Mathematical Research Letters **1**, 159–166 (1994)

THE FIRST EIGENVALUE OF ANALYTIC LEVEL SURFACES ON SPHERES

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Abstract. In this paper we establish a lower bound for the first eigenvalue of the Laplace-Beltrami operator of the level set of a real valued real-analytic function defined on spheres. The question of existence of such a lower bound was posed by P.Cordaro and J.Hounie and arose in their work on local solvability of systems of vector fields [CH].

0. Introduction

The aim of this note is to give a proof of a question raised by Paulo Cordaro and Jorge Hounie. Consider a real valued, real-analytic funtion f defined on the $n+1$ dimensional sphere S^{n+1}, with $n \geq 1$, i.e. $f : S^{n+1} \to \mathbb{R}, \;\; n \geq 1$. We next consider the non-singular connected level set V_t given by $V_t = f^{-1}(t)$, $t \in \mathbb{R}$. V_t is a manifold and furthermore V_t inherits the Riemannian structure of the sphere. Let t_0 denote a critical value of f. We are interested in the first eigenvalue $\lambda_1(V_t)$ as $t \to t_0$, of the Laplace-Beltrami operator on V_t. We henceforth choose the sign of the Laplace-Beltrami operator on V_t so that its eigenvalues are non-negative.

Theorem. *There exist constants (independent of t and t_0), $C = C(f) > 0$ and $\alpha = \alpha(f)$, such that*

$$\lambda_1(V_t) \geq C|t - t_0|^\alpha \qquad \textit{as } t \to t_0.$$

In the case where f is a polynomial, M. Gromov [G] has answered the Cordaro-Hounie question in the affirmative. However from the viewpoint of applications to local solvability which originally led Cordaro and Hounie to pose this question, it is essential that f be allowed to be at least real-analytic. When one wishes to study local solvability of a system of vector fields as in [CH] one is interested in appropriate bounds on the Green

Received September 2, 1993.
Supported by NSF grant DMS-9202051.

function, this is equivalent to our theorem. The function f arises from the first integrals of these vectors fields which a-priori need not be polynomials. The results of this paper can be used to give another proof of theorem (3.1) of [CH] wherein Cordaro and Hounie partially proved a conjecture of Treves [Tr] for the particular case of forms of top degree.

It is possible to prove the theorem by using elementary calculus but this approach leads to a very large value of α. For local solvability questions the value of α is immaterial. Since the value of α may be important for other problems we have approached this problem by using some differential geometry, though elementary.

In what follows C, C_1 denote constants which are different at different places but at all times independent of t and t_0. In particular C_1 will only depend on the dimension.

1. Proof of the theorem

We proceed by reducing the problem to essentially a Euclidean case. First notice, from the Lojasiewicz inequality,

$$\inf_{x\in V_t} |\nabla f(x)| \geq C\ |t-t_0|^{\alpha_1} \tag{1}$$

for $C = C(f) > 0$ and $\alpha_1 = \alpha_1(f)$ independent of t and t_0.

We now set up some notation. We will denote by $\{e_1, \ldots, e_n\}$ an orthonormal basis for the tangent space at a point $x \in V_t$. We denote the sectional curvature at x with respect to the tangent directions e_i and e_j by $\kappa(x)(e_i, e_j)$. Finally we set $|\kappa(x)| = \sup_{i,j} |\kappa(x)(e_i, e_j)|$.

Lemma 1. *Let $\kappa(x)$ be defined as above. Then for some $\beta < 0$, β independent of t and t_0, we have*

$$\sup_{x\in V_t} |\kappa(x)| \leq C|t-t_0|^{\beta}, \qquad C = C(f).$$

Proof. This follows from the definition of sectional curvature and (1). In fact one can set $\beta = -3\alpha_1$. □

Lemma 2. *Let $\tau = \tau(V_t)$ denote the injectivity radius of V_t. Then*

$$\tau(V_t) \geq C|t-t_0|^{\gamma}, \qquad C = C(f).$$

Proof. We first note that on V_t we have, in suitable local coordinates, $f(x_1, \ldots, x_{n+1}) = t$. Denote by z_0 any point of V_t that we keep fixed for the rest of the calculation. Assume, with no loss of generality in view of

(1), that $|\frac{\partial f}{\partial x_{n+1}}(z_0)| \geq C|t-t_0|^{\alpha_1}$ and x_{n+1} is the normal direction to the surface V_t at z_0. We will show that there is a neighbourhood of z_0 of radius at least $C|t-t_0|^{\alpha_1}$ such that for all points $z \in V_t$ in this neighborhood we also have $|\frac{\partial f}{\partial x_{n+1}}(z)| \geq C|t-t_0|^{\alpha_1}$. Thus the implicit function theorem will tell us that V_t is given by a graph in a neighborhood of z_0 of radius $C|t-t_0|^{\alpha_1}$.

