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SUBELLIPTIC ESTIMATES OF POLYNOMIAL DIFFERENTIAL OPERATORS AND APPLICATIONS TO RIGIDITY OF ABELIAN ACTIONS

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Abstract. We use subelliptic estimates for certain polynomial differential operators to show C^∞-regularity of distributions smooth "along" foliations which satisfy a certain non-degeneracy condition and whose sum is totally non-integrable. We use this to extend the cocycle trivialization theorem for Anosov actions of higher rank abelian groups [10] to certain partially hyperbolic actions of $\mathbb{Z}^k$ or $\mathbb{R}^k$ for $k \geq 2$. As a consequence, there are only trivial smooth time changes for these actions (up to an automorphism).

1. Introduction

A classical result from analysis asserts that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is C^∞ provided that $\frac{\partial^n}{\partial x^n} f$ and $\frac{\partial^n}{\partial y^n} f$ exist for all positive integers n. This follows from the ellipticity of the operator $\frac{\partial^n}{\partial x^n} + \frac{\partial^n}{\partial y^n}$, and standard regularity theory of elliptic operators. This easily generalizes to functions that are smooth along transverse smooth foliations on a manifold, and more deeply, functions that are smooth along transverse Hölder foliations with smooth leaves [5, 6, 7, 13].

In this note, we consider extensions of this phenomenon to smooth foliations or smooth plane fields D_i whose sum is a totally non-integrable plane field. Here one calls a plane field *totally non-integrable* if the tangent space T_pM at any point $p \in M$ is spanned by brackets of vectorfields tangent to the plane field. Suppose that a distribution f on M has continuous or locally L^2 partial derivatives in the directions of D_i of any order for each D_i. We then show that f is C^∞ on M, under some mild technical assumption on the distributions. In particular, f has derivatives in directions transverse to the sum of the D_i. This theorem relies heavily on deep results of L. Rothschild, B. Helffer and F. Nourrigat, G. Metivier and C. Rockland on the hypoellipticity of certain polynomial differential operators [15, 1, 16].

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We apply this regularity result to show rigidity of smooth cocycles of certain partially hyperbolic actions of $\mathbb{R}^k$ and $\mathbb{Z}^k$ for $k \geq 2$. They include the class of Anosov actions we introduced in [10]. This paper extends our results about the first cohomology of Anosov actions of higher rank abelian groups to certain partially hyperbolic actions.

As in [10], the main construction of the solution of the cohomology equation uses harmonic analysis and is very similar to [10]. The regularity properties of the solution rely on the subelliptic estimates mentioned above.

As an application, we obtain non-existence of smooth time changes for the standard partially hyperbolic actions.

2. Subelliptic estimates and regularity

We will consider totally non-integrable plane fields $\mathcal{D}$ on a manifold that satisfy the following technical condition:

> **($*$):** for each j, the dimension of the space spanned by the commutators of length at most j at each point is constant in a neighborhood.

Theorem 2.1. *Let $\mathcal{D}_1, \dots, \mathcal{D}_k$ be C^∞ plane fields on a manifold M such that their sum $\sum_{i=1}^k \mathcal{D}_i$ is totally non-integrable and satisfies condition ($*$). Let P be a distribution on M. Assume that for any positive integer p and C^∞ vectorfield X tangent to any $\mathcal{D}_j$, the p^{th} partial derivative $X^p(P)$ exists as a continuous or a local L^2 function . Then P is C^∞ on M.*

Note that we make no assumptions on the existence of mixed derivatives. Also note that a function P is C^∞ on M provided that it is C^∞ along k foliations for which the sum of its tangent distributions is totally non-integrable and satisfies condition ($*$). We do not know if condition ($*$) is merely technical.

Proof. The claim clearly follows from its local version. By condition ($*$) we may therefore assume that

(1) $X_1, \dots, X_l$ are linearly independent C^∞ vectorfields on a neighborhood U that, together with their commutators of length at most r, span the tangent space at every point of U,
(2) for each $j \leq r$, the dimension of the space spanned by their commutators of length at most j is constant in each neighborhood.

Since $\sum_{i=1}^l X_i^2(P)$ is continuous or L^2_{loc}, Hörmander's square theorem ensures that any distribution P is an L^2_{loc}-function [3, 17].

