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COHEN-MACAULAYNESS IN GRADED ALGEBRAS

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Cohen-Macaulayness of Rees algebras and associated graded rings of ideals has been actively investigated in recent years, cf. e.g., [AH], [AHT], [GN], [JK], and their references. We illustrate here how some methods introduced by Sancho de Salas in [SS] yield additional insights in this area. For one thing, they provide another approach to a basic result of Trung-Ikeda (Theorem 4.4). They also lead to new results. Theorem 4.1 relates the Cohen-Macaulay property of a Rees algebra to vanishing of cohomology of its Proj. As a special case, we get that in a normal local ring R, there exists an ideal I whose blowup desingularizes R and whose Rees algebra $R[It]$ is Cohen-Macaulay (CM) if and only if R has a rational singularity—at least in characteristic zero, and more generally if desingularization theorems such as have been announced by Spivakovsky hold up. (In contrast, Theorem 4.3, due to Sancho de Salas, implies that for *any* CM local ring on a variety over $\mathbb{C}$, there exists a desingularizing I whose associated graded ring is CM [SS, Thm. 2.8].) Theorem 5 gives an affirmative answer to a question of Huneke: it states that if R is pseudo-rational (for instance, R regular [LT, §4]) and if I is a non-zero R-ideal whose associated graded ring is CM, then $R[It]$ is CM too.

We begin with a most useful exact sequence due essentially to Sancho de Salas. Consider a noetherian graded ring

$$G = G_0 \oplus G_1 \oplus G_2 \oplus \cdots = R \oplus P$$

where $R := G_0$ and $P := G_1 \oplus G_2 \oplus \dots$ Let $N = \oplus_{n\in\mathbb{Z}} N_n$ be a graded G-module and let $\mathcal{N}_n$ $(n \in \mathbb{Z})$ be the quasi-coherent sheaf on $X := \operatorname{Proj}(G)$ corresponding to the graded G-module $N(n)$ $\big(N(n)_m = N_{n+m}$ $(m \in \mathbb{Z})\big)$.

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For any R-ideal $\mathfrak{m}$, and $E := X \otimes_R (R/\mathfrak{m})$, there is an exact sequence of graded G-module maps [SS, p. 202]:

(SS)
$$\cdots \to H^i_{\mathfrak{m}+P}(N) \to \bigoplus_{n\in\mathbb{Z}} H^i_{\mathfrak{m}}(N_n) \xrightarrow{\oplus\eta^i_n} \bigoplus_{n\in\mathbb{Z}} H^i_E(X,\mathcal{N}_n) \to H^{i+1}_{\mathfrak{m}+P}(N) \to \cdots$$

which we derive here somewhat differently, as follows.

Let $\mathbf{f} := (f_1,\dots,f_r)$ be a family of homogeneous elements in P such that the ideal $\mathbf{f}G$ contains some power of P. Let $\mathfrak{U}$ be the covering of X by its affine open subsets $\operatorname{Spec}(G_{(f_i)})$ $(1 \le i \le r)$ where $G_{(f_i)}$ is the ring consisting of degree 0 elements in the (graded) ring of fractions G_{f_i}, and let $\check{C}^\bullet(\mathfrak{U},\mathcal{N}_n)$ be the corresponding Čech complexes. Standard relations between Čech complexes and localizations (cf. e.g., [HIO, pp. 309–310]) translate into the exact sequence of graded complexes

$$(*) \qquad 0 \longrightarrow \oplus\check{C}^\bullet\big(\mathfrak{U},\mathcal{N}_n\big)[-1] \longrightarrow C^\bullet\big((\mathbf{f}),N\big) \longrightarrow \oplus N_n \longrightarrow 0$$

where $C^\bullet((\mathbf{f}),N)$, a direct limit of graded Koszul (co)complexes, is identical in degree $d \ge 1$ with $\oplus\check{C}^\bullet(\mathfrak{U},\mathcal{N}_n)[-1]$, the corresponding map in $(*)$ being multiplication by $(-1)^d$ (recall that the differentials in $\check{C}[-1]$ are the negatives of the corresponding differentials in $\check{C}$), and where in degree 0, $C^0\big((\mathbf{f}),N\big) \to \oplus N_n$ is the identity map. Then (SS) is essentially the graded hyperhomology sequence of the sequence $(*)$ with respect to the functor $\Gamma_{\mathfrak{m}} = \Gamma_{\mathfrak{m}G}$ (= elements annihilated by some power of $\mathfrak{m}$).

