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INFINITELY MANY SYNCHRONIZED SOLUTIONS TO A NONLINEARLY COUPLED SCHRÖDINGER EQUATIONS WITH NON-SYMMETRIC POTENTIALS*

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Abstract. We study a nonlinearly coupled Schrödinger equations in $\mathbb{R}^N(2 \leq N < 6)$. Assume that the potentials in the system are continuous functions satisfying some suitable decay assumptions but without any symmetric properties, and the parameters in the system satisfy some restrictions. Applying the Liapunov-Schmidt reduction methods twice and combining localized energy method, we prove that the problem has infinitely many positive synchronized solutions.

Key words. Nonlinear Schrödinger equations; non-symmetric potentials; synchronized solutions.

Mathematics Subject Classification. 35J10, 35B99, 35J60.

1. Introduction and the main result. We consider the following nonlinearly coupled Schrödinger equations in $\mathbb{R}^N(2 \leq N < 6)$,

$$\begin{cases} -\Delta u + (1+\epsilon P(x))u = \mu uv, & x \in \mathbb{R}^N, \\ -\Delta v + (1+\epsilon Q(x))v = \frac{\mu}{2}u^2 + \gamma v^2, & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where the potentials $P(x), Q(x)$ are continuous functions satisfying suitable decay assumptions, but without any symmetry properties, ϵ is a positive constant, μ and γ are some parameters.

These types of systems arise when one considers standing wave solutions to the following problem

$$\begin{cases} i\frac{\partial \psi_1}{\partial t} = -\Delta\psi_1 + P(x)\psi_1 - \alpha_1|\psi_1|\psi_2 - \beta_1|\psi_1|\psi_1, & (x,t) \in \mathbb{R}^N \times (0,+\infty), \\ i\frac{\partial \psi_2}{\partial t} = -\Delta\psi_2 + Q(x)\psi_2 - \frac{\alpha_2}{2}|\psi_1|\psi_1 - \beta_2|\psi_2|\psi_2, & (x,t) \in \mathbb{R}^N \times (0,+\infty), \end{cases} \tag{1.2}$$

where ψ_j are the complex wave functions which for solitary waves, i.e., localized solutions, must also decay at infinity.

Nonlinear Schrödinger equations appear in many contexts, for example, in photonics, plasmas, foundation of quantum mechanics, optics in nonlinear media or in mean-filed theory of Bose-Einstein condensates. In particular, in nonlinear media optic theory, the nonlinear Schrödinger equation appears as an asymptotic limit for a slowly varying dispersive wave envelope propagating in nonlinear medium. In nonlinear optic theory, the cubic nonlinear Schrödinger equation

$$i\frac{\partial \psi}{\partial z} + r\nabla^2\psi + \chi|\psi|^2\psi = 0, \tag{1.3}$$

is the basic equation describing the formation and propagation of optical solitons in Kerr-type material [28, 8]. Here ψ is a slowly varying envelope of electric filed, the

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real-valued parameters rand χ represent the relative strength and sign of dispersion/ diffraction and nonlinearity respectively, and z is the propagation distance coordinate. The Laplacian operator ∇^2 can either be $\frac{\partial^2}{\partial \nu^2}$ for temporal solitons, where ν is the normalized retarded time, or $\nabla^2 = \sum_{i=1}^{N} \frac{\partial^2}{\partial x_i^2}$ for spatial solitons, where $x = (x_1, x_2, ..., x_N)$ is the spatial coordinate with the spatial dimension $N \geq 1$. Here x is the direction orthogonal to z. Solitary wave solutions to (1.3) and its generation have been in, such as [5, 27].

The invention of lasers in 1960s enabled experimental physical scientists to obtain a powerful source of coherent light so nonlinear optical effects such as Second Harmonic Generation(SHG) were discovered when the optical material has a $\chi^{(2)}$(i.e. quatratic) nonlinear optical nonlinear response instead of conventional kerr$\chi^{(3)}$ material for which (1.3) is based on(see [6, 7]). Supposing that we consider a strong parametric interaction of three stationary quasi-plane monochromatic waves with frequencies $\omega_i (i = 1, 2, 3)$, the harmonic waves, the frequencies of interacting waves are matched exactly $(\omega_1 + \omega_2 = \omega_3)$, then with some conventional normalizations and the assumption that $\omega_1 = \omega_2 = \frac{\omega_3}{2}$, we can obtain the simplest case of type-I SHG (see [[6], p. 104])

$$\begin{cases} i\frac{\partial u}{\partial z} + r\frac{\partial^2 u}{\partial x^2} - u + vu = 0, & x \in \mathbb{R}^N, \\ i\sigma\frac{\partial v}{\partial z} + s\frac{\partial^2 v}{\partial x^2} - \alpha v + \frac{u^2}{2} = 0, & x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

where u is a senormalized slowly varying complex envelope of wave with frequency ω_1, v is the one with frequency $\omega_3, \sigma, \alpha > 0$ and $r, s = \pm 1$. In the spatial soliton case $r = s = 1$, while the temporal case all four combinations for $r, s = \pm 1$ are possible. The physically realistic spatial dimensions are $N = 1$ or $N = 2$. Then the chirp-free two-wave (symbiotic) solitons can be found as real-valued solutions of the steady state $(\frac{\partial}{\partial z} = 0)$ equation:

$$\begin{cases} \Delta u - u + vu = 0, & x \in \mathbb{R}^N, \\ \Delta v - \alpha v + \frac{u^2}{2} = 0, & x \in \mathbb{R}^N, \\ \lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = 0. \end{cases} \tag{1.5}$$

In the case $N = 1$, the existence of a non-trivial ground state solution of (1.5) was shown in [33] by using a variational approach. Multi-pulse solutions of (1.5) for $N = 1$ were first observed in numerical simulations (see [33]), and the existence of multi-pulse solutions was rigorously proved using singular perturbation theory in [32].

Similar to (1.4) and (1.5), the propagation of solitons in $\chi^{(3)}$ nonlinear fiber couplers can be described by a set of coupled nonlinear Schrödinger equations:

$$i\frac{\partial \psi_j}{\partial z} + r\nabla^2\psi_j + \chi\Big(\sum_{i=1}^{K} |\psi_i|^2\Big)\psi_j = 0 \tag{1.6}$$

for $j = 1, 2, ..., K$. Here the complex-valued ψ_j denotes the j-th component of the light beam, and $|\psi_j|^2$ is the change in refractive index profile created by all the incoherent components in the light beam. The solitary waves of (1.6) satisfies $\psi_j(t, x) = u_j(x)e^{i\mu_j t}$, and $u_j(x)$ satisfies

$$r\Delta u_j - \mu_j u_j + \chi\Big(\sum_{i=1}^{K} |u_i|^2\Big)u_j = 0, \tag{1.7}$$

for $j = 1, 2, ..., K$(see [17]). The existence of solitary wave solutions to (1.7) has been studied extensively in recent years, for example, [1, 13, 17, 19, 25, 26, 29] and the references therein, and the same system in a bounded domain was also considered in [11, 12, 22, 31].

In the mean-filed approximation, the system described by the Gross-Pitaevskii equations

$$\begin{cases} i\frac{\partial \psi_1}{\partial t} = (L_1 + U_{11}|\psi_1|^2 + U_{12}|\psi_2|^2)\psi_1 + \lambda\psi_2, & x \in \mathbb{R}^N, \\ i\frac{\partial \psi_2}{\partial t} = (L_2 + U_{21}|\psi_2|^2 + U_{22}|\psi_1|^2)\psi_2 + \lambda\psi_1, & x \in \mathbb{R}^N, \end{cases} \tag{1.8}$$

where $L_j = -\nabla^2 + V_j$ with $j = 1, 2$. The model which the propagation of solitons $\chi^{(3)}$ in nonlinear fiber couples based on is equivalent to (1.8). Here $\psi_j (j = 1, 2)$ denotes the j-th component of the light beam. This system has received a lot of attention both experimentally and theoretically. The standing-wave solutions $\psi_j (j = 1, 2)$, which are those of the form $\psi_j = u_j e^{iEt}(j = 1, 2)$ with u_j a real-valued function, satisfy

$$\begin{cases} -\Delta\psi_1 + V_1(x)\psi_1 = |\psi_1|^2\psi_2 + \lambda|\psi_2|^3, & x \in \mathbb{R}^N, \\ -\Delta\psi_2 + V_2(x)\psi_2 = |\psi_2|^2\psi_1 + \lambda|\psi_1|^3, & x \in \mathbb{R}^N. \end{cases} \tag{1.9}$$

The existence of solitary waves to (1.9) is well-studied and has been explored by many authors in recent years, for example,[1, 4, 13, 17, 18, 20, 21, 26, 29].

Compared with the well-studied $\chi^{(3)}$ nonlinear Schrödinger system, little attention is given to the $\chi^{(2)}$ SHG system. When $\alpha_1 = 1, \alpha_2 > 0, \beta_1 = \beta_2 = 0$, (1.2) is just the case considered by [34]. Very recently, Wang and Zhou in[24] using the finitely dimensional reduction method obtained infinitely many non-radial positive synchronized solutions of the system (1.1) under radial potentials satisfying some algebraic decay. In this paper, we consider the case when $\beta_1 = 0$ and $\alpha_j (j = 1, 2) > 0, \beta_2 \in \mathbb{R}$.

Inspired by [2, 24, 30], we want to investigate the existence of infinitely many positive synchronized solutions of the system (1.1). In order to state our main result, now we give the assumptions imposed on $P(x), Q(x)$ which are similar to those in [2],

(H_1) $P(x), Q(x)$ are positive and continuous functions in $\mathbb{R}^N$;

(H_2) $\lim_{|x|\to\infty} P(x) = \lim_{|x|\to\infty} Q(x) = 0$ as $|x| \to \infty$;

(H_3) $\exists\, 0 < \tau < 1,\ \lim_{|x|\to\infty} (\alpha^2 P(x) + \beta^2 Q(x))e^{\tau|x|} = +\infty$,

where α, β are defined in (1.12) below.

The energy functional associated with problem (1.1) is

$$\begin{aligned} J(u, v) = &\frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla u|^2 + (1 + \epsilon P(x))u^2 + |\nabla v|^2 + (1 + \epsilon Q(x))v^2\right] \\ &-\frac{\mu}{2}\int_{\mathbb{R}^N} u^2 v - \frac{\gamma}{3}\int_{\mathbb{R}^N} v^3 \qquad u, v \in H^1(\mathbb{R}^N). \end{aligned} \tag{1.10}$$

We will study $J(u, v)$ in Section 4. Let w be the unique solution of

$$\begin{cases} \Delta w - w + w^2 = 0,\ w > 0, & \text{in } \mathbb{R}^N, \\ w(0) = \max_{x\in\mathbb{R}^N}(x),\ w(x) \to 0, & \text{as } |x| \to \infty. \end{cases} \tag{1.11}$$

By the well-known result of Gidas, Ni and Nirenberg in [15], w is radially symmetric and strictly decreasing, $w'(r) < 0$ for $r > 0$. Moreover, from [15] we know the following asymptotic behavior of w:

$$\begin{cases} w(r) = A_N r^{-\frac{N-1}{2}} e^{-r}(1+O(\frac{1}{r})), \\ w'(r) = -A_N r^{-\frac{N-1}{2}} e^{-r}(1+O(\frac{1}{r})), \end{cases}$$

for $r > 0$, where A_N is a positive constant.

If $\mu > 0$ and $\mu > \gamma$, taking

$$\alpha = \frac{1}{\mu}\sqrt{\frac{2(\mu-\gamma)}{\mu}}, \quad \beta = \frac{1}{\mu}, \tag{1.12}$$

then

$$(U, V) = (\alpha w(x), \beta w(x)) \tag{1.13}$$

solves the following problem

$$\begin{cases} -\Delta u + u = \mu uv, & \text{in } \mathbb{R}^N, \\ -\Delta v + v = \frac{\mu}{2}u^2 + \gamma v^2, & \text{in } \mathbb{R}^N. \end{cases} \tag{1.14}$$

We will use (U, V) as the building blocks for the solutions of (1.1). Let $\varrho > 0$ be a real number such that $w(x) \le ce^{-|x|}$ for $|x| > \varrho$ and some positive constant c independent of ϱ large. Denote $\mathbf{O}_m = (O_1, \cdots, O_m)$. Now we define the configuration space

$$\Omega_1 = \mathbb{R}^N, \ \ \Omega_m = \Big\{\mathbf{O}_m \in \mathbb{R}^{mN} \Big| \min_{j\neq k} |O_j - O_k| \ge \varrho\Big\}, \ \forall m > 1.$$

For $\mathbf{O}_m \in \Omega_m$, we define

$$(U_{O_j}, V_{O_j}) = (U(x - O_j), V(x - O_j)) \tag{1.15}$$

and the approximate solutions to be

$$U_{\mathbf{O}_m} = \sum_{j=1}^{m} U_{O_j}, \ \ V_{\mathbf{O}_m} = \sum_{j=1}^{m} V_{O_j}. \tag{1.16}$$

Denote

$$G\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} \Delta u - (1+\epsilon P(x))u + \mu uv \\ \Delta v - (1+\epsilon Q(x))v + \frac{\mu}{2}u^2 + \gamma v^2 \end{pmatrix} \tag{1.17}$$

and for $f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$, $g = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, we denote $\langle f, g\rangle = \int_{\mathbb{R}^N} (f_1 g_1 + f_2 g_2)$.

Now we state our main result as follows:

THEOREM 1.1. *Assume that* $(H_1), (H_2)$ *and* (H_3) *hold and* $2 \le N < 6$. *Then there exists* ϵ_0 *such that for* $0 < \epsilon < \epsilon_0$, *if* $\mu > 0$ *and* $\mu > \gamma$, *problem* (1.1) *has infinitely many positive synchronized solutions.*

REMARK 1.2. *From the first equation of the system* (1.1), *it is easy to see that the system* (1.1) *has no segregated solutions.*

REMARK 1.3. *The system* (1.1) *possesses the symmetry that if* $(u(x), v(x))$ *is a solution of* (1.1)*, so is* $(-u(x), v(x))$.

In order to prove Theorem 1.1, we mainly use the Liapunov-Schmidt reduction method as in [2, 3, 16]. There are two main difficulties. Firstly, we need to show that the maximum points will not go to infinity (see Section 4). This is guaranteed by the slow decay assumption (H_3). Secondly, we have to detect the difference in the energy when the spikes move to the boundary of the configuration space. A crucial estimate is Lemma 3.1, in which we prove that the accumulated error can be controlled from step m to step $(m+1)$.

Our paper is organized as follows. In section 2, we perform the first finitely dimensional reduction. In Section 3, we show a key estimate which can recognize the differences between the m-th step and the $(m+1)$-th step which involves a secondary Liapunov-Schmidt reduction. We prove Theorem 1.1 in Section 4. Throughout this paper, denote $\mathbf{H} = H^1(\mathbb{R}^N) \times H^1(\mathbb{R}^N)$ and $\|(u,v)\|^2_{\mathbf{H}} = \|u\|^2_{H^1(\mathbb{R}^N)} + \|v\|^2_{H^1(\mathbb{R}^N)}$. c, C will always denote various generic constants that are independent of ϱ for ϱ large.

2. The first Liapunov-Schmidt reduction. In this section, we perform a finite-dimensional reduction.

For $\mathbf{O}_m \in \Omega_m$, we define the following functions:

$$D_{jk} = \begin{pmatrix} D_{jk,1} \\ D_{jk,2} \end{pmatrix} = \begin{pmatrix} \frac{\partial U_{O_j}}{\partial x_k}\zeta_j(x) \\ \frac{\partial V_{O_j}}{\partial x_k}\zeta_j(x) \end{pmatrix}, \text{ for } j = 1, \cdots, m, \ k = 1, \cdots, N, \tag{2.1}$$

where $\zeta_j(x) = \zeta\big(\frac{2|x-O_j|}{\varrho-1}\big)$ and $\zeta(t)$ is a cut-off function such that $\zeta(t) = 1$ for $|t| \le 1$ and $\zeta(t) = 0$ for $|t| \ge \frac{\varrho^2}{\varrho^2-1}$. So we know that the support of D_{jk} belongs to $B_{\frac{\varrho^2}{2(\varrho+1)}}(O_j)$.

