

LONG TIME BEHAVIOR OF DYNAMIC SOLUTION TO PEIERLS–NABARRO DISLOCATION MODEL*

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Abstract. In this paper we study the relaxation process of the Peierls-Nabarro dislocation model, which is a gradient flow with a singular nonlocal energy and a double well potential describing how the materials relax to its equilibrium with the presence of a dislocation. We prove the dynamic solution to the Peierls-Nabarro model will converge exponentially to a shifted steady profile which is uniquely determined.

Key words. Global attractor, spectral gap, nonlocal Ginzburg Landau, Omega limit set with vanishing dissipation.

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1. Introduction.

Motivation and Problem. Materials defects such as dislocations are important line defects in crystalline materials and they play essential roles in understanding materials properties like plastic deformation [28, 24].

As a line defect with line direction, a dislocation has a small region (called the dislocation core region) of heavily distorted atomistic structures with shear jump discontinuity across a slip plane $\{x \in \mathbb{R}, y = 0\}$; as illustrated in Fig 1. In this paper, we focus on a straight edge dislocation case, and suppose $u : \mathbb{R}^2 \rightarrow \mathbb{R}$ is the shear displacement of materials along the direction of the Burgers vector $\mathbf{b} = (b, 0)$.

Unlike the classical dislocation model [28, 24, 41], which assumes a uniform shear jump discontinuity across the slip plane, the true increment of the shear jump u at each position x is not simply a step function but depends on the atomistic misfit interaction across the slip plane of the dislocation [36, 39]. The Peierls-Nabarro (PN) model is used to describe the detailed structures in the dislocation core by introducing a nonlinear misfit potential $F(u)$ depending on the shear jump discontinuity u on the slip plane.

The simplest solvable nonlinear potential is introduced by Frenkel in 1926 to describe the misfit energy of the halite [18]. Setting some physical constants to be 1, under some symmetric assumption, a typical multi-well potential is

$$\begin{aligned} F(u) &:= \frac{1}{\pi^2}(1 + \cos(\pi u)), \\ f(u) &:= F'(u) = -\frac{1}{\pi} \sin(\pi u), \quad f'(u) = F''(u) = -\cos(\pi u), \end{aligned} \tag{1.1}$$

which phenomenologically reflects the lattice periodicity. To include the magnitude of Burgers vector, F with local minimums at $\pm \frac{b}{4}$ is $F(u) = \frac{c}{\pi^2}(1 + \cos(\frac{4\pi u}{b}))$. We remark there are also other kinds of double well potentials with two local minimums at ± 1 ; see (1.16). Due to the presence of an edge dislocation on the slip plane $\{x \in \mathbb{R}, y = 0\}$,

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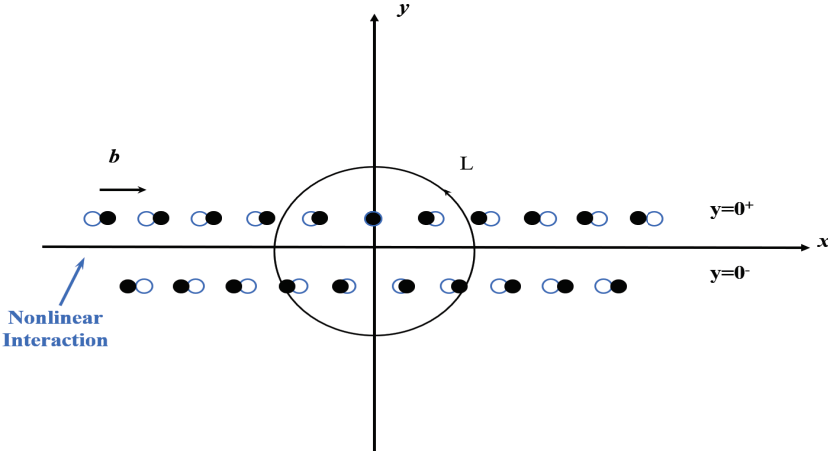


FIG. 1. Illustration of a straight edge dislocation at origin with the Burgers vector $\mathbf{b} = (b, 0)$ and the slip plane $\{x \in \mathbb{R}, y = 0\}$.

the total increment of the shear jump discontinuity from $-\infty$ to $+\infty$ is b , i.e., the magnitude of the Burgers vector; see Remark 1.

REMARK 1. In general, a Burgers vector, which indicates the magnitude and the direction of the lattice distortion resulting from a dislocation, is defined by a loop integration $\mathbf{b} := \oint_L \mathbf{d}\mathbf{u}$. Here L is a loop in x - y plane enclosing the dislocation line with a counterclockwise orientation; see Figure 1. Notice in the simplified case, u is the shear displacement, $\mathbf{b} = (b, 0)$ and $b = \oint_L du$. In two dimensions, we assume anti-symmetry with respect to the slip plane $\{x \in \mathbb{R}, y = 0\}$, i.e. $u^+(x, 0^+) = -u^-(x, 0^-)$. Due to Cauchy’s integral formula, the loop integrations, for the upper and lower half-spaces respectively, are both zero. Then the loop integration is reduced to x -axis and is given by $2 \int_\alpha^\beta u'(x, 0^+) dx$, where α and β are intersection points of the loop with x -axis. Therefore in the PN model, the distributional Burgers vector depends on the endpoints α, β we choose. However, since the total increment from $-\infty$ to $+\infty$ equals b , we have $2[u(+\infty, 0^+) - u(-\infty, 0^+)] = b$. If the bi-states at far fields are in a symmetric form, i.e. $u(+\infty, 0^+) = -u(-\infty, 0^+)$, the magnitude of the Burgers vector naturally gives the boundary condition of u at far fields, $u(\pm\infty) = \pm \frac{b}{4}$.

To find out the shear displacement u at each position x , the equilibrium of the PN model for a single edge dislocation is obtained by minimizing the total energy, including the elastic bulk energy E_{els} and the misfit interface energy $\int_{\mathbb{R}} F(u) dx$. By the Dirichlet to Neumann map and the elastic extension [20], the elastic bulk energy in the upper/lower plane E_{els} can be reduced equivalently to slip plane, which therefore becomes a nonlocal elastic energy on slip plane $\{x \in \mathbb{R}, y = 0\}$, $\int_{\mathbb{R}} \frac{1}{2} |(-\partial_{xx})^{\frac{1}{4}} u|^2 dx$; see (1.3) below. With this equivalence, from now on, we drop the second variable y in $u(x, y)$ and focus on the shear displacement on slip plane $u(x) := u(x, 0^+)$.

Denote $H^s(\mathbb{R})$ as the fractional Sobolev space with norm denoted as $\|\cdot\|_s$. Denote $\|\cdot\|$ as the standard $L^2(\mathbb{R})$ norm. We first give a singular integral definition, which is equivalent to the one using Fourier transformations [25]. For $0 < s < 1$, define the fractional Laplace operator L_s from $D(L_s) = H^{2s}(\mathbb{R}) \subset L^2(\mathbb{R})$ to $L^2(\mathbb{R})$

$$L_s v := (-\partial_{xx})^s v := C_s \text{P. V.} \int_{\mathbb{R}} \frac{v(x) - v(y)}{|x - y|^{1+2s}} dy, \tag{1.2}$$

where C_s is a normalizing constant to guarantee the symbol of the resulting operator is $|\xi|^{2s}$. Especially when $s = \frac{1}{2}$, $C_s = \frac{1}{\pi}$. Although there are different equivalent definitions, we clarify we use the singular integral definition above in the whole paper. Let us first express formally the problem we are interested in. Define the nonlocal energy for the Peierls-Nabarro model

$$E(u) := \int_{\mathbb{R}} \frac{1}{2} |(-\partial_{xx})^{\frac{1}{2}} u|^2 dx + \int_{\mathbb{R}} F(u) dx. \quad (1.3)$$

Alternatively, we can rewrite the nonlocal energy using a singular kernel

$$\begin{aligned} E(u) &= \frac{1}{4\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(u(x) - u(y))^2}{|x - y|^2} dx dy + \int_{\mathbb{R}} F(u) dx \\ &= \frac{1}{4} \int_{\mathbb{R}} \int_{\mathbb{R}} J(x - y)(u(x) - u(y))^2 dx dy + \int_{\mathbb{R}} F(u) dx, \end{aligned} \quad (1.4)$$

where $J(z) = \frac{1}{\pi z^2}$ and we used the identity

$$\frac{1}{2} \int_{\mathbb{R}} u(-\partial_{xx})^{\frac{1}{2}} u dx = \frac{1}{4\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(u(x) - u(y))^2}{|x - y|^2} dx dy.$$

Then the dynamic Peierls-Nabarro model is the following Allen-Cahn gradient flow

$$\partial_t u = -\frac{\delta E(u)}{\delta u} = -(-\partial_{xx})^{\frac{1}{2}} u - f(u) = -\mathcal{A}u, \quad (1.5)$$

where the nonlocal nonlinear operator \mathcal{A} formally defined as

$$\mathcal{A}u := (-\partial_{xx})^{\frac{1}{2}} u + f(u) = L_{\frac{1}{2}} u + f(u). \quad (1.6)$$

Due to the presence of a dislocation, with a magnitude of the Burgers vector $b = 4$ in Remark 1, we are interested in solutions with far field boundary conditions

$$u(\pm\infty, t) = \pm 1. \quad (1.7)$$

The readers may see three main issues here. First, the displacement function u is bounded but not vanish at far field. How does this boundary condition (1.7) at far field remain as time evolving? Second, can the nonlocal operator $(-\partial_{xx})^{\frac{1}{2}}$ defined above on $H^1(\mathbb{R})$ be extended to a $L^\infty(\mathbb{R})$ function with boundary condition (1.7)? Third, the non-vanishing boundary conditions at far field lead to an infinite nonlocal elastic energy $\int_{\mathbb{R}} \frac{1}{2} |(-\partial_{xx})^{1/4} u|^2 dx$ on the slip plane (see footnote¹ below), as well as an infinite elastic bulk energy in the upper and lower space, which is equivalently connected to the nonlocal elastic energy; see a precise statement in the perturbed sense established [20] by introducing a concept of the elastic extension. This singularity in energy is analogous to the vortex singularity in fluid mechanics or a single electron in electromagnetism, which inspires us to define a perturbed energy with respect to a reference state, steady profile described below.

¹There exists $A > 0$ such that $u > \frac{1}{2}$ for $x > A$ while $u < -\frac{1}{2}$ for $x < -A$. Therefore

$$\frac{1}{4\pi} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{(u(x) - u(y))^2}{|x - y|^2} dx dy \geq \frac{1}{4\pi} \int_{x>A} \int_{y<-A} \frac{1}{2(x^2 + y^2)} dx dy = \infty.$$

We observe the typical bistable steady solution to (1.12), which will be used as a reference profile later. Assume ϕ is a steady solution to (1.12) satisfying

$$\mathcal{A}\phi = 0, \quad \phi(\pm\infty) = \pm 1. \tag{1.8}$$

Since ϕ is smooth enough, we remark the operator $(-\partial_{xx})^{\frac{1}{2}}$ acting on ϕ is equivalent to $(-\partial_{xx})^{\frac{1}{2}}\phi = H(\phi')$ (see footnote² below), where H is the Hilbert transform

$$(Hu)(x) := \frac{1}{\pi} \text{P. V.} \int_{-\infty}^{+\infty} \frac{u(y)}{x-y} dy.$$

Indeed, $\phi(x) = \frac{2}{\pi} \arctan(x)$ is one special solution with fixed center at zero, i.e. $\phi(0) = 0$. Notice $f(\frac{2}{\pi} \arctan(x)) = -\frac{1}{\pi} \sin(2 \arctan x) = -\frac{2}{\pi} \frac{x}{1+x^2}$ and $(-\partial_{xx})^{\frac{1}{2}}\phi(x) = H(\phi'(x)) = H(\frac{2}{\pi} \frac{1}{1+x^2}) = \frac{2}{\pi} \frac{x}{1+x^2}$. We can check

$$\mathcal{A}\phi = (-\partial_{xx})^{\frac{1}{2}}\phi + f(\phi) = 0, \tag{1.9}$$

and the decay rate at far fields

$$\phi(x) \sim \pm 1 - \frac{2}{\pi x}, \quad \text{as } x \rightarrow \pm\infty. \tag{1.10}$$

In this paper, we consider the long time behavior of the solution to the dynamic equation (1.5) with initial data u_0 such that $u_0(\pm\infty) = \pm 1$. Our goal is to prove there is x_0 such that as $t \rightarrow \infty$

$$u(x, t) \rightarrow \phi(x - x_0)$$

uniformly with exponential decay rate.

To make the infinity integrals meaningful, we define the perturbed energy as

$$\mathcal{E}(u) := \frac{1}{2} \int_{\mathbb{R}} |(-\Delta)^{1/4}(u - \phi)|^2 - (u - \phi)f(\phi) + F(u) dx. \tag{1.11}$$

The motivations of choosing the steady profile ϕ as the reference profile are (i) it gives a finite energy integral; (ii) it is natural to see the steady state ϕ is a stable minimizer of \mathcal{E} , i.e. the second variation of \mathcal{E} near ϕ is nonnegative; (iii) the decay properties (1.10) will be used to obtain L^2 compactness later in Section 2.3. Precisely, we will study the existence and long time behaviors of solutions to

$$\partial_t u = -\frac{\delta \mathcal{E}(u)}{\delta u} = -\mathcal{A}u \tag{1.12}$$

with initial data $u(x, 0) = u_0(x)$ satisfying

$$(i) \mathcal{E}(u_0) < +\infty; \tag{1.13}$$

(ii) there exist constants $a \leq b$ such that

$$\phi(x - b) \leq u_0(x) \leq \phi(x - a). \tag{1.14}$$

Thanks to the theory of analytic semigroup, we first validate this dynamic equation for u by proving the existence of the global classical solution to the perturbation with respect to the reference profile, $v := u - \phi$; see more details in Section 2.1.

²Since ϕ is uniformly bounded, only $y = x$ is the singular point in the singular integral definition (1.2). Therefore $(-\partial_{xx})^{\frac{1}{2}}\phi := \frac{1}{\pi} \text{P. V.} \int_{\mathbb{R}} \frac{\phi(x) - \phi(y)}{|x-y|^2} dy = \lim_{\varepsilon \rightarrow 0} \frac{1}{\pi} \int_{|y-x| > \varepsilon} \frac{\phi(x) - \phi(y)}{|x-y|^2} dy = \lim_{\varepsilon \rightarrow 0} \int_{|y-x| > \varepsilon} \frac{\phi'(y)}{x-y} dy = H(\phi')$ due to integration by parts.

Main Results and Related References. Below, we state the main result for uniform exponential convergence of the dynamic solution to the PN model to its equilibrium profile.

THEOREM 1.1. *Assume initial data $u_0(x) - \phi(x) \in H^{\frac{1}{2}}(\mathbb{R})$ then (1.12) has a unique global smooth solution $u(x, t)$ with specific regularities in (2.7). Furthermore, if u_0 satisfies (1.13) and (1.14), then there exist constants x_0, c and μ such that*

$$|u(x, t) - \phi(x - x_0)| \leq c \min\left\{\frac{1}{1 + |x|}, e^{-\mu t}\right\} \quad \text{for any } t > 0, x \in \mathbb{R}. \quad (1.15)$$

REMARK 2. We remark that given an initial data, the equilibrium profile is uniquely determined. Although the steady solutions to the static equation are unique upto translations, the dynamic solution to the dynamic problem is unique. Therefore Theorem 1.1 actually proves that as $t \rightarrow +\infty$ the limiting steady profile $\phi(x - x_0)$ is uniquely determined.