More explicitly, since the abelian category of *graded* G-modules has enough injectives [HIO, (33.4)], we can resolve $(*)$ in that category by an exact sequence of injective complexes, then apply $\Gamma_{\mathfrak{m}}$ and take the homology sequence of the resulting short exact sequence of graded complexes. Suppose, e.g., that the first complex in $(*)$ is quasi-isomorphic to the injective complex $E^\bullet$. Since $H^j_{\mathfrak{m}}(E^i) = \oplus H^j_{\mathfrak{m}}(E^i_n)$ vanishes for $j > 0$, therefore the degree n components of the graded injective modules E^i form a $\Gamma_{\mathfrak{m}}$-acyclic complex quasi-isomorphic to $\check{C}^\bullet_n := \check{C}^\bullet\big(\mathfrak{U},\mathcal{N}_n\big)[-1]$; and so the degree n component of the graded hyperhomology of $\oplus_{n\in\mathbb{Z}}\check{C}^\bullet_n$ with respect to $\Gamma_{\mathfrak{m}}$ is just the hyperhomology of $\check{C}^\bullet_n$, i.e., it is $H^\bullet_E(X,\mathcal{N}_n)$. In a similar way, using the fact that the homology of $C^\bullet((\mathbf{f}),N)$ is the local cohomology $H^\bullet_P(N)$ [HIO, (35.18)], we get the other terms in (SS).[1]

[1] In fact, with $U := \operatorname{Spec}(G) - \operatorname{Spec}(G/P)$, $\tilde{N}$ the $\mathcal{O}_U$-module corresponding to N, $g\colon U \to X$ the obvious map, and $\Gamma_+(N) = H^0(U,\tilde{N}) = H^0(X, g_*\tilde{N}) = \oplus H^0(X,\mathcal{N}_n)$, $(*)$ is naturally associated with a canonical triangle $\mathbf{R}\Gamma_P N \to N \to \mathbf{R}\Gamma_+ N \to \mathbf{R}\Gamma_P N[1]$ in the derived category of graded G-modules; and consequently (SS) does not depend on the choice of $\mathbf{f}$. (We won't need this degree of refinement.)

Corollary 1. *For any $d \geq 0$, if the natural map $\alpha_n \colon N_n \to \Gamma(X, \mathcal{N}_n)$ is an isomorphism and $H^i(X, \mathcal{N}_n) = 0$ for $0 < i < d$, then the map η_n^i in* (SS) *is an isomorphism for $0 \leq i < d$ and injective for $i = d$; and therefore $(H^i_{\mathfrak{m}+P}(N))_n = 0$ for $0 \leq i \leq d$.*

Proof. Working in the derived category of R-modules, we find from the preceding description of (SS) that η_n^i is the homology map associated to the composition

$$\mathbf{R}\Gamma_{\mathfrak{m}} N_n \xrightarrow{\mathbf{R}\Gamma_{\mathfrak{m}}\alpha_n} \mathbf{R}\Gamma_{\mathfrak{m}}\Gamma\big(X, \mathcal{N}_n\big) \xrightarrow{\mathbf{R}\Gamma_{\mathfrak{m}}\beta_n} \mathbf{R}\Gamma_{\mathfrak{m}}\mathbf{R}\Gamma\big(X, \mathcal{N}_n\big) = \mathbf{R}\Gamma_E\big(X, \mathcal{N}_n\big)$$

with $\beta_n \colon \Gamma(X, \mathcal{N}_n) \to \mathbf{R}\Gamma(X, \mathcal{N}_n)$ the canonical map. The vanishing of $H^i(X, \mathcal{N}_n)$ for $0 < i < d$ implies that if V is the vertex of a triangle whose base is β_n then $H^i(V) = 0$ for $i < d$, so that $H^i(\mathbf{R}\Gamma_{\mathfrak{m}} V) = 0$ for $i < d$, i.e., $H^i(\mathbf{R}\Gamma_{\mathfrak{m}}\beta_n)$ is an isomorphism for $0 \leq i < d$ and injective for $i = d$, whence the conclusion. □