To see this, note that

$$|\frac{\partial f}{\partial x_{n+1}}(z_0)| \leq |\frac{\partial f}{\partial x_{n+1}}(z_0) - \frac{\partial f}{\partial x_{n+1}}(z)| + |\frac{\partial f}{\partial x_{n+1}}(z)|.$$

Now by (1) applied to the left side of the above inequality and the fact that f is real analytic, in particular C^2, we have

$$C|t-t_0|^{\alpha_1} \leq C|z_0 - z| + |\frac{\partial f}{\partial x_{n+1}}(z)|.$$

Thus for $|z-z_0| \leq \frac{C}{2}|t-t_0|^{\alpha_1}$ our claim follows by subtracting the first term on the right above from the left hand side of the inequality. Next note that the fact that $|\frac{\partial f}{\partial x_{n+1}}(z)| \geq C|t-t_0|^{\alpha_1}$ for all z for which $|z-z_0| \leq |t-t_0|^{\alpha_1}$ easily implies that the absolute value of the dot product of the unit normal vector to the graph at z and the unit vector along the positive x_{n+1} axis is bounded below by $C|t-t_0|^{\alpha_1}$ for all z for which $|z-z_0| \leq C|t-t_0|^{\alpha_1}$. Thus by a projection onto the tangent plane of V_t at z_0, the lemma follows in fact with the choice $\gamma = 2\alpha_1$. We point out that the lemma is stated in [G] with no proof. $\square$

We now select α_2 so that we have

$$r_0 = \frac{C|t-t_0|^{\alpha_2}}{100} \leq \min\left(\frac{\tau(V_t)}{100},\ C|t-t_0|^{-\beta/2}\right).$$

A choice $\alpha_2 = 2\alpha_1$ will suffice, for example. Consider the geodesic ball $B(x,s)$, $s \leq 10r_0$. By Lemma 2 and the choice for r_0, on this ball we may introduce geodesic polar coordinates. That is, for each $z \in B(x,s)$, we have $z = e^{r\xi}$, $r \in \mathbb{R}$, $0 \leq r \leq 10r_0$, $\xi \in S^{n-1}$. Let $d\mu$ denote the volume element on V_t, with respect to the induced metric. Then one has, in geodesic polar coordinates,

$$d\mu(r,\xi) = \sqrt{g(r,\xi)}\ dr\, d\sigma(\xi),$$

where $d\sigma(\xi)$ is the surface measure of the $n-1$ dimensional sphere.

Lemma 3. *For $r \le 10r_0$, there exists $C_1 > 0$ a dimensional constant such that*

$$C_1 r^{n-1} \le \sqrt{g(r,\xi)} \le C_1^{-1}\, r^{n-1}.$$

Proof. The left hand inequality follows by using Lemma 1 and the Bishop comparison theorem, eqn. (34), pg. 69 of [C]. We now prove the right hand inequality. First we denote the Ricci tensor by $\mathrm{Ric}(x)(\eta,\eta')$ where η and η' are two tangent vectors in the tangent space at $x \in V_t$. Set $\eta = |\eta| e_n$. Here $|\eta|$ denotes the metric length of the tangent vector η. Then observe that,

$$\mathrm{Ric}(x)(\eta,\eta) = |\eta|^2 \sum_{i=1}^{n-1} \kappa(x)(e_i, e_n).$$

Applying the lower bound of Lemma 1 we see at once that uniformly for $x \in V_t$ we have the following lower bound on the Ricci curvature,

$$\mathrm{Ric}(x)(\eta,\eta) \ \ge\ -C(n-1)|t-t_0|^{\beta}|\eta|^2.$$

Thus the second Bishop comparison theorem equation (42), pg. 72 of [C] applies and we immediately conclude the right hand inequality above. Notice the right hand inequality holds under a weaker hypothesis: just a lower bound for the Ricci curvature will suffice. Lemma 1 gives us a much stronger hypothesis than what is needed to apply the Bishop comparison theorem. □

Our next lemma is a local Poincaré inequality. We set up some notation. Let

$$|B(x,s)| = \frac{s^n}{n} \int_{S^{n-1}} d\sigma(\xi),$$

the "Euclidean" volume of $B(x,s)$. Further, for $z \in B(x,s)$, $z = e^{r\xi}$ we will write $h(z)$ as $h(r,\xi)$.

