METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 27, No. 2, pp. 125–152, June 2020

ON APPROXIMATION OF A HYPER-SINGULAR TRANSPORT OPERATOR AND EXISTENCE OF SOLUTIONS*

J. TERVO† AND M. HERTY‡

Abstract. The paper considers a class of linear Boltzmann transport equations containing hyper-singular integrals in their collision terms. A partial integro-differential approximation is derived and the approximation error is analysed. Existence results of solutions for the approximative initial inflow boundary value problem are shown. The approximative operator of this type may be used as a refined model for the dose calculation of radiation therapy. The paper continues authors' recent analytical results on the approximation of the linear transport equations.

Key words. Linear Boltzmann transport equation, hyper-singular integral operators, variational formulation, charged particle transport.

Mathematics Subject Classification. 35Q20, 35R09, 45E99.

1. Introduction. The Boltzmann transport equation (BTE) models the flux of evolving particles emerging from various sources and interactions. The equation BTE describes the evolution in the number density of particles in full phase space with position x, velocity direction ω and energy E and a detailed introduction as well as a discussion of many properties can be found e.g. in the recent book [25]. The equation plays an important role as a mathematical and physical model for a variety of transport phenomena including e.g. dose calculation models in radiation therapy [11, 17, 33], modelling of dispersion in clouds [12], solid state physics [22] or cosmic radiation [41]. The conventional equation has been extensively considered from an analytical point of view e.g. in [7], Chapter XXI and [1]. We also refer to [3, 9, 27] for further motivation of the equation from a physical point of view. Some more recent issues including e.g. spectral and certain inverse problems is exposed in [24] and general non-linear transport theory e.g. in [39], [2]. A mathematical survey of non-linear collision theory of particle physics (especially in dilute gases) is found in [40].

In the case of charged particle transport the linear BTE turns out to be of a type of a *partial hyper-singular integro-differential operator*. For theoretical and computational reasons it has been approximated by more simple models, including e.g. the Continuous Slowing Down Approximation (CSDA) [41, 11, 33] and a linear Fokker-Plank approximation e.g. [28, 16]. In those approximations the resulting operator is a partial integro-differential operator without hyper-singular integrals. In [33, 35] we provided a (non-conventional) partial integro-pseudo-differential approximation for certain transport operators related to charged particles. In this paper we further investigate the approximation of the exact partial hyper-singular integral operator emerging from the Møller scattering. The presented analysis can also cover analogous types of transport operators containing hyper-singularities.

As mentioned in the abstract above the present paper continues authors' recent analytical results on the approximation of the linear transport equations for charged particles. The main novelty of the present manuscript compared to [34], [35] is that

*Received March 6, 2019; accepted for publication April 2, 2020.
†University of Eastern Finland, Department of Applied Physics, Kuopio, Finland (040tervo@gmail.com).
‡RWTH Aachen University, IGPM, Templergraben 5, 52062, Germany (herty@igpm.rwth-aachen.de).

in the paper we derive and analyse an approximation of the exact hyper-singular transport operator, for which some error analysis can be carried out. In [34], [35] the Laplace-Beltrami like term is omitted (because we applied lower order Taylor's approximation therein). This is a shortcoming as to error analysis, such as mentioned at the end of section 3.2 of [35] and in its Conclusion. According to the analysis of [36] the Laplace-Beltrami term gets a relevant origin. The existence analysis of [34], [35] based on m-dissipality and the theory of evolution operators. In the present manuscript we apply Lions-Lax-Milgram Theorem together with generalized inflow trace results. Both methods need relevant a priori estimates and their proofs are technically quite a similar kind of. The presence of Laplace-Beltrami like term and inflow trace theory cause some additional considerations and efforts in this paper.

2. Linear Boltzman transport equation. The approximative BTE equation considered in this paper is of the form

$$T\psi := a(x,E)\frac{\partial \psi}{\partial E} + b(x,\omega,E,\partial_\omega)\psi + \omega\cdot\nabla_x\psi + \Sigma\psi - K_r\psi \tag{1}$$

for $(x,\omega,E)\in G\times S\times I$, where

$$b(x,\omega,E,\partial_\omega)\psi = c(x,E)\Delta_S\psi + d(x,\omega,E)\cdot\nabla_S\psi.$$