Corollary 2. *Assume that the following all hold:*

(i) *$N_n = 0$ for all $n < 0$,*

(ii) *$H^i_E\big(\mathcal{N}_n\big) = 0$ for all $n < 0$ and $i < d$,*

(iii) *$\alpha_n \colon N_n \to H^0\big(X, \mathcal{N}_n\big)$ is an isomorphism for all $n \geq 0$, and*

(iv) *$H^i\big(X, \mathcal{N}_n\big) = 0$ for all $n \geq 0$ and $0 < i < d$.*

Then $H^i_{\mathfrak{m}+P}(N) = 0$ for $0 \leq i \leq d$.

Proof. For $n < 0$ the map η_n^i in (SS) is an isomorphism when $i < d$ and injective when $i = d$. By Corollary 1, the same holds for $n \geq 0$. The conclusion follows. □

Remarks. Let $X := \operatorname{Proj}(G) \xrightarrow{f} \operatorname{Spec}(R)$ be the natural map. If $(R, \mathfrak{m})$ is local and has a dualizing complex $\mathcal{R}^\bullet$, normalized so that $\operatorname{Ext}^0(R/\mathfrak{m}, \mathcal{R}^\bullet) \neq 0$, and if $\widetilde{\mathcal{R}}^\bullet$ is the corresponding quasi-coherent complex on $\operatorname{Spec}(R)$, then $\mathcal{R}^\bullet_X := f^!\widetilde{\mathcal{R}}^\bullet$ is a dualizing complex on X (cf. [RD, p. 299]). Moreover, if X is locally Cohen-Macaulay with all components of the same dimension δ, then $\mathcal{R}^\bullet_X$ is isomorphic to $\omega_X[\delta]$ for some coherent $\mathcal{O}_X$-module ω_X, called *canonical;*[2] and with I an injective hull of $R/\mathfrak{m}$, there is a duality isomorphism for coherent $\mathcal{O}_X$-modules $\mathcal{M}$:

$$H^i_E(\mathcal{M}) \xrightarrow{\sim} \operatorname{Hom}_R\big(\operatorname{Ext}_X^{\delta-i}(\mathcal{M}, \omega_X), I\big)$$

(cf. [L2, p. 188]). Hence if $\mathcal{N}_n$ is an invertible $\mathcal{O}_X$-module for all $n < 0$,

[2]By local duality [RD, p. 280, Cor. 6.5], $H^i_{\mathfrak{m}}(\mathcal{O}_{X,x}) = 0 \implies H^{-i}(\mathcal{R}^\bullet_{X,x}) = 0$ $(x \in X)$.

then condition (ii) in Corollary 2 is equivalent to
(ii)$'$: $H^i\big(X, (\mathcal{N}_{-n})^{-1} \otimes \omega_X\big) = 0$ for all $n > 0$ and $i > \delta - d$.

If $\mathcal{N}$ is invertible, then there exists an integer $k > 0$ such that (ii)$'$, (iii) and (iv) will be satisfied for $n > 0$ if we replace G by $G^{(ke)} := \oplus_{m\in\mathbb{Z}} G_{mke}$ and N by $N^{(ke)}$ for any $e \gg 0$ [EGA II, (2.5.16)], [EGA III, (2.2.2), (2.3.1)]. (Such replacements don't change X or $\mathcal{N}$. For k such that $G^{(k)} = R[G_k]$, $\mathcal{O}_X(k)$ is an ample invertible O_X-module and $(\mathcal{N}_{-nke}) = \mathcal{N} \otimes \mathcal{O}_X(k)^{\otimes -ne}$.)

