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A SOBOLEV INEQUALITY AND NEUMANN HEAT KERNEL ESTIMATE FOR UNBOUNDED DOMAINS

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Abstract. Suppose D is an unbounded domain in $\mathbb{R}^d$ ($d \geq 2$) with compact boundary and that D satisfies a uniform interior cone property. We show that for $1 \leq p < d$, there exists a constant $c = c(D, p)$ such that for each $f \in W^{1,p}(D)$ the following Sobolev inequality holds:

$$\|f\|_q \leq c \, \|\nabla f\|_p,$$

where $1/q = 1/p - 1/d$ and for $r = p$, q, $\| \cdot \|_r$ denotes the norm in $L^r(D)$. As an application of this Sobolev inequality, assuming in addition that D is a Lipschitz domain in $\mathbb{R}^d$ with $d \geq 3$, we obtain a Gaussian upper bound estimate for the heat kernel on D with zero Neumann boundary condition.

1. Introduction

For a domain $U \subset \mathbb{R}^d$ and $p \in [1, \infty)$, we define $L^p(U)$ to be the space of real-valued functions defined on U that are L^p-integrable relative to Lebesgue measure on U. The norm on $L^p(U)$ is given by

$$\|f\|_p = \left(\int_U |f(x)|^p dx \right)^{\frac{1}{p}}.$$

We further define

$$W^{1,p}(U) = \left\{ f \in L^p(U) : \frac{\partial f}{\partial x_i} \in L^p(U) \text{ for } i = 1, \ldots, d \right\}, \tag{1}$$

with norm $\|f\|_{1,p} \equiv \|f\|_p + \|\nabla f\|_p$, where $\nabla f = \left(\frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_d} \right)$. Here the partial derivatives $\frac{\partial f}{\partial x_i}$ are understood in the distributional sense. Note that in the above we do not indicate the dependence of $\| \cdot \|_p$ and $\| \cdot \|_{1,p}$ on U, since usually there will only be one relevant domain U. If there is

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any chance of ambiguity, we shall explicitly indicate the domain U in the norm, for example, $\|\cdot\|_{U,p}$ for $\|\cdot\|_p$.

In the sequel, we assume that D is an unbounded domain in $\mathbb{R}^d$ with compact boundary and that D has the following uniform interior cone property, henceforth referred to simply as the cone property.

Cone Property. The domain D is said to have the cone property if there exists a finite cone

$$V = \left\{ x = (x_1, \cdots, x_d) \in \mathbb{R}^d : x_d > \alpha(x_1^2 + \cdots + x_{d-1}^2)^{1/2} \text{ and } \|x\| < \beta \right\}$$

for some $\alpha, \beta > 0$ such that each point $x \in D$ is the vertex of a finite cone V_x contained in D which is congruent to V. Here $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^d$.

Our main result is the following.

Theorem 1. *Suppose that D is an unbounded domain with compact boundary and that D has the cone property. Then for $1 \leq p < d$, there exists a constant $c = c(D, p)$ such that the following Sobolev inequality holds:*

$$\|f\|_q \leq c\,\|\nabla f\|_p \quad \text{for all } f \in W^{1,p}(D), \tag{2}$$

where $1/q = 1/p - 1/d$.

It is well known (cf. [1], [5]) that the above Sobolev inequality holds with $W_0^{1,p}(D)$ in place of $W^{1,p}(D)$ for arbitrary domains D, where $W_0^{1,p}(D)$ is the subspace of $W^{1,p}(D)$ obtained by completing the space of C^∞ functions having compact support in D with respect to the norm $\|\cdot\|_{1,p}$. However, the Sobolev inequality (2) on $W^{1,p}(D)$ cannot hold without any restrictions on D. For example, (2) cannot be true for a domain D with finite Lebesgue measure since in this case $1 \in W^{1,p}(D)$ and the right hand side of (2) vanishes.