Consider the following linear problem: given $h = \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}$, we find a function $\begin{pmatrix} \phi \\ \psi \end{pmatrix}$ satisfying

$$\begin{cases} L\begin{pmatrix} \phi \\ \psi \end{pmatrix} := \begin{pmatrix} \Delta\phi - (1+\epsilon P(x))\phi + \mu V_{\mathbf{O}_m}\phi + \mu U_{\mathbf{O}_m}\psi \\ \Delta\psi - (1+\epsilon Q(x))\psi + \mu U_{\mathbf{O}_m}\phi + 2\gamma V_{\mathbf{O}_m}\psi \end{pmatrix} = h + \sum_{j=1}^{m}\sum_{k=1}^{N} c_{jk}D_{jk}, \\ \Big\langle \begin{pmatrix} \phi \\ \psi \end{pmatrix}, \begin{pmatrix} D_{jk,1} \\ D_{jk,2} \end{pmatrix} \Big\rangle = 0 \text{ for } j = 1, \cdots, m, \ k = 1, \cdots, N. \end{cases} \tag{2.2}$$

Letting $0 < \sigma < 1$ and

$$E := \sum_{\mathbf{O}_m \in \Omega_m} e^{-\sigma|\cdot - O_j|},$$

we define the norm

$$\|h\|_* = \sup_{x\in\mathbb{R}^N} \big|E(x)^{-1}h_1(x)\big| + \sup_{x\in\mathbb{R}^N} \big|E(x)^{-1}h_2(x)\big|. \tag{2.3}$$

From [14], there holds the following relation

$$\|u\|_{L^\infty(\mathbb{R}^N)} \le C\|u\|_*, \tag{2.4}$$

where $C>0$ independent of ϱ, m and $\mathbf{O}_m$.

Firstly we give the following non-degeneracy result which will be used later.

Lemma 2.1 (Proposition 3.3, [24]). *If $\mu>0$ and $\mu>\gamma$, then (U,V) is non-degenerate for the system* (1.14) *in* $\mathbf{H}$ *in the sense that the kernel is given by*

$$Span\Big\{\Big(\frac{\partial U}{\partial x_j},\frac{\partial V}{\partial x_j}\Big)\,\Big|\,j=1,\cdots,N\Big\}.$$

In the following, θ will denote a positive constant depending on ϵ and σ but independent of $\varrho, m, \mathbf{O}_m$ and may vary from line to line.

Proposition 2.2. *Let h with $\|h\|_*$ norm bounded and assume that $\Big(\begin{pmatrix}\phi\\ \psi\end{pmatrix},\{c_{jk}\}\Big)$ is a solution to problem* (2.2). *Then there exist positive numbers ϵ_0, ϱ_0 such that for all $0<\epsilon<\epsilon_0$, $\varrho\ge\varrho_0$ and $\mathbf{O}_m\in\Omega_m$, we have*

$$\|(\phi,\psi)\|_*\le C\|h\|_*,$$

where C is a positive constant independent of ϱ, m and $\boldsymbol{O}_m\in\Omega_m$.

Proof. Similar to [3], we prove this proposition by contradiction. Assume that there exists a solution $\begin{pmatrix}\phi\\ \psi\end{pmatrix}$, such that $\|h\|_*\to 0$, and $\|(\phi,\psi)\|_*=1$.

Multiplying the frist system (2.2) by $D_{jk}=\begin{pmatrix}D_{jk,1}\\ D_{jk,2}\end{pmatrix}$ and integrating in $\mathbb{R}^N$, we get

$$\int_{\mathbb{R}^N} L\begin{pmatrix}\phi\\ \psi\end{pmatrix}\begin{pmatrix}D_{jk,1}\\ D_{jk,2}\end{pmatrix}=\int_{\mathbb{R}^N}\begin{pmatrix}h_1\\ h_2\end{pmatrix}\begin{pmatrix}D_{jk,1}\\ D_{jk,2}\end{pmatrix}+c_{jk}\int_{\mathbb{R}^N}D_{jk}^2.$$

By the definition of D_{jk}, we deduce

$$\begin{aligned}\int_{\mathbb{R}^N}D_{jk}^2&=\int_{\mathbb{R}^N}\big(D_{jk,1}^2+D_{jk,2}^2\big)=(\alpha^2+\beta^2)\int_{B_{\frac{\varrho^2}{2(\varrho+1)}}(0)}\Big|\frac{\partial w}{\partial x_k}\Big|^2\zeta_j^2(x+O_j)\\
&=(\alpha^2+\beta^2)\int_{\mathbb{R}^N}\Big|\frac{\partial w}{\partial x_k}\Big|^2+(\alpha^2+\beta^2)\int_{\mathbb{R}^N}\Big|\frac{\partial w}{\partial x_k}\Big|^2[\zeta_j^2(x+O_j)-1]\\
&\quad-(\alpha^2+\beta^2)\int_{\mathbb{R}^N\setminus B_{\frac{\varrho^2}{2(\varrho+1)}}(0)}\Big|\frac{\partial w}{\partial x_k}\Big|^2\zeta_j^2(x+O_j)\\
&=(\alpha^2+\beta^2)\int_{\mathbb{R}^N}\Big|\frac{\partial w}{\partial x_k}\Big|^2+O(e^{-\theta\varrho}),\end{aligned}$$

since

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big|\frac{\partial w}{\partial x_k}\Big|^2[\zeta_j^2(x+O_j)-1]=\int_{\mathbb{R}^N\setminus B_{\frac{\varrho-1}{2}}(0)}\Big|\frac{\partial w}{\partial x_k}\Big|^2\Big[\zeta^2\Big(\frac{2|x|}{\varrho-1}\Big)-1\Big]\\
&\le C\int_{\mathbb{R}^N\setminus B_{\frac{\varrho-1}{2}}(0)}\Big|\frac{\partial w}{\partial x_k}\Big|^2\le C\int_{\mathbb{R}^N\setminus B_{\frac{\varrho-1}{2}}(0)}e^{-2|x|}\le Ce^{-\theta\varrho}.\end{aligned}$$

On the other hand, from (2.4), we have

$$\Big|\int_{\mathbb{R}^N}\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}\begin{pmatrix} D_{jk,1} \\ D_{jk,2} \end{pmatrix}\Big| \le \Big|\int_{\mathbb{R}^N} h_1 D_{jk,1}\Big| + \Big|\int_{\mathbb{R}^N} h_2 D_{jk,2}\Big| \le C\|h\|_*. \tag{2.5}$$

Setting $\tilde{D}_{jk} = \begin{pmatrix} \frac{\partial U_{O_j}}{\partial x_k} \\ \frac{\partial V_{O_j}}{\partial x_k} \end{pmatrix}$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} L\begin{pmatrix} \phi \\ \psi \end{pmatrix}\begin{pmatrix} D_{jk,1} \\ D_{jk,2} \end{pmatrix} = \int_{\mathbb{R}^N} L\begin{pmatrix} D_{jk,1} \\ D_{jk,2} \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix} \\
&= \int_{\mathbb{R}^N}\begin{pmatrix} (\Delta\tilde{D}_{jk,1} - \tilde{D}_{jk,1} + \mu V_{O_j}\tilde{D}_{jk,1} + \mu U_{O_j}\tilde{D}_{jk,2})\zeta_j \\ (\Delta\tilde{D}_{jk,2} - \tilde{D}_{jk,2} + \mu U_{O_j}\tilde{D}_{jk,1} + 2\gamma V_{O_j}\tilde{D}_{jk,2})\zeta_j \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix} \\
&\quad + \int_{\mathbb{R}^N}\begin{pmatrix} \Delta\zeta_j\tilde{D}_{jk,1} + 2\nabla\tilde{D}_{jk,1}\cdot\nabla\zeta_j \\ \Delta\zeta_j\tilde{D}_{jk,2} + 2\nabla\tilde{D}_{jk,2}\cdot\nabla\zeta_j \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix} \\
&\quad + \int_{\mathbb{R}^N}\begin{pmatrix} \mu(V_{\mathbf{O}_m} - V_{O_j})\tilde{D}_{jk,1}\zeta_j + \mu(U_{\mathbf{O}_m} - U_{O_j})\tilde{D}_{jk,2}\zeta_j \\ \mu(U_{\mathbf{O}_m} - U_{O_j})\tilde{D}_{jk,1}\zeta_j + 2\gamma(V_{\mathbf{O}_m} - V_{O_j})\tilde{D}_{jk,2}\zeta_j \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix} \\
&\quad - \epsilon\int_{\mathbb{R}^N}\begin{pmatrix} P(x)\tilde{D}_{jk,1}\zeta_j \\ Q(x)\tilde{D}_{jk,2}\zeta_j \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix}.
\end{aligned} \tag{2.6}$$

Now we estimate all the terms in the right side of (2.6). Firstly, since (U_{O_j}, V_{O_j}) satisfies (1.14), we find that the first term is equal to 0. The second term can be estimated as follows:

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^N}\begin{pmatrix} \Delta\zeta_j\tilde{D}_{jk,1} + 2\nabla\tilde{D}_{jk,1}\cdot\nabla\zeta_j \\ \Delta\zeta_j\tilde{D}_{jk,2} + 2\nabla\tilde{D}_{jk,2}\cdot\nabla\zeta_j \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix}\Big| \\
&= \Big|\int_{B_{\frac{\varrho^2}{2(\varrho+1)}}(0)\setminus B_{\frac{\varrho-1}{2}}(0)}\begin{pmatrix} \Delta\zeta_j(x+O_j)\frac{\partial w}{\partial x_k} + 2\nabla\frac{\partial w}{\partial x_k}\nabla\zeta_i(x+O_j) \\ \Delta\zeta_j(x+O_j)\frac{\partial w}{\partial x_k} + 2\nabla\frac{\partial w}{\partial x_k}\nabla\zeta_i(x+O_j) \end{pmatrix}\begin{pmatrix} \phi \\ \psi \end{pmatrix}\Big| \\
&\le Ce^{-\frac{1}{2}\theta\varrho}\|(\phi,\psi)\|_*,
\end{aligned}$$

since

$$\begin{aligned}
&\Big|\int_{B_{\frac{\varrho^2}{2(\varrho+1)}}(0)\setminus B_{\frac{\varrho-1}{2}}(0)} \Delta\zeta_j(x+O_j)\frac{\partial w}{\partial x_k}\phi + 2\nabla\frac{\partial w}{\partial x_k}\cdot\nabla\zeta_j(x+O_j)\phi\Big| \\
&\le C\sup_{x\in\mathbb{R}^N}|\phi E^{-1}|\Big|\int_{B_{\frac{\varrho^2}{2(\varrho+1)}}(0)\setminus B_{\frac{\varrho-1}{2}}(0)}\sum_{l=1}^m e^{-\theta|x+O_j-O_l|}e^{-|x|}\Big| \\
&\le C\sup_{x\in\mathbb{R}^N}|\phi E^{-1}|e^{-\frac{1}{2}\theta\varrho}\int_{B_{\frac{\varrho^2}{2(\varrho+1)}}(0)\setminus B_{\frac{\varrho-1}{2}}(0)} e^{-|x|} \le Ce^{-\frac{1}{2}\theta\varrho}\sup_{x\in\mathbb{R}^N}|\phi E^{-1}| \\
&\le Ce^{-\frac{1}{2}\theta\varrho}\|(\phi,\psi)\|_*
\end{aligned}$$

and

$$\Big|\int_{B_{\frac{\varrho^2}{2(\varrho+1)}}(0)\setminus B_{\frac{\varrho-1}{2}}(0)} \Delta\zeta_j(x+O_j)\frac{\partial w}{\partial x_k}\psi + 2\nabla\frac{\partial w}{\partial x_k}\cdot\nabla\zeta_j(x+O_j)\psi\Big|$$
$$\leq Ce^{-\frac{1}{2}\theta\varrho}\sup_{x\in\mathbb{R}^N}|\psi E^{-1}| \leq Ce^{-\frac{1}{2}\theta\varrho}\|(\phi,\psi)\|_*,$$

for some $\theta > 0$. Similarly, we can deduce

$$\Big|\epsilon\int_{\mathbb{R}^N}\begin{pmatrix} P(x)\tilde{D}_{jk,1}\zeta_j \\ Q(x)\tilde{D}_{jk,2}\zeta_j\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix}\Big| \leq Ce^{-\frac{1}{2}\theta\varrho}\|(\phi,\psi)\|_*$$

and

$$\Big|\int_{\mathbb{R}^N}\begin{pmatrix} \mu(V_{\mathbf{O}_m}-V_{O_j})\tilde{D}_{jk,1}\zeta_j + \mu(U_{\mathbf{O}_m}-U_{O_j})\tilde{D}_{jk,2}\zeta_j \\ \mu(U_{\mathbf{O}_m}-U_{O_j})\tilde{D}_{jk,1}\zeta_j + 2\gamma(V_{\mathbf{O}_m}-V_{O_j})\tilde{D}_{jk,2}\zeta_j\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix}\Big|$$
$$\leq C\int_{B_{\frac{\varrho}{2}}(O_j)}\begin{pmatrix}\sum_{l\neq j} w(x-O_l)\frac{\partial w(x-O_j)}{\partial x_k} \\ \sum_{l\neq j} w(x-O_l)\frac{\partial w(x-O_j)}{\partial x_k}\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix} \leq Ce^{-\frac{1}{2}\theta\varrho}\|(\phi,\psi)\|_*$$

for some $\theta > 0$. So we can conclude that

$$|c_{jk}| \leq C(e^{-\frac{1}{2}\theta\varrho}\|(\phi,\psi)\|_* + \|h\|_*). \tag{2.7}$$

Let now $\vartheta \in (0,1)$. It is easy to check that the function E satisfies

$$L\begin{pmatrix}E\\ E\end{pmatrix} \leq \frac{1}{2}(\vartheta^2-1)\begin{pmatrix}E\\ E\end{pmatrix},$$

in $\mathbb{R}^N\setminus\cup_j^m B(O_j,\varrho_1)$ if ϱ_1 is large enough but independent of ϱ. Hence the function E can be used as a barrier to prove the pointwise estimate (similar to (3.11) in [3])

$$|(\phi,\psi)| \leq C\Big(\Big\|L\begin{pmatrix}\phi\\ \psi\end{pmatrix}\Big\|_* + \sup_j\|(\phi,\psi)\|_{L^\infty(\partial B(O_j,\varrho_1))}\Big)E(x), \tag{2.8}$$

for all $x\in\mathbb{R}^N\setminus\cup_j^m B(O_j,\varrho_1)$.