REMARK 3. We remark that the proof for the uniform convergence result above does not depend on the specific formulas of potential $F(u)$ and steady profile $\phi(x)$. Indeed, for a general potential $F \in C^{2,\alpha}(\mathbb{R})$ satisfying

$$F(v) > F(-1) = F(1) \quad \text{for } v \in (-1, 1), \quad F''(\pm 1) > 0 \quad (1.16)$$

[7, 8] obtained the existence of steady profile ϕ to (1.9) with the properties $\phi'(x) > 0$ and decay estimate (1.10). Any other assumptions for the behavior of F outside $[-1, 1]$ are not necessary because we will prove the Assumption (1.15), i.e. the initial data is sandwiched between two steady profiles, is persistent along time; see Proposition 2.5.

Below, we state the long time behavior result for the general potential $F(u)$ and delay the proof to the end of this paper.

THEOREM 1.2. *Assume $F \in C^{2,\alpha}(\mathbb{R})$ satisfies (1.16) and $\phi(x)$ is the corresponding steady profile to (1.9) with the properties $\phi'(x) > 0$ and (1.10). Then for initial data u_0 satisfying the same assumptions in Theorem 1.1, the unique global smooth solution $u(x, t)$ to (1.12) has the same uniform exponential convergence to its equilibrium profile, i.e. there exist constants x_1, c_1 and $\mu_1 > 0$ such that*

$$|u(x, t) - \phi(x - x_1)| \leq c_1 \min\left\{\frac{1}{1 + |x|}, e^{-\mu_1 t}\right\} \quad \text{for any } t > 0, x \in \mathbb{R}. \quad (1.17)$$

For the stationary solutions to the equilibrium PN model (1.8), [7] established the existence and uniqueness (upto a shift in x) of monotonic solutions by considering the corresponding local scalar problem by a harmonic extension; see also [8] for results of general nonlocal operators $(-\partial_{xx})^s, 0 < s < 1$. Recently, using a rearrangement method, [29] also obtained the existence and uniqueness of monotonic solutions and proved the monotonic solution is the global minimizer of the nonlocal Allen-Cahn energy (1.4) after a renormalization. To connect the nonlocal Allen-Cahn equation (1.5) to the true vector field solution rigorously, rather than the analogous scalar model, [20] prove the equivalence between the nonlocal problem and the corresponding extended problem by defining a perturbed elastic bulk energy and establishing the elastic extension analogue to the harmonic extension.

However, as far as we know the natural question proposed in the motivation subsection above has not been studied, i.e. whether the dynamic solution to (1.12) will converge exponentially to a uniquely determined steady profile as $t \rightarrow +\infty$. The difficulties are essentially the singularity in energy, the lack of uniform in time $H^1(\mathbb{R})$ bounds, as well as $L^2(\mathbb{R})$ estimates, and spectral gap analysis, which will be explained in details later.

Let us first compare with some results for dislocation models in larger scale, described by a dislocation density function. Analytic results such as well-posedness for the dislocation particle system, slow motion and concentration of transition layers are established in [13, 21, 14, 30, 31, 32]. More precisely, instead of (1.5), [21] chooses $v^\varepsilon(x, t) = u\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon^2}\right)$ and considers the corresponding equation for $v^\varepsilon(x, t)$ (with/without external stress)

$$\partial_t v^\varepsilon + \frac{1}{\varepsilon} \left[(-\partial_{xx})^{\frac{1}{2}} v^\varepsilon + \frac{1}{\varepsilon} F'(v^\varepsilon) \right] = 0 \quad (1.18)$$

with a well-prepared initial data

$$v_0^\varepsilon(x) = \frac{1 + (-1)^N}{2} + \sum_{i=1}^N \phi\left(\frac{x - h_i^0}{\varepsilon}\right), \quad (1.19)$$

where h_i^0 is the initial location of the i -th dislocation and there are totally N transition layers. They proved as $\varepsilon \rightarrow 0$, v^ε will converge to $v^0(t, x) = \frac{1+(-1)^N}{2} + \sum_{i=1}^N [2H(x - h_i(t)) - 1]$ (in the sense of viscosity solution) where h_i is driven by the particle system

$$\frac{d}{dt} h_i = c \sum_{j \neq i} \frac{1}{h_i - h_j}, \quad h_i(0) = h_i^0, \quad (1.20)$$

where c is a constant not depending on N . If we recast our convergence result into their scale but still assume there is only one transition layer $N = 1$, then we have

$$\left| v^\varepsilon(x, t) - \phi\left(\frac{x - h_0}{\varepsilon}\right) \right| \leq c e^{-\frac{ct}{\varepsilon^2}} \quad \text{for any } t > 0, x \in \mathbb{R}. \quad (1.21)$$

Physically speaking, we focus on the detailed relaxation of a single dislocation to its steady profile ϕ with exponential convergence rate. However, instead of caring about the detailed behavior for each dislocation, [21] focused on a larger scale behavior for several dislocations by zooming out in spatial variable x and waiting for a longer time t . We will discuss in Section 5 that the time scale for observing slow motion depends only on the tail decay rate, either algebraically or exponentially. In solid state physics, the algebraic decay and exponential decay are two typical tail estimates indicating the physical interactions between particles. For instant, for the $K(r) = \frac{1}{r} + \frac{1}{r^3}$, which shows elastic long range interaction between two steps in the epitaxial growth, [19] study the mean field limit of a similar particle system in a larger scale (taking particle number N goes to ∞). They prove if the tail estimate is faster than a quadratic decay rate, in the mean field limit, the corresponding continuum PDE from the particle system with only nearest-neighbor interactions is same as the one with global interactions.

Let us review some other related works among the vast literature of analysis for asymptotic behaviors. For the classical Allen-Cahn equation with double well

potential, [15] proved the global exponential stability of a traveling wave solution, which established the first framework to tackle the long time asymptotic behavior using spectral gap analysis for diffusion operator linearized along traveling waves; see also [11] for invariant manifold method. Under the small perturbation assumption, [42, 27] proved the multidimensional stability of traveling wave solutions. Furthermore, for a nonlocal Allen-Cahn equation with nonsingular kernel, [3] studied the properties of travelling wave solutions as well as the uniform asymptotic stability. For a class of integro-differential equations which contain a nonlocal term expressed by the convolution of u with some nonsingular kernel, [12] established an abstract theorem for uniqueness, existence and exponential stability of traveling wave solutions while [2] presented spectral analysis for linearized operators along traveling wave solutions and obtain multidimensional stability for small perturbations. We are unaware of any asymptotic stability results for nonlocal operators with singular kernels, whose steady profile has infinite energy. To obtain the exponential decay rate, we need to show 0 is the principal and simple eigenvalue for the linearized nonlocal operator, which is a nonlocal Schrödinger operator. For the estimates for the smallest eigenvalue of local or nonlocal Schrödinger operators, we refer to [16, 17, 26, 21, 9, 10] and references therein.

Strategy. The general idea is to first prove the dynamic solution will uniformly converge to a shifted steady profile $\phi(x - x_0)$. Then by the spectral analysis for the nonlocal Schrödinger operator, which is linearized along the steady profile, we obtain the exponential decay rate.

The essential difficulties for the uniform convergence are the compactness and the characterization of the ω -limit set. As shown in the footnote in the previous page, we have an infinite nonlocal energy, which is only meaningful with the perturbed definition (1.11). We introduce a special ω -limit set with vanishing dissipation; see Definition 1, which is shown to be nonempty. For this kind of ω -limit set, which takes advantage of the vanishing dissipation property for a sequence of solution $u(x, t_n)$, we have uniform estimate for $\|u(x, t_n)\|_{H^1}$ and can characterize the limit uniquely as a shifted steady profile $\phi(x - x_0)$; see Proposition 2.4. By imposing the initial condition (1.14) and thanks to the comparison principle and good decay properties for the steady profile ϕ , we obtain the compactness in Section 2.3. This, together with the characterization of ω -limit set, leads to a convergence from $u(x, t_n)$ to $\phi(x - x_0)$. Notice the vanishing dissipation property is valid only for the subsequence we extracted. By further proving for any t large enough, the solution will stay around the steady profile $\phi(x - x_0)$, we finally obtain the uniform convergence in Theorem 2.9.

As for the spectral gap analysis, to show 0 is the principal and simple eigenvalue for the linearized nonlocal Schrödinger operator, we give a new contradiction proof involving some particular global properties of the fractional Laplace operator and the concave part of the double well potential F ; see Proposition 3.3. The spectral gap obtained in Theorem 3.4 shows a lower bound for the norm of the linearized nonlocal operator for any u orthogonal to ϕ' . Using this property, we prove the exponential decay of dynamic solutions to its equilibrium in Section 4 and Theorem 1.1.

Outlines. The rest of this paper is organized as follows. In Section 2, we will first prove the uniform convergence of the dynamic solution $u(x, t)$ to its equilibrium, which is uniquely characterized as a shifted steady profile, i.e. $\phi(x - x_0)$. In Section 3, we establish the spectral decomposition for the linearized nonlocal Schrödinger operator, which leads to a spectral gap. All the proofs for the detailed spectral decomposition are given in Appendix B. In Section 4, we combine the spectral gap with the uniform

convergence to finally obtain the exponential decay of the dynamic solution to its equilibrium $\phi(x - x_0)$. Section 5 is the discussion on time scales for the slow motion.

2. Uniform convergence from the dynamic solution to the steady profile ϕ . This section will focus on the uniform convergence from the dynamic solution to its equilibrium, which involves essentially two main questions, compactness and characterization of the ω -limit set. Here the ω -limit set is a special one defined in Definition 1, which takes advantage of the property of solutions with a vanishing dissipation. For this kind of ω -limit set, we can characterize it uniquely as a shifted steady profile $\phi(x - x_0)$ in Section 2.2. Then thanks to the compactness and stability guaranteed by the comparison principle, we will obtain the uniform convergence to $\phi(x - x_0)$ in Section 2.4. We shall first clarify the existence and uniqueness of the global classical solution to the dynamic problem (1.12).

2.1. Global classical solution. Recall (1.12) and $\mathcal{A}\phi = 0$. Set a perturbation function as

$$v(x, t) := u(x, t) - \phi(x).$$

Then the dynamic equation for v is

$$\partial_t v = -L_{\frac{1}{2}} v - f(u) + f(\phi) \tag{2.1}$$

with initial data $v_0(x) = u_0(x) - \phi(x)$, where $u_0(x)$ satisfies (1.13) and (1.14). Notice that if $u_0(x)$ satisfies (1.13) and (1.14), then from $F(\cdot) \geq 0$ and $\|\phi(\cdot)\|_{L^\infty} < c$ we know $v_0(x) \in H^{\frac{1}{2}}(\mathbb{R})$. We will use the theory for contraction semigroup to first establish the existence and uniqueness of a global classical solution to (2.1). Define the free energy for v as

$$\mathcal{F}(v) := \int \frac{1}{2} |(-\partial_{xx})^{1/4} v|^2 - v f(\phi) + F(v + \phi) \, dx. \tag{2.2}$$

Denote

$$Av := (L_{\frac{1}{2}} + I)v, \quad G(v) := f(\phi) - f(v + \phi) + v. \tag{2.3}$$

Then (2.1) becomes

$$\partial_t v = \mathcal{A}\phi - \mathcal{A}u = -Av + G(v). \tag{2.4}$$

From now on, c and C will be genetic constants whose values may change from line to line. We have the following well-posedness result for (2.1). The proof is standard but to show the idea clearly, we give a brief proof in Appendix A for $v_0 \in H^1(\mathbb{R})$. For the case $v_0 \in H^{1/2}(\mathbb{R})$, the proof is similar by using analytic semigroup and we refer to [23].

THEOREM 2.1. *Assume initial data $v_0(x) := u_0(x) - \phi(x) \in H^1(\mathbb{R})$.*

(i) *There exists a unique global solution*

$$v \in C^1([0, \infty); L^2(\mathbb{R})) \cap C([0, \infty); H^1(\mathbb{R})) \tag{2.5}$$

to (2.4) such that $v(x, 0) = v_0(x)$ and $\partial_t v, Av, G(v) \in L^2(\mathbb{R})$ and the equation (2.4) is satisfied in $L^2(\mathbb{R})$ for any $t > 0$.

(ii) This solution can be expressed by

$$v(t) = e^{-At}v_0 + \int_0^t e^{-A(t-\tau)}G(v(\tau)) \, d\tau; \tag{2.6}$$

(iii) For any $k, j \in \mathbb{N}^+$ and $\delta > 0$ there exist $C_{\delta,k,j}$, c such that

$$\begin{aligned} v &\in C^k((0, \infty); H^j(\mathbb{R})); \\ \|\partial_t^k v(\cdot, t)\|_j &\leq C_{\delta,k,j}e^{ct}, \quad t \geq \delta. \end{aligned} \tag{2.7}$$

(iv) We have the energy identity

$$\frac{d\mathcal{F}(v(t))}{dt} = - \int [-(-\partial_{xx})^{1/2}v - f(v + \phi) + f(\phi)]^2 dx =: -\mathcal{Q}(v(t)) \leq 0. \tag{2.8}$$

2.2. Characterization of ω -limit set. In this section, we devote efforts to characterize the ω -limit set whenever it is not empty. We will characterize it for sequence $u(x, t_n)$ with vanishing dissipation.

LEMMA 2.2 (Vanishing sequence for dissipation). *Assume $\mathcal{F}(t)$ is bounded from below and $\mathcal{F}'(t) \leq 0$. Let $\mathcal{Q}(t) = -\mathcal{F}'(t)$ defined in (2.8). Then there exists a subsequence $t_n \rightarrow +\infty$ such that*

$$\mathcal{Q}(t_n) = -\mathcal{F}'(t_n) \rightarrow 0. \tag{2.9}$$

Proof. Notice that the conclusion in this lemma is equivalent to

For any $\varepsilon > 0$, any $T > 0$, there exists $t > T$ such that $-\varepsilon < \mathcal{F}'(t) \leq 0$.

Then we argue by contradiction. If not, there exists $\varepsilon_0 > 0$ and $T > 0$ such that for any $t > T$, $\mathcal{F}'(t) < -\varepsilon_0$. It implies $\mathcal{F}(t) \rightarrow -\infty$, which contradicts with $\mathcal{F}(t)$ is bounded from below. \square

Now we define the special ω -limit set below.

DEFINITION 1. Assume $v(x, t)$ is the dynamic solution to (2.1) with initial data $v_0 \in H^{\frac{1}{2}}(\mathbb{R})$. Let $\mathcal{Q}(t) = -\mathcal{F}'(t)$ defined in (2.8). We define the ω -limit set with vanishing dissipation as

$$\begin{aligned} \omega(v) &:= \{v^* \in L^2(\mathbb{R}); \text{ there exist } t_n \rightarrow +\infty \\ &\text{ such that } v(\cdot, t_n) \rightarrow v^*(\cdot) \text{ in } L^2(\mathbb{R}) \text{ and } \mathcal{Q}(t_n) \rightarrow 0\}. \end{aligned} \tag{2.10}$$

This is a subset of the usual ω -limit set in dynamics systems, which does not require additional dissipation property.

First we state a strict positivity property at global minima and global maxima for the nonlocal operator $(-\partial_{xx})^{\frac{1}{2}}$, which will be used later.

LEMMA 2.3 (Strict positivity property at global minima and global maxima). *For any function $g(x) \in C(\mathbb{R})$, assume $x_m, x_M \in (-\infty, +\infty)$ are the points where $g(x)$ attains its global minimum and maximum separately. Then we have*

$$(-\partial_{xx})^{\frac{1}{2}}g(x)|_{x=x_m} < 0, \quad (-\partial_{xx})^{\frac{1}{2}}g(x)|_{x=x_M} > 0 \tag{2.11}$$

provided $g(x)$ is not a constant.