In [9], using a form of capacity, Maz'ja characterizes the class $\mathcal{J}$ of open sets D for which the Sobolev inequality (2) holds. He also gives the best constant c in the Sobolev inequality (2) (see Theorem 4.7.4 of [9]). However we found Maz'ja's condition difficult to check in practice, despite the fact that the class $\mathcal{J}$ is closed under the operation of taking finite unions (by Theorem 4.7.4 and Proposition 4.3.1/1 in [9]). This motivated us to prove the Sobolev inequality (2) directly under the assumptions in Theorem 1. In particular, by Theorem 1 and Theorem 4 below, unbounded domains with compact boundary having the cone property and exteriors of closed convex sets are in $\mathcal{J}$.

As an application of the Sobolev inequality (2) for $p = 2$ and $d \geq 3$, we shall prove Theorem 2 below. This has been applied in [3] to the study of semilinear elliptic equations with Neumann boundary conditions. Before we can state Theorem 2, several notions need to be introduced.

A domain D is said to be Lipschitz (or $C^{0,1}$) if locally near ∂D, D can be represented as the region lying above the graph of a Lipschitz function (see, e.g., p.244 of [5]). For such a domain D, denote by $\mathcal{E}$ the quadratic form defined on $W^{1,2}(D)$ by:

$$\mathcal{E}(f,g) = \frac{1}{2}\int_D f(x)g(x)dx, \quad \text{for } f,g \in W^{1,2}(D).$$

There is a unique self-adjoint non-positive operator $\mathcal{A}$, with domain $\mathcal{D}(\mathcal{A})$, associated with $(W^{1,2}(D),\mathcal{E})$. In particular,

$$\mathcal{D}(\mathcal{A}) = \Big\{ f \in W^{1,2}(D) : \exists\, g \in L^2(D) \text{ s.t. } \mathcal{E}(f,h) = -\int_D gh\,dx \text{ for all } h \in W^{1,2}(D) \Big\},$$

and for f and g as in the description of $\mathcal{D}(\mathcal{A})$, $\mathcal{A}f = g$ (see [7]). The symmetric strongly continuous contraction semigroup $\{P_t\}_{t>0}$ associated with $(\mathcal{A},\mathcal{D}(\mathcal{A}))$ has a symmetric integral kernel $p(t,x,y)$ which is smooth on $(0,\infty)\times D\times D$ and such that $P_tf(x) = \int_D p(t,x,y)f(y)\,dy$ a.e. on D for $f \in L^2(D)$. See Lemma 2.11 of [6] for details on the existence of $p(t,x,y)$ (note that although in [6] domains are assumed to be bounded, the proof of the above fact works for unbounded domains as well). When ∂D is smooth, $p(t,x,y)$ can be shown to be the fundamental solution for the heat equation with zero Neumann boundary condition (see [12], for example). By analogy, when D is Lipschitz, we call $p(t,x,y)$ the heat kernel for $\frac{1}{2}\Delta$ on D with zero Neumann boundary condition.

Theorem 2. *Suppose D is an unbounded domain with compact Lipschitz boundary in $\mathbb{R}^d$ where $d \geq 3$. Then the heat kernel $p(t,x,y)$ of $\frac{1}{2}\Delta$ on D with zero Neumann boundary condition can be extended continuously to $(0,\infty)\times\overline{D}\times\overline{D}$; we still denote the extension by $p(t,x,y)$. Then there exist constants $c_1>0$ and $M>1$ such that*

$$p(t,x,y) \leq \frac{c_1}{t^{d/2}}\exp\left(-\frac{|x-y|^2}{Mt}\right), \quad \text{for all } t>0,\ x,\,y\in\overline{D}. \tag{3}$$

For $x,y\in\overline{D}$, let $G(x,y) = \int_0^\infty p(t,x,y)dt$. When $G(x,y)$ is finite for all $x,y\in\overline{D}$ with $x\neq y$, it is called the Green's function for $\frac{1}{2}\Delta$ on D with zero Neumann boundary condition. Integrating both sides of (3) gives:

Corollary 3. *The Green's function $G(x,y)$ for $\frac{1}{2}\Delta$ on D with zero Neumann boundary condition exists and is continuous on $\overline{D}\times\overline{D}$, except on the diagonal. Furthermore, there exists a constant $c_2 = c_2(D) > 0$ such that*

$$G(x,y) \le \frac{c_2}{|x-y|^{d-2}} \quad \text{for all } x,y\in\overline{D}. \tag{4}$$

2. Proof of Theorem 1

We begin by proving the Sobolev inequality (2) for the exterior of a closed convex set.