To show higher regularity, consider the polynomial differential operator $L = \sum_{i=1}^{l} X_i^m$ for any positive even integer m. Given a set of vector fields $X_1, \dots, X_l$ as above, Metivier attached a nilpotent Lie algebra to any point $x \in M$ that is generated by elements $\hat{X}_1, \dots, \hat{X}_l$ [14, 16]. Let G_x denote the simply- connected Lie group with that Lie algebra. If π is a unitary representation of G_x, let $\mathcal{S}_\pi$ denote the space of C^∞ vectors of π. By [15, Proposition 7.1] the operator $\pi(\sum_{i=1}^{l} \hat{X}_i^m)$ is injective as an operator on $\mathcal{S}_\pi$ for any non-trivial irreducible representation π. It follows from [16, Theorem 0.7] that L is maximally hypoelliptic. This means that for any multiindex α with $|\alpha| \leq m$ there exists $C_\alpha > 0$ such that in a neighborhood V of x

$$\|X_\alpha f\|_0^2 \leq C_\alpha \left(\|Lf\|_0^2 + \|f\|_0^2\right)$$

for all $f \in C_0^\infty(V)$ where $\| \, \|_0$ denotes the L^2-norm.

Denote by H^α the α^{th} Sobolev space of V with Sobolev norm $\| \, \|_\alpha$. By the above and [17, §16] there is a constant $C > 0$ such that for all $f \in C_0^\infty(V)$

$$\|f\|_{m/r} \leq C \left(\|Lf\|_0^2 + \|f\|_0^2\right)^{\frac{1}{2}}.$$

A. and J. Unterberger introduced mollifiers of the form

$$Op(\phi)u(x) = \int \phi(x, \eta, y) e^{-2i\pi(y-x)\cdot\eta} \, u(y) \, dy \, d\eta$$

where $\phi(x, \eta, y)$ is a C^∞-function on $\mathbb{R}^{3n}$ all of whose partial derivatives decay superpolynomially in η [18]. These mollifiers are infinitely smoothing. Let $\phi_t(x, \eta, y) = \phi(x, t\eta, y)$. Then for $0 < t \leq 1$ and any $s \in \mathbb{R}$, $Op(\phi_t)$ is in a bounded set of continuous linear endomorphisms of $H^s(\mathbb{R}^n)$ with the operator norm topology, and for every $u \in H^s(\mathbb{R}^n)$, $Op(\phi_t)u$ converges to the product of u by the function $\phi(x, 0, x)$ in $H^s(\mathbb{R}^n)$ as $t \to 0$ [18, Theorem 1.2]. Moreover, let X be a vectorfield. Suppose that for every y there is a compact set that contains the support of $\phi(\; , \eta, y)$ for all η. Then there is another symbol ψ with superpolynomial decay such that $[X, Op(\phi_t)] = Op(\psi_t)$ [18, Theorem 1.1]. They also satisfy the last property as one can see from the explicit formula given in [18].

Let us pick a mollifier ϕ as above where we identify V with $\mathbb{R}^n$. By the last assertion we can inductively find mollifiers $\phi^{i,j}$ with $\phi = \phi^{0,0}$ such that $[X_j, Op(\phi_t^{i,j})] = Op(\phi_t^{i+1,j})$. Then we easily see that

$$L \, Op(\phi_t) f = Op(\phi_t) Lf + \sum_{j=1}^{l} \sum_{i=1}^{m} \binom{m}{i} Op(\phi_t^{m-i,j}) \, (X_j)^i \, f.$$

By the a priori estimate we see that

$$\begin{aligned}\|Op(\phi_t)f\|^2_{\frac{m}{r}} &\leq C\ \left(\|LOp(\phi_t)f\|_0^2 + \|Op(\phi_t)f\|_0^2\right)\\ &= C\Bigg(\Bigg\|Op(\phi_t)Lf\\ &\qquad + \sum_{j=1}^{l}\sum_{i=1}^{m}\binom{m}{i}Op(\phi_t^{m-i,j})\,(X_j)^i\,f\Bigg\|_0^2\\ &\quad + \|Op(\phi_t)f\|_0^2\Bigg).\end{aligned}$$

By assumption all $(X_j)^i f$ are continuous or locally integrable and hence belong to $L^2(V)$ for small enough V. Hence by the properties of the mollifiers above $\|Op(\phi_t)f\|_{m/r}$ is bounded as $t \to 0$ and hence $f \in H^{m/r-\varepsilon}$ for every $\varepsilon > 0$ by Rellich's Lemma. Since this is true for all m, the Sobolev lemma shows that f is C^∞. □