Proof. From the definition of $(-\partial_{xx})^{\frac{1}{2}}$, since $g(x_m) \leq g(x)$ for all $x \in \mathbb{R}$, we have

$$(-\partial_{xx})^{\frac{1}{2}}g(x)|_{x=x_m} \leq 0$$

and the equality holds only when $g(x) \equiv g(x_m)$ for all $x \in \mathbb{R}$. The proof for $(-\partial_{xx})^{\frac{1}{2}}g$ at x_M is the same. \square

PROPOSITION 2.4 (Characterization of ω -limit set). *Let v be the dynamic solution to (2.1) with initial data $v_0 \in H^{\frac{1}{2}}(\mathbb{R})$. Assume $\omega(v) \neq \emptyset$ and let $v^* \in \omega(v)$ defined in (2.10). Then there exists $t_n \rightarrow +\infty$ such that*

- (i) $v(\cdot, t_n) \rightarrow v^*(\cdot)$ in $L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})$;
- (ii) $v^* \in H^1(\mathbb{R})$ is the steady solution to

$$(-\partial_{xx})^{\frac{1}{2}}v^* = f(v^* + \phi) - f(\phi), \quad (2.12)$$

- (iii) *in the sense that equation (2.12) holds in $L^2(\mathbb{R})$;*

$$\mathcal{F}(v^*) < +\infty, \quad \lim_{x \rightarrow \pm\infty} v^*(x) = 0;$$

- (iv) *moreover, there exists x_0 such that*

$$v^*(x) = \phi(x - x_0) - \phi(x), \quad x \in \mathbb{R}. \quad (2.13)$$

Proof. Step 1. Since $v^* \in \omega(v)$, we know there exist $t_n \rightarrow +\infty$ such that $v(\cdot, t_n) \rightarrow v^*(\cdot)$ in $L^2(\mathbb{R})$. Thus $\|v(t_n)\| \leq c$ and $\|v^*\| \leq c$. Recall

$$\mathcal{Q}(t_n) = -\mathcal{F}'(t_n) = - \int [(-\Delta)^{1/2}v - f(v + \phi) + f(\phi)]^2 dx \rightarrow 0. \quad (2.14)$$

Therefore, $|\mathcal{Q}(t_n)|$ is bounded by 1 for n large enough and thus

$$\begin{aligned} \|(-\partial_{xx})^{\frac{1}{2}}v(t_n)\|^2 &\leq -\|f(v(t_n) + \phi) - f(\phi)\|^2 \\ &\quad + 2 \int_{\mathbb{R}} |(-\partial_{xx})^{\frac{1}{2}}v(t_n)(f(v(t_n) + \phi) - f(\phi))| + 1 \\ &\leq \|f(v(t_n) + \phi) - f(\phi)\|^2 + \frac{1}{2}\|(-\partial_{xx})^{\frac{1}{2}}v(t_n)\|^2 + 1 \\ &\leq \max |f'| \|v(t_n)\|^2 + \frac{1}{2}\|(-\partial_{xx})^{\frac{1}{2}}v(t_n)\|^2 + 1, \end{aligned}$$

which implies

$$\|v(\cdot, t_n)\|_{\dot{H}^1} \leq c. \quad (2.15)$$

From Ladyzhenskaya's inequality, we have

$$\begin{aligned} \|v(\cdot, t_n)\|_{L^\infty} &\leq \sqrt{2}\|v(\cdot, t_n)\|^{1/2}\|v(\cdot, t_n)\|_{\dot{H}^1}^{1/2} \\ &\leq c\|v(\cdot, t_n)\|^{1/2}, \end{aligned} \quad (2.16)$$

which, after applying to $v(\cdot, t_n) - v^*(\cdot)$, concludes (i).

Step 2. Notice (2.15) and $\|v(t_n)\| \leq c$. We have $\|v(t_n)\|_1^2$ is bounded and there exists a subsequence such that $v(\cdot, t_n) \rightharpoonup v^*(\cdot)$ in $H^1(\mathbb{R})$ weakly. Thus from the lower semi continuity of norm and $v(\cdot, t_n) \rightarrow v^*(\cdot)$ in $L^\infty(\mathbb{R})$, we know

$$\int_{\mathbb{R}} [-(-\partial_{xx})^{\frac{1}{2}} v^*(x) - f(v^* + \phi) + f(\phi)]^2 dx \leq \liminf_{t_n \rightarrow \infty} \mathcal{Q}(t_n) \rightarrow 0, \quad (2.17)$$

which concludes v^* is the solution to (2.12). Since also $f(\phi) \in L^2(\mathbb{R})$, (2.12) holds in L^2 sense and we conclude (ii). Recall free energy $\mathcal{F}(v)$ in (2.2). We obtain the bound for $\mathcal{F}(v^*)$ from lower semi continuity of norm and $v(\cdot, t_n) \rightarrow v^*(\cdot)$ in $L^\infty(\mathbb{R})$. Moreover we know $v^* \in H^1(\mathbb{R}) \hookrightarrow C^{0,\alpha}(\mathbb{R})$, $\alpha < \frac{1}{2}$ so $\lim_{x \rightarrow \pm\infty} v^*(x) = 0$ and we conclude (iii).

Step 3. It remains to prove (iv) that all the steady solution $v^*(x)$ to (2.12) are of the form $\phi(x - x_0) - \phi(x)$ for some x_0 . Let $u^*(x) := v^*(x) + \phi(x)$. Since $\mathcal{A}\phi = 0$ in classical sense and $v^* \in H^1(\mathbb{R})$, we know from (2.12) $u^*(x)$ is the solution to

$$(-\partial_{xx})^{\frac{1}{2}} u^*(x) = -f(u^*(x)) \quad (2.18)$$

in the sense that equation holds in $L^2(\mathbb{R})$. In two cases below, we will first prove $v^*(x) = \phi(x - x_0) - \phi(x)$ if $u^* \in (-1, 1)$, then claim u^* must be in $(-1, 1)$ by contradiction argument.

Case 1. We assume $u^*(x) = v^*(x) + \phi(x) \in (-1, 1)$. For any $\varepsilon > 0$, since $v^*(\pm\infty) = 0$ and $u^*(\pm\infty) = \phi(\pm\infty) = \pm 1$, there exist x_ε and ξ_ε such that

$$v_\varepsilon(x) := u^*(x) - \phi(x - x_\varepsilon) + \varepsilon \geq 0 \text{ for any } x \in \mathbb{R} \quad (2.19)$$

and

$$v_\varepsilon(\xi_\varepsilon) = u^*(\xi_\varepsilon) - \phi(\xi_\varepsilon - x_\varepsilon) + \varepsilon = 0. \quad (2.20)$$

If $v_\varepsilon \equiv \text{const}$, then

$$u(x) \equiv \phi(x - x_\varepsilon) + \varepsilon$$

for any $x \in \mathbb{R}$, which contradicts with $u(\pm\infty) = \phi(\pm\infty) = \pm 1$. Thus v_ε is not constant.

Now we claim $x_\varepsilon, \xi_\varepsilon$ are both finite. Notice both $u^*(x)$ and $\phi(x - x_\varepsilon)$ satisfy (2.18). Since v_ε attains its minimum at ξ_ε , by Lemma 2.3 we have

$$0 > (-\partial_{xx})^{\frac{1}{2}} v_\varepsilon(x)|_{x=\xi_\varepsilon} = \left[-f(u^*(x)) + f(\phi(x - x_\varepsilon)) \right] \Big|_{x=\xi_\varepsilon} \quad (2.21)$$

$$= \left[-f(\phi(x - x_\varepsilon) - \varepsilon) + f(\phi(x - x_\varepsilon)) \right] \Big|_{x=\xi_\varepsilon} = f'(\eta)\varepsilon \quad (2.22)$$

with

$$\eta \in [u^*(\xi_\varepsilon), u^*(\xi_\varepsilon) + \varepsilon] = [\phi(\xi_\varepsilon - x_\varepsilon) - \varepsilon, \phi(\xi_\varepsilon - x_\varepsilon)].$$

Therefore η must locate in concave part of F , i.e. $\eta \in (-\frac{1}{2}, \frac{1}{2})$. Then

$$u^*(\xi_\varepsilon) \in (-\frac{1}{2} - \varepsilon, \frac{1}{2}) \subset [-\frac{3}{4}, \frac{1}{2}] \quad (2.23)$$

for $\varepsilon < \frac{1}{4}$. Since $u^*(\cdot) \in (-1, 1)$ is a continuous function, so ξ_ε is bounded uniformly for $\varepsilon < \frac{1}{4}$. On the other hand, we also have

$$\phi(\xi_\varepsilon - x_\varepsilon) \in \left(-\frac{1}{2}, \frac{1}{2} + \varepsilon\right) \subset \left[-\frac{1}{2}, \frac{3}{4}\right], \tag{2.24}$$

which implies $\xi_\varepsilon - x_\varepsilon \in [-2, 2]$. This concludes $x_\varepsilon, \xi_\varepsilon$ are both bounded uniformly for $\varepsilon < \frac{1}{4}$.

Take $\varepsilon \rightarrow 0$ and a convergent subsequence (still denote as $x_\varepsilon, \xi_\varepsilon$) such that $x_\varepsilon \rightarrow x_0$ and $\xi_\varepsilon \rightarrow \xi$ for some x_0 and ξ . Clearly we still know $\xi - x_0 \in [-2, 2]$. Then we have

$$\begin{aligned} u^*(x) - \phi(x - x_0) &\geq 0 \text{ for any } x \in \mathbb{R} \\ u^*(\xi) - \phi(\xi - x_0) &= 0. \end{aligned} \tag{2.25}$$

From (2.21), we know

$$0 \geq (-\partial_{xx})^{\frac{1}{2}}(u^*(x) - \phi(x - x_0))|_{x=\xi} = \lim_{\varepsilon \rightarrow 0} (-\partial_{xx})^{\frac{1}{2}}v_\varepsilon|_{x=\xi_\varepsilon} = \lim_{\varepsilon \rightarrow 0} f'(\eta)\varepsilon = 0 \tag{2.26}$$

This, together with ξ attains the minimum by (2.25), leads to

$$u^*(x) - \phi(x - x_0) \equiv \text{const} = 0 \text{ for all } x \in \mathbb{R},$$

which means $u^*(x) \equiv \phi(x - x_0)$ and $v^*(x) \equiv \phi(x - x_0) - \phi(x)$.

Case 2. We assume $u^*(x) = v^*(x) + \phi(x) \notin (-1, 1)$ for some x . We use a contradiction argument to show this case is not possible. We only deal with the left side, i.e. $u^*(x) = v^*(x) + \phi(x) \leq -1$ for some x . The argument for the other side $u^*(x) = v^*(x) + \phi(x) \geq 1$ is analogous.

Since u^* is continuous function connecting from -1 to -1 , then if $u^* \leq -1$, it can attain its minimal point at some finite x^* . Assume

$$u^*(x^*) = \min_{x \in \mathbb{R}} u^* \in (-1 - 2k, 1 - 2k] \text{ for some } k \in \mathbb{N}^+. \tag{2.27}$$

First, from (2.18), $u^*(x^*) \neq 1 - 2k$. Otherwise by Lemma 2.3,

$$0 = (-\partial_{xx})^{\frac{1}{2}}u^*(x)|_{x=x^*} + f(1 - 2k) = (-\partial_{xx})^{\frac{1}{2}}u^*(x)|_{x=x^*} < 0 \tag{2.28}$$

leads to a contradiction. Then we know $u^*(x^*) = \min_{x \in \mathbb{R}} u^*(x) \in (-1 - 2k, 1 - 2k)$. Therefore, we choose η such that $u^*(x) + 2k \geq \phi(x - \eta)$ for any $x \in \mathbb{R}$ and $u^*(x) + 2k$ touches $\phi(x - \eta)$ at the point x_1 , i.e.

$$\begin{cases} u^*(x) + 2k \geq \phi(x - \eta) & \text{for } x \in \mathbb{R}; \\ u^*(x_1) + 2k = \phi(x_1 - \eta). \end{cases} \tag{2.29}$$

Notice the minimal point x^* is finite so x_1, η are finite. Since f is $2k$ -periodic function, we have

$$\begin{aligned} 0 &= \left[(-\partial_{xx})^{\frac{1}{2}}(u^*(x) + 2k) + f(u^*(x) + 2k) - (-\partial_{xx})^{\frac{1}{2}}(\phi(x - \eta)) - f(\phi(x - \eta)) \right] \Big|_{x=x_1} \\ &= (-\partial_{xx})^{\frac{1}{2}}(u^*(x) + 2k - \phi(x - \eta)) \Big|_{x=x_1} < 0, \end{aligned}$$

where we used Lemma 2.3 again. This also gives a contradiction and we complete the proof of (iv). \square

2.3. Comparison Principle and Compactness. In the previous section, we have seen clearly the characterization of ω -limit set with vanishing dissipation whenever it is not empty. However, in order to extract such a sequence $v(t_n)$ with a limit in $\omega(v)$ defined in (2.10), we need compactness in $L^2(\mathbb{R})$. One possible way to achieve it is the comparison principle.

2.3.1. Comparison Principle . We have the following comparison principle.

PROPOSITION 2.5. *Let initial data satisfy assumption (1.14). Then*

$$\phi(x - b) \leq u(x, t) \leq \phi(x - a), \quad \forall x \in \mathbb{R}, t > 0, \tag{2.30}$$

where $b \geq a$ are constants given in (1.14).

Proof. We only prove the left hand side of (2.30). Denote $w(x, t) := u(x, t) - \phi(x - b)$. Then we know

$$\begin{aligned} \partial_t w &= -(-\partial_{xx})^{\frac{1}{2}} w + f(\phi(\cdot - b)) - f(w + \phi(\cdot - b)), \\ w(\cdot, 0) &\geq 0. \end{aligned}$$

Assume t^* is the first time such that w attains zero at some point x^* . Therefore

$$\begin{aligned} w(x, t) &\geq 0 \quad \text{for any } 0 \leq t \leq t^*, x \in \mathbb{R}; \\ w(x^*, t^*) &= 0. \end{aligned}$$

Then at $x = x^*, t = t^*$

$$\begin{aligned} \partial_t w|_{(x^*, t^*)} &= -(-\partial_{xx})^{\frac{1}{2}} w|_{(x^*, t^*)} + f(\phi(x^* - b)) - f(\phi(x^* - b) + w(x^*, t^*)) \\ &= -(-\partial_{xx})^{\frac{1}{2}} w|_{(x^*, t^*)} \geq 0, \end{aligned}$$

where we used $w(x^*, t^*)$ is the minimum. Moreover, since $w(\pm\infty) = 0$, w can not be a nontrivial constant. Therefore by Lemma 2.3 $\partial_t w|_{(x^*, t^*)} > 0$ and we conclude $w(x, t) = u(x, t) - \phi(x - b) \geq 0$ all the time. \square

LEMMA 2.6 (Basic decay estimate at far fields). *There exists a positive constant C such that for any dynamic solution $u(x, t)$ to (1.12) with initial data satisfying (1.13) and (1.14),*

$$\begin{aligned} |1 - u(x, t)|, |f(u)| &< \frac{C}{1 + |x|}, \quad x > 0, t > 0; \\ |1 + u(x, t)|, |f(u)| &< \frac{C}{1 + |x|}, \quad x < 0, t > 0. \end{aligned} \tag{2.31}$$

Proof. From (2.30), we obtain the basic estimate for u ,

$$|1 - u(x, t)| \leq |1 - \phi(x - b)| \leq \frac{C}{1 + |x|}, \quad x > 0,$$

where we use the asymptotic estimate (1.10). Similarly we have,

$$|-1 - u(x, t)| \leq \frac{C}{1 + |x|}, \quad x < 0.$$

Moreover, we obtain the basic estimate for nonlinear term

$$\begin{aligned} |f(u)| &= \frac{1}{\pi} |\sin(\pi u)| \\ &= \frac{1}{\pi} |\sin(\pi(1-u))| = \frac{1}{\pi} |\sin(\pi(1+u))| \\ &\leq \begin{cases} \frac{C}{1+|x|}, & \text{for } x > 0; \\ \frac{C}{1+|x|}, & \text{for } x < 0. \end{cases} \end{aligned}$$

□

2.3.2. Compactness. Now we turn to prove the compactness in $L^2(\mathbb{R})$, which is the key point to guarantee that the ω -limit set is not empty.