K_r is the so called *restricted collision operator* of the form

$$(K_r\psi)(x,\omega,E) = \int_{I'}\int_0^{2\pi}\hat\sigma(x,E',E)\psi(x,\gamma(E',E,\omega)(s),E')dsdE' \tag{2}$$

where $\gamma(E',E,\omega)(s)$ is the cosine of the incoming and leaving particles. Here $G\subset\mathbb{R}^3$ is the spatial domain. $S = S_2\subset\mathbb{R}^3$ is the unit sphere (scattering angle domain) and $I = [E_0, E_m]$ is the energy interval. Consequently, the phase space is $G\times S\times I$. Δ_S and ∇_S are the Laplace-Beltrami operator and gradient on S, respectively. In this paper (ω', E') refers to the angle and energy of the incoming particle whereas (ω, E) refers to the angle and energy of the leaving particle. The solution $\psi = \psi(x,\omega,E)$ is a real valued function which describes the radiation flux of particles under consideration. The relevant problem to be solved is an *initial inflow boundary value problem* of the form

$$T\psi = f,\ \ \psi_{|\Gamma_-} = g,\ \ \psi(.,.,E_m) = 0. \tag{3}$$

Here f represents possible (internal) sources and g (inflow) boundary sources. The set Γ_- is "the inflow boundary part of ∂G" (see section 3). E_m is the so called *cut-off energy.*

The (classical) example of the initial inflow boundary value problems of the form (3) is the case where $a(x,E)\frac{\partial}{\partial E} + b(x,\omega,E,\partial_\omega) = 0$. Existence of solutions of this transport problem (3) has been studied for single equations e.g. in [10, 7, 1] and for coupled systems [32]. In the references it is assumed that the restricted collision operator K_r satisfies a (partial) *Schur criterion* for boundedness, [15]. In [34, 35] we studied existence of solutions for the case

$$b(x,\omega,E,\partial_\omega)\psi = 0. \tag{4}$$

In [11] additionally it has been assumed that $a = a(E)$.

We also discuss related results for (deterministic and linear) Fokker-Planck type equations. In [8, Appendix A] and [37] existence of solutions for a linear non-stationary Fokker-Planck equation are given for the case $G=\mathbb{R}^n$ and without restricted collision term. Methods in [8] are closely related to our techniques in section 6 below. In [20] existence results of solutions are shown for a class of non-stationary Fokker-Planck equations with irregular (having only Sobolev regularity) coefficients. Results in [6] consider existence of a special form of stationary Fokker-Planck equation in appropriate (Maxwellian) weighted Sobolev spaces. Therein G is more general but the restricted collision term is not included and the spatial average of the solution is fixed instead of boundary conditions. In [16, 17] existence results are obtained in the context of dose calculation for optimal radiation treatment planning. In part of the references the state space is $G\times V$ where V is velocity space but they can be formulated for $G\times S\times I$ via the (local) diffeomorphism $h(\omega,E):=\sqrt{E}\omega$.

In [36] we derived a refined expression for the exact transport operator T and additionally we derived a variational formulation of the associated inflow initial boundary value problem. The analysis was carried out only for the Møller-type scattering which is a special prototype of hyper-singular interactions. This interaction models the electron's (and positron's) inelastic collisions and other type of collisions (such as Bremsstrahlung) could be handeled analogously. In section 3.3 we recall the hyper-singular partial integral expression of the exact transport operator obtained in [33] (or concisely in [35]). For a refined pseudo-differential like expression we refer to [36]. The restricted collision operator K_r is roughly speaking the residual when the singular part is separated from the collision operator. In [35] we showed that K_r is a bounded operator in $L^2(G\times S\times I)$. In addition, we showed that $\Sigma-K_r$ is *coercive (accretive) operator* in $L^2(G\times S\times I)$ under relevant assumptions.

In section 4 we deduce an approximation of the hyper-singular transport operator by decomposing *Hadamard finite part hyper-singular integrals* of the form p.f. $\int_E^{E_m}\frac{u(E,E')}{(E'-E)^j}dE',\ j=1,2$ to

$$\text{p.f.}\int_E^{E_m}\frac{u(E,E')}{(E'-E)^j}dE'=\text{p.f.}\int_E^{\kappa E}\frac{u(E,E')}{(E'-E)^j}dE'+\int_{\kappa E}^{E_m}\frac{u(E,E')}{(E'-E)^j}dE'$$

where the "cutting parameter" $\kappa>1$. In the first integral p.f. $\int_E^{\kappa E}\frac{u(E,E')}{(E'-E)^j}dE'$ we apply appropriate Taylor's approximations. This leads to an approximative transport operator T_κ of the form (1). In section 5 we verify that $T_\kappa\psi\to T\psi$ uniformly in $G\times S\times I$ as $\kappa\to 1^+$ when ψ is smooth enough. The analysis yields a foundation for the CSDA-Fokker-Planck type approximations (1) of transport operators.

Section 6 considers the existence of solutions (in relevant function spaces) of the initial inflow boundary value problem of the form (3). The problem is analysed in the variational (weak) form

$$\widetilde{B}(\psi,v)=Fv,\ v\in\widehat{\mathcal{H}}.$$

Here $\widehat{\mathcal{H}}$ is a suitable space of test functions and $\widetilde{B}(.,.)$ and F are bilinear and linear forms, respectively. The variational formulation is an essential step in order to define generalized solutions and to show existence results e.g. by applying Lax-Milgram Theorem based methods.