Example 3. Let R be a two-dimensional pseudo-rational local ring [LT, p. 103, Example (a)]. Let $I \neq (0)$ be an integrally closed R-ideal, let t be an indeterminate, and let $G := R[It] =: N$. Then condition (i) of Corollary 2 is obvious. Since all the powers I^n $(n > 0)$ are integrally closed [L1, (7.1)], therefore $X := \mathrm{Proj}(G)$ is normal and (iii) is satisfied. Also, with $d = 2$, (ii) holds by [L2, p. 177, (2.4)], and (iv) by [L1, (12.1)(ii)]. Hence $H^i_{\mathfrak{m}+P}(N) = 0$ for $i = 0, 1$, and so (by e.g., [HIO, (11.11)]) *$R[It]$ is Cohen-Macaulay*. This result was obtained differently in [GN, Prop. (8.5)], the converse part of which is contained in Theorem 4.1 below.

Example 4. Let $(R, \mathfrak{m})$ be a d-dimensional noetherian local ring, with a filtration $\mathbf{F}$: $R = F_0 \supset F_1 \supset F_2 \supset \dots$, i.e., a sequence of R-ideals such that $F_iF_j \subset F_{i+j}$ for all i, j. The graded ring

$$R_{\mathbf{F}} := R \oplus F_1 t \oplus F_2 t^2 \oplus \dots \qquad (t \text{ indeterminate})$$

is known as the *Rees algebra* of $\mathbf{F}$. We assume $\mathbf{F}$ to be such that $R_{\mathbf{F}}$ is noetherian, of dimension $d+1$, i.e., [GN, (2.1), (2.2)]:

(i) There is an integer $k > 0$ such that $F_{nk} = (F_k)^n$ for all $n > 0$, and

(ii) R has a prime ideal $p \not\supseteq F_1$ such that $\dim R/p = d$.

If conditions (i) and (ii) hold for $\mathbf{F}$, and if $e > 0$, then (i) and (ii) hold for $\mathbf{F}^{(e)}$: $R = F_0 \supset F_e \supset F_{2e} \supset \dots$

For example, $\mathbf{F}$ could consist of the powers I^n $(n \geq 0)$ of an ideal I; or of the integral closures $\overline{I^n}$.

In the foregoing, take $G := R_{\mathbf{F}} =: N$ and $X := \mathrm{Proj}(R_{\mathbf{F}}) = \mathrm{Proj}(R_{\mathbf{F}^{(e)}})$. If F_1 contains a regular element of R, and if K is the total ring of fractions of R, then

$$R \subset H^0(X, \mathcal{O}_X) = \bigcup_{n>0} \{\, x \in K \mid xF_{nk} \subset F_{nk} \,\} \qquad (k \text{ as in (i)}),$$

with equality when R is normal.

Theorem 4.1. *In the preceding situation, suppose that X is Cohen-Macaulay (CM for short), and that the canonical map $R \to \mathbf{R}\Gamma(\mathcal{O}_X)$ is an isomorphism (i.e., $H^0(X, \mathcal{O}_X) = R$ and $H^i(X, \mathcal{O}_X) = 0$ for all $i > 0$). Then for some $e > 0$, the Rees algebra $R_{\mathbf{F}^{(e)}}$ is CM (and is generated by its forms of degree one).*

Conversely, if $R_{\mathbf{F}}$ is CM then X is CM and $R \xrightarrow{\sim} \mathbf{R}\Gamma(\mathcal{O}_X)$.