LEMMA 2.7 (Compactness). *Assume $u(x, t)$ is the dynamic solution to (1.12) with initial data satisfying (1.13) and (1.14). For each $\delta > 0$ the set of functions*

$$\{u(\cdot, t) - \phi(\cdot); t \geq \delta\}$$

is relatively compact in $L^2(\mathbb{R})$.

Proof. Step 1. For any $\varepsilon > 0$, from Lemma 2.6, we can choose K such that for $|x| > K, t > 0$

$$\begin{aligned} &\|u - \phi\|_{L^2(|x|>K)} \\ &\leq \|u - \phi\|_{L^2(x>K)} + \|u - \phi\|_{L^2(x<-K)} \\ &\leq \|u - 1\|_{L^2(x>K)} + \|1 - \phi\|_{L^2(x>K)} + \|u + 1\|_{L^2(x<-K)} + \|-1 - \phi\|_{L^2(x<-K)} \\ &\leq c \left(\int_{|x|>K} \frac{1}{(1+|x|)^2} dx \right)^{\frac{1}{2}} < \frac{\varepsilon}{2}. \end{aligned}$$

Step 2. Recall free energy for $v = u - \phi$

$$\mathcal{F}(v) = \int \frac{1}{2} |(-\partial_{xx})^{1/4} v|^2 - v f(\phi) + F(v + \phi) dx \tag{2.32}$$

and energy identity (2.8). Since $F(v + \phi) \geq 0$ and $\mathcal{F}(v(t)) \leq \mathcal{F}(v_0)$, we know

$$\int_{\mathbb{R}} \frac{1}{2} |(-\partial_{xx})^{1/4} v|^2 dx \leq c + \|v\| \|f(\phi)\| \leq c, \tag{2.33}$$

where we also used $\|v\| \leq c$ by Lemma 2.6. Thus the compact embedding $H^{\frac{1}{2}}(-K, K) \hookrightarrow L^2(-K, K)$ shows there exists a subsequence $t_n \rightarrow +\infty$ such that $u(\cdot, t_n) - \phi(\cdot) \rightarrow u^*(\cdot) - \phi(\cdot)$ in $L^2(-K, K)$. Therefore, $\lim_{n \rightarrow \infty} u(x, t_n) - \phi(x) = u^*(x) - \phi$ in $L^2(\mathbb{R})$. □

REMARK 4. It worth to notice the initial condition (1.14) is only used to obtain the uniform in time estimate for u at far field. As we have seen in the proof of Lemma 2.7, the compactness result can be achieved as long as we have the uniform in time $L^2(\mathbb{R})$ bound. It is another possible way to relax the initial condition (1.14).

2.4. Stability and Uniform Convergence. We have obtained the compactness in $L^2(\mathbb{R})$ and the characterization of ω -limit set in previous preparations. Therefore, (i) we can first extract a sequence $u(x, t_n) - \phi(x)$ with vanishing dissipation $\mathcal{Q}(t_n)$ by Lemma 2.2, (ii) then by compactness Lemma 2.7 $u(x, t_n) - \phi(x)$ possesses further a subsequence such that the limit of $u(x, t_{n_k}) - \phi(x)$ is in $\omega(v)$, in other words, for any $v_0 \in H^{\frac{1}{2}}(\mathbb{R})$, $\omega(v) \neq \emptyset$. However those properties are only for some subsequence t_n . In this section, we are finally in the position to obtain the uniform convergence by proving the dynamic solution will stay close to the standing profile for all large time. First we list some properties for the double well function $F(x)$.

Since $f'(\pm 1) > 0$, there exist $\mu > 0, \delta > 0$ such that for $0 < q < \frac{\delta}{2}$,

$$f(\phi) - f(\phi - q) \geq \mu q \quad \text{for } 1 - \delta \leq \phi \leq 1 \text{ or } -1 \leq \phi \leq -1 + \delta. \tag{2.34}$$

Moreover, for $\phi \in [-1 + \delta, 1 - \delta]$, there exist $k > 0, \beta \geq 0$ such that

$$|f(\phi - q) - f(\phi)| \leq kq \quad \text{for any } 0 < q < \frac{\delta}{2}, \tag{2.35}$$

and

$$\phi'(x) \geq \beta > 0 \quad \text{for } x \text{ such that } \phi(x) \in [-1 + \delta, 1 - \delta]. \tag{2.36}$$

PROPOSITION 2.8 (Stability). *Assume $u(x, t)$ is a dynamic solution to (1.12) and for any $0 < \varepsilon < \frac{\delta}{2}$ there exists N such that*

$$\sup_{x \in \mathbb{R}} |u(\cdot, t_N) - \phi(\cdot - x_0)| < \varepsilon.$$

Then for any $t > t_N$, there exists C such that

$$\sup_{x \in \mathbb{R}} |u(x, t) - \phi(x - x_0)| < C\varepsilon.$$

Moreover

$$\begin{aligned} & \phi(x - x_0 - \frac{\mu + k}{\mu\beta}\varepsilon) - \varepsilon e^{-\mu(t-t_N)} \\ \leq u(x, t) & \leq \phi(x - x_0 + \frac{\mu + k}{\mu\beta}\varepsilon) + \varepsilon e^{-\mu(t-t_N)}, \quad \forall x \in \mathbb{R}, t > t_N. \end{aligned} \tag{2.37}$$

Proof. We will use the comparison principle to prove that for $t > t_N$ the solution still stay close to $\phi(x - x_0)$. First we prove the lower bound for u . Notice

$$\phi(x - x_0) - \varepsilon \leq u(x, t_N) \quad \text{for any } x \in \mathbb{R}. \tag{2.38}$$

We construct a subsolution

$$\underline{u}(x, t) := \max\{-1, \phi(x - \xi(t)) - q(t)\} \in [-1, 1] \tag{2.39}$$

by choosing $\xi(t)$ and $q(t)$ such that $q(t) := \varepsilon e^{-\mu(t-t_N)}$, $\xi(t) := c_1 + c_2 e^{-\mu(t-t_N)}$ with $c_1 = x_0 - c_2$ and $c_2 < 0$ to be determined.

Define

$$N(u) := \partial_t u + \mathcal{A}u = \partial_t u + (-\partial_{xx})^{\frac{1}{2}} u + f(u) \tag{2.40}$$

and divide $[-1, 1]$ into several sets

$$I_1 := \{(x, t); \phi(x - \xi(t)) \in [-1, -1 + q(t)]\},$$

$$I_2 := \{(x, t); \phi(x - \xi(t)) \in [-1 + q(t), -1 + \delta]\},$$

$$I_3 := \{(x, t); \phi(x - \xi(t)) \in [-1 + \delta, 1 - \delta]\},$$

$$I_4 := \{(x, t); \phi(x - \xi(t)) \in [1 - \delta, 1]\}.$$

- (1) If $(x, t) \in I_1$, then $\phi(x - \xi) - q(t) \leq -1$ and $N(\underline{u}) = 0$.
- (2) If $(x, t) \in I_2$, since $\mathcal{A}\phi = 0$, $\xi' \geq 0$ and (2.34), we know

$$\begin{aligned} N(\underline{u}) &= -\phi'(x - \xi(t))\xi' - q' + (-\partial_{xx})^{\frac{1}{2}}\phi(x - \xi(t)) + f(\phi(x - \xi(t)) - q) \\ &= -\phi'(x - \xi(t))\xi' - q' - f(\phi(x - \xi(t))) + f(\phi(x - \xi(t)) - q) \\ &\leq -\phi'(x - \xi(t))\xi' - q' - \mu q \\ &\leq -q' - \mu q = 0. \end{aligned}$$

The situation for $(x, t) \in I_4$ is exactly the same.

- (3) For $(x, t) \in I_3$, i.e. $-1 + \delta \leq \phi(x - \xi(t)) \leq 1 - \delta$, from (2.36) and (2.35) we know

$$\begin{aligned} N(\underline{u}) &\leq -\phi'(x - \xi(t))\xi' - q' + kq \\ &\leq -\beta\xi' - q' + kq. \end{aligned}$$

Set $\xi' = \frac{-q'+kq}{\beta} = \frac{\mu+k}{\beta}q > 0$, we have

$$c_1 = x_0 - c_2, \quad c_2 = -\frac{\mu + k}{\mu\beta}\varepsilon.$$

Then $N(\underline{u}) \leq 0$ and \underline{u} is a subsolution satisfying

$$u(x, t) \geq \underline{u}(x, t) \geq \phi(x - \xi(t)) - q(t) \tag{2.41}$$

due to the comparison principle. Therefore we have

$$\phi(x - x_0 - \frac{\mu + k}{\mu\beta}\varepsilon) - \varepsilon e^{-\mu(t-t_N)} \leq u(x, t), \quad \forall x \in \mathbb{R}, t > t_N. \tag{2.42}$$

Similarly, we can obtain the upper bound for u

$$u(x, t) \leq \phi(x - x_0 + \frac{\mu + k}{\mu\beta}\varepsilon) + \varepsilon e^{-\mu(t-t_N)}, \quad \forall x \in \mathbb{R}, t > t_N. \tag{2.43}$$

Hence we know

$$|u(x, t) - \phi(x - x_0)| \leq \max_{x \in \mathbb{R}} \phi'(x) \cdot \frac{\mu + k}{\mu\beta}\varepsilon + \varepsilon, \quad \forall x \in \mathbb{R}, t > t_N,$$

which concludes for $C = 1 + \frac{2}{\pi} \frac{\mu+k}{\mu\beta}$ we have

$$\sup_{x \in \mathbb{R}} |u(x, t) - \phi(x - x_0)| < C\varepsilon$$

for any $t > t_N$. \square

After all the preparations above, we can first extract a time sequence t_n with vanishing dissipation $\mathcal{Q}(t_n)$ by Lemma 2.2 and then by the compactness Lemma 2.7 we can further extract a subsequence such that the limit of $u(x, t_n) - \phi(x)$ is in $\omega(v)$. Moreover, $u(x, t) - \phi(x)$ will stay close to its limit for any t large enough.

THEOREM 2.9 (Uniform Convergence). *Assume $u(x, t)$ is the dynamic solution to (1.12) with initial data satisfying (1.13) and (1.14). Then there exists a value x_0 such that*

$$\lim_{t \rightarrow +\infty} b(t) = 0, \quad b(t) := \max_{x \in \mathbb{R}} |u(x, t) - \phi(x - x_0)|. \tag{2.44}$$

Proof. Recall $v(x, t) = u(x, t) - \phi(x)$ with the free energy

$$\mathcal{F}(v) = \frac{1}{2} \int |(-\Delta)^{1/4} v|^2 - v f(\phi) + F(v + \phi) dx.$$

Then by Lemma 2.6 we know $\|v\| \leq c$ and thus $\mathcal{F}(v)$ is bounded from below. Therefore, combining energy identity (2.8) and Lemma 2.2 leads to a vanishing sequence for \mathcal{Q} , i.e. there exists a time sequence $t_n \rightarrow +\infty$ such that

$$\mathcal{Q}(t_n) = -\mathcal{F}'(t_n) \rightarrow 0. \tag{2.45}$$

For such a sequence t_n , from Lemma 2.7 we know

$$\{u(\cdot, t_n) - \phi(\cdot), \quad t_n \geq \delta\}$$

is relative compact in $L^2(\mathbb{R})$. Therefore we know the ω -limit set $\omega(v) \neq \emptyset$ and the limit of the subsequence (still denote as t_n) $v(x, t_n) = u(x, t_n) - \phi(x) \rightarrow v^*$ can be characterized by Proposition 2.4 (iv), i.e. $v^*(x) = \phi(x - x_0) - \phi(x)$ and thus

$$u(x, t_n) - \phi(x - x_0) = v(x, t_n) + \phi(x) - \phi(x - x_0) \rightarrow 0 \quad \text{in } L^\infty(\mathbb{R}).$$

Next, from the stability Proposition 2.8, we conclude the uniform convergence (2.44). \square

3. Spectral decomposition for the linearized nonlocal Schrödinger operator. In this section, we will study detailed structures for spectrum of the linearized nonlocal Schrödinger operator and prove the spectral gap in Proposition 3.4. Note $f'(\phi) = -\cos(\pi\phi) = \frac{x^2-1}{x^2+1}$. The linearized operator along the steady profile ϕ is $L : D(L) \subset L^2 \rightarrow L^2(\mathbb{R})$ with

$$Lu := (-\partial_{xx})^{\frac{1}{2}} u + \frac{x^2 - 1}{x^2 + 1} u. \tag{3.1}$$

Denote σ_p , σ_r and σ_c as the point spectrum, the residual spectrum and the continuous spectrum separately. Then

$$\mathbb{C} = \rho(L) \cup \sigma(L) = \rho(L) \cup \sigma_p(L) \cup \sigma_c(L) \cup \sigma_r(L).$$

We will first prove there is no residual spectrum and all the continuous spectrum locate in $[1, +\infty)$, see Proposition 3.1 and Proposition 3.2 separately. Although the

proof is standard but for completeness we put them in Appendix B.1 and Appendix B.2.

PROPOSITION 3.1. *For linear operator L in (3.1), the spectrum $\sigma(L) = \sigma_p(L) \cup \sigma_c(L) \subset [-1, +\infty)$.*

PROPOSITION 3.2. *For linear operator L in (3.1), the continuous spectrum $\sigma_c(L) \subset [1, +\infty)$.*

Next proposition is the key procedure to prove 0 is the principle eigenvalue and there is no other kinds of spectra near zero. The proof is the standard contradiction argument but it takes advantage of strict positivity property at global minima and global maxima for the nonlocal operator (see Lemma 2.3), which allow us to construct a sequence of eigenfunctions with minimal points locating in the concave part of the double well potential F .

PROPOSITION 3.3. *For linear operator L in (3.1), the point spectrum $\sigma_p(L) \subset [0, +\infty)$ and 0 is simple eigenvalue with eigenfunction $\phi'(x)$.*

Proof. Step 1. We prove 0 is a simple eigenvalue with eigenfunction $\phi'(x)$. First, by differentiating $\mathcal{A}\phi = 0$ once, it is straightforward that $\phi'(x) = \frac{2}{\pi} \frac{1}{1+x^2}$ is an eigenfunction corresponding to the eigenvalue 0.

Assume there is another eigenfunction g corresponding to 0 such that $g \in L^2(\mathbb{R})$. By the elliptic regularity of the steady solution to $((-\partial_{xx})^{\frac{1}{2}} + I)g + \frac{-2}{x^2+1}g = 0$, we know for any $k > 0$, $g \in H^k(\mathbb{R})$ thus g is smooth function. Without loss of generality, we assume g takes positive values at some x_0 (otherwise we can always construct such a function with some positive points by linear combination). Below, we will show g is linearly dependent on ϕ' .

Define

$$\phi_\beta := \phi' + \beta g, \quad \beta \in \mathbb{R}. \tag{3.2}$$

Define the set

$$D_1 := \{\beta < 0; \phi_\beta(\xi) < 0 \text{ for some } \xi\}.$$

Let

$$\bar{\beta} := \sup D_1.$$

Such a $\bar{\beta}$ is well-defined. Indeed, since g is positive at x_0 , we know $\bar{\beta} \in [\beta_1, 0]$ with $\beta_1 = -\frac{\phi'(x_0)}{g(x_0)} < 0$.

Notice that if ϕ_β is a constant, since $\phi_\beta \in L^2(\mathbb{R})$, we know $\phi_\beta \equiv 0$, which concludes ϕ' and g are linearly dependent. Therefore, we can simply assume ϕ_β is not a constant.