3. Preliminary Analysis and Notations.

3.1. Basic notations. We assume that G is an open bounded set in $\mathbb{R}^3$ such that $\overline{G}$ is a C^1-manifold with boundary [21]. In particular, it follows from this definition that G lies on one side of its boundary. The unit outward (with respect to G) pointing normal on ∂G is denoted by ν and the surface measure (induced by the Lebesgue measure dx) on ∂G is denoted as $d\sigma$. We let $S = S_2$ be the unit sphere in $\mathbb{R}^3$ equipped with the standard surface measure $d\omega$. Furthermore, let $I = [E_0, E_{\rm m}]$ where $0 \leq E_0 < E_{\rm m} < \infty$. We could replace I by $I = [E_0, \infty[$ but we neglect this case here. We shall denote by I° the interior of I and I is equipped with the Lebesgue measure dE. All functions considered in this paper are real-valued, and all Hilbert spaces are real.

For $(x,\omega) \in G \times S$ the *escape time* $t(x,\omega) = t_-(x,\omega)$ is defined as $t(x,\omega) := \inf\{s > 0 \mid x - s\omega \notin G\} = \sup\{T > 0 \mid x - s\omega \in G \text{ for all } 0 < s < T\}$. We define

$$\Gamma' := (\partial G) \times S, \quad \Gamma := \Gamma' \times I,$$

and their subsets

$$\begin{aligned}
\Gamma_0' &:= \{(y,\omega) \in \Gamma' \mid \omega \cdot \nu(y) = 0\}, \quad \Gamma_0 := \Gamma_0' \times I,\\
\Gamma_-' &:= \{(y,\omega) \in \Gamma' \mid \omega \cdot \nu(y) < 0\}, \quad \Gamma_- := \Gamma_-' \times I,\\
\Gamma_+' &:= \{(y,\omega) \in \Gamma' \mid \omega \cdot \nu(y) > 0\}, \quad \Gamma_+ := \Gamma_+' \times I.
\end{aligned}$$

Note that $\Gamma = \Gamma_0 \cup \Gamma_- \cup \Gamma_+$.

In the sequel we denote for $k \in \mathbb{N}_0$, $C^k(\overline{G} \times S \times I) := \{\psi \in C^k(G \times S \times I^\circ) \mid \psi = f_{|G\times S\times I^\circ},\ f \in C_0^k(\mathbb{R}^3 \times S \times \mathbb{R})\}$, where for a C^k-manifold M without boundary, the set $C_0^k(M)$ denotes the set of all C^k-functions on M with compact support. Define the (Sobolev) space $W^2(G \times S \times I)$ by

$$W^2(G \times S \times I) := \{\psi \in L^2(G \times S \times I) \mid \omega \cdot \nabla_x \psi \in L^2(G \times S \times I)\}.$$

The space $W^2(G\times S\times I)$ is a Hilbert space equipped with the standard inner product

$$\langle \psi, v\rangle_{W^2(G\times S\times I)} := \langle \psi, v\rangle_{L^2(G\times S\times I)} + \langle \omega\cdot\nabla_x\psi, \omega\cdot\nabla_x v\rangle_{L^2(G\times S\times I)}.$$

Moreover, the space $C^1(\overline{G} \times S \times I)$ is a dense subspace of $W^2(G \times S \times I)$ [13].

Let $T^2(\Gamma)$ be the weighted Lebesgue space $L^2(\Gamma, |\omega\cdot\nu| d\sigma d\omega dE)$. The trace $\gamma(\psi) := \psi_{|\Gamma}$ is well-defined for $\psi \in W^2(G \times S \times I)$ and $\gamma(\psi) \in L^2_{\rm loc}(\Gamma_\pm, |\omega\cdot\nu| d\sigma d\omega dE)$ ([4], [33, Section 2.2]). The space

$$\widetilde{W}^2(G \times S \times I) := \{\psi \in W^2(G \times S \times I) \mid \gamma(\psi) \in T^2(\Gamma)\}$$

is a Hilbert space equipped with the inner product

$$\begin{aligned}
\langle \psi, v\rangle_{\widetilde{W}^2(G\times S\times I)} &:= \langle \psi, v\rangle_{W^2(G\times S\times I)} + \langle \gamma(\psi), \gamma(v)\rangle_{T^2(\Gamma)};\\
\langle h_1, h_2\rangle_{T^2(\Gamma)} &:= \int_\Gamma h_1(x,\omega,E) h_2(x,\omega,E) |\omega\cdot\nu| d\sigma d\omega dE.
\end{aligned}$$

We recall the Green's formula [7, p. 225] for every $\psi,\ v \in \widetilde{W}^2(G \times S \times I)$

$$\begin{aligned}
&\int_{G\times S\times I} (\omega\cdot\nabla_x\psi) v\, dx d\omega dE + \int_{G\times S\times I} \psi(\omega\cdot\nabla_x v)\, dx d\omega dE\\
&= \int_{\partial G\times S\times I} (\omega\cdot\nu)\psi v d\sigma d\omega dE. \qquad (5)
\end{aligned}$$