Proof. With $G = N = R_{\mathbf{F}^{(e)}}$, condition (i) of Corollary 2 holds, (iii) and (iv) hold for $n = 0$ by hypothesis, and as above, for $n > 0$, (ii)′, (iii) and (iv) hold for some $e \gg 0$ such that $R_{\mathbf{F}^{(e)}} \cong R[F_e t^e]$; hence $H^i_{\mathfrak{m}+P^{(e)}}(R_{\mathbf{F}^{(e)}}) = 0$ for all $i \le d$, and so $R_{\mathbf{F}^{(e)}}$ is CM (cf. [MR] or [HR, Prop. 4.10] or [HIO, (11.11)]).

If $R_{\mathbf{F}}$ is CM, then so is X [HIO, (12.19)]. Also, (SS) (and its derivation) shows that CM-ness of $R_{\mathbf{F}}$, i.e., vanishing of $H^i_{\mathfrak{m}+P^{(e)}}(R_{\mathbf{F}^{(e)}})$ for $i \le d$, implies that the homology maps η_0^i $(i \ge 0)$ associated to the composition

(4.1.1)
$$\mathbf{R}\Gamma_{\mathfrak{m}} R \xrightarrow{\alpha_0} \mathbf{R}\Gamma_{\mathfrak{m}}\Gamma\big(X, \mathcal{O}_X\big) \xrightarrow{\mathbf{R}\Gamma_{\mathfrak{m}}\beta_0} \mathbf{R}\Gamma_{\mathfrak{m}}\mathbf{R}\Gamma\big(X, \mathcal{O}_X\big) = \mathbf{R}\Gamma_E\big(X, \mathcal{O}_X\big)$$

are isomorphisms for $i < d$ and injective for $i = d$. In fact since η_0^d is surjective ([LT, p. 103, remark (b)], and note that $H^d_{\mathfrak{m}}(R) \to H^d_{\mathfrak{m}}\big(H^0(X, \mathcal{O}_X)\big)$ is surjective), it too is an isomorphism. In other words, the composed map (4.1.1) is an isomorphism. Thus the second assertion follows from the next Lemma, which generalizes [K, p. 50].

Lemma 4.2. *Let $(R, \mathfrak{m})$ be a noetherian local ring, let $f \colon X \to \operatorname{Spec}(R)$ be a proper map, and set $E := f^{-1}\{\mathfrak{m}\}$. Then the following are equivalent:*

(i) *The canonical map $R \to \mathbf{R}\Gamma(X, \mathcal{O}_X)$ is an iso(morphism).*

(ii) *The canonical map $\mathbf{R}\Gamma_{\mathfrak{m}}(R) \to \mathbf{R}\Gamma_{\mathfrak{m}}\mathbf{R}\Gamma(\mathcal{O}_X) = \mathbf{R}\Gamma_E(X, \mathcal{O}_X)$ is an iso.*

Moreover, if R has a dualizing complex $\mathcal{R}^\bullet$, giving rise to dualizing complexes $\widetilde{\mathcal{R}}^\bullet$ on $\operatorname{Spec}(R)$ and $\mathcal{R}^\bullet_X := f^!\widetilde{\mathcal{R}}^\bullet$ on X, then these two conditions are equivalent to:

(iii) *The canonical map $\mathbf{R}\Gamma(X, \mathcal{R}^\bullet_X) \to \mathcal{R}^\bullet$ is an iso.*

Proof. (Sketch.) (i) $\Rightarrow$ (ii) is obvious; and to prove (ii) $\Rightarrow$ (i), we may replace R by its completion, so we may assume that $\mathcal{R}^\bullet$ exists.

(ii) $\Leftrightarrow$ (iii): Using local-global duality [L2, p. 188], one checks that the functor $\operatorname{Hom}_R(-, I)$ (I = injective hull of $R/\mathfrak{m}$) takes the map in (iii) to the map in (ii). If the latter is an iso, then so is the former, since that is determined by the induced homology maps, and the homology of $\mathbf{R}\Gamma(X, \mathcal{R}^\bullet_X)$ and of $\mathcal{R}^\bullet$ is finitely generated.