Lemma 4. *For any function $h \in C^1(V_t)$, we have, for $0 \le s \le 10r_0$ and $C_1 = C_1(n)$,*

$$\int_{B(x,s)} |h - h_B|^2 \, d\mu \le C_1 s^2 \int_{B(x,s)} |\nabla h|^2 \, d\mu$$

where $h_B = \frac{1}{|B(x,s)|} \int_{B(x,s)} h(r,\xi) r^{n-1} \, dr \, d\sigma(\xi)$.

Proof. Since s is less than the injectivity radius of V_t we may express the left side in geodesic polar coordinates and use the right hand inequality of

Lemma 3 to get

$$\int_{B(x,s)} |h - h_B|^2 \sqrt{g(r,\xi)}\, dr\, d\sigma(\xi) \leq C_1 \int_{B(x,s)} |h - h_B|^2\ r^{n-1}\, dr\, d\sigma(\xi).$$

Applying the regular Euclidean Poincaré inequality to the right side, the right side is bounded by

$$C_1 s^2 \int_{B(x,s)} |\nabla h|^2\ r^{n-1}\, dr\, d\sigma(\xi).$$

Applying the left hand inequality of Lemma 3 to the integral above we immediately get our lemma. □

We next need a covering lemma the statement of which was motivated by the pictures in Gromov's paper [G].

Lemma 5. *There exists a family of geodesic balls* $\{B(x_1, r_0)\}_{i=1}^k$ *such that*

(a) $V_t = \bigcup_{i=1}^{k} B(x_i, r_0)$
(b) *For* $i \neq j$, $B(x_i, \frac{r_0}{2}) \cap B(x_j, \frac{r_0}{2})$ *is a set of measure zero.*
(c) $k \leq C r_0^{-n}$, $C = C(f)$ *independent of* t *and* t_0.

Proof. Parts (a) and (b) are trivial to accomplish. For part (c) we use a result of R. Hardt [H2], which is based on [H1]. Particularly elegant proofs have also been given by H. Sussmann [S] and B. Teissier [T]. Their results state that $\mathcal{H}^n(V_t)$, the n-dimensional Hausdorff measure of V_t, satisfies

$$\mathcal{H}^n(V_t) \leq C, \qquad C = C(f), \tag{2}$$

independent of t and t_0.

Part (c) immediately follows from (b) and (2), for $V_t \supset \bigcup_{i-1}^{k} B(x_i, \frac{r_0}{2})$ and thus by (b), (2) and Lemma 3,

$$k r_0^n \leq C \mathcal{H}^n(V_t) \leq C.$$

This gives us (c). □

We can now prove our theorem.

Proof of the theorem. It will be enough to show that one can find a constant $c_h = c(h)$, such that for any $h \in C^1(V_t)$,

$$\int_{V_t} |h - c_h|^2 \, d\mu \leq C|t - t_0|^{-\alpha} \int_{V_t} |\nabla h|^2 \, d\mu. \tag{3}$$

We now denote the balls $B(x_i, r_0)$ in Lemma 5 by B_i. We simply set

$$c_h = \frac{1}{|B_1|} \int_{B_1} h(r, \xi) r^{n-1} \, dr \, d\sigma(\xi).$$

By (a) of Lemma 5,

$$\int_{V_t} |h - c_h|^2 \, d\mu \leq \sum_{i=1}^{k} \int_{B_i} |h - c_h|^2 \, d\mu. \tag{4}$$

Using the notation of Lemma 4, we set

$$h_{B_i} = \frac{1}{|B(x_i, r_0)|} \int_{B(x_i, r_0)} h(r, \xi) r^{n-1} \, dr \, d\sigma(\xi).$$

The right side of (4) is thus at most

$$A + B = 4 \sum_{i=1}^{k} \int_{B_i} |h - h_{B_i}|^2 + 4 \sum_{i=1}^{k} |h_{B_i} - c_h|^2 \mu(B_i). \tag{5}$$

