof.

1. This is an immediate corollary of Proposition 6.2 and Theorem 4.

2. Suppose that $f \in \mathcal{F}$ is a solution of (6.4) of the form $f(x) = x^\nu +$ lower order terms. Expanding coefficients of Macdonald's operators in Laurent series, we find the highest term of $M_r f$:

$$(M_r f)(x) = \sum_{I:|I|=r} q^{2\sum_{i\in I}\rho_i} T_{q^2, x_I} f = \sum_{I:|I|=r} q^{2\sum_{i\in I}(\nu+\rho)_i} x^\nu + \cdots .$$

Thus $f(x)$ can be a solution only if for any r,

$$\sum_{I:|I|=r} q^{2\sum_{i\in I}(\nu+\rho)_i} = \sum_{I:|I|=r} q^{2\sum_{i\in I}(\lambda+\rho)_i},$$

which is only possible if $\nu + \rho = \sigma(\lambda + \rho)$ for some $\sigma \in S_n$. Then the highest term of f coincides with the highest term of $f_{\sigma(\lambda+\rho)-\rho}$. Considering $f - f_{\sigma(\lambda+\rho)-\rho}$ and repeating the same arguments, we finally see that f is a linear combination of the functions $f_{\sigma(\lambda+\rho)-\rho}$. □

This theorem can be generalized to the case of arbitrary k; in this case one must replace the module U by the module W_k defined in Section 5.

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