3. Partially hyperbolic actions

Definition 3.1. Let A be $\mathbb{R}^k$ or $\mathbb{Z}^k$. Suppose A acts C^∞ and locally freely on a manifold M with a Riemannian norm $\|\ \|$. Call an element $g \in A$ *normally hyperbolic* if there exist real numbers $\lambda > \mu > 0,\ \ C, C' > 0$ and a continuous splitting of the tangent bundle

$$TM = E_g^+ + E_g^0 + E_g^-$$

such that E_g^+ and E_g^- have positive dimension and for all $p \in M$, for all $v \in E_g^+(p)$ ($v \in E_g^-(p)$ respectively) and $n > 0$ ($n < 0$ respectively) we have for the differential $g_* : TM \to TM$

$$\|g_*^n(v)\| \leq Ce^{-\lambda|n|}\|v\|$$

and for all $n \in \mathbb{Z}$ and $v \in E^0$ we have

$$(**) \qquad \|g_*^n(v)\| \geq C'e^{-\mu|n|}\|v\|.$$

Definition 3.2. Call an A-action *partially hyperbolic* if it contains a normally hyperbolic element such that the subbundle E^0 is uniquely integrable and that the growth in E^0 is subexponential, i.e. the estimate $(**)$ takes place for any $\mu > 0$ for a constant that depends on μ.

We call E_g^+ and E_g^- its stable and unstable subbundle respectively. Call $g \in A$ *regular* if the neutral direction for g is contained in the neutral direction of any other element.

If M is compact, these notions do not depend on the ambient Riemannian metric. Note that the splitting and the constants in the definition above depend on the normally hyperbolic element, and that the subbundle E^0 contains the tangent subbundle $T\mathcal{O}$ to the A-orbits. If A acts partially hyperbolically, we will call the foliation into the integral manifolds of E^0 the *neutral foliation.*

It is an interesting and apparently open problem whether any topologically transitive partially hyperbolic action contains a regular element.

Hirsch, Pugh and Shub developed the basic theory of partially hyperbolic transformations in [2].

Theorem 3.3. *Suppose $g \in A$ acts partially hyperbolically on a manifold M. Then there are Hoelder foliations W_g^s and W_g^u tangent to the subbundles E_g^+ and E_g^- respectively. We call these foliations the* stable *and* unstable foliations *of g. The individual leaves of these foliations are C^∞-immersed submanifolds of M.*

Note that all Anosov actions are partially hyperbolic [10]. In particular, the standard Anosov actions introduced in [10] are all partially hyperbolic. We will now extend this class by including certain genuine partially hyperbolic actions. All examples of $\mathbb{R}^k$-actions in this class come from the following unified algebraic construction. Let G be a connected Lie group, $A \subset G$ a closed Abelian subgroup which is isomorphic with $\mathbb{R}^k$, S a compact subgroup of the centralizer $Z(A)$ of A, and Γ a cocompact lattice in G. Then A acts by left translation on the compact space $M \stackrel{def}{=} S \setminus G/\Gamma$.

Example 3.4 (Symmetric space examples). Let G be a semisimple connected real Lie group of the noncompact type and of $\mathbb{R}$-rank at least 2. Let A be the connected component of a split Cartan subgroup of G. Suppose Γ is an irreducible torsion-free cocompact lattice in G. Then the action of A on G/Γ is partially hyperbolic.

Indeed, recall that the centralizer $Z(A)$ of A splits as a product $Z(A) = M\,A$ where M is compact. Let Σ denote the restricted root system of G. Then the Lie algebra $\mathcal{G}$ of G decomposes

$$\mathcal{G} = \mathcal{M} + \mathcal{A} + \sum_{\alpha \in \Sigma} \mathcal{G}^\alpha$$

where g^α is the root space of α and $\mathcal{M}$ and $\mathcal{A}$ are the Lie algebras of M and A. Fix an ordering of Σ. If X is any element of the positive Weyl chamber $\mathcal{C}_p \subset \mathcal{A}$ then $\alpha(X)$ is nonzero and real for all $\alpha \in \Sigma$. Hence $\exp X$

acts normally hyperbolically on G with respect to the foliation given by the MA-orbits.

Since A commutes with M, A acts on $N \stackrel{def}{=} M \setminus G/\Gamma$. We called this action the *Weyl chamber flow* of A in [10]. We will call the Weyl chamber flows as well as the actions on the cover G/Γ or any intermediate cover *standard*. For the partially hyperbolic actions we will also assume that the Lie algebra $\mathcal{G}$ of G does not have factors isomorphic with $so(n,1)$ or $su(n,1)$.

Let us note that we only need Γ to be torsion-free to assure that G/Γ is a manifold. All of our arguments in this paper directly generalize to the orbifold case.