Now we assume that there exist a sequence $\{\varrho^n\}$ tending to ∞ and sequences $\{h^n\}$, $\begin{pmatrix}\phi^n\\ \psi^n\end{pmatrix}$, $\{c_{jk}^n\}$ such that

$$\|h^n\|_* \to 0, \text{ and } \|(\phi^n,\psi^n)\|_* = 1.$$

By (2.7), we can get

$$\Big\|\sum_{j,k} c_{jk}^n D_{jk}\Big\|_* \to 0.$$

Then (2.8) implies that there exists $\{O_j^n\}\subset\Omega_m$ such that

$$\|(\phi^n,\psi^n)\|_{L^\infty(B(O_j^n,\varrho^n/2))} \geq C, \tag{2.9}$$

for some constant $C > 0$. Using elliptic estimates with Ascoli-Arzela's theorem, we can find a subsequence of $\{O_j^n\}$ and we can extract, from the sequence $(\phi^n(\cdot -$

$O_j^n), \psi^n(\cdot - O_j^n))$ a subsequence which will converge (on compact sets) to $(\phi_\infty, \psi_\infty)$ a solution of

$$\begin{cases} \Delta\phi_\infty - \phi_\infty + \mu V\phi_\infty + \mu U\psi_\infty = 0, & \text{in } \mathbb{R}^N, \\ \Delta\psi_\infty - \psi_\infty + \mu U\phi_\infty + 2\gamma V\psi_\infty = 0, & \text{in } \mathbb{R}^N. \end{cases} \tag{2.10}$$

Moreover, recall that (ϕ^n, ψ^n) satisfies the orthogonal condition in (2.2). So,

$$\int_{\mathbb{R}^N} \left(\phi_\infty \frac{\partial U}{\partial x_i} + \psi_\infty \frac{\partial V}{\partial x_i}\right) = 0 \quad i = 1, 2, ..., N. \tag{2.11}$$

By the non-degeneracy of (U, V), we have $(\phi_\infty, \psi_\infty) \equiv (0, 0)$, which contradicts to (2.9). The proof is complete. □

Applying Proposition 2.2, we get the following result at once.

PROPOSITION 2.3. *Given* $0 < \sigma < 1$, *there exist positive numbers* ϵ_0, ϱ_0, C *such that for all* $0 < \epsilon < \epsilon_0$, $\varrho \geq \varrho_0$ *and for any given* h *with* $\|h\|_*$ *norm bounded, there is a unique solution* $\left(\begin{pmatrix} \phi \\ \psi \end{pmatrix}, \{c_{jk}\}\right)$ *to problem* (2.2)*. Furthermore,*

$$\|(\phi, \psi)\|_* \leq C\|h\|_*. \tag{2.12}$$

Proof. Consider the space

$$\mathcal{H} = \left\{(u, v) \in \mathbf{H} \,\middle|\, \left\langle \begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} D_{jk,1} \\ D_{jk,2} \end{pmatrix} \right\rangle = 0, \ \mathbf{O}_m \in \Omega_m \right\}.$$

Since the problem (2.2) can be rewritten as

$$\begin{pmatrix} \phi \\ \psi \end{pmatrix} + \begin{pmatrix} \mathcal{K}_1 & \mathcal{K} \\ \mathcal{K} & \mathcal{K}_2 \end{pmatrix} \begin{pmatrix} \phi \\ \psi \end{pmatrix} = \bar{h} \text{ in } \mathcal{H}, \tag{2.13}$$

where $\bar{h}$ is defined by duality and $\mathcal{K}, \mathcal{K}_1, \mathcal{K}_2 : \mathcal{H} \to \mathcal{H}$ are linear compact operators. By Fredholm's alternative theorem, we know that (2.13) has a unique solution for each $\bar{h}$ is equivalent to showing that the system has a unique solution for $\bar{h} = 0$, which in turn follows from Proposition 2.2. This concludes the proof of Proposition 2.3. □

In the following, if (ϕ, ψ) is the unique solution given by Proposition 2.3, we denote

$$(\phi, \psi) = \mathcal{A}(h) \tag{2.14}$$

and (2.12) yields

$$\|\mathcal{A}(h)\|_* \leq C\|h\|_*. \tag{2.15}$$

Now we reduce (1.1) to a finite-dimensional one. For large ϱ and fixed $\mathbf{O}_m \in \Omega_m$, we are going to find a function $(\phi_{\mathbf{O}_m}, \psi_{\mathbf{O}_m})$ such that for some $\{c_{jk}\}, j = 1, \cdots, m, k =$

$1,\cdots,N$, the following nonlinear projected problem holds true

$$\begin{cases} \begin{pmatrix} \Delta(U_{\mathbf{O}_m}+\phi_{\mathbf{O}_m})-(1+\epsilon P(x))(U_{\mathbf{O}_m}+\phi_{\mathbf{O}_m})+\mu(U_{\mathbf{O}_m}+\phi_{\mathbf{O}_m})(V_{\mathbf{O}_m}+\psi_{\mathbf{O}_m}) \\ \Delta(V_{\mathbf{O}_m}+\psi_{\mathbf{O}_m})-(1+\epsilon Q(x))(V_{\mathbf{O}_m}+\psi_{\mathbf{O}_m})+\frac{\mu}{2}(U_{\mathbf{O}_m}+\phi_{\mathbf{O}_m})^2+\gamma(V_{\mathbf{O}_m}+\psi_{\mathbf{O}_m})^2 \end{pmatrix} \\ =\begin{pmatrix} \sum_{j=1}^m\sum_{k=1}^N c_{jk}D_{jk,1}, \\ \sum_{i=1}^m\sum_{k=1}^N c_{jk}D_{jk,2} \end{pmatrix}, \\ \Big\langle \begin{pmatrix} \phi_{\mathbf{O}_m} \\ \psi_{\mathbf{O}_m} \end{pmatrix}, \begin{pmatrix} D_{jk,1} \\ D_{jk,2} \end{pmatrix} \Big\rangle = 0 \text{ for } j=1,\cdots,m,\ k=1,\cdots,N. \end{cases} \tag{2.16}$$

It is obvious that the first system in (2.16) can be rewritten as

$$L\begin{pmatrix} \phi_{\mathbf{O}_m} \\ \psi_{\mathbf{O}_m} \end{pmatrix} = -G\begin{pmatrix} U_{\mathbf{O}_m} \\ V_{\mathbf{O}_m} \end{pmatrix} - M\begin{pmatrix} \phi_{\mathbf{O}_m} \\ \psi_{\mathbf{O}_m} \end{pmatrix} + \sum_{j=1}^m\sum_{k=1}^N c_{jk}\begin{pmatrix} D_{jk,1} \\ D_{jk,2} \end{pmatrix}, \tag{2.17}$$

where

$$M\begin{pmatrix} \phi_{\mathbf{O}_m} \\ \psi_{\mathbf{O}_m} \end{pmatrix} = \begin{pmatrix} \mu\phi_{\mathbf{O}_m}\psi_{\mathbf{O}_m} \\ \frac{\mu}{2}\phi^2_{\mathbf{O}_m}+\gamma\psi^2_{\mathbf{O}_m} \end{pmatrix}. \tag{2.18}$$

Now we come to the main result in this section.

PROPOSITION 2.4. *There exist positive numbers ϱ_0, C and θ such that for all $\varrho > \varrho_0$, and for any $\boldsymbol{O}_m \in \Omega_m$, $\epsilon < e^{-2\varrho}$, there is a unique solution $\Big(\begin{pmatrix} \phi_{\boldsymbol{O}_m} \\ \psi_{\boldsymbol{O}_m} \end{pmatrix}, \{c_{jk}\}\Big)$ to problem (2.16). Furthermore, $(\phi_{\boldsymbol{O}_m}, \psi_{\boldsymbol{O}_m})$ is $C^1\times C^1$ in Ω_m and*

$$\|(\phi_{\boldsymbol{O}_m}, \psi_{\boldsymbol{O}_m})\|_* \le Ce^{-\theta\varrho},\quad |c_{j,k}| \le Ce^{-\theta\varrho}. \tag{2.19}$$

In order to apply the contraction mapping theorem to prove Proposition 2.4, firstly we have to obtain the following two lemmas.

LEMMA 2.5. *For any $0 < \sigma < 1$, there exists $\varrho_0 > 0$ such that for $\varrho > \varrho_0$ large, and for any $\boldsymbol{O}_m \in \Omega_m$, $\epsilon < e^{-2\varrho}$, the following estimate holds*

$$\Big\|G\begin{pmatrix} U_{\boldsymbol{O}_m} \\ V_{\boldsymbol{O}_m} \end{pmatrix}\Big\|_* \le Ce^{-\theta\varrho} \tag{2.20}$$

for some positive constants θ and C independent of ϱ, m and $\boldsymbol{O}_m$.

Proof. Using the system (1.14) satisfied by (U_{O_j}, V_{O_j}), $j=1,\cdots,m$, we have

$$\begin{aligned} G\begin{pmatrix} U_{\mathbf{O}_m} \\ V_{\mathbf{O}_m} \end{pmatrix} &= \begin{pmatrix} \Delta U_{\mathbf{O}_m}-(1+\epsilon P(x))U_{\mathbf{O}_m}+\mu U_{\mathbf{O}_m}V_{\mathbf{O}_m} \\ \Delta V_{\mathbf{O}_m}-(1+\epsilon Q(x))V_{\mathbf{O}_m}+\frac{\mu}{2}U^2_{\mathbf{O}_m}+\gamma V^2_{\mathbf{O}_m} \end{pmatrix} \\ &= \begin{pmatrix} \mu(U_{\mathbf{O}_m}V_{\mathbf{O}_m}-\sum_{j=1}^m U_{O_j}V_{O_j}) \\ \frac{\mu}{2}(U^2_{\mathbf{O}_m}-\sum_{j=1}^m U^2_{O_j})+\gamma(V^2_{\mathbf{O}_m}-\sum_{j=1}^m V^2_{O_j}) \end{pmatrix} - \begin{pmatrix} \epsilon P(x)U_{\mathbf{O}_m} \\ \epsilon Q(x)V_{\mathbf{O}_m} \end{pmatrix} \\ &= \begin{pmatrix} \mu\sum_{i\ne j}U_{O_i}V_{O_j} \\ \frac{\mu}{2}\sum_{i\ne j}U_{O_i}U_{O_j}+\gamma\sum_{i\ne j}V_{O_i}V_{O_j} \end{pmatrix} - \begin{pmatrix} \epsilon P(x)U_{\mathbf{O}_m} \\ \epsilon Q(x)V_{\mathbf{O}_m} \end{pmatrix}. \end{aligned}$$

Fix $j \in \{1, \cdots, m\}$ and consider the region $|x - O_j| \le \varrho/2$. In this region, we have

$$\Big|\sum_{i\neq j} U_{O_i} V_{O_j}\Big| \le Ce^{-\frac{\varrho}{2}} V_{O_j} \le Ce^{-\frac{\varrho}{2}} e^{-\sigma|x-O_j|}$$

and similarly,

$$\Big|\sum_{i\neq j} U_{O_i} U_{O_j}\Big|, \Big|\sum_{i\neq j} V_{O_i} V_{O_j}\Big| \le Ce^{-\frac{\varrho}{2}} e^{-\sigma|x-O_j|}.$$

Consider the region $|x - O_j| > \varrho/2$ for all $j \in \{1, \cdots, m\}$. We have

$$\begin{aligned}\Big|\sum_{i\neq j} U_{O_i} V_{O_j}\Big| &\le C\sum_{i=1}^{m} U_{O_i}^2 \le C\sum_{i=1}^{m} e^{-2|x-O_i|}\\ &\le C\sum_{i=1}^{m} e^{-\sigma|x-O_i|} e^{-(2-\sigma)\frac{\varrho}{2}} \le Ce^{-\theta\varrho}\sum_{i=1}^{m} e^{-\sigma|x-O_i|}.\end{aligned}$$

Similarly, we also have

$$\Big|\sum_{i\neq j} U_{O_i} U_{O_j}\Big|, \Big|\sum_{i\neq j} V_{O_i} V_{O_j}\Big| \le Ce^{-\theta\varrho}\sum_{i=1}^{m} e^{-\sigma|x-O_i|}$$

for a proper choice of $\theta > 0$. Now, under the assumption on ϵ, it is easy to see that

$$|\epsilon P(x) U_{\mathbf{O}_m}| \le Ce^{-\theta\varrho}\sum_{i=1}^{m} e^{-\sigma|x-O_i|}$$

and

$$|\epsilon Q(x) V_{\mathbf{O}_m}| \le Ce^{-\theta\varrho}\sum_{i=1}^{m} e^{-\sigma|x-O_i|}.$$

Thus using the above estimates, we have

$$\Big\|G\Big(\begin{array}{c} U_{\mathbf{O}_m} \\ V_{\mathbf{O}_m}\end{array}\Big)\Big\|_* \le Ce^{-\theta\varrho}$$

for some $\theta > 0$. □

Define

$$\mathcal{B} = \Big\{(\phi, \psi) \in \mathbf{H} \,:\, \|(\phi,\psi)\|_* \le e^{-(\theta-\iota)\varrho},\ \Big\langle\Big(\begin{array}{c}\phi\\ \psi\end{array}\Big), \Big(\begin{array}{c} D_{jk,1}\\ D_{jk,2}\end{array}\Big)\Big\rangle = 0\Big\},$$

where $\iota > 0$ is small enough.

LEMMA 2.6. *For any $\boldsymbol{O}_m \in \Omega_m$, if $(\phi, \psi) \in \mathcal{B}$, then we have*

$$\Big\|M\Big(\begin{array}{c}\phi\\ \psi\end{array}\Big)\Big\|_* \le Ce^{-\theta\varrho} \tag{2.21}$$

for some positive constants θ and C independent of ϱ, m and $\boldsymbol{O}_m$.

Proof. By the definition of M, we have

$$\begin{aligned}\Big\|M\begin{pmatrix}\phi\\ \psi\end{pmatrix}\Big\|_* &\le C\|\phi\|_*|\psi| + C\|\phi\|_*|\phi| + C\|\psi\|_*|\psi|\\ &\le C\|\phi\|_*\|\psi\|_* + C\|\phi\|_*^2 + C\|\psi\|_*^2\\ &\le C\|(\phi,\psi)\|_*^2 \le Ce^{-2(\theta-\iota)\varrho} \le Ce^{-\theta\varrho}\end{aligned}$$

for a proper θ independent of ϱ, m and $\mathbf{O}_m$. □

Now we are in a position to prove Proposition 2.4.