For any $\beta \in D_1$, since ϕ_β is also an eigenfunction corresponding to eigenvalue 0,

$$L\phi_\beta = (-\partial_{xx})^{\frac{1}{2}}\phi_\beta + f'(\phi)\phi_\beta = 0. \tag{3.3}$$

Let $\xi_\beta \in [-\infty, +\infty]$ be a point such that ϕ_β attains its minimum. Thus we know $\phi_\beta(\xi_\beta) < 0$. Consider two cases (i) $\xi_\beta \in (-\infty, +\infty)$; (ii) $\xi_\beta = -\infty$ or $+\infty$. For case (ii), since $\phi_\beta \in L^2(\mathbb{R})$ and $\phi_\beta \in H^1(\mathbb{R}) \hookrightarrow C(\mathbb{R})$, $\phi_\beta(\pm\infty)$ must be zero, which contradicts with $\phi_\beta(\xi_\beta) < 0$.

For case (i), by Lemma 2.3 we have

$$(-\partial_{xx})^{\frac{1}{2}}\phi_\beta|_{x=\xi_\beta} = \frac{1}{\pi} \text{P.V.} \int \frac{\phi_\beta(x) - \phi_\beta(y)}{|x - y|^2} dy \Big|_{x=\xi_\beta} < 0.$$

From (3.3) we know

$$f'(\phi)\phi_\beta|_{x=\xi_\beta} > 0, \tag{3.4}$$

which, together with $\phi_\beta(\xi_\beta) < 0$, leads to

$$f'(\phi)|_{x=\xi_\beta} < 0.$$

Due to the concave part of F is bounded between $-\frac{1}{2}$ and $\frac{1}{2}$, we know the set of ξ_β is bounded. Indeed, $f'(\phi)(x) = \frac{x^2-1}{x^2+1} < 0$ if and only if $x \in (-1, 1)$.

Take a convergent subsequence (still denote as β) with limit $\beta \rightarrow \bar{\beta}$ and $\xi_\beta \rightarrow \bar{\xi}$ for some $\bar{\xi} \in [-1, 1]$. From the definition of $\bar{\beta}$, we know

$$\phi_{\bar{\beta}}(\bar{\xi}) = 0 \leq \phi_{\bar{\beta}}(\xi) \quad \text{for any } \xi \in \mathbb{R}. \tag{3.5}$$

Therefore from $L\phi_{\bar{\beta}} = 0$ we have

$$(-\partial_{xx})^{\frac{1}{2}}\phi_{\bar{\beta}}|_{x=\bar{\xi}} = -f(\phi)\phi_{\bar{\beta}}|_{x=\bar{\xi}} = 0.$$

However by Lemma 2.3

$$(-\partial_{xx})^{\frac{1}{2}}\phi_{\bar{\beta}}|_{x=\bar{\xi}} = \frac{1}{\pi} \text{P.V.} \int \frac{\phi_{\bar{\beta}}(x) - \phi_{\bar{\beta}}(y)}{|x - y|^2} dy \Big|_{x=\bar{\xi}} \leq 0. \tag{3.6}$$

Therefore

$$\phi_{\bar{\beta}} \equiv \text{const} = 0,$$

which means ϕ' and g are linearly dependent.

Step 2. We prove 0 is the principle eigenvalue. Assume $\lambda < 0$ is the eigenvalue such that

$$Lu = \lambda u$$

for some $u \in L^2(\mathbb{R})$ and $u \neq 0$. By the elliptic regularity of the steady solution, we know for any $k > 0$, $u \in H^k(\mathbb{R})$ thus u is smooth function and $|u|$ is continuous function. Indeed, for $A := (-\partial_{xx})^{\frac{1}{2}} + I$, we know g satisfies $Ag + \frac{-2}{x^2+1}g = 0$. Then by the general Leibniz rule and bootstrap, one can obtain the regularities $u \in H^k(\mathbb{R})$ for any k . Then

$$(-\partial_{xx})^{\frac{1}{2}}|u| + f(\phi)|u| \leq \text{sgn}u \cdot [(-\partial_{xx})^{\frac{1}{2}}u + f(\phi)u] = \lambda|u| \leq 0. \tag{3.7}$$

Similarly, define

$$\phi_\beta := \phi' + \beta|u|, \quad \beta \in \mathbb{R} \tag{3.8}$$

and the set

$$D_1 := \{\beta < 0; \phi_\beta(\xi) < 0 \text{ for some } \xi\}.$$

Let

$$\bar{\beta} = \sup D_1.$$

which is well-defined since $|u|$ is positive at x_0 and we know $\bar{\beta} \in [\beta_1, 0]$ with $\beta_1 = -\frac{\phi'(x_0)}{|u|(x_0)} < 0$.

Notice if ϕ_β is a constant, since $\phi_\beta \in L^2(\mathbb{R})$, we know $\phi_\beta \equiv 0$, which concludes ϕ' and $|u|$ are linearly dependent, i.e. $L|u| = 0$. However from (3.7), $L|u| \leq \lambda|u| \leq 0$ and thus $\lambda = 0$. It contradicts with $\lambda < 0$. Therefore we can simply assume ϕ_β is not a constant.

For $\beta \in D_1$, from (3.7)

$$L\phi_\beta = \beta L|u| = \beta[(-\partial_{xx})^{\frac{1}{2}}|u| + f(\phi)|u|] \geq \beta\lambda|u| \geq 0, \text{ for all } x \in \mathbb{R}. \quad (3.9)$$

Let $\xi_\beta \in [-\infty, +\infty]$ be a point such that ϕ_β attains its minimum. Thus $\phi_\beta(\xi_\beta) < 0$. Consider two cases (i) $\xi_\beta \in (-\infty, +\infty)$; (ii) $\xi_\beta = -\infty$ or $+\infty$. For case (ii), since $\phi_\beta \in L^2(\mathbb{R})$ and $\phi_\beta \in H^1(\mathbb{R}) \hookrightarrow C(\mathbb{R})$, $\phi_\beta(\pm\infty)$ must be zero, which contradicts with $\phi_\beta(\xi_\beta) < 0$.

For case (ii), by Lemma 2.3 we have

$$(-\partial_{xx})^{\frac{1}{2}}\phi_\beta|_{x=\xi_\beta} = \frac{1}{\pi} \text{P. V.} \int \frac{\phi_\beta(x) - \phi_\beta(y)}{|x - y|^2} dy \Big|_{x=\xi_\beta} < 0.$$

This, together with (3.9), we know

$$f'(\phi)\phi_\beta|_{x=\xi_\beta} > 0. \quad (3.10)$$

Notice also $\phi_\beta(\xi_\beta) < 0$, thus

$$f'(\phi)|_{x=\xi_\beta} < 0.$$

Due to the concave part of F is bounded between $-\frac{1}{2}$ and $\frac{1}{2}$, we know the set of ξ_β is bounded, especially, $f'(\phi)(x) = \frac{x^2-1}{x^2+1} < 0$ if and only if $x \in (-1, 1)$.

Take a convergent subsequence (still denote as β) with limit $\beta \rightarrow \bar{\beta}$ and $\xi_\beta \rightarrow \xi^*$ for some $\xi^* \in [-1, 1]$. From the definition of $\bar{\beta}$,

$$\phi_{\bar{\beta}}(\xi^*) = 0, \quad \phi_{\bar{\beta}}(x) \geq 0 \text{ for any } x \in \mathbb{R}.$$

Then the limit of (3.9) shows that

$$0 \leq L\phi_{\bar{\beta}} = (-\partial_{xx})^{\frac{1}{2}}\phi_{\bar{\beta}} + f(\phi)\phi_{\bar{\beta}}.$$

However at $x = \xi^*$, the RHS is

$$(-\partial_{xx})^{\frac{1}{2}}\phi_{\bar{\beta}}|_{x=\xi^*} + f(\phi)\phi_{\bar{\beta}}|_{x=\xi^*} = (-\partial_{xx})^{\frac{1}{2}}\phi_{\bar{\beta}}|_{x=\xi^*} \leq 0.$$

Therefore $(-\partial_{xx})^{\frac{1}{2}}\phi_{\bar{\beta}}|_{x=\xi^*} = 0$ and thus

$$\phi_{\bar{\beta}} \equiv \phi_{\bar{\beta}}(\xi^*) = 0,$$

which means λ could only be zero and contradicts with $\lambda < 0$. \square

From the Proposition 3.2 and 3.3 above, we know 0 is the principle, simple eigenvalue of L and the continuous spectrum $\sigma_c(L) \subset [1, +\infty)$. Thus we obtain spectral gap for the nonlocal Schrödinger operator below.

THEOREM 3.4 (Spectral gap). *For linear operator L in (3.1), there exists a constant $\lambda_2 > 0$ such that for any $u \perp \text{Null}(L)$, i.e. $\int_{\mathbb{R}} u(x)\phi'(x) dx = 0$, we have*

$$\langle Lu, u \rangle \geq \lambda_2 \|u\|^2. \tag{3.11}$$

REMARK 5 (A Hardy type functional inequality and the best constant). Recall Hardy’s inequality for the homogeneous Sobolev space in one dimension. For $0 < s < \frac{1}{2}$

$$\|u\|_{\dot{H}^s}^2 \geq C_s \int_{\mathbb{R}} |x|^{-2s} |u(x)|^2 dx, \tag{3.12}$$

with sharp constant

$$C_s = 2^{2s} \frac{(\Gamma(1 + 2s)/4)^2}{(\Gamma(1 - 2s)/4)^2}.$$

As a consequence of Proposition 3.3, we have the following Hardy’s type functional inequality at critical index $s = \frac{1}{2}$.

COROLLARY 3.5. *For any $u \in H^{1/2}(\mathbb{R})$, we have*

$$\int_{\mathbb{R}} \frac{1 - x^2}{1 + x^2} u^2(x) dx \leq \|u\|_{\dot{H}^{\frac{1}{2}}}^2. \tag{3.13}$$

Moreover, the equality holds if and only if $u(x) = \frac{C}{1+x^2}$.

REMARK 6. Notice by the harmonic extension of the steady profile in the upper half plane is $\phi(x, y) := \frac{2}{\pi} \arctan \frac{x}{1+y}$, which has the harmonic conjugate $g(x, y) := \frac{1}{\pi} \ln(x^2 + (1 + y)^2)$. So $z(x, t) := \phi(x, y) + ig(x, y)$ is the holomorphic extension in the upper half-space \mathbb{C}_+ of $\phi(x) = \frac{2}{\pi} \arctan x$. For the linearized problem, a related holomorphic eigenvalue problem in \mathbb{C}_+ is

$$-i\partial_z w - \frac{2i}{i+z} w = \lambda w, \tag{3.14}$$

whose restriction on the real line becomes a nonlocal eigenvalue problem

$$(-\partial_{xx})^{\frac{1}{2}} u + \frac{x^2 - 1}{x^2 + 1} u + \frac{2x}{x^2 + 1} H(u) = (\lambda + 1)u. \tag{3.15}$$

4. Exponential decay to steady profile. Next we will use the spectral gap Theorem 3.4 to prove the exponential decay rate for $u(x, t)$. To take advantage the lower bound of the linearized nonlocal operator L for functions orthogonal to its null space $\text{Null}(L)$, we need to first shift the standing profile in terms of a dynamic coordinate. We construct a shift function $\alpha(t)$ such that

$$\phi_\alpha(x, t) := \phi(x - x_0 - \alpha(t)), \quad v_\alpha(x, t) := u(x, t) - \phi_\alpha(x, t) \tag{4.1}$$

satisfy

$$v_\alpha \perp \text{Null}(\mathcal{L}_\alpha), \quad \mathcal{L}_\alpha := (-\partial_{xx})^{\frac{1}{2}} + f'(\phi_\alpha)$$

i.e.

$$\begin{aligned} & \int_{-\infty}^{\infty} (u(x, t) - \phi(x - x_0 - \alpha(t)))\phi'(x - x_0 - \alpha(t)) \, dx \\ &= \int_{-\infty}^{\infty} v_\alpha(x, t)\phi'(x - x_0 - \alpha(t)) \, dx = 0. \end{aligned} \tag{4.2}$$

Notice that $\int_{-\infty}^{\infty} \phi(x)\phi'(x) \, dx = 0$. Define a functional of α as

$$W(t, \alpha) := \int_{-\infty}^{\infty} u(x, t)\phi'(x - x_0 - \alpha) \, dx. \tag{4.3}$$

The following proposition is to clarify the existence, uniqueness and properties of $\alpha(t)$ and it also provides an elementary proof for the implicit function theorem in the unbounded domain.

PROPOSITION 4.1. *For $W(t, \alpha)$ in (4.2), there exist $T > 0$ large enough and a unique $\alpha(t)$ such that*

- (i) $W(t, \alpha(t)) = 0$ for $t > T$;
- (ii) $\alpha(t) \rightarrow 0$ as $t \rightarrow +\infty$;
- (iii) $\alpha(t) \in C^1(T, +\infty)$.

Proof. Step 1. We prove the existence and bound of $\alpha(t)$. Using the intermediate value theorem and (2.37), we will first prove there exists $T > 0$ such that for any $t > T$ there exist at least one $\alpha(t)$ such that $W(t, \alpha(t)) = 0$ for $t > T$. Moreover, for all the solutions to $W(t, \alpha(t)) = 0$, there exist a_T, b_T such that $\alpha(t) \in [a_T, b_T]$ for $t > T$.

By (2.37), for $t > T$ large enough and ε small enough, we know that there exists x_0 such that

$$\int_{-\infty}^{\infty} (\phi(x - x_0 - \frac{\mu + k}{\mu\beta}\varepsilon) - \varepsilon e^{-\mu t})\phi'(x - x_0 - \alpha) \, dx \leq W(t, \alpha),$$

$$W(t, \alpha) \leq \int_{-\infty}^{\infty} (\phi(x - x_0 + \frac{\mu + k}{\mu\beta}\varepsilon) + \varepsilon e^{-\mu t})\phi'(x - x_0 - \alpha) \, dx,$$

or equivalently

$$\int_{-\infty}^{\infty} (\phi(x + \alpha - \frac{\mu + k}{\mu\beta}\varepsilon) - \varepsilon e^{-\mu t})\phi'(x) \, dx \leq W(t, \alpha), \tag{4.4}$$

$$W(t, \alpha) \leq \int_{-\infty}^{\infty} (\phi(x + \alpha + \frac{\mu + k}{\mu\beta}\varepsilon) + \varepsilon e^{-\mu t})\phi'(x) \, dx. \tag{4.5}$$

We choose T such that $\varepsilon e^{-\mu T} < \frac{1}{2}$. Therefore

$$1 < \int_{-\infty}^{\infty} (1 - \varepsilon e^{-\mu t})\phi'(x) \, dx \leq \lim_{\alpha \rightarrow \infty} W(t, \alpha)$$

$$\lim_{\alpha \rightarrow -\infty} W(t, \alpha) \leq \int_{-\infty}^{\infty} (-1 + \varepsilon e^{-\mu t}) \phi'(x) \, dx < -1,$$

for any $t > T$. Hence by the intermediate value theorem there is at least one $\alpha(t)$.

Next, define b_T as the solution of

$$\int_{-\infty}^{\infty} \phi(x + b_T - \frac{\mu + k}{\mu\beta} \varepsilon) \phi'(x) \, dx = 2\varepsilon e^{-\mu T} < 1$$

and a_T as the solution of

$$\int_{-\infty}^{\infty} \phi(x + a_T + \frac{\mu + k}{\mu\beta} \varepsilon) \phi'(x) \, dx = -2\varepsilon e^{-\mu T} > -1.$$

From (4.4),

$$\begin{aligned} 0 = W(t, \alpha(t)) &\geq \int_{-\infty}^{\infty} \phi\left(x + \alpha(t) - \frac{\mu + k}{\mu\beta} \varepsilon\right) \phi'(x) \, dx - 2\varepsilon e^{-\mu t} \\ &\geq \int_{-\infty}^{\infty} \phi\left(x + \alpha(t) - \frac{\mu + k}{\mu\beta} \varepsilon\right) \phi'(x) \, dx - 2\varepsilon e^{-\mu T} \\ &\geq \int_{-\infty}^{\infty} \phi\left(x + \alpha(t) - \frac{\mu + k}{\mu\beta} \varepsilon\right) \phi'(x) \, dx - \int_{-\infty}^{\infty} \phi\left(x + b_T - \frac{\mu + k}{\mu\beta} \varepsilon\right) \phi'(x) \, dx. \end{aligned}$$

This implies $\alpha(t) \leq b_T$ since $\int_{-\infty}^{\infty} \phi(x + \alpha - \frac{\mu+k}{\mu\beta}\varepsilon)\phi'(x) \, dx$ is increasing with respect to α . Similarly, we can use (4.5) to obtain $\alpha(t) \geq a_T$ so $a_T \leq \alpha(t) \leq b_T$.