3.2. Definition of Hadamard finite part integrals. We also recall some standard concepts from analysis. The Taylor's expansion (of order $r \in \mathbb{N}_0$) for sufficiently smooth functions $f : U \to \mathbb{R}$ on an open set $U \subset \mathbb{R}^N$ is given by

$$f(x) = \sum_{|\alpha| \leq r} \frac{1}{\alpha!} \frac{\partial^\alpha f}{\partial x^\alpha}(x_0)(x - x_0)^\alpha + \sum_{|\alpha|=r+1} R_\alpha(x)(x - x_0)^\alpha \tag{6}$$

where the residual term (one of its variant forms) is

$$R_\alpha(x) := \frac{|\alpha|}{\alpha!} \int_0^1 (1-t)^{|\alpha|-1} \frac{\partial^\alpha f}{\partial x^\alpha}(x_0 + t(x - x_0))dt.$$

For the definitions of Hadamard finite part integrals (for discontinuous functions) $f : [a, b] \to \mathbb{R}$ we refer to [23, pp. 5 and 32, Equations (14), (32)] or [31, p. 104], [5], [18, section 3.2]. Applying these definitions (for a fixed x) to the function $F_x(t) := \chi_{[x,b]}(t)f(t)$, where $f \in C([a, b])$ and where $\chi_{[x,b]}(t)$ is the characteristic function of the interval $[x, b]$, leads to

$$\text{p.f.} \int_a^b \frac{F_x(t)}{t - x} dt = \text{p.f.} \int_x^b \frac{f(t)}{t - x} dt = \lim_{\epsilon \to 0} \Big(\int_{x+\epsilon}^b \frac{f(t)}{t - x} dt + f(x^+) \ln(\epsilon) \Big) \tag{7}$$

and

$$\begin{aligned} \text{p.f.} \int_a^b \frac{F_x(t)}{(t - x)^2} dt =& \text{p.f.} \int_x^b \frac{f(t)}{(t - x)^2} dt \\ &= \lim_{\epsilon \to 0} \Big(\int_{x+\epsilon}^b \frac{f(t)}{(t - x)^2} dt + f'(x^+) \ln(\epsilon) - \frac{1}{\epsilon} f(x^+) \Big). \end{aligned} \tag{8}$$

Analogously we define

$$\text{p.f.} \int_a^x \frac{f(t)}{t - x} dt = \lim_{\epsilon \to 0} \Big(\int_a^{x-\epsilon} \frac{f(t)}{t - x} dt - f(x^-) \ln(\epsilon) \Big) \tag{9}$$

or equivalently

$$\text{p.f.} \int_a^x \frac{f(t)}{x - t} dt = \lim_{\epsilon \to 0} \Big(\int_a^{x-\epsilon} \frac{f(t)}{x - t} dt + f(x^-) \ln(\epsilon) \Big)$$

and

$$\text{p.f.} \int_a^x \frac{f(t)}{(t - x)^2} dt = \lim_{\epsilon \to 0} \Big(\int_a^{x-\epsilon} \frac{f(t)}{(t - x)^2} dt - f'(x^-) \ln(\epsilon) - \frac{1}{\epsilon} f(x^-) \Big) \tag{10}$$

In particular, these formulas give

$$\text{p.f.} \int_x^b \frac{1}{t - x} dt = \ln(b - x), \tag{11}$$

$$\text{p.f.} \int_x^b \frac{1}{(t - x)^2} dt = -\frac{1}{b - x}. \tag{12}$$

Note that p.f. $\int_x^b \frac{f(t)}{t-x} dt$ is well-defined (at least) for all $f \in C^\alpha([a, b])$, $\alpha > 0$ and (cf. [5])

$$\text{p.f.} \int_x^b \frac{f(t)}{t - x} dt = \int_x^b \frac{f(t) - f(x)}{t - x} dt + f(x) \ln(b - x). \tag{13}$$

3.3. Hyper-singular transport operator for Møller scattering. The Møller collision operator has been introduced e.g. in [33, 35]. It is written explicitly for $E \geq E_0$ as

$$\begin{aligned}(K\psi)(x,\omega,E) = \text{p.f.}\int_E^{E_m} \hat{\sigma}_2(x,E',E)\frac{1}{(E'-E)^2}\int_0^{2\pi}\psi(x,\gamma(E',E,\omega)(s),E')dsdE' \\ - \text{p.f.}\int_E^{E_m} \hat{\sigma}_1(x,E',E)\frac{1}{E'-E}\int_0^{2\pi}\psi(x,\gamma(E',E,\omega)(s),E')dsdE' \\ + (K_0\psi)(x,\omega,E) \qquad (14)\end{aligned}$$