Proof of Proposition 2.4. We will use the contraction mapping theorem to prove it. Notice that $(\phi_{\mathbf{O}_m}, \psi_{\mathbf{O}_m})$ solves (2.16) if and only if

$$\begin{pmatrix}\phi_{\mathbf{O}_m}\\ \psi_{\mathbf{O}_m}\end{pmatrix} = \mathcal{A}\Big(-G\begin{pmatrix}U_{\mathbf{O}_m}\\ V_{\mathbf{O}_m}\end{pmatrix} - M\begin{pmatrix}\phi_{\mathbf{O}_m}\\ \psi_{\mathbf{O}_m}\end{pmatrix}\Big),$$

where $\mathcal{A}$ is the operator given by (2.14). In other words, $(\phi_{\mathbf{O}_m}, \psi_{\mathbf{O}_m})$ solves (2.16) if and only if $(\phi_{\mathbf{O}_m}, \psi_{\mathbf{O}_m})$ is a fixed point for the operator

$$\mathcal{T}\begin{pmatrix}\phi\\ \psi\end{pmatrix} =: \mathcal{A}\Big(-G\begin{pmatrix}U_{\mathbf{O}_m}\\ V_{\mathbf{O}_m}\end{pmatrix} - M\begin{pmatrix}\phi\\ \psi\end{pmatrix}\Big).$$

We will prove that $\mathcal{T}$ is a contraction mapping from $\mathcal{B}$ to itself. On one hand, by (2.15), Lemmas 2.5 and 2.6, we have for any $(\phi,\psi)\in\mathcal{B}$,

$$\Big\|\mathcal{T}\begin{pmatrix}\phi\\ \psi\end{pmatrix}\Big\|_* \le C\Big\|G\begin{pmatrix}U_{\mathbf{O}_m}\\ V_{\mathbf{O}_m}\end{pmatrix} + M\begin{pmatrix}\phi\\ \psi\end{pmatrix}\Big\|_* \le Ce^{-\theta\varrho} \le e^{(-\theta-\iota)\varrho}.$$

On the other hand, taking $\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}$ and $\begin{pmatrix}\phi_2\\ \psi_2\end{pmatrix}$ in $\mathcal{B}$, we have

$$\begin{aligned}\Big\|\mathcal{T}\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix} - \mathcal{T}\begin{pmatrix}\phi_2\\ \psi_2\end{pmatrix}\Big\|_* &\le C\Big\|M\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix} - M\begin{pmatrix}\phi_2\\ \psi_2\end{pmatrix}\Big\|_*\\ &= C\Big\|\begin{pmatrix}\mu\phi_1(\psi_1-\psi_2) + \mu(\phi_1-\phi_2)\psi_2\\ \frac{\mu}{2}(\phi_1+\phi_2)(\phi_1-\phi_2) + \gamma(\psi_1+\psi_2)(\psi_1-\psi_2)\end{pmatrix}\Big\|_*\\ &\le C\|\psi_1-\psi_2\|_*\|\phi_1\|_* + C\|\phi_1-\phi_2\|_*\|\psi_2\|_*\\ &\quad + C(\|\phi_1\|_* + \|\phi_2\|_*)\|\phi_1-\phi_2\|_* + C(\|\psi_1\|_* + \|\psi_2\|_*)\|\psi_1-\psi_2\|_*\\ &\le \frac{1}{2}(\|\phi_1-\phi_2\|_* + \|\psi_1-\psi_2\|_*)\\ &= \frac{1}{2}\Big\|\begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix} - \begin{pmatrix}\phi_2\\ \psi_2\end{pmatrix}\Big\|_*.\end{aligned}$$

This means that $\mathcal{T}$ is a contraction mapping from $\mathcal{B}$ to itself. It follows from the contraction mapping theorem that there exists a unique $\begin{pmatrix}\phi_{\mathbf{O}_m}\\ \psi_{\mathbf{O}_m}\end{pmatrix}\in\mathcal{B}$ such that (2.16) holds. So,

$$\Big\|\begin{pmatrix}\phi_{\mathbf{O}_m}\\ \psi_{\mathbf{O}_m}\end{pmatrix}\Big\|_* = \Big\|\mathcal{T}\begin{pmatrix}\phi_{\mathbf{O}_m}\\ \psi_{\mathbf{O}_m}\end{pmatrix}\Big\|_* \le Ce^{-\theta\varrho}.$$

Furthermore, combining (2.7), (2.20) and (2.21), we find

$$|c_{j,k}| \le Ce^{-\theta\varrho}.$$

□

3. A secondary Liapunov-Schmidt reduction. In this section, we prove a key estimate on the difference between the solutions in the m-th step and the $(m+1)$-th step. This second Liapunov-Schmidt reduction has been used in [2, 3, 30]. For $\mathbf{O}_m \in \Omega_m$, we denote

$$\begin{pmatrix} u_{\mathbf{O}_m} \\ v_{\mathbf{O}_m} \end{pmatrix} := \begin{pmatrix} U_{\mathbf{O}_m} + \phi_{\mathbf{O}_m} \\ V_{\mathbf{O}_m} + \psi_{\mathbf{O}_m} \end{pmatrix},$$

where $\begin{pmatrix} \phi_{\mathbf{O}_m} \\ \psi_{\mathbf{O}_m} \end{pmatrix}$ is the unique solution given by Proposition 2.4.

We now write

$$\begin{pmatrix} u_{\mathbf{O}_{m+1}} \\ v_{\mathbf{O}_{m+1}} \end{pmatrix} = \begin{pmatrix} u_{\mathbf{O}_m} \\ v_{\mathbf{O}_m} \end{pmatrix} + \begin{pmatrix} U_{O_{m+1}} \\ V_{O_{m+1}} \end{pmatrix} + \varphi_{m+1} = \begin{pmatrix} \bar{U} \\ \bar{V} \end{pmatrix} + \begin{pmatrix} \varphi_{m+1,1} \\ \varphi_{m+1,2} \end{pmatrix}, \tag{3.1}$$

where $\begin{pmatrix} \bar{U} \\ \bar{V} \end{pmatrix} = \begin{pmatrix} u_{\mathbf{O}_m} \\ v_{\mathbf{O}_m} \end{pmatrix} + \begin{pmatrix} U_{O_{m+1}} \\ V_{O_{m+1}} \end{pmatrix}$.

By Proposition 2.4, we can easily derive that

$$\|(\varphi_{m+1,1}, \varphi_{m+1,2})\|_* \le Ce^{-\theta\varrho}. \tag{3.2}$$

But the estimate is not sufficient, we need a key estimate for $\begin{pmatrix} \varphi_{m+1,1} \\ \varphi_{m+1,2} \end{pmatrix}$ which will be given later. In the following we will always assume that $\sigma > \frac{1}{2}$.

LEMMA 3.1. *Letting ϱ and ϵ be as in Proposition 2.4, then it holds*

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(|\nabla\varphi_{m+1,1}|^2 + |\varphi_{m+1,1}|^2 + |\nabla\varphi_{m+1,2}|^2 + |\varphi_{m+1,2}|^2\right) \\ &\le Ce^{-\theta\varrho}\sum_{j=1}^{m} w(|O_{m+1} - O_j|) + C\epsilon^2\Big[\Big(\int_{\mathbb{R}^N} \left(|P(x)|U_{O_{m+1}} + |Q(x)|V_{O_{m+1}}\right)\Big)^2 \\ &\quad + \int_{\mathbb{R}^N} \Big(|P(x)|^2U^2_{O_{m+1}} + |Q(x)|^2V^2_{O_{m+1}}\Big)\Big] \end{aligned} \tag{3.3}$$

for some positive constants C, θ independent of ϱ, m and $\mathbf{O}_{m+1} \in \Omega_{m+1}$.

Proof. In order to prove (3.3), we need to perform a further decomposition. From the non-degeneracy result of (U, V), we know that there are finitely many positive eigenvalues to the following linearized operators

$$\begin{pmatrix} \Delta\eta_{l,1} - \eta_{l,1} + \mu V\eta_{l,1} + \mu U\eta_{l,2} \\ \Delta\eta_{l,2} - \eta_{l,2} + \mu U\eta_{l,1} + 2\gamma V\eta_{l,2} \end{pmatrix} = \lambda_l \begin{pmatrix} \eta_{l,1} \\ \eta_{l,2} \end{pmatrix} \tag{3.4}$$

and the eigenfunctions $\eta_{l,i} (i = 1, 2)$ are exponential decay. Assume that $\lambda_l > 0$ for $l = 1, \cdots, K$. Let $\omega_{jl} = \zeta_j\eta_l(x - O_j)$, where ζ_j is given in Section 2 and $\eta_l = \begin{pmatrix} \eta_{l,1} \\ \eta_{l,2} \end{pmatrix}$.

It follows from (3.1) that

$$\varphi_{m+1} = \begin{pmatrix} u_{\mathbf{O}_{m+1}} \\ v_{\mathbf{O}_{m+1}} \end{pmatrix} - \begin{pmatrix} u_{\mathbf{O}_m} \\ v_{\mathbf{O}_m} \end{pmatrix} - \begin{pmatrix} U_{O_{m+1}} \\ V_{O_{m+1}} \end{pmatrix}$$

and then

$$\bar{L}\varphi_{m+1} = -\bar{G} + \sum_{j=1}^{m+1}\sum_{k=1}^{N} c_{jk}D_{jk} \text{ for some } c_{jk}, \tag{3.5}$$

where

$$\bar{L}\begin{pmatrix} \varphi_{m+1,1} \\ \varphi_{m+1,2} \end{pmatrix} = \begin{pmatrix} \Delta\varphi_{m+1,1} - (1+\epsilon P(x))\varphi_{m+1,1} + \mu\widetilde{V}\varphi_{m+1,1} + \mu\tilde{U}\varphi_{m+1,2} \\ \Delta\varphi_{m+1,2} - (1+\epsilon Q(x))\varphi_{m+1,2} + \mu\widetilde{U}\varphi_{m+1,1} + 2\gamma\tilde{V}\varphi_{m+1,2} \end{pmatrix},$$

$$\widetilde{U} = \begin{cases} \dfrac{(\bar{U}+\varphi_{m+1,1})^2 - \bar{U}^2}{2\varphi_{m+1,1}}, & \text{if } \varphi_{m+1,1} \neq 0, \\ \bar{U}, & \text{if } \varphi_{m+1,1} = 0, \end{cases}$$

$$\widetilde{V} = \begin{cases} \dfrac{(\bar{V}+\varphi_{m+1,2})^2 - \bar{V}^2}{2\varphi_{m+1,2}}, & \text{if } \varphi_{m+1,2} \neq 0, \\ \bar{V}, & \text{if } \varphi_{m+1,2} = 0, \end{cases}$$

and

$$\bar{G} = \begin{pmatrix} \mu(\bar{U}\bar{V} - u_{\mathbf{O}_m}v_{\mathbf{O}_m} - U_{O_{m+1}}V_{O_{m+1}}) \\ \frac{\mu}{2}(\bar{U}^2 - u^2_{\mathbf{O}_m} - U^2_{O_{m+1}}) + \gamma(\bar{V}^2 - v^2_{\mathbf{O}_m} - V^2_{O_{m+1}}) \end{pmatrix} - \epsilon\begin{pmatrix} P(x)U_{O_{m+1}} \\ Q(x)V_{O_{m+1}} \end{pmatrix}. \tag{3.6}$$

We proceed the proof into a few steps. First we estimate the L^2 norm of $\bar{G}$. Noting that $\sigma > \frac{1}{2}$, then we have

$$\begin{aligned} &\int_{\mathbb{R}^N} \left|\mu(\bar{U}\bar{V} - u_{\mathbf{O}_m}v_{\mathbf{O}_m} - U_{O_{m+1}}V_{O_{m+1}})\right|^2 \le C\int_{\mathbb{R}^N}\left(u^2_{\mathbf{O}_m}V^2_{O_{m+1}} + v^2_{\mathbf{O}_m}U^2_{O_{m+1}}\right) \\ &\le C\int_{\mathbb{R}^N}\left(U^2_{\mathbf{O}_m}V^2_{O_{m+1}} + \phi^2_{\mathbf{O}_m}V^2_{O_{m+1}} + V^2_{\mathbf{O}_m}U^2_{O_{m+1}} + \psi^2_{\mathbf{O}_m}U^2_{O_{m+1}}\right) \\ &\le Ce^{-\theta\varrho}\sum_{j=1}^{m} w(|O_{m+1} - O_j|), \end{aligned} \tag{3.7}$$

and similarly

$$\int_{\mathbb{R}^N} \left|\frac{\mu}{2}(\bar{U}^2 - u^2_{\mathbf{O}_m} - U^2_{O_{m+1}}) + \gamma(\bar{V}^2 - v^2_{\mathbf{O}_m} - V^2_{O_{m+1}})\right|^2 \le Ce^{-\theta\varrho}\sum_{j=1}^{m} w(|O_{m+1} - O_j|). \tag{3.8}$$

It follows from (3.6), (3.7) and (3.8) that

$$\|\bar{G}\|^2_{L^2(\mathbb{R}^N)} \le Ce^{-\theta\varrho}\sum_{j=1}^{m} w(|O_{m+1} - O_j|) + \epsilon^2\int_{\mathbb{R}^N}(|P(x)|^2U^2_{O_{m+1}} + |Q(x)|^2V^2_{O_{m+1}}). \tag{3.9}$$

Now we decompose φ_{m+1} as

$$\varphi_{m+1} = \Phi + \sum_{j=1}^{m+1}\sum_{l=1}^{K} r_{jl}\omega_{jl} + \sum_{j=1}^{m+1}\sum_{k=1}^{N} d_{jk}D_{jk} \tag{3.10}$$

for some r_{jl}, d_{jk} such that

$$\langle \Phi, \omega_{jl}\rangle = \langle \Phi, D_{jk}\rangle = 0,\ \ j = 1,\cdots, m+1, k = 1,\cdots, N, l = 1,\cdots, K. \tag{3.11}$$

Since

$$\varphi_{m+1} = \begin{pmatrix} \phi_{\mathbf{O}_{m+1}} \\ \psi_{\mathbf{O}_{m+1}} \end{pmatrix} - \begin{pmatrix} \phi_{\mathbf{O}_m} \\ \psi_{\mathbf{O}_m}, \end{pmatrix},$$

we have for $j = 1, \cdots, m$,

$$\begin{aligned} d_{jk} &= \langle \varphi_{m+1}, D_{jk}\rangle + \sum_{l=1}^{K} r_{jl}\, \langle \omega_{jl}, D_{jk}\rangle \\ &= \Big\langle \begin{pmatrix} \phi_{\mathbf{O}_{m+1}} \\ \psi_{\mathbf{O}_{m+1}} \end{pmatrix} - \begin{pmatrix} \phi_{\mathbf{O}_m} \\ \psi_{\mathbf{O}_m} \end{pmatrix}, D_{jk}\Big\rangle + \sum_{l=1}^{K} r_{jl}\, \langle \omega_{jl}, D_{jk}\rangle \\ &= \sum_{l=1}^{K} r_{jl}\, \langle \omega_{jl}, D_{jk}\rangle = Ce^{-\theta\varrho}\sum_{l=1}^{K} r_{jl}, \end{aligned}$$

where we used the fact that

$$\begin{aligned} \langle \omega_{jl}, D_{jk}\rangle &= \int \begin{pmatrix} \omega_{jl,1} \\ \omega_{jl,2} \end{pmatrix}\begin{pmatrix} D_{jk,1} \\ D_{jk,2} \end{pmatrix} \\ &= \int \zeta_j^2 \eta_{l,1}(x - O_j)\frac{\partial U_{O_j}}{\partial x_k} + \int \zeta_j^2 \eta_{l,2}(x - O_j)\frac{\partial V_{O_j}}{\partial x_k} \\ &= \int \zeta_j^2(x + O_j)\eta_{l,1}(x)\frac{\partial U(x)}{\partial x_k} + \int \zeta_j^2(x + O_j)\eta_{l,2}(x)\frac{\partial V}{\partial x_k} \\ &\le C\int_{\frac{\varrho-1}{2}}^{\frac{\varrho}{2}} e^{-2r}r^{N-1}dr = Ce^{-\theta\varrho}, \end{aligned}$$

which can be derived from (3.4) and preliminary calculation.

For $j = m+1$, there holds

$$\begin{aligned} d_{m+1,k} =& \langle \varphi_{m+1}, D_{m+1,k}\rangle + \sum_{l=1}^{K} r_{m+1,l}\, \langle \phi_{m+1,l}, D_{m+1,k}\rangle \\ &= \Big\langle \begin{pmatrix} \phi_{\mathbf{O}_{m+1}} \\ \psi_{\mathbf{O}_{m+1}} \end{pmatrix} - \begin{pmatrix} \phi_{\mathbf{O}_m} \\ \psi_{\mathbf{O}_m} \end{pmatrix}, D_{m+1,k}\Big\rangle + \sum_{l=1}^{K} r_{m+1,l}\, \langle \omega_{m+1,l}, D_{m+1,k}\rangle \\ &= -\Big\langle \begin{pmatrix} \phi_{\mathbf{O}_m} \\ \psi_{\mathbf{O}_m} \end{pmatrix}, D_{m+1,k}\Big\rangle + \sum_{l=1}^{K} r_{m+1,l}\, \langle \omega_{m+1,l}, D_{m+1,k}\rangle, \end{aligned}$$

where we used the orthogonality conditions satisfied by $\begin{pmatrix} \phi_{\mathbf{O}_m} \\ \psi_{\mathbf{O}_m} \end{pmatrix}$ and $\begin{pmatrix} \phi_{\mathbf{O}_{m+1}} \\ \psi_{\mathbf{O}_{m+1}} \end{pmatrix}$.