Step 2. Uniqueness of $\alpha(t)$. Differentiating G with respect α yields

$$\begin{aligned} \partial_{\alpha} W &= \int_{-\infty}^{\infty} -u(x, t) \phi''(x - x_0 - \alpha) \, dx \\ &= \int_{-\infty}^{\infty} \phi'(x + \alpha) \phi'(x) \, dx - \int_{-\infty}^{\infty} (u(x, t) - \phi(x - x_0)) \phi''(x - x_0 - \alpha) \, dx \\ &\geq \int_{-\infty}^{\infty} \phi'(x + \alpha) \phi'(x) \, dx - \max_x |u(x, t) - \phi(x - x_0)| \int_{-\infty}^{\infty} |\phi''(x)| \, dx > 0 \end{aligned}$$

for large $t > T_2$. Here we used $b(t) = \max_{x \in \mathbb{R}} |u(x, t) - \phi(x - x_0)| \rightarrow 0$ as $t \rightarrow +\infty$ from Theorem 2.9.

Step 3. We prove $\alpha(t) \rightarrow 0$ as $t \rightarrow +\infty$.

If $\alpha(t) \not\rightarrow 0$ as $t \rightarrow +\infty$, then there are constant $a > 0$ and a sequence $t_k \rightarrow \infty$ as $k \rightarrow \infty$ such that $b_T \geq \alpha(t_k) \geq a$ (or $a_T \leq \alpha(t_k) \leq -a$). Then we have a subsequence (still denote as t_k) and $a^* > 0$ such that $\alpha(t_k) \rightarrow a^*$. Recall (4.2), which shows

$$\begin{aligned} &\int_{-\infty}^{\infty} (\phi(x - x_0) - \phi(x - x_0 - \alpha(t_k))) \phi'(x - x_0 - \alpha(t_k)) \, dx \\ &= \int_{-\infty}^{\infty} (\phi(x - x_0) - u(x, t_k)) \phi'(x - x_0 - \alpha(t_k)) \, dx. \end{aligned}$$

Taking limit as $t_k \rightarrow \infty$ in

$$\begin{aligned} & \int_{-\infty}^{\infty} (\phi(x - x_0) - \phi(x - x_0 - \alpha(t_k)))\phi'(x - x_0 - \alpha(t_k)) \, dx \\ &= \int_{-\infty}^{\infty} (\phi(x - x_0) - u(x, t_k))\phi'(x - x_0 - \alpha(t_k)) \, dx \\ &\leq \max_x |\phi(x - x_0) - u(x, t_k)| \int_{-\infty}^{\infty} \phi'(x - x_0 - \alpha(t_k)) \, dx \\ &= 2 \max_x |\phi(x - x_0) - u(x, t_k)| \rightarrow 0, \end{aligned}$$

leads to

$$\int_{-\infty}^{\infty} (\phi(x - x_0) - \phi(x - x_0 - a^*))\phi'(x - x_0 - a^*) \, dx \leq 0. \tag{4.6}$$

On the other hand, since $a^* > 0$

$$\phi(x - x_0) - \phi(x - x_0 - a^*) \geq 0. \tag{4.7}$$

Then due to $\phi' > 0$, (4.6) and (4.7) lead to

$$\int_{-\infty}^{\infty} (\phi(x - x_0) - \phi(x - x_0 - a^*))\phi'(x - x_0 - a^*) \, dx = 0,$$

which is a contradiction due to $a^* > 0$.

Step 4. $\alpha(t) \in C^1(T, +\infty)$ is directly from the implicit function theorem. \square

Next, we prove the shift $\alpha(t)$ introduced above contributes an exponentially small error.

LEMMA 4.2. *For $\alpha(t)$ and $v_\alpha(x, t)$ defined in (4.1), there are constants C and μ such that*

- (i) $\|v_\alpha\| \leq Ce^{-\mu t}$;
- (ii) $|\alpha(t)| \leq Ce^{-\mu t}$.

Proof. Step 1. Decay of $\|v_\alpha\|$. From Theorem 2.9, we have $b(t) = \max_{x \in \mathbb{R}} |u(x, t) - \phi(x - x_0)| \rightarrow 0$. Since

$$\max_{x \in \mathbb{R}} |v_\alpha(x, t)| \leq b(t) + c_1 \alpha(t) \tag{4.8}$$

for $c_1 := \max_{x \in \mathbb{R}} \phi'(x) = \frac{2}{\pi}$, we have

$$\max_{x \in \mathbb{R}} |v_\alpha(x, t)| \rightarrow 0, \quad \text{as } t \rightarrow +\infty \tag{4.9}$$

due to Proposition 4.1 (ii). From the definition of v_α , for any $x \in \mathbb{R}$,

$$\begin{aligned} \partial_t v_\alpha &= \partial_t u + \alpha' \partial_x \phi_\alpha \\ &= -\mathcal{A}u + \mathcal{A}\phi_\alpha + \alpha' \partial_x \phi_\alpha \\ &= -\mathcal{L}_\alpha u - f(u) + f'(\phi_\alpha)u + \mathcal{A}\phi_\alpha + \alpha' \partial_x \phi_\alpha \\ &= -\mathcal{L}_\alpha v_\alpha - f(\phi_\alpha + v_\alpha) + f(\phi_\alpha) + f'(\phi_\alpha)v_\alpha + \alpha' \partial_x \phi_\alpha \\ &= -\mathcal{L}_\alpha v_\alpha - \frac{1}{2} f''(\xi)v_\alpha^2 + \alpha' \partial_x \phi_\alpha, \end{aligned} \tag{4.10}$$

where $\xi := \xi(x)$ locates between $\phi_\alpha(x)$ and $\phi_\alpha(x) + v_\alpha(x, t)$.

Since the shift $\alpha(t) \rightarrow 0$ and the upper/lower bound of $f'(\phi_\alpha) = -\cos(\pi\phi_\alpha)$ remains same, we can directly apply the spectral gap Theorem 3.4 to \mathcal{L}_α to obtain

$$\langle \mathcal{L}_\alpha v_\alpha, v_\alpha \rangle \geq \lambda_2 \|v_\alpha\|^2.$$

Therefore, multiplying v_α to both sides of (4.10) and integrating with respect to x lead to

$$\frac{d}{dt} \frac{1}{2} \|v_\alpha(\cdot, t)\|^2 \leq -\lambda_2 \|v_\alpha(\cdot, t)\|^2 - \frac{1}{2} \int_{\mathbb{R}} f''(\xi(x)) v_\alpha^3(x, t) dx, \quad (4.11)$$

where we used $\langle \phi'_\alpha, v_\alpha \rangle = 0$. For the second term $\int f''(\xi(x)) v_\alpha(x, t)^3 dx$, from (4.9),

$$\left| \int_{\mathbb{R}} f''(\xi(x)) v_\alpha^3(x, t) dx \right| \leq \max_{x \in \mathbb{R}} (v_\alpha(x, t) f''(\xi(x))) \|v_\alpha(\cdot, t)\|^2 \leq \frac{\lambda_2}{2} \|v_\alpha(\cdot, t)\|^2$$

for t large enough. Therefore, (4.11) gives the exponential decay rate for v_α

$$\|v_\alpha(\cdot, t)\| \leq C e^{-\mu t}. \quad (4.12)$$

Step 2. Decay of $\alpha(t)$. Multiply (4.10) by $\partial_x \phi_\alpha$, then we have

$$\begin{aligned} \langle \partial_x \phi_\alpha, \partial_t v_\alpha \rangle &= \langle -\mathcal{L}_\alpha v_\alpha + \alpha' \partial_x \phi_\alpha - \frac{1}{2} f''(\xi) v_\alpha^2, \partial_x \phi_\alpha \rangle \\ &= \alpha' \|\partial_x \phi_\alpha\|^2 - \int_{\mathbb{R}} \frac{1}{2} f''(\xi) v_\alpha^2 \partial_x \phi_\alpha dx, \end{aligned} \quad (4.13)$$

where we used $\mathcal{L}_\alpha \partial_x \phi_\alpha = \mathcal{L}_\alpha \phi'_\alpha = 0$. Differentiating the relation (4.2) with respect to t leads to

$$\int_{\mathbb{R}} \phi'_\alpha \partial_t v_\alpha dx = \alpha' \int_{\mathbb{R}} \phi''_\alpha v_\alpha dx, \quad (4.14)$$

which is the left-hand-side of (4.13). Thus (4.13) becomes

$$\alpha' \|\phi'_\alpha\|^2 = \alpha' \int_{\mathbb{R}} \phi''_\alpha v_\alpha dx + \int_{\mathbb{R}} \frac{1}{2} f''(\xi) v_\alpha^2 \phi'_\alpha dx.$$

This, together with the decay of $\|v_\alpha\|$ in Step 1, shows

$$|\alpha'| \|\phi'_\alpha\|^2 \leq |\alpha'| \|\phi''_\alpha\| \|v_\alpha\| + C \max |\phi'_\alpha| \|v_\alpha\|^2 \leq C e^{-\mu t}.$$

Notice also $\|\phi'_\alpha\|^2 = \|\phi'\|^2 = \int \frac{4}{2\pi^2} \frac{1}{(1+x^2)^2} dx$ and $\alpha(+\infty) = 0$. Then standard calculus gives the exponential decay of $|\alpha(t)|$. \square

Finally we collect all the results above and complete the proof of Theorem 1.1.

Proof of Theorem 1.1. In Lemma 4.2, we proved

$$\|v_\alpha\| \leq C e^{-\mu t}.$$

Since $\phi \in H^1(\mathbb{R})$, then from Theorem 2.1 we know $u \in C(0, \infty; H^1(\mathbb{R}))$ and thus $v_\alpha \in C(0, \infty; H^1(\mathbb{R}))$. However, the uniform $H^1(\mathbb{R})$ bound for u , as well as v_α , is only valid for some sequence t_n . Therefore we need to use the same trick in the proof of Theorem 2.9 as explained below. From Theorem 2.9, there exists a time sequence

$t_n \rightarrow +\infty$ such that $\mathcal{Q}(t_n) \rightarrow 0$ and thus $\|u(\cdot, t_n)\|_{\dot{H}^1} \leq c$ due to (2.15). Applying Ladyzhenskaya's inequality to $v_\alpha(\cdot, t_n)$, we have

$$\begin{aligned} \|v_\alpha(\cdot, t_n)\|_{L^\infty} &\leq \sqrt{2}\|v_\alpha(\cdot, t_n)\|^{1/2}\|v_\alpha(\cdot, t_n)\|_{\dot{H}^1}^{1/2} \\ &\leq c\|v_\alpha(\cdot, t_n)\|^{1/2}. \end{aligned} \tag{4.15}$$

Hence we obtain the pointwise decay rate for the subsequence $v_\alpha(x, t_n)$

$$\|v_\alpha(\cdot, t_n)\|_{L^\infty} \leq Ce^{-\mu t_n} \tag{4.16}$$

for t_n large enough. Notice $\phi'(x) = \frac{2}{\pi} \frac{1}{1+x^2}$, which has a maximum $\frac{2}{\pi}$. Thus

$$\begin{aligned} |u(x, t_n) - \phi(x, x_0)| &\leq |v_\alpha(x, t_n)| + |\phi(x - x_0) - \phi(x - x_0 - \alpha(t_n))| \\ &\leq |v_\alpha(x, t_n)| + \max_{x \in \mathbb{R}} |\phi'| |\alpha(t_n)| \leq |v_\alpha(x, t_n)| + \frac{2}{\pi} |\alpha(t_n)| \leq Ce^{-\mu t_n}, \end{aligned}$$

uniformly in x due to Lemma 4.2 and (4.16). Then by the stability result Lemma 2.8, we know for any t large enough,

$$|u(x, t) - \phi(x, x_0)| < ce^{-\mu t} \quad \text{uniformly in } x \in \mathbb{R}. \tag{4.17}$$

Notice also the basic estimate for u in Lemma 2.6 which gives

$$\sup_{x \in \mathbb{R}} |u(x, t) - \phi(x - x_0)| \leq \frac{c}{1 + |x|}, \quad \text{for any } t > 0.$$

We complete the proof of the main Theorem 1.1. \square

Proof of Theorem 1.2. The proof of Theorem 1.1 for the special potential $F(u) = \frac{1}{\pi^2}(1 + \cos(\pi u))$ relies only on the concave part of F is $(-\frac{1}{2}, \frac{1}{2})$. Therefore, after replacing the concave part of F in Proposition 2.4 and Proposition 3.3 by $[a_1, a_2] \subset (-1, 1)$, the proof of Theorem 1.2 is exactly the same as the proof of Theorem 1.1. \square

REMARK 7. To the end, we discuss the relation to the classical Benjamin-Ono equation. Benjamin-Ono equation is a nonlinear partial integrodifferential equation describing one dimensional internal waves in deep water. Consider

$$h_t = (-\partial_{xx})^{\frac{1}{2}}h_x + h_x - 2hh_x. \tag{4.18}$$

Denote

$$E_B(h) := \frac{1}{2} \langle (-\partial_{xx})^{\frac{1}{2}}h, h \rangle + \frac{h^2}{2} - \frac{h^3}{3}$$

with

$$\frac{\delta E_B}{\delta h} = -(-\partial_{xx})^{\frac{1}{2}}h + h - h^2.$$

Then (4.18) becomes

$$h_t = \partial_x \left(\frac{\delta E_B}{\delta h} \right),$$

which is a Hamiltonian system. If we consider a special one-parameter family transformation T_c such that

$$h(x, t) = T_c u := u(x - ct),$$

then we have the traveling wave form of the Benjamin-Ono equation

$$(-\partial_{xx})^{\frac{1}{2}}u = u^2 + (-1 - c)u.$$

Let $W := (1 + c)\frac{u^2}{2} - \frac{u^3}{3}$ with $W' = (1 + c)u - u^2$ and $W'' = 1 + c - 2u$. The special traveling wave form of the Benjamin-Ono equation is

$$(-\partial_{xx})^{\frac{1}{2}}u = u^2 - (1 + c)u = -W'(u), \quad (4.19)$$

which is closely related to our static equation (1.9). Benjamin [4] found that $\Phi_c = \frac{2(c+1)}{1+(1+c)^2x^2}$ is a solitary solution to (4.19) which, apart from the periodic solution, is unique up to translations [1]. For instance for $c = 0$, notice $\Phi^2 - \Phi = \frac{2(1-x^2)}{(1+x^2)^2}$ and $(H(\Phi))' = (\frac{2x}{1+x^2})'$, then $\Phi = \frac{2}{1+x^2}$ is a solitary solution.

Define the linearized operator of the Benjamin-Ono equation along its solitary profile Φ_c as

$$L_B u := (-\partial_{xx})^{\frac{1}{2}}u + W''(\Phi)u = (-\partial_{xx})^{\frac{1}{2}}u + \frac{(1+c)^3x^2 - 3(c+1)}{(1+c)^2x^2 + 1}u, \quad (4.20)$$

whose potential $\frac{(1+c)^3x^2 - 3(c+1)}{(1+c)^2x^2 + 1}$ is very similar to our problem $\frac{x^2 - 1}{x^2 + 1}$ in Section 3; with lower bound $-3(c+1)$ and upper bound $1 + c$. The spectral analysis for this kind of self-adjoint operators like L_B and L defined in (3.1) is standard. But for completeness, we give a new proof involving some particular global properties of the fractional Laplace operator; see Proposition 3.3.