Restrict any such $\mathbb{R}^k$-action to any lattice $\mathbb{Z}^k \subset \mathbb{R}^k$. Then we get partially hyperbolic $\mathbb{Z}^k$-actions which satisfy our properties above. We may also restrict to any $\mathbb{R}^m$ or discrete $\mathbb{Z}^m$ in $\mathbb{R}^k$ which contains a regular element. Again, these examples are called *standard*. Note that again we do not consider $\mathcal{G}$ with factors isomorphic with $so(n,1)$ or $su(n,1)$.

Example 3.5 (Twisted symmetric space examples). Assume the notations of Example 2.7. Let $\rho : \Gamma \to SL(n,\mathbb{Z})$ be a representation of Γ which is irreducible over $\mathbb{Q}$. Then Γ acts on the n-torus T^n via ρ and hence on $(M \setminus G) \times T^n$ via

$$\gamma(x,t) = (x\gamma^{-1}, \rho(\gamma)(t)).$$

Let $N \stackrel{def}{=} M \setminus G \times_\Gamma T^n \stackrel{def}{=} (M \setminus G \times T^n)/\Gamma$ be the quotient of this action. As the action of A on $M \setminus G \times T^n$ given by $a(x,t) = (ax,t)$ commutes with the Γ-action, it induces an action of A on N.

This example generalizes by taking an action on an intermediate cover between G/Γ and $M \setminus G/\Gamma$ as the base space of the twisting. We may also restrict the action of $\mathbb{R}^k$ to a closed subgroup isomorphic to either $\mathbb{R}^m$ or $\mathbb{Z}^m$ with $m \geq 2$ as long as at least one element acts partially hyperbolically with neutral foliation given by the quotient of the MA-orbit foliation. Note that such a partially hyperbolic element is a regular element of the split Cartan subgroup. For the non-Anosov examples, we will again assume that the Lie algebra $\mathcal{G}$ of G does not have factors isomorphic with $so(n,1)$ or $su(n,1)$.

The above construction can be generalized considering toral extensions of other higher rank actions for which one of the monodromy elements is Anosov. For example, using a twisted Weyl chamber flow as above as the base we obtain nilmanifold extensions of the Weyl chamber flow. As A. Starkov pointed out, one can also start with the product of a Weyl chamber flow with a transitive action of some $\mathbb{R}^l$ on a torus and produce a toral extension which is Anosov and no finite cover splits as a product. These two extension constructions can be combined and iterated. This is our last class of *standard examples.*

Let us emphasize that for all standard actions, the splitting $TM = E_g^+ + E_g^0 + E_g^-$ is smooth, and that the following property holds:

> The sum of the subbundles E_g^+ and E_g^- is totally non-integrable, i.e. the vectorfields belonging to them and their brackets span TM.

The next theorem generalizes the cocycle triviality theorem for Anosov actions from [10].

Theorem 3.6. *Consider a standard partially hyperbolic A-action on a manifold M where A is isomorphic to $\mathbb{R}^k$- or $\mathbb{Z}^k$ with $k \geq 2$. Then any C^∞-cocycle $\beta : A \times M \to \mathbb{R}^l$ is C^∞-cohomologous to a constant cocycle.*

As in [10, Proof of Theorem 2.12 in Section 5], we get the

Corollary 3.7. *All C^∞-time changes of a standard partially hyperbolic $\mathbb{R}^k$-action with $k \geq 2$ are C^∞-conjugate to the original action up to an automorphism.*

Proof of Theorem 3.6. We may and will always assume that $l = 1$. Pick a regular element $a \in A$ for which the sum of the subbundles E_a^+ and E_a^- is totally non-integrable. For the symmetric space and twisted symmetric space actions any element regular for the ambient $\mathbb{R}^k$-action will work.

We will show that β is cohomologous to $\rho(b) = \int_M \beta(b,x)dx$, or that $\beta - \rho$ is cohomologous to 0. Thus we may assume that β has 0 averages.

Define the function f by $f(x) = \beta(a, x)$. Now we can define formal solutions of the cohomology equation by

$$P_a^+ = \sum_{k=0}^{\infty} a_1^k f \qquad \text{and} \qquad P_a^- = -\sum_{k=-\infty}^{-1} a^k f.$$

The first step is to show that P_a^+ and P_a^- are distributions. Let us consider the symmetric space case first. We use the exponential decay of matrix coefficients for irreducible unitary representations of G. In Section 3 of [10], we established a specific form of such estimates. Let $g \in C^\infty(G/\Gamma)$.