By the definition of D_{jk}, we have

$$\begin{aligned}\left\langle\begin{pmatrix}\phi_{\mathbf{O}_m}\\ \psi_{\mathbf{O}_m}\end{pmatrix}, D_{m+1,k}\right\rangle &= \int_{\mathbb{R}^N}\Big(\phi_{\mathbf{O}_m}\zeta_{m+1}\frac{\partial U_{O_{m+1}}}{\partial x_k}+\psi_{\mathbf{O}_m}\zeta_{m+1}\frac{\partial V_{O_{m+1}}}{\partial x_k}\Big)\\ &\le Ce^{-\theta\varrho}\int_{\mathbb{R}^N}\sum_{j=1}^{m}e^{-\sigma|x-O_j|}e^{-\sigma|x-O_{m+1}|}e^{-(1-\sigma)|x-O_{m+1}|}\\ &\le Ce^{-\theta\varrho}\sum_{j=1}^{m}e^{-\sigma|O_{m+1}-O_j|}.\end{aligned}$$

So we can deduce that

$$\begin{cases}|d_{m+1,k}|\le Ce^{-\theta\varrho}\displaystyle\sum_{j=1}^{m}e^{-\sigma|O_{m+1}-O_j|}+Ce^{-\theta\varrho}\sum_{l=1}^{K}r_{m+1,l},\\ |d_{jk}|\le Ce^{-\theta\varrho}\displaystyle\sum_{l=1}^{K}r_{jl},\quad \text{for } j=1,\cdots,m.\end{cases}\tag{3.12}$$

It follows from (3.10) that (3.5) can be rewritten as

$$\bar{L}\Phi+\sum_{j=1}^{m+1}\sum_{l=1}^{K}r_{jl}\bar{L}\omega_{jl}+\sum_{j=1}^{m+1}\sum_{k=1}^{N}d_{jk}\bar{L}D_{jk}=-\bar{G}+\sum_{j=1}^{m+1}\sum_{k=1}^{N}c_{jk}D_{jk}.\tag{3.13}$$

To estimate the coefficients r_{jl}, $l\in\{1,\cdots K\}$, multiplying (3.13) by ω_{jl} and integrating over $\mathbb{R}^N$, we have

$$\begin{aligned}r_{jl}\left\langle\bar{L}\omega_{jl},\omega_{jl}\right\rangle=&-\sum_{k=1}^{N}d_{jk}\left\langle\bar{L}D_{jk},\omega_{jl}\right\rangle-\left\langle\bar{G},\omega_{jl}\right\rangle-\sum_{s\neq l}r_{js}\left\langle\bar{L}\omega_{js},\omega_{jl}\right\rangle\\ &+\sum_{k=1}^{N}c_{jk}\left\langle D_{jk},\omega_{jl}\right\rangle-\left\langle\bar{L}\Phi,\omega_{jl}\right\rangle.\end{aligned}\tag{3.14}$$

By the definition of $\bar{G}$, it is easy to verify that for $j=1,\cdots,m$,

$$|\left\langle\bar{G},\omega_{jl}\right\rangle|\le Ce^{-\theta\varrho}e^{-\sigma|O_{m+1}-O_j|}+\Big|\left\langle\begin{pmatrix}\epsilon P(x)U_{O_{m+1}}\\ \epsilon Q(x)V_{O_{m+1}}\end{pmatrix},\begin{pmatrix}\omega_{jl,1}\\ \omega_{jl,2}\end{pmatrix}\right\rangle\Big|,$$

since

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N}\mu(\bar{U}\bar{V}-u_{\mathbf{O}_m}v_{\mathbf{O}_m}-U_{O_{m+1}}V_{O_{m+1}})\omega_{jl,1}\Big|\\ &\le C\int_{B_{\frac{\varrho}{2}}(O_j)}\big(|u_{\mathbf{O}_m}|V_{O_{m+1}}+|v_{\mathbf{O}_m}|U_{O_{m+1}}\big)\zeta_j\eta_{l,1}(x-O_j)\\ &\le Ce^{-\theta\varrho}e^{-\sigma|O_{m+1}-O_j|},\end{aligned}$$

and

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N}\big[\frac{\mu}{2}(\bar{U}^2-u^2_{\mathbf{O}_m}-U^2_{O_{m+1}})+\gamma(\bar{V}^2-v^2_{\mathbf{O}_m}-V^2_{O_{m+1}})\big]\omega_{jl,2}\Big|\\ &\le C\int_{B_{\frac{\varrho}{2}}(O_j)}\big(|u_{\mathbf{O}_m}|U_{O_{m+1}}+|v_{\mathbf{O}_m}|V_{O_{m+1}}\big)\zeta_j\eta_{l,2}(x-O_j)\\ &\le Ce^{-\theta\varrho}e^{-\sigma|O_{m+1}-O_j|}.\end{aligned}$$

Similarly, one has

$$\big|\left\langle \bar{G}, \omega_{m+1,l}\right\rangle\big| \leq Ce^{-\theta\varrho}\sum_{j=1}^{m} e^{-\sigma|O_{m+1}-O_j|} + \Big|\Big\langle\begin{pmatrix} \epsilon P(x)U_{O_{m+1}} \\ \epsilon Q(x)V_{O_{m+1}}\end{pmatrix}, \begin{pmatrix} \omega_{m+1,l,1} \\ \omega_{m+1,l,2}\end{pmatrix}\Big\rangle\Big|.$$

Moreover, it follows from (2.7) that

$$\Big|\sum_{k=1}^{N} c_{jk}\left\langle D_{jk}, \omega_{jl}\right\rangle\Big| \leq C|c_{jk}| \leq Ce^{-\theta\varrho} + C\|\bar{G}\|_* \leq Ce^{-\theta\varrho}. \tag{3.15}$$

Applying the system (3.4), integration by parts and by direct computation, we have

$$\begin{aligned}
&\big|\left\langle \bar{L}\Phi, \omega_{jl}\right\rangle\big| = \big|\left\langle \bar{L}\omega_{jl}, \Phi\right\rangle\big| = \int_{\mathbb{R}^N} \bar{L}\begin{pmatrix} \omega_{jl,1} \\ \omega_{jl,2}\end{pmatrix}\begin{pmatrix} \Phi_1 \\ \Phi_2\end{pmatrix} \\
&= \int_{\mathbb{R}^N}\begin{pmatrix} \Delta\omega_{l,1} - (1+\epsilon P(x))\omega_{l,1} + \mu\tilde{V}\omega_{l,1} + \mu\tilde{U}\omega_{l,2} \\ \Delta\omega_{l,2} - (1+\epsilon Q(x))\omega_{l,2} + \mu\tilde{U}\omega_{l,1} + 2\gamma\tilde{V}\omega_{l,2})\end{pmatrix}\begin{pmatrix} \Phi_1 \\ \Phi_2\end{pmatrix} \\
&= \int_{\mathbb{R}^N}\begin{pmatrix} \lambda_l\eta_{l,1}\zeta_j \\ \lambda_l\eta_{l,2}\zeta_j\end{pmatrix}\begin{pmatrix} \Phi_1 \\ \Phi_2\end{pmatrix} + \int_{\mathbb{R}^N}\begin{pmatrix} \Delta\zeta_j\eta_{l,1} + 2\nabla\eta_{l,1}\cdot\nabla\zeta_j \\ \Delta\zeta_j\eta_{l,2} + 2\nabla\eta_{l,2}\cdot\nabla\zeta_j\end{pmatrix}\begin{pmatrix} \Phi_1 \\ \Phi_2\end{pmatrix} \\
&\quad + \int_{\mathbb{R}^N}\begin{pmatrix} \mu(\tilde{V}-V)\eta_{l,1}\zeta_j + \mu(\tilde{U}-U)\eta_{l,2}\zeta_j \\ \mu(\tilde{U}-U)\eta_{l,1}\zeta_j + 2\gamma(\tilde{V}-V)\eta_{l,2}\zeta_j\end{pmatrix}\begin{pmatrix} \Phi_1 \\ \Phi_2\end{pmatrix} - \epsilon\int_{\mathbb{R}^N}\begin{pmatrix} P(x)\eta_{l,1}\zeta_j \\ Q(x)\eta_{l,2}\zeta_j\end{pmatrix}\begin{pmatrix} \Phi_1 \\ \Phi_2\end{pmatrix} \\
&\leq Ce^{-\theta\varrho}\|\Phi\|_{H^1(B_{\frac{\varrho}{2}}(O_j))}.
\end{aligned}$$

Similarly, we can deduce

$$\left\langle \bar{L}\omega_{jl}, \omega_{js}\right\rangle = \delta_{ls}\lambda_s\left\langle \eta_l, \eta_s\right\rangle + O(e^{-\theta\varrho}).$$

So, from the above estimates, we can infer that

$$\begin{cases}
|r_{m+1,l}| \leq Ce^{-\theta\varrho}\displaystyle\sum_{j=1}^{m} e^{-\sigma|O_{m+1}-O_j|} + Ce^{-\theta\varrho}\|\Phi\|_{H^1(B_{\frac{\varrho}{2}}(O_{m+1}))} \\
\qquad\qquad + C\Big|\Big\langle\begin{pmatrix} \epsilon P(x)U_{O_{m+1}} \\ \epsilon Q(x)V_{O_{m+1}}\end{pmatrix}, \begin{pmatrix} \omega_{m+1,l,1} \\ \omega_{m+1,l,2}\end{pmatrix}\Big\rangle\Big|, \\
|r_{i,l}| \leq Ce^{-\theta\varrho}e^{-\sigma|O_{m+1}-O_j|} + Ce^{-\theta\varrho}\|\Phi\|_{H^1(B_{\frac{\varrho}{2}}(O_j))} \\
\qquad\qquad + C\Big|\Big\langle\begin{pmatrix} \epsilon P(x)U_{O_{m+1}} \\ \epsilon Q(x)V_{O_{m+1}}\end{pmatrix}, \begin{pmatrix} \omega_{jl,1} \\ \omega_{jl,2}\end{pmatrix}\Big\rangle\Big|
\end{cases} \tag{3.16}$$

and then

$$\begin{cases}
|d_{k+1,j}| \leq Ce^{-\theta\varrho}\displaystyle\sum_{j=1}^{m} e^{-\sigma|O_{m+1}-O_j|} + Ce^{-\theta\varrho}\|\Phi\|_{H^1(B_{\frac{\varrho}{2}}(O_{m+1}))}, \\
|d_{i,j}| \leq Ce^{-\theta\varrho}e^{-\sigma|O_{m+1}-O_j|} + Ce^{-\theta\varrho}\|\Phi\|_{H^1(B_{\frac{\varrho}{2}}(O_j))}
\end{cases} \tag{3.17}$$

for $j = 1, \cdots, m, k = 1, \cdots, N, l = 1, \cdots, K$.

Finally we need to estimate Φ. Multiplying (3.13) by Φ and integrating over $\mathbb{R}^N$, we have

$$\left\langle \bar{L}\Phi, \Phi\right\rangle = -\left\langle \bar{G}, \Phi\right\rangle - \sum_{j=1}^{m+1}\sum_{k=1}^{N} d_{jk}\left\langle \bar{L}D_{jk}, \Phi\right\rangle - \sum_{j=1}^{m+1}\sum_{l=1}^{K} r_{jl}\left\langle \bar{L}\omega_{jl}, \Phi\right\rangle. \tag{3.18}$$

We claim that

$$-\left\langle \bar{L}\Phi, \Phi \right\rangle \geq c_0 \|\Phi\|^2_{H^1(\mathbb{R}^N)} \tag{3.19}$$

for some constant $c_0 > 0$ (independent of $\mathbf{O}_{m+1}$).

Indeed, since the approximate solution is exponentially decaying away from the point O_j, we have

$$-\int_{\mathbb{R}^N \setminus \cup_j B_{\frac{\varrho}{2}}(O_j)} (\bar{L}\Phi)\Phi \geq \frac{1}{2}\int_{\mathbb{R}^N \setminus \cup_j B_{\frac{\varrho}{2}}(O_j)} \left(|\nabla\Phi_1|^2 + \Phi_1^2 + |\nabla\Phi_2|^2 + \Phi_2^2\right). \tag{3.20}$$

So we only need to prove (3.20) in the domain $\cup_j B_{\frac{\varrho}{2}}(O_j)$. Here we prove it by contradiction. Assume that there exist a sequence $\varrho_n \to +\infty$, and O_j^n such that as $n \to \infty$,

$$\int_{B_{\frac{\varrho_n}{2}}(O_j^n)} \left(|\nabla\Phi_1^n|^2 + |\Phi_1^n|^2 + |\nabla\Phi_2^n|^2 + |\Phi_2^n|^2\right) = 1 \tag{3.21}$$

and

$$\int_{B_{\frac{\varrho_n}{2}}(O_j^n)} (\bar{L}\Phi^n)\Phi^n \to 0. \tag{3.22}$$

Then we can extract from the sequence $\Phi^n(x - O_j^n)$ a subsequence which will converge weakly in $\mathbf{H}$ to Φ^∞ satisfying

$$\begin{aligned}&\int_{\mathbb{R}^N} \left(|\nabla\Phi_{\infty,1}|^2 + \Phi_{\infty,1}^2 - \mu V\Phi_{\infty,1}^2 - \mu U\Phi_{\infty,2}\Phi_{\infty,1}\right)\\ &\quad + \int_{\mathbb{R}^N} \left(|\nabla\Phi_{\infty,2}|^2 + \Phi_{\infty,2}^2 - \mu U\Phi_{\infty,1}\Phi_{\infty,2} - 2\gamma V\Phi_{\infty,2}^2\right) = 0\end{aligned}$$

and from (3.11), we can find that

$$\left\langle \Phi_\infty, \begin{pmatrix} \eta_{l,1} \\ \eta_{l,2} \end{pmatrix} \right\rangle = \left\langle \Phi_\infty, \begin{pmatrix} \frac{\partial U}{\partial x_k} \\ \frac{\partial V}{\partial x_k} \end{pmatrix} \right\rangle = 0$$

for $l = 1, \cdots, K, k = 1, \cdots, N$. So we infer that $\Phi_\infty = 0$. Hence

$$\Phi^n \rightharpoonup 0 \text{ weakly in } \mathbf{H}.$$

As a result, as $n \to \infty$,

$$\int_{B_{\frac{\varrho_n}{2}}(O_j^n)} \left[\mu\widetilde{V}(\Phi_1^n)^2 + \mu\tilde{U}\Phi_2^n\Phi_1^n + \mu\widetilde{U}\Phi_1^n\Phi_2^n + 2\gamma\tilde{V}(\Phi_2^n)^2\right] \to 0$$

and then by (3.22), one has as $n \to +\infty$,

$$\|\Phi^n\|_{H^1\left(B_{\frac{\varrho_n}{2}}(O_j^n)\right)} \to 0,$$

which contradicts to (3.21). Thus (3.19) holds.