One may also notice that unlike the solitary profile to Benjamin-Ono equation which vanishes at far field, in the PN model the steady profile to (1.9) is a transition connecting from -1 to 1 due to the double well potential. The dynamic PN model is a gradient flow while the Benjamin-Ono equation is a Hamiltonian flow. However, the steady profile are closely related, the derivative of $\pi\phi$ is exactly $\Phi(x)$; see more connections in [38]. We refer to [5] for the orbit stability of solitary solutions to (4.18); see also [40, 6, 22] for more general integrodifferential equations.

5. Discussion on slow motion of dislocations. Now we explain how to choose the slow time scale in order to observe the slow motion behavior after typical N -transition layers pattern formation. Recall the rescaled PDE (1.18)

$$\partial_t v^\varepsilon + \frac{1}{\varepsilon} \left[(-\partial_{xx})^{\frac{1}{2}}v^\varepsilon + \frac{1}{\varepsilon}F'(v^\varepsilon) \right] = 0 \quad (5.1)$$

Using the abstract framework of approximated invariant manifold [34], denote

$$\mathcal{M} := \left\{ v^\varepsilon(x; h) = \frac{1 + (-1)^N}{2} + \sum_{i=1}^N \phi\left(\frac{x - h_i}{\varepsilon}\right); h := \{h_i\}_{i=1}^N \in \mathbb{R}^N \right\}. \quad (5.2)$$

We will show formally if \mathcal{M} is an N -dimensional approximated invariant manifold, i.e. we have a gradient flow restricted to this finite dimensional manifold, then it leads to the expected particle system describing the motion of N transition layers. More explicitly, denote the tangent space as $T_{\mathcal{M}}$ then any $v_\alpha \in T_{\mathcal{M}}$ can be expressed as

$$v_\alpha = \sum_{i=1}^N \alpha_i \frac{\partial v^\varepsilon(x; h)}{\partial h_i} = -\frac{1}{\varepsilon} \sum_{i=1}^N \alpha_i \phi'\left(\frac{x - h_i}{\varepsilon}\right), \quad \alpha := \{\alpha_i\}_{i=1}^N \in \mathbb{R}^N. \quad (5.3)$$

Define the Riemannian metrics $g_h : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ as

$$g_h(\alpha, \beta) := \langle v_\alpha, v_\beta \rangle = \sum_{i,j} g_{ij} \alpha_i \beta_j \tag{5.4}$$

with

$$g_{ij} = \frac{1}{\varepsilon^2} \int_{\mathbb{R}} \phi' \left(\frac{x - h_i}{\varepsilon} \right) \phi' \left(\frac{x - h_j}{\varepsilon} \right) dx \approx \begin{cases} 0, & i \neq j, \\ \frac{1}{\varepsilon} \int_{\mathbb{R}} |\phi'|^2 dx, & i = j, \end{cases} \tag{5.5}$$

due to the tail estimate (1.10). Then the gradient flow of the total energy of (1.18)

$$\mathcal{F}_\varepsilon(v^\varepsilon) := \int_{\mathbb{R}} \frac{1}{2\varepsilon} |(-\partial_{xx})^{\frac{1}{4}} v^\varepsilon|^2 + \frac{1}{\varepsilon^2} F(v^\varepsilon) dx \tag{5.6}$$

is given by

$$g_h(\alpha, \partial_t h) \approx - \frac{d}{d\tau} \Big|_{\tau=0} \mathcal{F}_\varepsilon(v^\varepsilon(x; h + \tau\alpha)) \quad \text{for any } \alpha \in \mathbb{R}^N. \tag{5.7}$$

Thus some straightforward calculations yield

$$\begin{aligned} \frac{1}{\varepsilon} \int_{\mathbb{R}} |\phi'|^2 dx \sum_{j=1}^N \alpha_j h'_j &\approx - \frac{1}{\varepsilon^2} \left\langle \left[\sum_{i=1}^N (-\partial_{xx})^{\frac{1}{2}} \phi \Big|_{\frac{x-h_i(t)}{\varepsilon}} + F'(v^\varepsilon) \right], v_\alpha \right\rangle \\ &= - \frac{1}{\varepsilon^2} \left\langle \left[- \sum_{i=1}^N F' \left(\phi \left(\frac{x - h_i(t)}{\varepsilon} \right) \right) + F'(v^\varepsilon) \right], v_\alpha \right\rangle. \end{aligned}$$

Equivalently, taking $v_\alpha = -\frac{1}{\varepsilon} \phi' \left(\frac{x-h_j(t)}{\varepsilon} \right)$, we have for $j = 1, \dots, N$,

$$\begin{aligned} \int_{\mathbb{R}} |\phi'|^2 dx h'_j &\approx \frac{1}{\varepsilon^2} \left\langle \left[- \sum_{i=1}^N F' \left(\phi \left(\frac{x - h_i(t)}{\varepsilon} \right) \right) \right. \right. \\ &\quad \left. \left. + F' \left(\frac{1 + (-1)^N}{2} + \sum_{i=1}^N \phi \left(\frac{x - h_i(t)}{\varepsilon} \right) \right) \right], \phi' \left(\frac{x - h_j(t)}{\varepsilon} \right) \right\rangle. \end{aligned} \tag{5.8}$$

As for how much longer we shall wait to observe an $O(1)$ particle system (1.20) indicating the motion of each transition layer $h_i(t)$, we show below heuristically that the time scale depends only on the tail (decay rate) of steady profile ϕ . Notice the decay estimate (1.10) shows that

$$\phi \left(\frac{h_i - h_j}{\varepsilon} \right) \sim 2H(x) - 1 - \frac{c\varepsilon}{h_i - h_j}, \quad \text{as } \varepsilon \rightarrow 0. \tag{5.9}$$

To be more general, assume the tail estimate (5.9) is replaced by

$$\phi \left(\frac{h_i - h_j}{\varepsilon} \right) \sim 2H(x) - 1 - K \left(\frac{h_i - h_j}{\varepsilon} \right), \quad \text{as } \varepsilon \rightarrow 0. \tag{5.10}$$

To calculate leading behavior of the particle system for h_j , from $F(\pm 1) = 0$, F is 2-periodic and the tail estimate (5.9), we also know

$$\left\langle \sum_{i=1}^N F' \left(\phi \left(\frac{x - h_i(t)}{\varepsilon} \right) \right), \phi' \left(\frac{x - h_j(t)}{\varepsilon} \right) \right\rangle \approx - \sum_{i \neq j} c\varepsilon K \left(\frac{h_j - h_i}{\varepsilon} \right), \tag{5.11}$$

$$\begin{aligned} & \left\langle F' \left(\frac{1 + (-1)^N}{2} + \sum_{i=1}^N \phi \left(\frac{x - h_i(t)}{\varepsilon} \right) \right), \phi' \left(\frac{x - h_j(t)}{\varepsilon} \right) \right\rangle \\ &= \left\langle F' \left(\phi' \left(\frac{x - h_j(t)}{\varepsilon} \right) + \left[\frac{1 + (-1)^N}{2} + \sum_{i \neq j}^N \phi \left(\frac{x - h_i(t)}{\varepsilon} \right) \right] \right), \phi' \left(\frac{x - h_j(t)}{\varepsilon} \right) \right\rangle \approx 0 \end{aligned}$$

due to $\int F'(\phi)\phi' dx = 0$. Thus the leading behavior of the particle system is

$$h'_j \approx \frac{c}{\varepsilon} \sum_{i \neq j} K \left(\frac{h_j - h_i}{\varepsilon} \right), \quad j = 1, \dots, N, \quad (5.12)$$

which is a $O(1)$ particle system independent of ε if $K(r) = \frac{1}{r}$. This $\frac{1}{r}$ interaction is a typical example (edge dislocations with same orientations) of the Peach-Koehler force, which in our case is a repulsive force acting on dislocations [33]. It shows the time scale $\frac{t}{\varepsilon^2}$ depends only on K , i.e. how fast ϕ decay at far field. To rigorously prove the above approximate, one needs add a corrector ψ to the ansatz such that $\langle \psi, \phi' \rangle = 0$, i.e. a corrector perpendicular to the ground state ϕ' . The existence of such a corrector is ensured by the spectral gap analysis (Theorem 3.4), or equivalently the coercivity in Lax-Milgram theorem [21, Theorem 3.2]. For more general fractional operator $(-\partial_{xx})^s$ with $0 < s < 1$, in order to observe the motion of N transition layers, the slow time scale should be $\frac{t}{\varepsilon^{1+2s}}$ due to the tail estimate for the corresponding steady profile $\phi \sim H(x) - \frac{cx}{|x|^{2s+1}}$; see details in [13, 14].

It is worth to point out that sometimes we can not obtain an $O(1)$ particle system independent of ε by choosing proper slow time scale. Indeed, if the tail estimate is not algebraic decay, for example $K(r) = e^{-|r|}$, [11] study the motion of metastable patterns for $u_t = \varepsilon^2 \partial_{xx} u - W'(u)$ at original time scale, where the double well potential is $W(u) = \frac{1}{8}(u^2 - 1)^2$. Then the corresponding steady profile, still denoted as ϕ , satisfies

$$\phi(x) = \tanh(x/2), \quad \int_{\mathbb{R}} |\phi'|^2 dx = \frac{2}{3}, \quad \phi(x) \sim \pm 1 \mp 2e^{-|x|} \text{ as } x \rightarrow \pm\infty. \quad (5.13)$$

We consider a new approximated invariant manifold,

$$\mathcal{M} := \left\{ v^\varepsilon = \frac{1 + (-1)^N}{2} + \sum_{i=1}^N (-1)^i \phi \left(\frac{x - h_i}{\varepsilon} \right), h \in \mathbb{R}^N \right\}. \quad (5.14)$$

This manifold describes a typical pattern alternating between ± 1 with $v^\varepsilon(-\infty) = 1$, $v^\varepsilon(+\infty) = (-1)^N$. Then by the same calculations, (5.8) becomes

$$\begin{aligned} \int_{\mathbb{R}} |\phi'|^2 dx h'_j &\approx \frac{1}{\varepsilon^2} \left\langle \left[- \sum_{i=1}^N F'((-1)^i \phi \left(\frac{x - h_i(t)}{\varepsilon} \right)) \right. \right. \\ &\quad \left. \left. + F' \left(\sum_{i=1}^N (-1)^i \phi \left(\frac{x - h_i(t)}{\varepsilon} \right) \right) \right], (-1)^j \phi' \left(\frac{x - h_j(t)}{\varepsilon} \right) \right\rangle. \end{aligned} \quad (5.15)$$

Since $W'(\pm 1) = W''(\pm 1) = 0$, we have

$$\begin{aligned} & \left\langle \sum_{i=1}^N F'((-1)^i \phi \left(\frac{x - h_i(t)}{\varepsilon} \right)), (-1)^j \phi' \left(\frac{x - h_j(t)}{\varepsilon} \right) \right\rangle \\ &\approx - \sum_{i>j} (-1)^{i+j} 4e^{-|h_i - h_j|/\varepsilon} + \sum_{i<j} (-1)^{i+j} 4e^{-|h_i - h_j|/\varepsilon}, \end{aligned} \quad (5.16)$$

$$\begin{aligned}
& \left\langle F' \left(\sum_{i=1}^N (-1)^i \phi \left(\frac{x - h_i(t)}{\varepsilon} \right) \right), (-1)^j \phi' \left(\frac{x - h_j(t)}{\varepsilon} \right) \right\rangle \\
& \approx \left\langle F' \left(\pm 1 - \sum_{i>j} (-1)^i 2e^{-|h_i - h_j|/\varepsilon} + \sum_{i<j} (-1)^i 2e^{-|h_i - h_j|/\varepsilon} \right), (-1)^j \phi' \left(\frac{x - h_j(t)}{\varepsilon} \right) \right\rangle \\
& \approx - \sum_{i>j} (-1)^{i+j} 4e^{-|h_i - h_j|/\varepsilon} + \sum_{i<j} (-1)^{i+j} 4e^{-|h_i - h_j|/\varepsilon}.
\end{aligned} \tag{5.17}$$

From this, together with (5.13) and (5.15), we obtain

$$h'_k \approx 12\varepsilon \left(\sum_{i<k} (-1)^{i+k} e^{-|h_i - h_k|/\varepsilon} - \sum_{i>k} (-1)^{i+k} e^{-|h_i - h_k|/\varepsilon} \right), \quad k = 1, \dots, N, \tag{5.18}$$

using proper boundary data. For fixed N , [11] also proved the global interactions in (5.18) can be controlled by the nearest-neighbor interaction for small ε , which leads to

$$h'_k \approx 12\varepsilon \left(e^{-(h_{k+1} - h_k)/\varepsilon} - e^{-(h_k - h_{k-1})/\varepsilon} \right), \quad k = 1, \dots, N. \tag{5.19}$$

Here we set $h_{N+1} = +\infty, h_0 = -\infty$.

In solid state physics, the algebraic decay and exponential decay are two typical tail estimates indicating the physical interactions between particles. For instant, for the $K(r) = \frac{1}{r} + \frac{1}{r^3}$, which shows elastic long range interaction between two steps in the epitaxial growth, [19] study the mean field limit of a similar particle system in a larger scale (taking particle number N goes to ∞). They prove if the tail estimate is faster than a quadratic decay rate, in the mean field limit, the corresponding continuum PDE from (5.18) with only nearest-neighbor interactions is same as the mean field limit of (5.18) with global interactions.

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Appendix A. Proof of Theorem 2.1.

Proof of Theorem 2.1. Step 1. We collect some properties for G defined in (2.3).
 (a) $G : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ is global Lipschitz, i.e.

$$\|G(v_1) - G(v_2)\| \leq (1 + \max |f'|)\|v_1 - v_2\| \leq 2\|v_1 - v_2\|. \tag{A.1}$$

(b) If $v(\cdot) \in H^1(\mathbb{R})$, then $G(v(\cdot)) \in H^1(\mathbb{R})$. Indeed,

$$\|\partial_x G(v)\| \leq \|\partial_x v\| + \pi\|v\|,$$

which implies

$$\|G(v)\|_1 \leq (\pi + 2)\|v\|_1. \tag{A.2}$$

Step 2. First it is easy to check that operator A defined in (2.3) is m -accretive in $L^2(\mathbb{R})$. Indeed we know $\text{Re}\langle Ax, x \rangle \geq 0$ for all $x \in D(A)$ and from Lemma B.6, we know $\sigma(A) = [1, +\infty)$. Therefore A is an infinitesimal generator of a linear strongly continuous semigroup of contractions and $\|e^{-At}\| \leq 1$. Second from the global Lipschitz condition (A.1), there exists a unique mild solution expressed by (2.6) and $v \in C([0, +\infty); L^2(\mathbb{R}))$.