By Corollary 3.2 in [10], there is a positive integer m and constant $E > 0$ such that $\left|\langle a_1^k f, g\rangle\right| \leq E e^{-k\rho} \|f\|_m \|g\|_m$ where $\|\ \|_m$ is the Sobolev norm. Hence $\sum_{k=0}^{\infty} \langle a_1^k f, g\rangle$ converges absolutely, and there is a constant $A > 0$ such that $\left|\sum_{k=0}^{\infty} \langle a_1^k f, g\rangle\right| \leq A \|g\|_m$. Thus P_+ and similarly P_- are distributions. In fact, they are elements of the Sobolev space H^{-m}.

The following lemma contains the key to the trivialization of cohomology for the higher rank abelian actions. It shows that the natural dual obstructions for the solvability of the cohomology equation disappear for smooth cocycles with 0 averages. A similar argument for actions by toral automorphisms is Lemma 4.3 in [10]. The same key idea is used in [8, Proposition 2.3] to show the n^{th} cohomology of a $\mathbb{Z}^k$-action by hyperbolic toral automorphisms trivializes for all $1 \leq n \leq k-1$.

Lemma 3.8. *The distributions P_a^+ and P_a^- coincide.*

Proof. For $a \in A$, denote by Δ_a the difference operator $(\Delta_a f)(x) = f(a\,x) - f(x)$. Set $a_1 = a$, and let a_2 be $\mathbb{R}$-linearly independent from $a = a_1$. As in the proof of Proposition 4.2 we have the difference equations

$$\Delta_{a_j^{-1}} f_i = \Delta_{a_i^{-1}} f_j.$$

Hence we get

$$\sum_{k=-l}^{l} a_1^k a_2 f_1 - \sum_{k=-l}^{l} a_1^k f_1 = \sum_{k=-l}^{l} a_1^{k+1} f_2 - a_1^k f_2 = a_1^{l+1} f_2 - a_1^{-l} f_2.$$

Since Γ is an irreducible lattice the matrix coefficients of elements in $L^2(G/\Gamma)$ orthogonal to the constants vanish [20, ch. 2]. Hence we see that for $g \in C^\infty(G/\Gamma)$

$$\sum_{k=-\infty}^{\infty} \langle a_1^k f_1, a_2^{-1} g\rangle - \sum_{k=-\infty}^{\infty} \langle a_1^k f_1, g\rangle = \lim_{l\to\infty} \langle a_1^{l+1} f_2 - a_1^{-l} f_2, g\rangle = 0.$$

Since $a_1^k a_2^m \to \infty$ as $(k, m) \to \infty$ and the matrix coefficients decay exponentially, the sum

$$\sum_{m=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \langle a_1^k f_1, a_2^m g\rangle = \lim_{m\to\infty} 2m \sum_{k=-\infty}^{\infty} \langle a_1^k f_1, g\rangle$$

converges absolutely. Thus we get $\sum_{k=-\infty}^{\infty} \langle a_1^k f_1, g\rangle = 0$. $\square$

For the twisted symmetric space examples, this technique is combined with the superexponential decay of Fourier coefficients of smooth functions in the toral fiber direction. For a detailed treatment, we refer to Section 4.4 of [10].

Hyperbolicity implies that the distribution P_a^+ has continuous derivatives of any order along the stable manifolds while P_a^- has continuous derivatives of any order along the unstable manifolds. Note that for Anosov actions by $\mathbb{Z}$ and $\mathbb{R}$, P_a^+ and P_a^- in general do not coincide even if they are distributions. In the higher rank case however, they do coincide, and thus P_a^+ is differentiable along both stable and unstable manifolds which is the basis for proving P_a^+ is a smooth function.

The stable and unstable directions are totally non-integrable and satisfy the technical condition in Theorem 2.1. Thus P_a^+ is smooth. Once it is known that the solution P_a^+ is at least a measurable function, an ergodicity argument shows that it is a solution of the coboundary equation for the whole group. □

Remark 3.9. Let us note that the cocycle trivialization theorem is also correct for partially hyperbolic actions of toral automorphisms. More precisely, we need to assume that the action is irreducible, i.e. no finite cover splits as a product, and that $\mathbb{Z}^k$ contains a $\mathbb{Z}^2$ such that every non-trivial element of $\mathbb{Z}^2$ acts ergodically with respect to Haar measure. In this case, the formal solutions are constructed using Fourier analysis. The counterpart of the key lemma above holds with the same proof as that of Lemma 4.3 in [10]. The smoothness of P follows from Proposition 5.8 in [19]. Diophantine properties of eigenspaces play the role of the uniform estimates in the hyperbolic case.

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