It follows from (3.18) and (3.19) that

$$\begin{aligned}\|\Phi\|^2_{H^1(\mathbb{R}^N)} &\le C\Big(\sum_{jk}|d_{jk}||\big\langle\bar{L}D_{jk},\Phi\big\rangle| + \sum_{jl}|r_{jl}||\big\langle\bar{L}\omega_{jl},\Phi\big\rangle| + |\big\langle\bar{G},\Phi\big\rangle|\Big)\\ &\le C\Big(\sum_{jk}|d_{jk}|\|\Phi\|_{H^1(B_{\frac{\varrho}{2}}(O_j))} + \sum_{jl}|r_{jl}|\|\Phi\|_{H^1(B_{\frac{\varrho}{2}}(O_j))}\\ &\qquad + \|\bar{G}\|_{L^2(\mathbb{R}^N)}\|\Phi\|_{H^1(\mathbb{R}^N)}\Big).\end{aligned} \tag{3.23}$$

Using (3.23), (3.10), (3.16) and (3.17), we get that

$$\begin{aligned}\|\varphi_{m+1}\|_{H^1(\mathbb{R}^N)} \le C\Big\{&e^{-\theta\varrho}\sum_{j=1}^m e^{-\sigma|O_j-O_{m+1}|} + \int_{\mathbb{R}^N}\epsilon\Big(|P(x)|U_{O_{m+1}} + |Q(x)|V_{O_{m+1}}\Big)\\ &+\Big(\int_{\mathbb{R}^N}\epsilon^2\Big(|P(x)|^2U^2_{O_{m+1}} + |Q(x)|^2V^2_{O_{m+1}}\Big)\Big)^{\frac{1}{2}}\\ &+\Big(e^{-\theta\varrho}\sum_{j=1}^m w(|O_j-O_{m+1}|)\Big)^{\frac{1}{2}}\Big\}.\end{aligned} \tag{3.24}$$

Since we choose $\sigma > 1/2$, we have

$$\Big(\sum_{j=1}^m e^{-\sigma|O_j-O_{m+1}|}\Big)^2 \le C\sum_{j=1}^m w(|O_j-O_{m+1}|). \tag{3.25}$$

From (3.24) and (3.25), we infer that

$$\begin{aligned}\|\varphi_{m+1}\|_{H^1(\mathbb{R}^N)} \le C\Big\{&\Big(e^{-\theta\varrho}\sum_{j=1}^m w(|O_j-O_{m+1}|)\Big)^{\frac{1}{2}}\\ &+\Big(\int_{\mathbb{R}^N}\epsilon^2(|P(x)|^2U^2_{O_{m+1}} + |Q(x)|^2V^2_{O_{m+1}})\Big)^{\frac{1}{2}}\\ &+\int_{\mathbb{R}^N}\epsilon\big(|P(x)|U_{O_{m+1}} + |Q(x)|V_{O_{m+1}}\big)\Big\}.\end{aligned} \tag{3.26}$$

Furthermore, from the estimates (3.12) and (3.16), and taking into consideration that ζ_j is supposed in $B_{\frac{\varrho}{2}}(O_j)$, using Hölder inequality, we can get an accurate estimate on φ_{m+1},

$$\begin{aligned}&\|\varphi_{m+1}\|_{H^1(\mathbb{R}^N)}\\ &\le C\Big\{\Big(e^{-\theta\varrho}\sum_{j=1}^m w(|O_j-O_{m+1}|)\Big)^{\frac{1}{2}}\\ &\quad+\Big(\int_{\mathbb{R}^N}\epsilon^2(|P(x)|^2U^2_{O_{m+1}} + |Q(x)|^2V^2_{O_{m+1}})\Big)^{\frac{1}{2}}\\ &\quad+\sum_{j=1}^m\Big(\int_{B_{\frac{\varrho}{2}}(O_j)}\epsilon^2|P(x)|^2U^2_{O_{m+1}}\Big)^{\frac{1}{2}} + \sum_{j=1}^m\Big(\int_{B_{\frac{\varrho}{2}}(O_j)}\epsilon^2|Q(x)|^2V^2_{O_{m+1}}\Big)^{\frac{1}{2}}\Big\}.\end{aligned} \tag{3.27}$$

This concludes the proof of Lemma 3.1. □

4. Proof of our main result. In this section, first we study a maximization problem, then we prove our main result.

Fixing $\mathbf{O}_m \in \Omega_m$, we define a new functional

$$\mathcal{N}(\mathbf{O}_m) = J(u_{\mathbf{O}_m}, v_{\mathbf{O}_m}) : \Omega_m \to \mathbb{R} \tag{4.1}$$

and

$$\mathcal{R}_m := \sup_{\mathbf{O}_m \in \Omega_m} \mathcal{N}(\mathbf{O}_m). \tag{4.2}$$

Observe that $\mathcal{N}(\mathbf{O}_m)$ is continuous in $\mathbf{O}_m$. We will prove below that the maximization problem has a solution. Let $\mathcal{N}(\bar{\mathbf{O}}_m)$ be the maximum, where $\bar{\mathbf{O}}_m = (\bar{O}_1, \cdots, \bar{O}_m) \in \bar{\Omega}_m$, that is

$$\mathcal{N}(\bar{\mathbf{O}}_m) = \max_{\mathbf{O}_m \in \Omega_m} \mathcal{N}(\mathbf{O}_m)$$

and we denote the corresponding solution by $u_{\bar{\mathbf{O}}_m}$.

First we give a result which follows from Lemma 2.4 in [2] and will be used later.

LEMMA 4.1. *For $|O_j - O_k| \geq \varrho$ large, we have*

$$\int_{\mathbb{R}^N} w^2(x - O_j) w(x - O_k) = (\gamma_1 + o(1)) w(|O_j - O_k|),$$

where

$$\gamma_1 = \int_{\mathbb{R}^N} w^2(x) e^{-x_1} > 0.$$

Now we prove that the maximum can be attained at finite points for each $\mathcal{R}_m$.

LEMMA 4.2. *Assume that $(H_1), (H_2)$, (H_3) and the assumptions in Proposition 2.4 hold. Then for all m:*
(i) There exists $\boldsymbol{O}_m \in \Omega_m$ such that

$$\mathcal{R}_m = \mathcal{N}(\boldsymbol{O}_m);$$

(ii) There holds

$$\mathcal{R}_{m+1} > \mathcal{R}_m + I(U, V),$$

where $I(U, V)$ is the energy of (U, V),

$$I(U,V) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\nabla U|^2 + U^2 + |\nabla V|^2 + V^2\right) - \frac{\mu}{2} \int_{\mathbb{R}^N} U^2 V - \frac{\gamma}{3} \int_{\mathbb{R}^N} V^3.$$

Proof. Here we follow the proofs in [2, 9] and we need to use the estimate in Lemma 4.1. To prove this lemma, we divide the proof into the following two steps.

Step 1: We first show that $\mathcal{R}_1 > I(U, V)$ and $\mathcal{R}_1$ can be attained at finite points. Similar to the proof of Lemma 3.1, we have

$$\|(\phi_O, \psi_O)\|_{H^1(\mathbb{R}^N)} \leq C\epsilon \Big(\int_{\mathbb{R}^N} \left(|P(x)|^2 U_O^2 + |Q(x)|^2 V_O^2\right) \Big)^{\frac{1}{2}} \tag{4.3}$$

for some $C > 0$ independent of O.

Assuming that $|O|$ large enough, we have

$$
\begin{aligned}
J(u_O, v_O) &= \frac{1}{2}\int_{\mathbb{R}^N} [|\nabla(U_O+\phi_O)|^2 + (1+\epsilon P(x))(U_O+\phi_O)^2] \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} [|\nabla(V_O+\psi_O)|^2 + (1+\epsilon Q(x))(V_O+\psi_O)^2] \\
&\quad - \frac{\mu}{2}\int_{\mathbb{R}^N} (U_O+\phi_O)^2(V_O+\psi_O) - \frac{\gamma}{3}\int_{\mathbb{R}^N}(V_O+\psi_O)^3 \\
&\geq I(U_O,V_O) + \frac{\epsilon}{2}\int_{\mathbb{R}^N}(P(x)U_O^2 + Q(x)V_O^2) - C\|(\epsilon P(x)U_O, \epsilon Q(x)V_O)\|^2_{L^2(\mathbb{R}^N)} \\
&\geq I(U_O,V_O) + \frac{\epsilon}{4}\int_{\mathbb{R}^N}(P(x)U_O^2 + Q(x)V_O^2) \\
&\geq I(U_O,V_O) + \frac{\epsilon}{4}\int_{B_{\frac{\varrho}{2}}(O)}(P(x)U_O^2 + Q(x)V_O^2) \\
&\quad - \frac{\epsilon}{4}\sup_{B_{\frac{|O|}{4}}(0)}(U_O^{\frac{3}{2}} + V_O^{\frac{3}{2}})\int_{supp(\alpha^2 P(x)+\beta^2 Q(x))^-}(|P(x)|U_O^{\frac{1}{2}} + |Q(x)|V_O^{\frac{1}{2}}) \\
&\geq I(U_O,V_O) + \frac{\epsilon}{4}\int_{B_{\frac{\varrho}{2}}(O)}(P(x)U_O^2 + Q(x)V_O^2) - O\big(\epsilon e^{-\frac{9}{8}|O|}\big).
\end{aligned}
$$

By the slow decay assumption (H_3), we have

$$
\frac{\epsilon}{4}\int_{B_{\frac{\varrho}{2}}(O)}(P(x)U_O^2 + Q(x)V_O^2) - O(\epsilon e^{-\frac{9}{8}|O|}) > 0.
$$

So we can deduce that

$$
\mathcal{R}_1 \geq J(U_O, V_O) > I(U,V). \tag{4.4}
$$

Let us prove now that $\mathcal{R}_1$ can be attained at finite points. Let $\{O_j\}$ be a sequence such that $\lim_{j\to\infty}\mathcal{N}(O_j) = \mathcal{R}_1$, and assume that $|O_j| \to \infty$ as $j \to \infty$. Then from the system satisfied by (U_{O_j}, V_{O_j}), (3.3), Hölder inequality and the Sobolev embedding theorem, we have

$$\begin{aligned}
J(u_{O_j}, v_{O_j}) &= \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla(U_{O_j} + \phi_{O_j})|^2 + (1+\epsilon P(x))(U_{O_j} + \phi_{O_j})^2) \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla(V_{O_j} + \psi_{O_j})|^2 + (1+\epsilon Q(x))(V_{O_j} + \psi_{O_j})^2) \\
&\quad - \frac{\mu}{2}\int_{\mathbb{R}^N} (U_{O_j} + \phi_{O_j})^2(V_{O_j} + \psi_{O_j}) - \frac{\gamma}{3}\int_{\mathbb{R}^N} (V_{O_j} + \psi_{O_j})^3 \\
&= I(U_{O_j}, V_{O_j}) + \frac{\epsilon}{2}\int_{\mathbb{R}^N} (P(x)U_{O_j}^2 + Q(x)V_{O_j}^2) + \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla \phi_{O_j}|^2 + \phi_{O_j}^2) \\
&\quad + \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla \psi_{O_j}|^2 + \psi_{O_j}^2) + \epsilon \int_{\mathbb{R}^N} (P(x)U_{O_j}\phi_{O_j} + Q(x)V_{O_j}\psi_{O_j}) \\
&\quad + \frac{\epsilon}{2}\int_{\mathbb{R}^N} (P(x)\phi_{O_j}^2 + Q(x)\psi_{O_j}^2) - \frac{\mu}{2}\int_{\mathbb{R}^N} (\phi_{O_j}^2 V_{O_j} + \phi_{O_j}^2\psi_{O_j} + 2U_{O_j}^2\psi_{O_j}) \\
&\quad - \frac{\gamma}{3}\int_{\mathbb{R}^N} (\psi_{O_j}^3 + 3V_{O_j}\psi_{O_j}^2) \\
&\le I(U_{O_j}, V_{O_j}) + \frac{\epsilon}{2}\int_{\mathbb{R}^N} (P(x)U_{O_j}^2 + Q(x)V_{O_j}^2) \\
&\quad + O\Big(\epsilon^2 \int (P^2(x)U_{O_j}^2 + Q^2(x)V_{O_j}^2)\Big).
\end{aligned}$$

By the decay assumptions on $P(x)$ and $Q(x)$, we have as $|O_j| \to \infty$,

$$\frac{\epsilon}{2}\int_{\mathbb{R}^N} (P(x)U_{O_j}^2 + Q(x)V_{O_j}^2) + O\Big(\epsilon^2 \int (P^2(x)U_{O_j}^2 + Q^2(x)V_{O_j}^2)\Big) \to 0.$$

So it follows that

$$\mathcal{R}_1 = \lim_{j\to\infty} \mathcal{N}(O_j) \le I(U, V), \tag{4.5}$$

which yields a contradiction to (4.4). Then $\mathcal{R}_1$ can be attained at finite points.

Step 2: Assume that there exists $\bar{\mathbf{O}}_m = (\bar{O}_1, \cdots, \bar{O}_m) \in \Omega_m$ such that $\mathcal{R}_m = \mathcal{N}(\bar{\mathbf{O}}_m)$. Next, we prove that there exists $\mathbf{O}_{m+1} \in \Omega_{m+1}$ such that $\mathcal{R}_{m+1}$ can be attained. Let $(O_1^n, \cdots, O_{m+1}^n)$ be a sequence such that

$$\mathcal{R}_{m+1} = \lim_{n\to\infty} \mathcal{N}(O_1^n, \cdots, O_{m+1}^n). \tag{4.6}$$

We claim that $(O_1^n, \cdots, O_{m+1}^n)$ is bounded. Here we prove it by an indirect method. Without loss of generality, we assume that $|O_{m+1}^n| \to \infty$ as $n \to \infty$. In the following, we omit the index n for simplicity. By Lemma 3.1, we have

$$\begin{aligned} J(u_{\mathbf{O}_{m+1}}, v_{\mathbf{O}_{m+1}}) &= J\Big(\binom{u_{\mathbf{O}_m}}{v_{\mathbf{O}_m}} + \binom{U_{O_{m+1}}}{V_{O_{m+1}}} + \binom{\varphi_{m+1,1}}{\varphi_{m+1,2}}\Big) \\ = J\Big(\binom{u_{\mathbf{O}_m}}{v_{\mathbf{O}_m}} + \binom{U_{O_{m+1}}}{V_{O_{m+1}}}\Big) &+ \sum_{j=1}^{m}\sum_{k=1}^{N} c_{jk}\int_{\mathbb{R}^N} D_{jk}\varphi_{m+1} - \langle \bar{G}, \varphi_{m+1}\rangle \\ &+O(\|\varphi_{m+1}\|^2_{H^1(\mathbb{R}^N)}) \\ = J\Big(\binom{u_{\mathbf{O}_m}}{v_{\mathbf{O}_m}} + \binom{U_{O_{m+1}}}{V_{O_{m+1}}}\Big) \\ &+O\Big\{e^{-\theta\varrho}\sum_{j=1}^{m} w(|O_{m+1}-O_j|) + \epsilon^2\int_{\mathbb{R}^N}(|P(x)|^2U^2_{O_{m+1}} + |Q(x)|^2V^2_{O_{m+1}}) \\ &+\Big(\epsilon\int_{\mathbb{R}^N}\Big(|P(x)|U_{O_{m+1}} + |Q(x)|V_{O_{m+1}}\Big)\Big)^2\Big\}. \end{aligned} \tag{4.7}$$

Next we estimate $J\Big(\begin{pmatrix} u_{\mathbf{O}_m} \\ v_{\mathbf{O}_m}\end{pmatrix} + \begin{pmatrix} U_{O_{m+1}} \\ V_{O_{m+1}}\end{pmatrix}\Big)$. By direct computation, we can find

$$\begin{aligned} &J\Big(\binom{u_{\mathbf{O}_m}}{v_{\mathbf{O}_m}} + \binom{U_{O_{m+1}}}{V_{O_{m+1}}}\Big) \\ &= J\binom{u_{\mathbf{O}_m}}{u_{\mathbf{O}_m}} + I(U_{O_{m+1}}, V_{O_{m+1}}) + \int_{\mathbb{R}^N}(\nabla u_{\mathbf{O}_m}\nabla U_{O_{m+1}} + \nabla v_{\mathbf{O}_m}\nabla V_{O_{m+1}}) \\ &\quad+\int_{\mathbb{R}^N}\big[(1+\epsilon P(x))u_{\mathbf{O}_m}U_{O_{m+1}} + (1+\epsilon Q(x))v_{\mathbf{O}_m}V_{O_{m+1}}\big] \\ &\quad-\frac{\mu}{2}\int_{\mathbb{R}^N}(u^2_{\mathbf{O}_m}V_{O_{m+1}} + v_{\mathbf{O}_m}U^2_{O_{m+1}} + 2u_{\mathbf{O}_m}v_{\mathbf{O}_m}U_{O_{m+1}} + 2u_{\mathbf{O}_m}U_{O_{m+1}}V_{O_{m+1}}) \\ &\quad-\frac{\gamma}{3}\int_{\mathbb{R}^N}(3v^2_{\mathbf{O}_m}V_{O_{m+1}} + 3v_{\mathbf{O}_m}V^2_{O_{m+1}}) \\ &= J\binom{u_{\mathbf{O}_m}}{u_{\mathbf{O}_m}} + I(U_{O_{m+1}}, V_{O_{m+1}}) + \sum_{j=1}^{m}\sum_{k=1}^{N} c_{jk}\Big\langle D_{jk}, \binom{U_{O_{m+1}}}{V_{O_{m+1}}}\Big\rangle \\ &\quad+\frac{\epsilon}{2}\int_{\mathbb{R}^N}(P(x)U^2_{O_{m+1}} + Q(x)V^2_{O_{m+1}}) - \frac{\mu}{2}\int_{\mathbb{R}^N}(v_{\mathbf{O}_m}U^2_{O_{m+1}} + 2u_{\mathbf{O}_m}U_{O_{m+1}}V_{O_{m+1}}) \\ &\quad-\gamma\int_{\mathbb{R}^N} v_{\mathbf{O}_m}V^2_{O_{m+1}}. \end{aligned} \tag{4.8}$$