Theorem 4. *Suppose that U is the exterior of a closed convex set in $\mathbb{R}^d$. Then for $1\le p<d$ there exists a constant $c=c(U,p)$ such that for $f\in W^{1,p}(U)$,*

$$\|f\|_q \le c\,\|\nabla f\|_p, \tag{5}$$

where $1/q = 1/p - 1/d$. In particular, the above inequality holds for the exterior of a bounded closed ball.

Remark. In the above theorem, U may have non-compact boundary.

Proof. Since U is the exterior of a closed convex set, U is Lipschitz (see, for example, Theorem 4.2 of Ch.V in [5]). Therefore by Theorem 4.7 of Chapter V in [5], the set of restrictions to U of all C^∞ functions with compact support in $\mathbb{R}^d$ is $\|\cdot\|_{1,p}$-dense in $W^{1,p}(U)$ for $p\ge 1$. Hence it suffices to prove (5) for all functions f in $W^{1,p}(D)$ that are smooth in D and such that $f(x)$ vanishes when $\|x\|$ is sufficiently large. Since $\mathbb{R}^d\setminus U$ is convex, for $x\in U$ and each $i\in\{1,\cdots,d\}$, there is a half-line in U which is parallel to the ith coordinate axis and has x as its initial point. Thus one has, for all $x\in U$,

$$|f(x)| \le \int_{-\infty}^{\infty} 1_D(\xi)\left|\frac{\partial f}{\partial \xi_i}(\xi)\right| d\xi_i, \quad i=1,2,\ldots,d. \tag{6}$$

Inequality (5) then follows from the standard argument for proving the corresponding Sobolev inequality in $W_0^{1,p}(D)$ (see the proof of Theorem 3.6 in Ch.V of [5], for example). □

For $r>0$, denote by B_r the ball $\{x\in\mathbb{R}^d:\ \|x\|<r\}$. Let $D_r = D\cap B_r$. Define $W_r^{1,p}(D_r)$ to be the closure in $W^{1,p}(D_r)$ of the set of restrictions to D_r of all $C^\infty(\mathbb{R}^d)$ functions having compact support in B_r. Intuitively, $W_r^{1,p}(D_r)$ contains those functions in $W^{1,p}(D_r)$ that vanish on ∂B_r.

We have the following Poincaré inequality on $W_r^{1,p}(D_r)$ for $r>0$ such that $B_r\supset\partial D$.

Lemma 5. *Suppose that* $1 \le p < d$ *and* $r > 0$ *such that* $B_r \supset \partial D$. *There exists a constant* $c = c(D, r, p) > 0$ *such that for each* $f \in W_r^{1,p}(D_r)$,

$$\|f\|_p \le c\,\|\nabla f\|_p. \tag{7}$$

Proof. Let

$$\lambda = \inf\left\{\frac{\|\nabla f\|_p}{\|f\|_p} :\ f \in W_r^{1,p}(D_r)\right\}.$$

There exists a sequence $\{f_n\}_{n\ge1} \subset W_r^{1,p}(D_r)$ such that $\|f_n\|_p = 1$ for all n and $\|\nabla f\|_p$ decreases to λ as $n \to \infty$. Since D_r is a bounded domain with the cone property, by the Rellich-Kondrachov compactness theorem (see [1], p.144), the imbedding $W^{1,p}(D_r) \hookrightarrow L^p(D_r)$ is compact for $1 \le p < d$ (note that for this one uses the fact that $p < q \equiv pd/(d-p)$). Therefore, without loss of generality, we may assume that $\{f_n\}_{n\ge1}$ converges in $L^p(D_r)$ to some f with $\|f\|_p = 1$. Now suppose that $\lambda = 0$. Then $\|\nabla f_n\|_p$ decreases to zero as $n \to \infty$, and for any smooth function ψ with compact support in D_r and $i = 1, 2, \ldots, d$, using integration by parts we have

$$\begin{aligned}\int_{D_r} f(x)\frac{\partial\psi}{\partial x_i}(x)\,dx &= \lim_{n\to\infty}\int_{D_r} f_n(x)\frac{\partial\psi}{\partial x_i}(x)\,dx\\ &= -\lim_{n\to\infty}\int_{D_r}\frac{\partial f_n}{\partial x_i}(x)\psi(x)\,dx\\ &= 0.\end{aligned}$$