where

$$(K_0\psi)(x,\omega,E) := \int_{I'}\int_0^{2\pi}\chi(E',E)\hat{\sigma}_0(x,E',E)\psi(x,\gamma(E',E,\omega)(s),E')dsdE' \qquad (15)$$

and where $\chi(E',E) := \chi_{\mathbb{R}_+}(E'-E)$ is the characteristic function on $\mathbb{R}_+$. Denote as in [33], [35]

$$\begin{aligned}&(K_j\psi)(x,\omega,E) \\ &:= \text{p.f.}\int_E^{E_m}\hat{\sigma}_j(x,E',E)\frac{1}{(E'-E)^j}\int_0^{2\pi}\psi(x,\gamma(E',E,\omega)(s),E')dsdE',\ j=1,2 \qquad (16)\end{aligned}$$

and

$$(\overline{\mathcal{K}}_j\psi)(x,\omega,E',E) := \hat{\sigma}_j(x,E',E)\int_0^{2\pi}\psi(x,\gamma(E',E,\omega)(s),E')ds. \qquad (17)$$

Above $\gamma = \gamma(E',E,\omega) : [0,2\pi] \to S$ is a parametrization of the curve

$$\Gamma(E',E,\omega) = \{\omega' \in S \mid \omega'\cdot\omega - \mu(E',E) = 0\},\ \mu(E',E) := \sqrt{\frac{E(E'+2)}{E'(E+2)}}.$$

We choose

$$\gamma(E',E,\omega)(s) = R(\omega)\big(\sqrt{1-\mu^2}\cos(s), \sqrt{1-\mu^2}\sin(s), \mu\big), \quad s \in [0,2\pi],$$

where $R(\omega)$ is a rotation (unitary) matrix mapping the vector $e_3 = (0,0,1)$ onto ω.

We see that

$$(K_j\psi)(x,\omega,E) = \mathcal{H}_j\big((\overline{\mathcal{K}}_j\psi)(x,\omega,\cdot,E)\big)(E)$$

where $\mathcal{H}_j$ are the Hadamard finite part singular integral operators ([33], section 3.2). The corresponding transport operator is

$$\begin{aligned}T\psi = &-\mathcal{H}_2\big((\overline{\mathcal{K}}_2\psi)(x,\omega,\cdot,E)\big)(E) + \mathcal{H}_1\big((\overline{\mathcal{K}}_1\psi)(x,\omega,\cdot,E)\big)(E) \\ &+ \omega\cdot\nabla_x\psi + \Sigma\psi - K_r\psi. \qquad (18)\end{aligned}$$

A refined pseudo-differential like form of the operator (18) is derived in [36], Theorem 4.17 (together with Theorem 4.9, Lemma 4.10, Remark 4.11 therein).

4. Asymptotic analysis of the hyper–singular kernels. In this section we present the truncation of the hyper-singular kernels. The computations will be used in the forthcoming section to establish existence results on the truncated problem.

4.1. Approximation by truncation. We shall below apply the strongly hyper-singular form (18) of T. The operator K_0 given by (15) is an ordinary partial Schur integral operator and so it suffices only to truncate the first and second terms $(K_j\psi)(x,\omega,E)$ of equation (14). In the following we assume that the 'cut-off energy for primary particles' is $E'=\kappa E$ where $\kappa>1$. We decompose the integration in (16) as follows for $j=1,2$:

$$
\begin{aligned}
&(K_j\psi)(x,\omega,E)=\text{p.f.}\int_E^{\kappa E}\hat{\sigma}_j(x,E',E)\frac{1}{(E'-E)^j}\int_0^{2\pi}\psi(x,\gamma(E',E,\omega)(s),E')dsdE' \\
&+\int_{\kappa E}^{E_m}\hat{\sigma}_j(x,E',E)\frac{1}{(E'-E)^j}\int_0^{2\pi}\psi(x,\gamma(E',E,\omega)(s),E')dsdE' \qquad (19)\\
&=:(K_{j,1,\kappa}\psi)(x,\omega,E)+(K_{j,0,\kappa}\psi)(x,\omega,E) \qquad (20)
\end{aligned}
$$

where we noticed that the last integral $(K_{j,0,\kappa}\psi)$ does **not** involve a singular kernel. Then,

$$
T\psi=-K_{2,1,\kappa}\psi+K_{1,1,\kappa}\psi+\omega\cdot\nabla_x\psi+\Sigma\psi-(K_{2,0,\kappa}\psi-K_{1,0,\kappa}\psi+K_0\psi). \qquad (21)
$$

Since

$$
\begin{aligned}
&(K_{j,0,\kappa}\psi)(x,\omega,E)\\
&=\int_{I'}\chi_{\mathbb{R}_+}(E'-\kappa E)\hat{\sigma}_j(x,E',E)\frac{1}{(E'-E)^j}\int_0^{2\pi}\psi(x,\gamma(E',E,\omega)(s),E')dsdE' \qquad (22)
\end{aligned}
$$

we find that the operators $K_{j,0,\kappa}$ are partial Schur integral operators and hence they are bounded operators $L^2(G\times S\times I)\to L^2(G\times S\times I)$(section 6.1 below). Hence it suffices to consider approximations only for the operators $K_{j,1,\kappa}$.