(iii) $\Leftrightarrow$ (i): Global duality [RD, p. 379] implies that the derived-category maps (i) and (iii) correspond via the dualizing functor $\mathbf{R}\mathrm{Hom}_R(-, \mathcal{R}^\bullet)$. $\square$

The next result is due to Sancho de Salas [SS]. His proof, though given in a less general context, applies here too *mutatis mutandis.*[3]

Theorem 4.3. *With notation as above, if the ring* $G_{\mathbf{F}} := \oplus_{n\geq 0} F_n/F_{n+1}$ *is CM then* X *and* R *are CM, and* $H^i_E(X, \mathcal{O}_X) = 0$ *for all* $i < d := \dim R$.

Conversely, if X *and* R *are CM, and* $H^i_E(X, \mathcal{O}_X) = 0$ *for all* $i < d$, *then for some* $e > 0$, $G_{\mathbf{F}^{(e)}}$ *is CM.*

Remark. If R has a dualizing complex $\mathcal{R}^\bullet$ and $\omega_X := H^{-d}\mathcal{R}^\bullet_X$ (cf. Lemma 4.2) then by duality, $H^i_E(X, \mathcal{O}_X) = 0$ for all $i < d := \dim R$ iff $H^i(X, \omega_X) = 0$ for all $i > 0$.

The methods of Sancho de Salas's proof of Theorem 4.3 can readily be adapted to prove the following basic result of Goto-Nishida and Viêt, (first proved by Trung-Ikeda for filtrations by powers of an ideal) [GN, Thm. (1.1)].

Theorem 4.4. *The following are equivalent:*

(i) $R_{\mathbf{F}}$ *is CM.*

(ii) *If* $i < d$ *and* $n \neq -1$ *or if* $i = d$ *and* $n \geq 0$, *then* $H^i_{\mathfrak{m}+P}(G_{\mathbf{F}})_n = 0$.

When these conditions hold, there is for each $i \neq d$ *a natural isomorphism* $H^i_{\mathfrak{m}}(R) \xrightarrow{\sim} H^i_{\mathfrak{m}+P}(G_{\mathbf{F}})$.

Here is a sketch of an argument. Keeping in mind that the natural composition $H^d_{\mathfrak{m}}(F_n) \to H^d_{\mathfrak{m}}\big(H^0(X, \mathcal{O}_X(n))\big) \to H^d_E(X, \mathcal{O}_X(n))$ is surjective (cf. [LT, p. 103, remark (b)]), we see that (i) is equivalent to:

(a) for $n \geq 0$ the map η^i_n in (SS) (with $G = R_{\mathbf{F}}$) is bijective for $i \leq d$, and

(b) for $n < 0$ and $i < d$, $H^i_E(\mathcal{O}_X(n)) = 0$;

while (ii) is equivalent to:

(a)$'$ for $n \geq 0$ the map η^i_n in (SS) (with $G = G_{\mathbf{F}}$) is bijective for $i < d$ and injective for $i = d$, and

(b)$'$ for $n < -1$ and $i < d-1$, $H^i_E(\mathcal{O}_{\overline{X}}(n)) = 0$ $\quad\big(\overline{X} := \mathrm{Proj}(G_{\mathbf{F}})\big)$.

Recalling that η^i_{ke} is an isomorphism for some $k > 0$ and all $e \gg 0$ (cf. Corollary 1 and remarks following Corollary 2), one gets (a) $\Leftrightarrow$ (a)$'$

[3]To see that CM-ness of $G_{\mathbf{F}}$ implies that of R, deduce from (SS) that $G_{\mathbf{F}^{(e)}}$ is CM for all $e > 0$; and then since for some e, $G_{\mathbf{F}^{(e)}}$ is generated by its one-forms, [HIO, (11.16)] applies.