Thus $\nabla f = 0$ and f is a constant function on D_r. But f is a limit in $W^{1,p}(D_r)$ of functions in $W_r^{1,p}(D_r)$ and hence $f \in W_r^{1,p}(D_r)$. The only constant function in this space is identically zero, which contradicts the fact that $\|f\|_p = 1$. Therefore $\lambda > 0$ and (7) is established with $c = 1/\lambda$. □

Proof of Theorem 1. In this proof we have functions defined on different domains, U_1, U_2, D. To clearly indicate which domain applies for integration, for this proof only we shall use the notation

$$\|g\|_{U,p} = \left(\int_U |g(x)|^p dx\right)^{1/p}$$

for a domain U and $g \in L^p(U)$. In this proof, c will denote various constants.

Let $r > 1$ such that $B_{r-1} \supset \partial D$. Let ϕ be a C^∞ function on $\mathbb{R}^d$ with compact support in B_r such that $0 \le \phi \le 1$ and $\phi = 1$ on $B_{r-\frac{1}{2}}$. Let $U_1 = D \cap B_r$ and $U_2 = \{x \in \mathbb{R}^d :\ \|x\| > r-1\}$. For $f \in W^{1,p}(D)$, set

$f_1 = f\,\phi$ and $f_2 = f\,(1-\phi)$ on D. Then $f = f_1 + f_2$. We next consider f_1 as an element of $W_r^{1,p}(U_1)$ and f_2 as an element of $W_0^{1,p}(U_2)$. Since U_1 is bounded and has the cone property, by the Sobolev imbedding theorem for $W^{1,p}(U_1)$ (cf. Theorem 5.4 of [1]),

$$\|f_1\|_{U_1,q} \le c\,\left(\|\nabla f_1\|_{U_1,p} + \|f_1\|_{U_1,p}\right),$$

which by Lemma 5 implies

$$\|f_1\|_{U_1,q} \le c\,\|\nabla f_1\|_{U_1,p} \le c\,\left(\|\nabla f\|_{D,p} + \|f\,\nabla\phi\|_{D,p}\right). \tag{8}$$

By Theorem 4,

$$\|f_2\|_{U_2,q} \le c\,\|\nabla f_2\|_{U_2,p} \le c\,\left(\|\nabla f\|_{D,p} + \|f\,\nabla\phi\|_{D,p}\right). \tag{9}$$

Since $\nabla\phi$ is supported in $B_r \setminus B_{r-1}$, by Hölder's inequality,

$$\|f\,\nabla\phi\|_{D,p} = \|f\nabla\phi\|_{U_2,p} \le \|\nabla\phi\|_{D,d}\,\|f\|_{U_2,q}. \tag{10}$$

Note that the restriction of f to U_2 is in $W^{1,p}(U_2)$ and so by Theorem 4,

$$\|f\|_{U_2,q} \le c\,\|\nabla f\|_{U_2,p} \le c\,\|\nabla f\|_{D,p}. \tag{11}$$

Since $f = f_1 + f_2$ where f_1 has support in U_1 and f_2 has support in U_2, combining (8)-(11) proves the Sobolev inequality (2). □