4.2. Approximation of hyper-singular integrals for primary particles. Let

$$
f_1(x,\omega,E',E):=\hat{\sigma}_1(x,E',E)\int_0^{2\pi}\psi(x,\gamma(E',E,\omega)(s),E')ds.
$$

For $K_{1,1,\kappa}$ we apply the approximation

$$
f_1(x,\omega,E',E)\approx f_1(x,\omega,E,E) \qquad (23)
$$

on the interval $[E,\kappa E]$ which leads to the approximation

$$
(K_{1,1,\kappa}\psi)(x,\omega,E)\approx(\widetilde{K}_{1,1,\kappa}\psi)(x,\omega,E):=2\pi\ \ln((\kappa-1)E)\hat{\sigma}_1(x,E,E)\psi(x,\omega,E) \qquad (24)
$$

where we recalled that $\gamma(E,E,\omega)(s)=\omega$ and by (11)

$$
\text{p.f.}\int_E^{\kappa E}\frac{1}{E'-E}dE'=\ln(\kappa E-E).
$$

To approximate $K_{2,1,\kappa}$ we recall the next result

Theorem 4.1. *Let $\psi\in C(\overline{G},C^2(I,C^3(S)))$. Then for fixed $x,\ \omega,\ E$ the mapping*

$$F(E'):=\int_0^{2\pi}\psi(x,\gamma(E',E,\omega)(s),E')ds$$

is differentiable and
(1) *for $E'\neq E$*

$$\begin{aligned}F'(E')&=\frac{\partial}{\partial E'}\big(\int_0^{2\pi}\psi(x,\gamma(E',E,\omega)(s),E')ds\big)\\&=\int_0^{2\pi}\Big\langle(\nabla_S\psi)(x,\gamma(E',E,\omega)(s),E'),\frac{\partial\gamma}{\partial E'}(E',E,\omega)(s)\Big\rangle ds\\&+\int_0^{2\pi}\frac{\partial\psi}{\partial E}(x,\gamma(E',E,\omega)(s),E')ds\end{aligned}\tag{25}$$

(2) *for $E'=E$*

$$\begin{aligned}F'(E)&=\frac{\partial}{\partial E'}\big(\int_0^{2\pi}\psi(x,\gamma(E',E,\omega)(s),E')ds\big)\Big|_{E'=E}\\&=(A(x,\omega,E,\partial_\omega)\psi)(x,\omega,E)+2\pi\frac{\partial\psi}{\partial E}(x,\omega,E)\end{aligned}\tag{26}$$

where $A(\omega,E,\partial_\omega)\psi$ is the second order partial differential operator $\sum_{|\alpha|\leq 2}a_\alpha(\omega,E)\partial_\omega^\alpha$ (here $\partial_\omega^\alpha=\partial_{\omega_1}^{\alpha_1}\partial_{\omega_2}^{\alpha_2}$ where $\partial_{\omega_j}:=\frac{\partial}{\partial\omega_j},\ j=1,2$ are local tangent vectors in the tangent space (bundle) $T(S)$ of S). In more detail, it can be shown that ([36], Remark 4.11)

$$A(\omega,E,\partial_\omega)\psi=-\pi\ (\partial_{E'}\mu)(E,E)\Delta_S\psi\tag{27}$$

where Δ_S is the Laplace-Beltrami operator on S.

The proof follows by Theorem 4.9, Lemma 4.10 and Remark 4.11 given in [36]. Let $A^*(\omega,E,\partial_\omega)$ be the formal adjoint of $A(\omega,E,\partial_\omega)$. Since $\Delta_S^*=\Delta_S$ we see that $A^*(\omega,E,\partial_\omega)=A(\omega,E,\partial_\omega)$. We also notice that $(\partial_{E'}\mu)(E,E)=\frac{1}{E+2}-\frac{1}{E}<0$.