Step 3. Lipschitz continuity in t of v and $G(v)$.

$$\begin{aligned} & v(t+h) - v(t) \tag{A.3} \\ &= e^{-At}(e^{-Ah}v_0 - v_0) + \int_0^{t+h} e^{-A(t+h-\tau)}G(v(\tau)) \, d\tau - \int_0^t e^{-A(t-\tau)}G(v(\tau)) \, d\tau \\ &= e^{-At}[(e^{-Ah}v_0 - v_0) + \int_0^h e^{-A(h-\tau)}G(v(\tau)) \, d\tau] \\ &\quad + \int_0^t e^{-A(t-\tau)}[G(v(\tau+h)) - G(v(\tau))] \, d\tau \\ &= e^{-At}(v(h) - v_0) + \int_0^t e^{-A(t-\tau)}[G(v(\tau+h)) - G(v(\tau))] \, d\tau. \end{aligned}$$

Since $\|e^{-At}\| \leq 1$,

$$\|v(t+h) - v(t)\| \leq \|v(h) - v_0\| + \int_0^t 2\|v(\tau+h) - v(\tau)\| \, d\tau.$$

Then by Grönwall’s inequality, we have

$$\|v(t+h) - v(t)\| \leq \|v(h) - v_0\|e^{2t}. \tag{A.4}$$

On the other hand,

$$v(h) - v_0 = (e^{-Ah} - I)v_0 + \int_0^h e^{-A(h-\tau)} [G(v(\tau)) - G(v_0) + G(v_0)] d\tau. \quad (\text{A.5})$$

Then from (A.1) and $\|e^{-At}\| \leq 1$ we know

$$\begin{aligned} \|v(h) - v_0\| &\leq h\|Av_0\| + 2 \int_0^h \|v(\tau) - v_0\| d\tau + 2h\|v_0\| \\ &= h(2\|v_0\| + \|Av_0\|) + 2 \int_0^h \|v(\tau) - v_0\| d\tau. \end{aligned} \quad (\text{A.6})$$

Thus Grönwall's inequality gives us

$$\|v(h) - v_0\| \leq h(2\|v_0\| + \|Av_0\|)e^{2h}, \quad (\text{A.7})$$

which, together with (A.4), leads to the Lipschitz continuity of $v(t)$

$$\left\| \frac{v(t+h) - v(t)}{h} \right\| \leq 2\|v_0\|_1 e^{2t+2h}. \quad (\text{A.8})$$

Then from (A.1) we concludes the Lipschitz continuity of $G(v(t))$

$$\left\| \frac{G(v(t+h)) - G(v(t))}{h} \right\| \leq 4\|v_0\|_1 e^{2t+2h}. \quad (\text{A.9})$$

Moreover, from (A.3) we know

$$\frac{v(t+h) - v(t)}{h} = e^{-At} \frac{v(h) - v_0}{h} + \int_0^t e^{-A(t-\tau)} \frac{G(v(\tau+h)) - G(v(\tau))}{h} d\tau. \quad (\text{A.10})$$

On one hand, by the reflexivity of L^2 space and generalized Rademacher's theorem, there exists $g(t) \in L^1(0, T; L^2(\mathbb{R}))$ such that for a.e. $t \geq 0$,

$$\lim_{h \rightarrow 0} \frac{G(v(t+h)) - G(v(t))}{h} = g(t), \quad (\text{A.11})$$

i.e. $g(t) = \partial_t G(v(t))$ which is the Fréchet derivative of G . Then by Lebesgue's dominated convergence theorem and (A.9), we know the limit for the second term on the right hand side of (A.10) exists. On the other hand, from (A.5),

$$\begin{aligned} \frac{v(h) - v_0}{h} &= \frac{e^{-Ah} - I}{h} v_0 + \frac{1}{h} \int_0^h e^{-A(h-\tau)} [G(v(\tau)) - G(v_0) + G(v_0)] d\tau \\ &\rightarrow -Av_0 + G(v_0) \end{aligned}$$

due to continuity of G , so the first term on the right hand side of (A.10) converges. Then by Lebesgue's dominated convergence theorem and (A.8), we know the limit for the second term on the right hand side of (A.10) also exists. Therefore we know

$$\partial_t v(t) = e^{-At}(-Av_0 + G(v_0)) + \int_0^t e^{-A(t-\tau)} g(\tau) d\tau, \quad (\text{A.12})$$

which concludes $\partial_t v \in C([0, T]; L^2(\mathbb{R}))$. Plugging in the formula (A.11), (A.12) becomes

$$\begin{aligned} \partial_t v(t) &= e^{-At}(-Av_0 + G(v_0)) + \int_0^t e^{-A(t-\tau)} \partial_\tau G(v(\tau)) \, d\tau \\ &= e^{-At}(-Av_0 + G(v_0)) + e^{-A(t-\tau)} G(v(\tau))|_0^t - A \int_0^t e^{-A(t-\tau)} G(v(\tau)) \, d\tau \\ &= G(v(t)) - Ae^{-At}v_0 - A \int_0^t e^{-A(t-\tau)} G(v(\tau)) \, d\tau \\ &= -Av(t) + G(v(t)) \end{aligned}$$

due to (2.6). Then since $G(v(t)) \in C([0, T]; L^2(\mathbb{R}))$ we have

$$Av \in C([0, T]; L^2(\mathbb{R})). \tag{A.13}$$

Step 4. Higher order regularities.

Set $w_1 = \partial_t v$ and $w_2 = \partial_x v$. Then

$$\partial_t G(v) = G'(v)\partial_t v \in C([0, T]; L^2(\mathbb{R}))$$

and

$$\partial_x G(v) = (1 - f'(\phi + v))\partial_x v - (f'(\phi + v) - f'(\phi))\partial_x \phi \in C([0, T]; L^2(\mathbb{R})).$$

Therefore we can repeat Step 1 and 2 for

$$\partial_t w_1 + Aw_1 = G'(v)w_1 \tag{A.14}$$

and

$$\partial_t w_2 + Aw_2 = (1 - f'(v + \phi))w_2 - (f'(\phi + v) - f'(\phi))\partial_x \phi \tag{A.15}$$

to obtain

$$\begin{aligned} w_1, w_2 &\in C((0, T]; L^2(\mathbb{R})) \cap C((0, T]; H^1(\mathbb{R})) \\ \partial_t w_1, \partial_t w_2 &\in C((0, T]; L^2(\mathbb{R})) \end{aligned}$$

which concludes v is a global classical solution to (1.12) and satisfies (2.7).

Step 5. (2.8) is directly from (1.12) and above regularity properties. Then we have

$$\mathcal{F}(v(t)) \leq \mathcal{F}(v_0).$$

□

Appendix B. Proof of key propositions in Section 3.

B.1. The spectrum $\sigma(L) \subset [-1, +\infty)$. **LEMMA B.1.** *The operator L in (3.1) is a densely defined, self-adjoint linear operator.*

Proof. First, it is obvious L is linear operator and $D(L) = H^1(\mathbb{R})$ is dense in $L^2(\mathbb{R})$.

Second, denote $q(x) := \frac{x^2-1}{x^2+1}$ which is bounded function. For any $y, u \in H^1$,

$$\langle Lu, y \rangle = \langle (-\partial_{xx})^{\frac{1}{2}}u + q(x)u, y \rangle = \langle u, (-\partial_{xx})^{\frac{1}{2}}y + q(x)y \rangle = \langle u, Ly \rangle.$$

Hence L is self-adjoint. \square

LEMMA B.2. *Assume the operator L is a densely defined, self-adjoint linear operator, then*

$$\text{Ker}(\lambda I - L) = \text{Ran}(\lambda I - L)^\perp.$$

Proof. For any $y \in D(\lambda I - L)$, any $u \in \text{Ker}(\lambda I - L)$, we have

$$0 = \langle (\lambda I - L)u, y \rangle = \langle u, (\lambda I - L)y \rangle.$$

\square

LEMMA B.3. *The operator L in (3.1) is closed.*

Proof. Assume we have $u_n \rightarrow u$ in $D(L)$ and $y_n := Lu_n \rightarrow y$ in $L^2(\mathbb{R})$. Since $D(L)$ is dense in $L^2(\mathbb{R})$, we first choose any test function $\varphi \in D(L)$. Then

$$\langle \varphi, y \rangle \leftarrow \langle \varphi, y_n \rangle = \langle \varphi, Lu_n \rangle = \langle L\varphi, u_n \rangle \rightarrow \langle L\varphi, u \rangle = \langle \varphi, Lu \rangle,$$

and dense argument shows that $y = Lu$. Hence L is closed. \square

DEFINITION 2. Let T be a closed operator on a Hilbert space X . A complex number λ is in the resolvent set $\rho(T)$ if $\lambda I - T$ is bijection of $D(T)$ onto X with a bounded inverse. If $\lambda \in \rho(T)$, $R_\lambda(T) = (\lambda I - T)^{-1}$ is called the resolvent of T .

REMARK 8. If $\lambda I - T$ is a bijection of $D(T)$ onto X , by the closed-graph theorem, its inverse is automatically bounded.

LEMMA B.4. *For linear operator L in (3.1) and any $\lambda \in \mathbb{C} \setminus [-1, +\infty)$, the range $\text{Ran}(\lambda I - L)$ is closed.*

Proof. Notice the lower bound for potential $q(x) := \frac{x^2-1}{x^2+1}$ is -1 . For $\lambda = a + bi$ with $b \neq 0$, we have

$$\|(\lambda I - L)u\|^2 = \|bu\|^2 + \|(aI - L)u\|^2 \geq b^2\|u\|^2 \tag{B.1}$$

For $\lambda < -1$, we have

$$\|(\lambda I - L)u\|^2 = \|(\lambda + 1)u\|^2 - 2(\lambda + 1)\langle (L + I)u, u \rangle + \|(L + I)u\|^2 \geq (\lambda + 1)^2\|u\|^2, \tag{B.2}$$

where we used $\langle (L + I)u, u \rangle \geq 0$ and $\lambda + 1 < 0$.

Now, we show that $\text{Ran}(\lambda I - L)$ is closed. For any $y_n \in \text{Ran}(\lambda I - L)$ with $y_n = (\lambda I - L)u_n$, if $y_n \rightarrow y$, from the lower bound estimate in (B.1) and (B.2), $u_n \rightarrow u$. Therefore from Lemma B.3 we know $y = (\lambda I - L)u$. Thus $\text{Ran}(\lambda I - L)$ is closed. \square

LEMMA B.5. *For linear operator L in (3.1), $\sigma(L) = \sigma_p(L) \cup \sigma_c(L) \subset [-1, +\infty)$.*

Proof. (1) For a self-adjoint operator, the spectrum locates on the real line. Indeed, for any $b \neq 0$, (B.1) implies there is a lower bound for $\lambda I - L$. To obtain $\lambda \in$

$\rho(L)$, it remains to prove $\lambda I - L$ is onto. If it is not onto and we assume $\text{Ran}(\lambda I - L) \neq L^2$. Then by Lemma B.1, B.2 and B.4, $\text{Ker}(\lambda I - L) = \text{Ran}(\lambda I - L)^\perp = \overline{\text{Ran}(\lambda I - L)}^\perp$ is not empty, which means there exists $u^* \neq 0$ such that $(\lambda I - L)u^* = 0$ and it contradicts with (B.1).

(2) For self-adjoint operator, the residual spectrum is empty. Indeed, if $\lambda \in \sigma_r$, we concludes a contradiction from the fact

$$\text{Ker}(\lambda I - L) = \text{Ran}(\lambda I - L)^\perp \supseteq \overline{\text{Ran}(\lambda I - L)}^\perp \neq \emptyset.$$

(3) $\sigma_p \subset [-1, +\infty)$. Otherwise, if $\lambda_p = a < -1$ is a point spectrum, then (B.2) leads to an contradiction.

(4) $\sigma_c \subset [-1, +\infty)$. Otherwise, if $\lambda_c = a < -1$ is a continuous spectrum, then

$$L^2 = \overline{\text{Ran}(\lambda I - L)} = \text{Ran}(\lambda I - L),$$

which contradicts with the definition of continuous spectrum. \square

B.2. The continuous spectrum $\sigma_c(L) \subset [1, +\infty)$. It worth to mention that the proof relies only on the lower and upper bound of the potential $f'(\phi) = \frac{x^2-1}{x^2+1}$, which are -1 and 1 separately.

LEMMA B.6. *For linear operator L in (3.1), $\sigma_c(L) \subset [1, +\infty)$. Besides, for the linear operator A in (2.3), $\sigma(A) = [1, +\infty)$.*

Proof. Recall the perturbation theorem for spectrum in [37, Theorem XIII 14 and Corollary 2].

Let A be a self-adjoint operator and let C be a relatively compact perturbation of A . Then $L := A + C$ has the same essential spectrum with A .

In our case, notice the upper bound for potential $q(x) := \frac{x^2-1}{x^2+1}$ is 1 . Taking $A = (-\partial_{xx})^{\frac{1}{2}} + I$ and $C := v(x) - I$, we will first prove C is a relatively compact perturbation of A , i.e. $C(A + i)^{-1}$ is compact, and then we prove $\sigma(A) = [1, +\infty)$.

(1) First we prove $C(A + i)^{-1}$ is compact. Assume $u_j \in L^2(\mathbb{R})$ satisfying $\|u_j\| \leq M$ for any j . Denote $w_j := (A + i)^{-1}u_j = ((1 + i)I + (-\partial_{xx})^{\frac{1}{2}})^{-1}u_j$. We want to prove for any $\varepsilon > 0$ there exist J and a subsequence (still denoted as j) such that for any $j \geq J$ and $\ell \geq 0$, $(q(x) - 1)(w_j - w_{j+\ell})$ are Cauchy sequence in $L^2(\mathbb{R})$.

(1.a) For $\varepsilon > 0$, $n := [1/\varepsilon]$, there exists R_n , such that for any $|x| > R_n$,

$$\int_{|x| > R_n} (q - 1)(w_j - w_{j+\ell}) \, dx \leq \|q - 1\|_{L^2(|x| > R_n)} \cdot \|w_j - w_{j+\ell}\|_{L^2(|x| > R_n)} \leq \frac{\varepsilon}{2}. \tag{B.3}$$

(1.b) For $|x| \leq R_n$, we claim $w_j = ((1 + i)I + (-\partial_{xx})^{\frac{1}{2}})^{-1}u_j$ is bounded in $H^1(|x| \leq R_n)$. Indeed from $((1 + i)I + (-\partial_{xx})^{\frac{1}{2}})w_j = u_j$ and the Fourier transform, we know

$$\hat{w}_j(\xi) = \frac{\hat{u}_j(\xi)}{1 + i + |\xi|}.$$

Then by Parserval's identity

$$\begin{aligned} \|w_j\|_{H^1}^2 &= \|w_j\|^2 + \|w'_j\|^2 = \|\hat{w}_j\|^2 + \|\xi \hat{w}_j\|^2 \\ &= c \int \frac{1 + |\xi|^2}{1 + (1 + |\xi|)^2} \hat{u}_j^2(\xi) \, d\xi \leq c \|u_j\|^2. \end{aligned}$$

Since $H^1(|x| \leq R_n) \hookrightarrow L^2(|x| \leq R_n)$ compactly, we obtain a subsequence (still denoted as w_j) of w_j which strongly converges in $L^2(|x| \leq R_n)$ and $\|(q(x) - 1)(w_j - w_{j+\ell})\|_{L^2(|x| \leq R_n)} \leq \frac{\varepsilon}{2}$. Combining (1.a) and (1.b) gives a Cauchy sequence in $L^2(\mathbb{R})$ and we conclude $C(\hat{A} + i)^{-1}$ is compact.

(2) We turn to prove $\sigma(A) = [1, +\infty)$. First notice the lower bound now is 1, so by Lemma B.5, $\sigma(A) \subset [1, +\infty)$. It remains to prove $[0, +\infty) \subset \sigma((-\partial_{xx})^{\frac{1}{2}})$ due to A is a shift of $(-\partial_{xx})^{\frac{1}{2}}$ with constant 1. For any $\lambda \geq 0$, we will prove $\text{Ran}(\lambda I - (-\partial_{xx})^{\frac{1}{2}}) \neq L^2(\mathbb{R})$. Set $f := e^{i\xi_0 x} N(0, 1)$ where $N(0, 1)$ is the normal distribution. Then $f \in L^2(\mathbb{R})$ and $\hat{f}(\xi) = N(\xi_0, 1)$. Then by the Fourier transformation, if there exists a solution to $(\lambda I - (-\partial_{xx})^{\frac{1}{2}})u = f$, then $\hat{u}(\xi) = \frac{\hat{f}(\xi)}{\lambda - |\xi|}$. Therefore u is the inverse Fourier transform of $\frac{N(\xi_0, 1)}{\lambda - |\xi|}$ which is not integrable. Thus we have $[1, +\infty) \subset \sigma(A) \subset [1, +\infty)$ and $\sigma_{ess}(A) = \sigma(A) = [1, +\infty)$.

Finally we conclude $\sigma_c(L) \subset \sigma_{ess}(L) = \sigma_{ess}(A) = [1, +\infty)$. \square