3. Proof of Theorem 2

In this section we assume that D is an unbounded domain in $\mathbb{R}^d$ with compact Lipschitz boundary and $d \ge 3$. Thus, the Sobolev inequality (2) yields

$$\|f\|_p \le c\,\|\nabla f\|_2 \quad \text{for all } f \in W^{1,2}(D), \tag{12}$$

with $p = 2d/(d-2)$. Let $p(t,x,y)$ be the heat kernel for $\frac{1}{2}\Delta$ on D with zero Neumann boundary condition as described in Section 1. Recall that $p(t,x,y)$ is symmetric in x, y and is smooth on $(0,\infty)\times D\times D$. By using the standard method in [4] (more specifically, Theorems 2.4.6, 2.2.3, and a straightforward adaptation of Section 3.2 to the case of zero Neumann boundary conditions), one can use (12) to show that

(13)
$$p(t,x,y) \le \frac{c_3}{t^{d/2}}\,\exp\left(-\frac{|x-y|^2}{Mt}\right), \quad \text{for all } (t,x,y)\in(0,\,\infty)\times D\times D,$$

for some constants $c_3 > 0$ and $M > 1$.

We are going to show now that $p(t,x,y)$ can be extended continuously to $(0,\,\infty)\times \overline{D}\times\overline{D}$. Let $r > 0$ such that $B_r \supset \partial D$. Denote by $h(t,x,y)$

the (symmetric) heat kernel for $\frac{1}{2}\Delta$ on $D_r = D \cap B_r$ with zero Neumann boundary condition. From [2] we have that h can be extended continuously to $(0, \infty) \times \overline{D}_r \times \overline{D}_r$ and that for each $T > 0$ there exist constants $c_r = c_r(D, r, T) > 0$ and $M_r = M_r(D, r, T) > 1$ such that

(14)
$$h(t, x, y) \leq \frac{c_r}{t^{d/2}} \exp\left(-\frac{|x-y|^2}{M_r t}\right) \quad \text{for all } (t, x, y) \in (0, T] \times \overline{D}_r \times \overline{D}_r.$$

Let $(Y, \{Q_x,\ x \in \overline{D}_r\})$ be the continuous strong Markov process that is (normally) reflecting Brownian motion on the bounded Lipschitz domain D_r (see [2]). It follows from [2] that this process has $h(t, x, y)$ as its transition density function. Denote by $p_r(t, x, y)$ the symmetric integral kernel on $\overline{D} \cap B_r$ for the semigroup of Y killed on hitting ∂B_r. The existence of p_r follows from the strong Markov property of Y in a similar manner to that in [10], p.33; in particular we have

$$p_r(t, x, y) = h(t, x, y) - E^{Q_x}[h_r(t - \tau_r, Y_{\tau_r}, y); t > \tau_r] \tag{15}$$

for $(t, x, y) \in (0, \infty) \times (\overline{D} \cap B_r) \times (\overline{D} \cap B_r)$, where $\tau_r = \inf\{t \geq 0 : Y_t \in \partial B_r\}$. Intuitively $p_r(t, x, y)$ is the heat kernel for $\frac{1}{2}\Delta$ on D_r with zero Neumann boundary condition on ∂D and zero Dirichlet boundary condition on ∂B_r. For $\epsilon \in (0, r)$, by (14),

$$|h(t, x, y)| \leq \frac{c_r}{t^{d/2}} \exp\left(-\frac{\epsilon^2}{M_r t}\right) \quad \text{for all } (t, x, y) \in (0, T] \times \partial B_r \times (\overline{D} \cap \overline{B}_{r-\epsilon}).$$

Thus as $t \to 0$, $h(t, x, y)$ converges to zero uniformly for $(x, y) \in \partial B_r \times (\overline{D} \cap \overline{B}_{r-\epsilon})$. Therefore $\{h(t, x, y)\}_{x \in \partial B_r}$ as a family of functions of (t, y) is equi-continuous on $(0, T] \times (\overline{D} \cap \overline{B}_{r-\epsilon})$. Since $Y_{\tau_r} \in \partial B_r$,

$$\left\{E^{Q_x}\left[h(t - \tau_r, Y_{\tau_r}, y);\ t > \tau_r\right]\right\}_{x \in \overline{D} \cap B_r}$$