Let

$$f_2(x,\omega,E',E):=\hat\sigma_2(x,E',E)\int_0^{2\pi}\psi(x,\gamma(E',E,\omega)(s),E')ds.$$

For $K_{2,1,\kappa}$ we apply the approximation

$$f_2(x,\omega,E',E)\approx f_2(x,\omega,E,E)+\frac{\partial f_2}{\partial E'}(x,\omega,E,E)(E'-E)\tag{28}$$

on the interval $[E,\kappa E]$. Since by Theorem 4.1 (recall that $\gamma(E,E,\omega)(s)=\omega$)

$$\begin{aligned}\frac{\partial f_2}{\partial E'}(x,\omega,E,E)=2\pi\ &\frac{\partial\hat\sigma_2}{\partial E'}(x,E,E)\psi(x,\omega,E)\\&+\hat\sigma_2(x,E,E)(A(x,\omega,E,\partial_\omega)\psi)(x,\omega,E)\\&+2\pi\ \hat\sigma_2(x,E,E)\frac{\partial\psi}{\partial E}(x,\omega,E)\end{aligned}$$

we obtain the approximation

$$
\begin{aligned}
&(K_{2,1,\kappa}\psi)(x,\omega,E) = \text{p.f.}\int_E^{\kappa E} \frac{1}{(E'-E)^2} f_2(x,\omega,E',E)dE' \\
&\approx (\widetilde{K}_{2,1,\kappa}\psi)(x,\omega,E) := -2\pi\ \frac{1}{\kappa E - E}\hat{\sigma}_2(x,E,E)\psi(x,\omega,E) \\
&+ \ln(\kappa E - E)\hat{\sigma}_2(x,E,E)(A(x,\omega,E,\partial_\omega)\psi)(x,\omega,E) \\
&+ 2\pi\ \hat{\sigma}_2(x,E,E)\ln(\kappa E - E)\frac{\partial\psi}{\partial E}(x,\omega,E) \\
&+ 2\pi\ \ln(\kappa E - E)\frac{\partial\hat{\sigma}_2}{\partial E'}(x,E,E)\psi(x,\omega,E)
\end{aligned} \tag{29}
$$

where we again recalled that $\gamma(E,E,\omega)(s) = \omega$ and by (11), (12)

$$
\text{p.f.}\int_E^{\kappa E} \frac{1}{E'-E}dE' = \ln(\kappa E - E),\ \ \text{p.f.}\int_E^{\kappa E} \frac{1}{(E'-E)^2}dE' = -\frac{1}{\kappa E - E}.
$$

To simplify notation we introduce the following definitions.

$$
\begin{aligned}
S_\kappa(x,E) &:= 2\pi\ \hat{\sigma}_2(x,E,E)\ln(\kappa E - E), \\
\Sigma_\kappa(x,E) &:= \Sigma(x,E) + 2\pi\ \hat{\sigma}_2(x,E,E)\frac{1}{\kappa E - E} \\
&\quad - 2\pi\ \ln(\kappa E - E)\frac{\partial\hat{\sigma}_2}{\partial E'}(x,E,E) \\
&\quad + 2\pi\ \ln(\kappa E - E)\hat{\sigma}_1(x,E,E), \\
(Q_\kappa(x,\omega,E,\partial_\omega)\psi)(x,\omega,E) &:= \ln(\kappa E - E)\hat{\sigma}_2(x,E,E)(A(x,\omega,E,\partial_\omega)\psi)(x,\omega,E), \\
K_{r,\kappa} &:= K_0 + K_{2,0,\kappa} - K_{1,0,\kappa}.
\end{aligned}
$$

Then, the truncated and approximated transport operator, say T_κ, is given by

$$
\begin{aligned}
T_\kappa\psi &:= -\widetilde{K}_{2,1,\kappa}\psi + \widetilde{K}_{1,1,\kappa}\psi + \omega\cdot\nabla_x\psi + \Sigma\psi - K_{r,\kappa}\psi \\
&= -S_\kappa(x,E)\frac{\partial\psi}{\partial E} - Q_\kappa(x,\omega,E,\partial_\omega)\psi + \omega\cdot\nabla_x\psi + \Sigma_\kappa\psi - K_{r,\kappa}\psi.
\end{aligned} \tag{30}
$$

The restricted collision operator $K_{r,\kappa}$ is

$$
\begin{aligned}
&(K_{r,\kappa}\psi)(x,\omega,E) = ((K_0 + K_{2,0,\kappa} - K_{1,0,\kappa})\psi)(x,\omega,E) \\
&= \int_{I'} \chi_{\mathbb{R}_+}(E'-E)\hat{\sigma}_0(x,E',E)\int_0^{2\pi} \psi(x,\gamma(E',E,\omega)(s),E')dsdE' \\
&- \int_{I'} \chi_{\mathbb{R}_+}(E'-\kappa E)\hat{\sigma}_1(x,E',E)\frac{1}{E'-E}\int_0^{2\pi} \psi(x,\gamma(E',E,\omega)(s),E')dsdE' \\
&+ \int_{I'} \chi_{\mathbb{R}_+}(E'-\kappa E)\hat{\sigma}_2(x,E',E)\frac{1}{(E'-E)^2}\int_0^{2\pi} \psi(x,\gamma(E',E,\omega)(s),E')dsdE' \\
&= \int_{I'} \hat{\sigma}_{r,\kappa}(x,E',E)\int_0^{2\pi} \psi(x,\gamma(E',E,\omega)(s),E')dsdE'
\end{aligned} \tag{31}
$$

where

$$
\begin{aligned}
\hat{\sigma}_{r,\kappa}(x,E',E) := {}& \chi_{\mathbb{R}_+}(E'-E)\hat{\sigma}_0(x,E',E) - \chi_{\mathbb{R}_+}(E'-\kappa E)\hat{\sigma}_1(x,E',E)\frac{1}{E'-E} \\
&+ \chi_{\mathbb{R}_+}(E'-\kappa E)\hat{\sigma}_2(x,E',E)\frac{1}{(E'-E)^2}.
\end{aligned}
$$