We apply Lemma 4 to the sum A to get

$$A \leq \sum_{i=1}^{k} 4r_0^2 \int_{B_i} |\nabla h^2| \, d\mu \leq 4k_0^2 \int_{V_t} |\nabla h|^2 \, d\mu.$$

Using (c) of Lemma 5 and inserting the estimate for k, we do get the right side of (3). We now estimate B. First note we can connect any ball B_i to the ball B_1, by a chain of balls $B_{i,1} = B_1, \ldots, B_{i,m}, \ldots, B_{i,m_0} = B_i$, where the balls $B_{i,m}$ are members of the cover in Lemma 5. Thus, keeping in mind $c_h = h_{B_{i,1}}$, we have

$$|h_{B_i} - c_h| \leq \sum_{m=1}^{m_0} |h_{B_{i,m}} - h_{B_{i,m+1}}|. \tag{6}$$

The balls $B_{i,m}$ and $B_{i,m+1}$ are adjacent and have the same radii, r_0. Thus there exists a ball $B_m = B(x, 5r_0)$ such that

a) $B_m \supset B_{i,m}$ and $B_m \supset B_{i,m+1}$; and
b) $|B_m| \leq C_1 |B_{i,m}|$ and $|B_m| \leq C_1 |B_{i,m+1}|$.

Thus,

$$|h_{B_{i,m}} - h_{B_{i,m+1}}| \le |h_{B_{i,m}} - h_{B_m}| + |h_{B_m} - h_{B_{i,m+1}}|. \tag{7}$$

Now, by Schwarz's inequality

$$|h_{B_{i,m}} - h_{B_m}| \le \frac{1}{|B_{i,m}|}\int_{B_m} |h - h_{B_m}| \le \left(\frac{1}{|B_{i,m}|}\int_{B_m} |h - h_{B_m}|^2\right)^{1/2}.$$

By (a) and (b) above $|B_{i,m}| \approx |B_m|$. Thus applying Lemma 3 to the last integral on the right above we may replace the last integral by

$$\left(\frac{1}{|B_m|}\int_{B_m} |h - h_{B_m}|^2\, d\mu\right)^{1/2}.$$

Since $5r_0 \le \tau(V_t)$ we may apply Lemma 4 to upper bound the expression above by

$$C_1 r_0 \left(\frac{1}{|B_m|}\int_{B_m} |\nabla h|^2\, d\mu\right)^{1/2}.$$

Thus, we have the pair of inequalities

$$|h_{B_{i,m}} - h_{B_m}| \le C_1 r_0 \left(\frac{1}{|B_m|}\int_{B_m} |\nabla h|^2\, d\mu\right)^{1/2}$$

and

$$|h_{B_{i,m+1}} - h_{B_m}| \le C_1 r_0 \left(\frac{1}{|B_m|}\int_{B_m} |\nabla h|^2\, d\mu\right)^{1/2}.$$

From (7) and the pair of inequalities above we get

$$|h_{B_{i,m+1}} - h_{B_{i,m}}| \le C_1 r_0 |B_m|^{-1/2} \left(\int_{B_m} |\nabla h|^2\, d\mu\right)^{1/2}.$$

Inserting this in (6), and remembering that $|B_m| \approx C_1 r_o^n$ and $m_0 \le k$, we get

$$|h_{B_i} - c_h| \le C r_0^{-(n-2)/2} k \left(\int_{V_t} |\nabla h|^2\, d\mu\right)^{1/2}.$$

We now insert this into the expression B in (5), and recalling from Lemma 3 that $\mu(B_i) \approx C_1 r_0^n$, we get

$$B \le C_1 r_0^2 k^3 \int_{V_t} |\nabla h|^2\, d\mu.$$

Thus summing up A and B,

$$\int_{V_t} |h - c_h|^2\, d\mu \le C_1 r_0^2 k^3 \int_{V_t} |\nabla h|^2\, d\mu.$$

Since $k \le C r_0^{-n}$ and $r_0 = \frac{C|t-t_0|^{-\beta}}{100}$, we easily have (3) with $\alpha = \alpha_2(3n-2)$, $\alpha_2 = 2\alpha_1$. This proves the theorem. $\square$

Acknowledgement

I am deeply indebted to Paulo Cordaro for bringing this problem to my attention. I also wish to thank F. Treves for his encouragement and interest in this work.

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