via descending induction and by chasing around the natural commutative diagram (with exact rows)

$$\begin{array}{ccccccccc}
\longrightarrow & H^i_{\mathfrak m}(F_{n+1}) & \longrightarrow & H^i_{\mathfrak m}(F_n) & \longrightarrow & H^i_{\mathfrak m}(F_n/F_{n+1}) & \longrightarrow & H^{i+1}_{\mathfrak m}(F_{n+1}) & \longrightarrow \\
 & \eta\downarrow & & \eta\downarrow & & \eta\downarrow & & \eta\downarrow & \\
\longrightarrow & H^i_E(\mathcal{O}_X(n+1)) & \longrightarrow & H^i_E(\mathcal{O}_X(n)) & \longrightarrow & H^i_E(\mathcal{O}_{\overline{X}}(n)) & \longrightarrow & H^{i+1}_E(\mathcal{O}_X(n+1)) & \longrightarrow
\end{array}$$

Recalling similarly that $H^i_E\big(\mathcal{O}_X(-ke)\big) = 0$ for some $k > 0$ and all $e \gg 0$, one gets (b) $\Leftrightarrow$ (b)$'$ from the second row of the diagram.

The rest is left to the reader.

The next theorem answers a question of Huneke. A local ring $(R, \mathfrak{m})$ of dim. d is *pseudo-rational* if R is normal and CM, $\hat{R}$ is reduced, and for any $X := \mathrm{Proj}(R_{\mathbf F})$ and E as above, the natural map $\delta_X\colon H^d_{\mathfrak m}(R) \to H^d_E(X, \mathcal{O}_X)$ is injective (hence bijective) [LT, §2]. (In *loc. cit.,* X is restricted to be normal; but if $\pi\colon Z \to X$ is the normalization of an arbitrary X, then δ_Z factors as

$$H^d_{\mathfrak m}(R) \xrightarrow{\delta_X} H^d_E(X, \mathcal{O}_X) \xrightarrow{\text{natural}} H^d_E(X, \pi_*\mathcal{O}_Z) = H^d_{\pi^{-1}E}(Z, \mathcal{O}_Z)$$

and so δ_Z injective $\Rightarrow$ δ_X injective.) If a d-dimensional CM local ring R has a dualizing complex $\mathcal{R}^\bullet$—i.e., R has a canonical module ω_R and $\mathcal{R}^\bullet$ is isomorphic to the complex $\omega_R[d]$—then δ_X is dual to the natural inclusion $\tau_X\colon H^{-d}(X, \mathcal{R}^\bullet_X) \hookrightarrow \omega_R$; so R is pseudo-rational iff τ_X is an isomorphism, and consequently if R is pseudo-rational then the localization R_p at any prime ideal $p \subset R$ is pseudo-rational too.

Let $\mathbf{F}$ consist of the powers I^n of an R-ideal I, and set $R_I := R_{\mathbf F} = R[It]$, $G_I := G_{\mathbf F} = \oplus_{n\ge 0} I^n/I^{n+1}$. Let $\overline{X} := \mathrm{Proj}(G_I) \xrightarrow{f} Y := \mathrm{Spec}(R/I)$ be the natural map. For any prime $p \supset I$, $\ell_p(I) := 1 + \dim f^{-1}\{p/I\}$ is the *analytic spread* of IR_p.

Theorem 5. *With preceding notation, assume that* $\dim R/I < d$ *and that the localization* R_p *is pseudo-rational for every prime ideal* $p \supset I$ *in* R *such that* $\ell_p(I) = \dim R_p = d - \dim R/p$. *If* G_I *is CM then* R_I *is CM.*

Remark. When R itself is pseudo-rational it follows from Theorems 4.1–4.3 that if G_I is CM then R_{I^e} is CM for some $e > 0$. After seeing a preprint of this paper, S. Goto observed that together with Theorem 4.4 above, arguments such as are used in proving [GN, Lemma 3.4] get us from e down to 1—even for arbitrary $\mathbf{F}$ as in Example 4.