is equi-continuous for $(t, y) \in (0, T] \times (\overline{D} \cap \overline{B}_{r-\epsilon})$. It follows from (15) that $\{p_r(t, x, y)\}_{x \in \overline{D} \cap \overline{B}_{r-\epsilon}}$ is equi-continuous for $(t, y) \in [\epsilon, T] \times (\overline{D} \cap \overline{B}_{r-\epsilon})$ for any $\epsilon \in (0, T)$. Since $p_r(t, x, y)$ is symmetric in (x, y), $p_r(t, x, y)$ is uniformly continuous on $[\epsilon, T] \times (\overline{D} \cap \overline{B}_{r-\epsilon}) \times (\overline{D} \cap \overline{B}_{r-\epsilon})$. Therefore $p_r(t, x, y)$ is continuous on $(0, T) \times (\overline{D} \cap B_r) \times (\overline{D} \cap B_r)$ for each $T > 0$.

On the other hand, if $(X, \{P_x, x \in \overline{D}\})$ denotes the continuous strong Markov process that is (normally) reflecting Brownian motion on $\overline{D}$, then similar to (15) we have for $(t, x, y) \in (0, \infty) \times D_r \times D_r$,

$$p_r(t, x, y) = p(t, x, y) - E^{P_x}[p(t - \tau_r, X_{\tau_r}, y);\ t > \tau_r], \tag{16}$$

where $\tau_r = \inf\{t \geq 0 : X_t \in \partial B_r\}$. Since $X_{\tau_r} \in \partial B_r$, it follows from the inequality (13) that for each positive integer k, as a function of (t, x, y) in $(0, \infty) \times D_k \times D_k$, $E^{P_x}[p(t - \tau_r, X_{\tau_r}, y);\ t > \tau_r]$ converges to zero uniformly as $r \to \infty$, where $D_k = D \cap B_k$. Therefore for each $k > 0$, $p_r(t, x, y)$ converges to $p(t, x, y)$ uniformly on $(0, \infty) \times D_k \times D_k$ as $r \to \infty$. Hence $p(t, x, y)$ can be extended continuously to $(0, \infty) \times \overline{D}_k \times \overline{D}_k$ and therefore on $(0, \infty) \times \overline{D} \times \overline{D}$. Thus inequality (13) holds for $(t, x, y) \in (0, \infty) \times \overline{D} \times \overline{D}$. Theorem 2 is now proved. □

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References

1. R. A. Adams, *Sobolev Spaces.* Academic Press, 1975.
2. R. F. Bass and P. Hsu, *Some potential theory for reflecting Brownian motion in Hölder and Lipschitz domains.* Ann. Probab. **19** (1991), 486–508.
3. Z. Q. Chen, R. J. Williams and Z. Zhao, *On the existence of positive solutions for semilinear elliptic equations with Neumann boundary conditions.* Preprint.
4. E. B. Davies, *Heat Kernels and Spectral Theory.* Cambridge University Press, Cambridge, U.K., 1989.
5. D. E. Edmunds and W. D. Evans, *Spectral Theory and Differential Operators.* Clarendon Press, Oxford, 1987.
6. M. Fukushima, *A construction of reflecting barrier Brownian motions for bounded domains.* Osaka J. Math. **4** (1967), 183–215.
7. M. Fukushima, *Dirichlet Forms and Markov Processes.* North-Holland, Amsterdam, 1980.
8. P. Li and S.-T. Yau, *Asymptotically flat complete Kähler manifolds.* In Pure and Applied Mathematics **143**, edited by G. Komatsu and V. Sakane, Marcel Dekker, New York-Basel-Hong Kong, 1992, 131–143.
9. V. G. Maz'ja, *Sobolev Spaces.* Springer-Verlag, Berlin-Heidelberg-New York, 1985.
10. S. C. Port and C. J. Stone, *Brownian Motion and Classical Potential Theory.* Academic Press, New York, 1978.
11. D. W. Stroock, *Diffusion semigroups corresponding to uniformly elliptic divergence form operators.* In Lecture Notes in Mathematics **1321**, Springer-Verlag, Berlin-Heidelberg-New York, 1988, 316–347.
12. K. Sato and T. Ueno, *Multi-dimensional diffusion and the Markov process on the boundary.* J. Math. Kyoto Univ. **4-5** (1965), 529–605.

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