By (3.15) and the definition of D_{jk}, we have

$$\sum_{j=1}^{m}\sum_{k=1}^{N} c_{jk}\Big\langle D_{jk}, \binom{U_{O_{m+1}}}{V_{O_{m+1}}}\Big\rangle \le Ce^{-\theta\varrho}\sum_{j=1}^{m} w(|O_{m+1}-O_j|). \tag{4.9}$$

Moreover, by Lemma 4.1, we find

$$
\begin{aligned}
&\frac{\mu}{2}\int_{\mathbb{R}^N}(v_{\mathbf{O}_m}U_{O_{m+1}}^2+2u_{\mathbf{O}_m}U_{O_{m+1}}V_{O_{m+1}})+\gamma\int_{\mathbb{R}^N}v_{\mathbf{O}_m}V_{O_{m+1}}^2\\
&=\frac{\mu}{2}\int_{\mathbb{R}^N}(V_{\mathbf{O}_m}U_{O_{m+1}}^2+\psi_{\mathbf{O}_m}U_{O_{m+1}}^2+2U_{\mathbf{O}_m}U_{O_{m+1}}V_{O_{m+1}}+2\phi_{\mathbf{O}_m}U_{O_{m+1}}V_{O_{m+1}})\\
&\quad+\gamma\int_{\mathbb{R}^N}(V_{\mathbf{O}_m}V_{O_{m+1}}^2+\psi_{\mathbf{O}_m}V_{O_{m+1}}^2)\\
&=\big(\gamma\beta^3+\frac{3\mu}{2}\alpha^2\beta\big)\int_{\mathbb{R}^N}\sum_{j=1}^m w_{O_{m+1}}^2 w_{O_j}+\mu\int_{\mathbb{R}^N}U_{O_{m+1}}V_{O_{m+1}}\phi_{\mathbf{O}_m}\\
&\quad+\int_{\mathbb{R}^N}\big(\frac{\mu}{2}U_{O_{m+1}}^2+\gamma V_{O_{m+1}}^2\big)\psi_{\mathbf{O}_m}\\
&=\big(\gamma\beta^3+\frac{3\mu}{2}\alpha^2\beta\big)\gamma_1\sum_{j=1}^m w(|O_{m+1}-O_j|)+\mu\int_{\mathbb{R}^N}U_{O_{m+1}}V_{O_{m+1}}\phi_{\mathbf{O}_m}\\
&\quad+\int_{\mathbb{R}^N}\big(\frac{\mu}{2}U_{O_{m+1}}^2+\gamma V_{O_{m+1}}^2\big)\psi_{\mathbf{O}_m}+O(e^{-\theta\varrho})\sum_{j=1}^m w(|O_{m+1}-O_j|).
\end{aligned}
$$

Therefore it follows from (4.8) that

$$
\begin{aligned}
&J\Big(\binom{u_{\mathbf{O}_m}}{v_{\mathbf{O}_m}}+\binom{U_{O_{m+1}}}{V_{O_{m+1}}}\Big)\\
&\le J\binom{u_{\mathbf{O}_m}}{v_{\mathbf{O}_m}}+I(U_{O_{m+1}},V_{O_{m+1}})+\frac{\epsilon}{2}\int_{\mathbb{R}^N}(P(x)U_{O_{m+1}}^2+Q(x)V_{O_{m+1}}^2)\\
&\quad-\mu\int_{\mathbb{R}^N}U_{O_{m+1}}V_{O_{m+1}}\phi_{\mathbf{O}_m}-\int_{\mathbb{R}^N}\big(\frac{\mu}{2}U_{O_{m+1}}^2+\gamma V_{O_{m+1}}^2\big)\psi_{\mathbf{O}_m}\\
&\quad-\big(\gamma\beta^3+\frac{3\mu}{2}\alpha^2\beta\big)\gamma_1\sum_{i=1}^k w(|O_{m+1}-O_j|)+O\Big(e^{-\theta\varrho}\sum_{i=1}^k w(|O_{m+1}-O_j|)\Big).
\end{aligned}
\tag{4.10}
$$

By the systems (1.14) and (2.16), we see that

$$
\begin{aligned}
&\mu\int_{\mathbb{R}^N}U_{O_{m+1}}V_{O_{m+1}}\phi_{\mathbf{O}_m}+\int_{\mathbb{R}^N}\big(\frac{\mu}{2}U_{O_{m+1}}^2+\gamma V_{O_{m+1}}^2\big)\psi_{\mathbf{O}_m}\\
&=\int_{\mathbb{R}^N}(-\Delta U_{O_{m+1}}+U_{O_{m+1}})\phi_{\mathbf{O}_m}+\int_{\mathbb{R}^N}(-\Delta V_{O_{m+1}}+V_{O_{m+1}})\psi_{\mathbf{O}_m}\\
&=\int_{\mathbb{R}^N}(-\Delta\phi_{\mathbf{O}_m}+\phi_{\mathbf{O}_m})U_{O_{m+1}}+\int_{\mathbb{R}^N}(-\Delta\psi_{\mathbf{O}_m}+\psi_{\mathbf{O}_m})V_{O_{m+1}}\\
&=\int_{\mathbb{R}^N}\begin{pmatrix}\sum_{j=1}^m\sum_{k=1}^N c_{jk}D_{jk,1}-\mu\Big(\sum_{j=1}^m U_{O_j}V_{O_j}-u_{\mathbf{O}_m}v_{\mathbf{O}_m}\Big)\\ \sum_{j=1}^m\sum_{k=1}^N c_{jk}D_{jk,2}-\frac{\mu}{2}\Big(\sum_{j=1}^m U_{O_j}^2-u_{\mathbf{O}_m}^2\Big)-\gamma\Big(\sum_{j=1}^m V_{O_j}^2-v_{\mathbf{O}_m}^2\Big)\end{pmatrix}\begin{pmatrix}U_{O_{m+1}}\\ V_{O_{m+1}}\end{pmatrix}\\
&\quad-\int_{\mathbb{R}^N}\begin{pmatrix}\epsilon P(x)u_{\mathbf{O}_m}\\ \epsilon Q(x)v_{\mathbf{O}_m}\end{pmatrix}\begin{pmatrix}U_{O_{m+1}}\\ V_{O_{m+1}}\end{pmatrix}.
\end{aligned}
$$

Using (2.19), we have

$$\Big|\int_{\mathbb{R}^N}\epsilon P(x)u_{\mathbf{O}_m}U_{O_{m+1}}\Big| = \Big|\int_{\mathbb{R}^N}\epsilon P(x)\Big(\sum_{j=1}^{m}U_{O_j}+\phi_{\mathbf{O}_m}\Big)U_{O_{m+1}}\Big|$$

$$\le Ce^{-\theta\varrho}\int_{\mathbb{R}^N}|P(x)|\sum_{j=1}^{m}e^{-|x-O_j|}U_{O_{m+1}}+C\|\phi_{\mathbf{O}_m}\|_*\int_{\mathbb{R}^N}|P(x)|\sum_{j=1}^{m}e^{-\sigma|x-O_j|}U_{O_{m+1}}$$

$$\le Ce^{-\theta\varrho}\sum_{j=1}^{m}\int_{\mathbb{R}^N}|P(x)|e^{-\sigma|x-O_j|}U_{O_{m+1}},$$

and similarly

$$\Big|\int_{\mathbb{R}^N}\epsilon Q(x)v_{\mathbf{O}_m}V_{O_{m+1}}\Big| \le Ce^{-\theta\varrho}\sum_{j=1}^{m}\int_{\mathbb{R}^N}|Q(x)|e^{-\sigma|x-O_j|}V_{O_{m+1}}.$$

Moreover, applying (2.9) and by direct computations we can check that

$$\Big|\int_{\mathbb{R}^N}\Big(\sum_{j=1}^{m}U_{O_j}V_{O_j}-u_{\mathbf{O}_m}v_{\mathbf{O}_m}\Big)U_{O_{m+1}}\Big| \le Ce^{-\theta\varrho}\sum_{j=1}^{m}w(|O_{m+1}-O_j|)$$

and

$$\Big|\int_{\mathbb{R}^N}\Big[\big(\sum_{j=1}^{m}U_{O_j}^2-u_{\mathbf{O}_m}^2\big)+\big(\sum_{j=1}^{m}V_{O_j}^2-v_{\mathbf{O}_m}^2\big)\Big]V_{O_{m+1}}\Big| \le Ce^{-\theta\varrho}\sum_{j=1}^{m}w(|O_{m+1}-O_j|),$$

since $\sigma > \frac{1}{2}$.

So from (4.9), (4.10) and the above estimates, one has

$$\begin{aligned}
&J\Big(\binom{u_{\mathbf{O}_m}}{v_{\mathbf{O}_m}}+\binom{U_{O_{m+1}}}{V_{O_{m+1}}}\Big)\\
&\le \mathcal{R}_m+I(U,V)+\frac{\epsilon}{2}\int_{\mathbb{R}^N}(P(x)U_{O_{m+1}}^2+Q(x)V_{O_{m+1}}^2)\\
&\quad-(\gamma\beta^3+\frac{3\mu}{2}\alpha^2\beta)\gamma_1\sum_{j=1}^{m}w(|O_{m+1}-O_j|)\\
&\quad+C\Big\{e^{-\theta\varrho}\sum_{j=1}^{m}\int_{\mathbb{R}^N}\Big[e^{-\sigma|x-O_j|}\big(|P(x)U_{O_{m+1}}+|Q(x)|V_{O_{m+1}}\big)+w(|O_{m+1}-O_j|)\Big]\Big\}.
\end{aligned} \tag{4.11}$$

As a result, we have

$$
\begin{aligned}
&J\binom{u_{\mathbf{O}_{m+1}}}{v_{\mathbf{O}_{m+1}}}\\
&\leq \mathcal{R}_m + I(U,V) + \frac{\epsilon}{2}\int_{\mathbb{R}^N}(P(x)U^2_{O_{m+1}} + Q(x)V^2_{O_{m+1}})\\
&\quad -(\gamma\beta^3 + \frac{3\mu}{2}\alpha^2\beta)\gamma_1\sum_{j=1}^{m} w(|O_{m+1}-O_j|)\\
&\quad +C\Big\{e^{-\theta\varrho}\sum_{j=1}^{m}\int_{\mathbb{R}^N}\Big[e^{-\sigma|x-O_j|}\big(|P(x)|U_{O_{m+1}} + |Q(x)|V_{O_{m+1}}\big) + w(|O_{m+1}-O_j|)\Big]\\
&\qquad +\Big(\int_{\mathbb{R}^N}\epsilon(|P(x)|U_{O_{m+1}} + |Q(x)|V_{O_{m+1}})\Big)^2\\
&\qquad +\epsilon^2\int_{\mathbb{R}^N}\big(|P(x)|^2U^2_{O_{m+1}} + |Q(x)|^2V^2_{O_{m+1}}\big)\Big\}.
\end{aligned} \tag{4.12}
$$

Since we assume that $|O^n_{m+1}| \to \infty$, we deduce that

$$
\begin{aligned}
&\epsilon\int_{\mathbb{R}^N}(P(x)U^2_{O_{m+1}} + Q(x)V^2_{O_{m+1}}) + C\Big\{\epsilon^2\int_{\mathbb{R}^N}(|P(x)|^2U^2_{O_{m+1}} + |Q(x)|^2V^2_{O_{m+1}})\\
&+ e^{-\theta\varrho}\sum_{j=1}^{m}\int_{\mathbb{R}^N}(|P(x)|e^{-\sigma|x-O_j|}U_{O_{m+1}} + |Q(x)|e^{-\sigma|x-O_j|}V_{O_{m+1}})\\
&+ \Big(\epsilon\int_{\mathbb{R}^N}(|P(x)|U_{O_{m+1}} + |Q(x)|V_{O_{m+1}})\Big)^2\Big\} \to 0,\ \text{as}\, n\to\infty
\end{aligned}
$$

and

$$
-(\gamma\beta^3 + \frac{3\mu}{2}\alpha^2\beta)\gamma_1\sum_{j=1}^{m} w(|O_{m+1}-O_j|) + O(e^{-\theta\varrho})\sum_{j=1}^{m} w(|O_{m+1}-O_j|) < 0.
$$

Thus we can deduce

$$
\mathcal{R}_{m+1} \leq \mathcal{R}_m + I(U,V). \tag{4.13}
$$

On the other hand, by the assumption $\mathcal{R}_m$ can be attained at $(\bar{O}_1,\cdots,\bar{O}_m)$. So there exists other point O_{m+1} which is far away from the m points and determined later. Let us consider the solution concentrating at the point $(\bar{O}_1,\cdots,\bar{O}_m,O_{m+1})$. We denote the solution by $(u_{\bar{\mathbf{O}}_m,O_{m+1}}, v_{\bar{\mathbf{O}}_m,O_{m+1}})$. By similar argument as above, using the estimate (3.27) instead of (3.26), we have

$$
\begin{aligned}
&J\begin{pmatrix} u_{\bar{\mathbf{O}}_m,O_{m+1}} \\ v_{\bar{\mathbf{O}}_m,O_{m+1}} \end{pmatrix} \\
&= J(u_{\bar{\mathbf{O}}_m}, v_{\bar{\mathbf{O}}_m}) + I(U,V) + \frac{\epsilon}{2}\int_{\mathbb{R}^N}(P(x)U^2_{O_{m+1}} + Q(x)V^2_{O_{m+1}}) \\
&\quad -\big(\gamma\beta^3 + \frac{3\mu}{2}\alpha^2\beta\big)\gamma_1\sum_{j=1}^m w(|O_{m+1} - \bar{O}_j|) \\
&\quad +C\Big\{e^{-\theta\varrho}\sum_{j=1}^m\int_{\mathbb{R}^N}|P(x)|e^{-\sigma|x-\bar{O}_j|}U_{O_{m+1}} \\
&\qquad +e^{-\theta\varrho}\sum_{j=1}^m\int_{\mathbb{R}^N}|Q(x)|e^{-\sigma|x-\bar{O}_j|}V_{O_{m+1}} + e^{-\theta\varrho}\sum_{j=1}^m w(|O_{m+1} - \bar{O}_j|) \\
&\qquad +\Big(\epsilon\sum_{j=1}^m\Big(\int_{B_{\frac{\varrho}{2}}(\bar{O}_j)}|P(x)|^2U^2_{O_{m+1}}\Big)^{\frac{1}{2}}\Big)^2 + \Big(\epsilon\sum_{j=1}^m\Big(\int_{B_{\frac{\varrho}{2}}(\bar{O}_j)}|Q(x)|^2U^2_{O_{m+1}}\Big)^{\frac{1}{2}}\Big)^2 \\
&\qquad +\Big(\epsilon\int_{\mathbb{R}^N}(|P(x)|U_{O_{m+1}} + |Q(x)|V_{O_{m+1}})\Big)^2\Big\}.
\end{aligned}
\tag{4.14}
$$