Proof. We may assume $d > 1$. By Theorem 4.4, it's enough that

$$H^d_{\mathfrak{m}+P}(G_I)_n = 0 \qquad \text{for all } n \ge 0.$$

Since $\dim R/I < d$, therefore $H^d_{\mathfrak{m}}(I^n/I^{n+1}) = 0$, so by (SS) it suffices that the natural map $H^{d-1}_{\mathfrak{m}}(I^n/I^{n+1}) \to H^{d-1}_E\big(\overline{X}, \mathcal{O}_{\overline{X}}(n)\big)$ be surjective for all $n \geq 0$. A simple spectral sequence argument will yield this if for all $e > 0$,

$$H^{d-1-e}_{\mathfrak{m}}\big(H^e(\overline{X}, \mathcal{O}_{\overline{X}}(n))\big) = 0; \tag{5.1}$$

and (5.1) holds if the support of $R^e f_* \mathcal{O}_{\overline{X}}(n)$ has dimension $< d - 1 - e$.

Since G_I is CM, therefore R is CM [HIO, (11.16)], hence universally catenary [EGA IV, (6.3.7)]. We have $\dim f^{-1}\{y\} \geq e$ for any point y in the support of $R^e f_* \mathcal{O}_{\overline{X}}(n)$ [EGA III, (4.2.2)]. So if $p/I \in \operatorname{Spec}(R/I)$ is a maximal such point, and if Y is an irreducible component of $f^{-1}\big(\operatorname{Spec}(R/p)\big)$ containing an $(\ell_p(I) - 1)$-dimensional component of $f^{-1}\{p/I\}$, then

$$d - 1 = \dim \overline{X} \geq \dim Y = \dim(R/p) + \ell_p(I) - 1 \geq \dim(R/p) + e,$$

the second equality being given by [EGA IV, (5.6.6)]; and hence (5.1) holds unless for some such p there is equality throughout, i.e.,

$$\ell_p(I) = e + 1 = d - \dim R/p. \tag{5.2}$$

We can finish by showing that (5.2) implies $H^e(\overline{X}, \mathcal{O}_{\overline{X}}(n))_p = 0$—in other words, $p \notin \operatorname{Supp}\big(R^e f_* \mathcal{O}_{\overline{X}}(n)\big)$—so that (5.2) cannot hold for any p. It's clear that $\ell_p(I) \leq \dim R_p \leq d - \dim R/p$, so (5.2) entails $\ell_p(I) = \dim R_p$, and then by assumption, R_p is pseudo-rational. Changing notation, we have reduced to showing that if R is pseudo-rational, of dimension d, then $H^{d-1}\big(\overline{X}, \mathcal{O}_{\overline{X}}(n)\big) = 0$.

Since the fibers of $X := \operatorname{Proj}(R_I) \to \operatorname{Spec}(R)$ have dimension $< d$, so that $H^d(X, -)$ vanishes on coherent $\mathcal{O}_X$-modules, and since $\mathcal{O}_{\overline{X}}(n)$ is a homomorphic image of $\mathcal{O}_X^N$ for some $N > 0$, it's enough to show that $H^{d-1}(X, \mathcal{O}_X) = 0$. This can be done, upon completing R, via Theorem 4.3 and (i)⇔(iii) in Theorem 4.2. Here's a simpler way. With $U := \operatorname{Spec}(R) - \{\mathfrak{m}\}$ and $V := X - E$, consider the standard exact sequence

$$\cdots \to H^i_E(\mathcal{O}_X) \to H^i(X, \mathcal{O}_X) \to H^i(V, \mathcal{O}_V) \xrightarrow{\lambda^i} H^{i+1}_E(\mathcal{O}_X) \to \cdots$$

By Theorem 4.3, $H^{d-1}_E(\mathcal{O}_X) = 0$, so we just need λ^{d-1} to be injective. But δ_X factors naturally as

$$H^d_{\mathfrak{m}}(R) = H^{d-1}(U, \mathcal{O}_U) \xrightarrow{\mu} H^{d-1}(V, \mathcal{O}_V) \xrightarrow{\lambda^{d-1}} H^d_E(X, \mathcal{O}_X)$$

with μ surjective since the fibers of $V \to U$ have dimension $< d-1 = \dim U$ (cf. again [LT, p. 103, remark (b)]); and since δ_X is injective, so is λ^{d-1}. □

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