By (H_3), choosing $\sigma > \tau$ and O_{m+1} such that $|O_{m+1}| >> \frac{\max_{j=1}^m |\bar{O}_j| - \ln\epsilon}{\sigma-\tau}$, we can get that

$$
\begin{aligned}
&\frac{\epsilon}{2}\int_{\mathbb{R}^N}(P(x)U^2_{O_{m+1}} + Q(x)V^2_{O_{m+1}}) - \big(\gamma\beta^3 + \frac{3\mu}{2}\alpha^2\beta\big)\gamma_1\sum_{j=1}^m w(|O_{m+1} - \bar{O}_j|) \\
&\quad + C\Big\{e^{-\theta\varrho}\sum_{j=1}^m\int_{\mathbb{R}^N}(|P(x)|e^{-\sigma|x-\bar{O}_j|}U_{O_{m+1}} + |Q(x)|e^{-\sigma|x-\bar{O}_j|}V_{O_{m+1}}) \\
&\qquad + \Big(\epsilon\int_{\mathbb{R}^N}(|P(x)|U_{O_{m+1}} + |Q(x)|V_{O_{m+1}})\Big)^2 + e^{-\theta\varrho}\sum_{j=1}^m w(|O_{m+1} - \bar{O}_j|) \\
&\qquad + \Big(\epsilon\sum_{j=1}^m\Big(\int_{B_{\frac{\varrho}{2}}(\bar{O}_j)}|P(x)|^2U^2_{O_{m+1}}\Big)^{\frac{1}{2}}\Big)^2 + \Big(\epsilon\sum_{j=1}^m\Big(\int_{B_{\frac{\varrho}{2}}(\bar{O}_j)}|Q(x)|^2U^2_{O_{m+1}}\Big)^{\frac{1}{2}}\Big)^2\Big\} \\
&> C\epsilon e^{-\tau|O_{m+1}|} - C\sum_{j=1}^m e^{-\sigma|O_{m+1}-\bar{O}_j|} > 0.
\end{aligned}
$$

As s consequence,

$$
\mathcal{R}_{m+1} \geq J\begin{pmatrix} u_{\bar{\mathbf{O}}_m,O_{m+1}} \\ v_{\bar{\mathbf{O}}_m,O_{m+1}} \end{pmatrix} > \mathcal{R}_m + I(U,V),
$$

which contradicts to (4.13). Then $\mathcal{R}_{m+1}$ can be attained at finite points in Ω_{m+1}.

Moreover, from the proof above, we can infer that

$$
\mathcal{R}_{m+1} \geq \mathcal{R}_m + I(U,V). \tag{4.15}
$$

□

Next we have the following proposition:

Proposition 4.3. *The maximization problem*

$$\max_{\boldsymbol{O}\in\bar{\Omega}_m} \mathcal{N}(\boldsymbol{O}) \tag{4.16}$$

has a solution $\boldsymbol{O}\in\Omega_m^0$*, i.e., the interior of* Ω_m.

Proof. We prove it by contradiction. If $\bar{\mathbf{O}}_m=(\bar{O}_1,\cdots,\bar{O}_m)\in\partial\Omega_m$, then there exists (j,k) such that $|\bar{O}_j-\bar{O}_k|=\varrho$. Without loss of generality, we assume that $(j,k)=(j,m)$. It follows from (4.12) that

$$\begin{aligned}
\mathcal{R}_m &= J\binom{u_{\bar{\mathbf{O}}_m}}{v_{\bar{\mathbf{O}}_m}}\\
&\le \mathcal{R}_{m-1}+I(U,V)+\frac{\epsilon}{2}\int_{\mathbb{R}^N}(P(x)U^2_{O_{m+1}}+Q(x)V^2_{O_{m+1}})\\
&\quad -(\gamma\beta^3+\frac{3\mu}{2}\alpha^2\beta)\gamma_1\sum_{i=1}^{m-1}w(|\bar{O}_m-\bar{O}_i|)\\
&\quad +C\Big(e^{-\theta\varrho}\sum_{i=1}^{m-1}\int_{\mathbb{R}^N}\Big[e^{-\sigma|x-\bar{O}_i|}(|P(x)|U_{\bar{O}_m}+|Q(x)|V_{\bar{O}_m})+w(|\bar{O}_m-\bar{O}_i|)\Big]\\
&\quad +\epsilon^2\int_{\mathbb{R}^N}(|P(x)|^2U^2_{\bar{O}_m}+|Q(x)|^2V^2_{\bar{O}_m})+\Big(\int_{\mathbb{R}^N}\epsilon(|P(x)|U_{\bar{O}_m}+|Q(x)|V_{\bar{O}_m})\Big)^2\Big).
\end{aligned} \tag{4.17}$$

By the definition of the configuration set, we observe that given a ball of size ϱ, there are at most $C_N:=6^N$ number of non-overlapping balls of size ϱ surrounding this ball. Using $|\bar{O}_j-\bar{O}_m|=\varrho$, we have

$$\sum_{i=1}^{m-1}w(|\bar{O}_m-\bar{O}_i|)=w(|\bar{O}_m-\bar{O}_j|)+\sum_{i\neq j}w(|\bar{O}_m-\bar{O}_i|)\le w(\varrho)+Ce^{-\varrho},$$

since

$$\begin{aligned}
\sum_{i\neq j}w(|\bar{O}_m-\bar{O}_i|) &\le Ce^{-\varrho}+C_Ne^{-\varrho-\frac{1}{2}\varrho}+\cdots+(C_N)^ie^{\varrho-\frac{i}{2}\varrho}+\cdots\\
&\le Ce^{-\varrho}\sum_{i=0}^{\infty}e^{i\ln C_N-\frac{1}{2}\varrho}\le Ce^{-\varrho},
\end{aligned}$$

if $C_N<e^{\frac{\varrho}{2}}$ and ϱ large enough.

So,

$$\begin{aligned}
\mathcal{R}_m &\le \mathcal{R}_{m-1}+I(U,V)+C\epsilon-(\gamma\beta^3+\frac{3\mu}{2}\alpha^2\beta)\gamma_1w(\varrho)-(\gamma\beta^3+\frac{3\mu}{2}\alpha^2\beta)\gamma_1e^{-\varrho}\\
&\quad +O(e^{-\theta\varrho})w(\varrho)+O(e^{-(1+\theta)\varrho})\\
&< \mathcal{R}_{m-1}+I(U,V),
\end{aligned}$$

which yields a contradiction to Lemma 4.2. Then the proof is complete. □

Now we apply all the results obtained before to prove Theorem 1.1.

Proof of Theorem 1.1. By Proposition 2.4, there exists ϱ_0 such that for $\varrho>\varrho_0$, we have a $C^1\times C^1$ map $(\phi_{\mathbf{O}^0},\psi_{\mathbf{O}^0})$ for any $\mathbf{O}^0\in\Omega_m$ such that

$$G\begin{pmatrix} U_{\mathbf{O}^0}+\phi_{\mathbf{O}^0} \\ V_{\mathbf{O}^0}+\psi_{\mathbf{O}^0} \end{pmatrix} = \sum_{s=1}^{m}\sum_{l=1}^{N} c_{sl} D_{sl}, \ \Big\langle \begin{pmatrix} \phi_{\mathbf{O}^0} \\ \psi_{\mathbf{O}^0} \end{pmatrix}, D_{sl} \Big\rangle = 0, \tag{4.18}$$

for some constants $\{c_{sl}\} \in \mathbb{R}^{m\times N}$.

From Proposition 4.3, there is $\mathbf{O}^0 \in \Omega_m^0$ that achieves the maximum of $\mathcal{N}(\mathbf{O})$ given by Lemma 4.2. Letting $\begin{pmatrix} u_{\mathbf{O}^0} \\ v_{\mathbf{O}^0} \end{pmatrix} = \begin{pmatrix} U_{\mathbf{O}^0}+\phi_{\mathbf{O}^0} \\ V_{\mathbf{O}^0}+\psi_{\mathbf{O}^0} \end{pmatrix}$, we have

$$D_{O_{jk}}|_{O_j=O_j^0}\mathcal{N}(\mathbf{O}^0) = 0, \ j=1,\cdots,m, k=1,\cdots,N.$$

Hence we have

$$\begin{aligned} &\int_{\mathbb{R}^N}\Big(\nabla u_{\mathbf{O}}\nabla\frac{\partial(U_{\mathbf{O}}+\phi_{\mathbf{O}})}{\partial O_{jk}}\Big|_{O_j=O_j^0} + (1+\epsilon P(x))u_{\mathbf{O}}\frac{\partial(U_{\mathbf{O}}+\phi_{\mathbf{O}})}{\partial O_{jk}}\Big|_{O_j=O_j^0}\Big) \\ &+\int_{\mathbb{R}^N}\Big(\nabla v_{\mathbf{O}}\nabla\frac{\partial(V_{\mathbf{O}}+\psi_{\mathbf{O}})}{\partial O_{jk}}\Big|_{O_j=O_j^0} + (1+\epsilon Q(x))v_{\mathbf{O}}\frac{\partial(V_{\mathbf{O}}+\psi_{\mathbf{O}})}{\partial O_{jk}}\Big|_{O_j=O_j^0}\Big) \\ &-\mu\int_{\mathbb{R}^N} u_{\mathbf{O}}v_{\mathbf{O}}\frac{\partial(U_{\mathbf{O}}+\phi_{\mathbf{O}})}{\partial O_{jk}}\Big|_{O_j=O_j^0} \\ &-\frac{\mu}{2}\int_{\mathbb{R}^N} u_{\mathbf{O}}^2\frac{\partial(V_{\mathbf{O}}+\psi_{\mathbf{O}})}{\partial O_{jk}}\Big|_{O_j=O_j^0} - \gamma\int_{\mathbb{R}^N} v_{\mathbf{O}}^2\frac{\partial(V_{\mathbf{O}}+\psi_{\mathbf{O}})}{\partial O_{jk}}\Big|_{O_j=O_j^0} = 0, \end{aligned}$$

which yields that

$$\sum_{s=1}^{m}\sum_{l=1}^{N} c_{sl}\int_{\mathbb{R}^N}\Big(D_{sl,1}\frac{\partial(U_{\mathbf{O}}+\phi_{\mathbf{O}})}{\partial O_{jk}}\Big|_{O_j=O_j^0} + D_{sl,2}\frac{\partial(V_{\mathbf{O}}+\psi_{\mathbf{O}})}{\partial O_{jk}}\Big|_{O_j=O_j^0}\Big) = 0. \tag{4.19}$$

We claim that (4.19) is a diagonally dominant system. Indeed, since

$$\int_{\mathbb{R}^N}(\phi_{\mathbf{O}}D_{sl,1}+\psi_{\mathbf{O}}D_{sl,2})|_{O_j=O_j^0} = 0,$$

we have

$$\int_{\mathbb{R}^N}\Big(D_{sl,1}\frac{\partial\phi_{\mathbf{O}}}{\partial O_{jk}}\Big|_{O_j=O_j^0} + D_{sl,2}\frac{\partial\psi_{\mathbf{O}}}{\partial O_{jk}}\Big|_{O_j=O_j^0}\Big) = 0, \ \text{if } s\neq j.$$

For $s=j$, we can get that

$$\begin{aligned} &\Big|\int_{\mathbb{R}^N}\Big(D_{sl,1}\frac{\partial\phi_{\mathbf{O}}}{\partial O_{jk}}\Big|_{O_j=O_j^0} + D_{sl,2}\frac{\partial\psi_{\mathbf{O}}}{\partial O_{jk}}\Big|_{O_j=O_j^0}\Big)\Big| \\ &=\Big|-\int_{\mathbb{R}^N}\Big(\phi_{\mathbf{O}}\frac{\partial D_{sl,1}}{\partial O_{jk}}\Big|_{O_j=O_j^0} + \psi_{\mathbf{O}}\frac{\partial D_{sl,2}}{\partial O_{jk}}\Big|_{O_j=O_j^0}\Big)\Big| \le C\|(\phi_{\mathbf{O}},\psi_{\mathbf{O}})\|_* \le Ce^{-\theta\varrho}. \end{aligned}$$

For $s\neq j$, we have

$$\begin{aligned} &\Big|\int_{\mathbb{R}^N}\Big(D_{sl,1}\frac{\partial U_{\mathbf{O}}}{\partial O_{jk}} + D_{sl,2}\frac{\partial V_{\mathbf{O}}}{\partial O_{jk}}\Big)\Big| \\ &\le C\Big|\int_{\mathbb{R}^N}\Big(\frac{\partial U(x-O_s)}{\partial x_l}\frac{\partial U(x-O_j)}{\partial x_k} + \frac{\partial V(x-O_s)}{\partial x_l}\frac{\partial V(x-O_j)}{\partial x_k}\Big)\Big| \\ &\le C\int_{\mathbb{R}^N} e^{-|x-O_s|}e^{-|x-O_j|} \le Ce^{-\frac{|O_j-O_s|}{2}} \le Ce^{-\frac{\varrho}{2}}. \end{aligned}$$

For $s = j$, letting $y = x - O_j$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big(D_{sl,1} \frac{\partial U_{\mathbf{O}}}{\partial O_{jk}} + D_{sl,2} \frac{\partial V_{\mathbf{O}}}{\partial O_{jk}} \Big) \\
&= \int_{\mathbb{R}^N} \Big(\zeta_j \frac{\partial U(x-O_j)}{\partial x_l} \frac{\partial U(x-O_j)}{\partial O_{jk}} + \zeta_j \frac{\partial V(x-O_j)}{\partial x_l} \frac{\partial V(x-O_j)}{\partial O_{jk}} \Big) \\
&= -\int_{B_{\frac{\varrho^2}{2(\varrho+1)}}(0)} \Big(\zeta_j(y+O_j) \frac{\partial U(y)}{\partial y_l} \frac{\partial U(y)}{\partial y_k} + \zeta_j(y+O_j) \frac{\partial V(y)}{\partial y_l} \frac{\partial V(y)}{\partial y_k} \Big) \\
&= -\delta_{lk} \int_{\mathbb{R}^N} \Big((\frac{\partial U}{\partial y_l})^2 + (\frac{\partial V}{\partial y_k})^2 \Big) + O(e^{\theta \varrho}).
\end{aligned} \tag{4.20}$$

Thus, from each (s, l), the off-diagonal term gives

$$\begin{aligned}
&\sum_{s \neq j} \int_{\mathbb{R}^N} \Big(D_{sl,1} \frac{\partial (U_{\mathbf{O}} + \phi_{\mathbf{O}})}{\partial O_{jk}} \Big|_{O_j = O_j^0} + D_{sl,2} \frac{\partial (V_{\mathbf{O}} + \psi_{\mathbf{O}})}{\partial O_{jk}} \Big|_{O_j = O_j^0} \Big) \\
&+ \sum_{s = j, l \neq k} \int_{\mathbb{R}^N} \Big(D_{sl,1} \frac{\partial (U_{\mathbf{O}} + \phi_{\mathbf{O}})}{\partial O_{jk}} \Big|_{O_j = O_j^0} + D_{sl,2} \frac{\partial (V_{\mathbf{O}} + \psi_{\mathbf{O}})}{\partial O_{jk}} \Big|_{O_j = O_j^0} \Big) \\
&= O(e^{-\frac{\varrho}{2}}) + O(e^{-\theta \varrho}) = O(e^{-\theta \varrho})
\end{aligned} \tag{4.21}$$

for some $\theta > 0$. So from (4.20) and (4.21), we see that $c_{sl} = 0$ for $s = 1, \cdots, m, l = 1, \cdots, N$. Hence $(u_{\mathbf{O}^0}, v_{\mathbf{O}^0})$ is a solution of (1.1). By our construction and the maximum principle, it is easy to see that $u_{\mathbf{O}^0} > 0$ and $v_{\mathbf{O}^0} > 0$. This concludes the proof of Theorem 1.1. □

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