5. Analysis of the approximation error. In the previous computations the only approximations used are

$$K_{1,1,\kappa} \approx \widetilde{K}_{1,1,\kappa} \text{ and } K_{2,1,\kappa} \approx \widetilde{K}_{2,1,\kappa}$$

where $\widetilde{K}_{j,1,\kappa},\ j = 1,2$ are given by (24) and (29), respectively. Hence, the exact transport operator T is approximated by T_κ which is recalled here as

$$T_\kappa = -\widetilde{K}_{2,1,\kappa} + \widetilde{K}_{1,1,\kappa} + \omega \cdot \nabla_x + \Sigma - (K_0 + K_{2,0,\kappa} - K_{1,0,\kappa}).$$

Therefore, the following operator needs to be estimated to control the error in the approximation.

$$T - T_\kappa = -(K_{2,1,\kappa} - \widetilde{K}_{2,1,\kappa}) + (K_{1,1,\kappa} - \widetilde{K}_{1,1,\kappa}). \tag{32}$$

We need the following technical lemma that is proven in [36, Corollary 4.14].

LEMMA 5.1. *Suppose that $\psi \in C(\overline{G}, C^1(S \times I))$. Let*

$$h_1(x,\omega,E',E) := \int_0^{2\pi} \psi(x, \gamma(E',E,\omega)(s), E')ds.$$

Then h_1 obeys

$$|h_1(x,\omega,E',E) - h_1(x,\omega,E,E)| \leq C_1 \|\psi\|_{C(\overline{G},C^1(S\times I)} (E'-E)^{\frac{1}{2}},\ E' \geq E. \tag{33}$$

Proof. See [36, Corollary 4.14]. □

For $E' \neq E$ we define

$$h_2(x,\omega,E',E) := \int_0^{2\pi} \left\langle \nabla_S \psi(x,\gamma(E',E,\omega)(s),E'), \frac{\partial \gamma}{\partial E'}(E',E,\omega)(s) \right\rangle ds.$$

By the proof of [36, Theorem 4.9] h_2 is defined also in the limit $E' \to E$ and

$$h_2(x,\omega,E,E) := \lim_{E'\to E} h_2(x,\omega,E',E) = (A(x,\omega,E,\partial_\omega)\psi)(x,\omega,E).$$

Due to [36, Corollary 4.16] we control also h_2 as

LEMMA 5.2. *Suppose that $\psi \in C(\overline{G}, C^2(I, C^3(S)))$. Then the mapping h_2 fulifIls*

$$|h_2(x,\omega,E',E) - h_2(x,\omega,E,E)| \leq C_2 \|\psi\|_{C(\overline{G},C^2(I,C^3(S)))} (E'-E)^{\frac{1}{2}},\ E' \geq E. \tag{34}$$

Proof. See [36, Corollary 4.16]. □

The preliminary discussion allows now to control the error in the approximation in the following Theorem.

THEOREM 5.3. *Suppose that $\hat{\sigma}_1 \in C(\overline{G}, C^1(I' \times I))$ and $\hat{\sigma}_2 \in C(\overline{G}, C^2(I' \times I))$. Then for $\psi \in C(\overline{G}, C^2(I, C^3(S))$*

$$\|T\psi - T_\kappa\psi\|_{L^\infty(G\times S\times I)} \leq C \|\psi\|_{C(\overline{G},C^2(I,C^3(S))} (\kappa - 1)^{\frac{1}{2}}. \tag{35}$$

Hence $T_\kappa\psi \to T\psi$ uniformly in $\overline{G} \times S \times I$ as $\kappa \to 1^+$.

Proof. Recall that

$$(K_{j,1,\kappa}\psi)(x,\omega,E)=\text{p.f.}\int_E^{\kappa E}\frac{1}{(E'-E)^j}f_j(x,\omega,E',E)dE'$$

where

$$f_j(x,\omega,E',E)=\hat{\sigma}_j(x,E',E)\int_0^{2\pi}\psi(x,\gamma(E',E,\omega)(s),E').$$

For $K_{1,1,\kappa}$ we used the approximation

$$f_1(x,\omega,E',E)\approx f_1(x,\omega,E,E) \tag{36}$$

on the interval $[E,\kappa E]$. Hence the error is

$$\begin{aligned}&|(K_{1,1,\kappa}\psi)(x,\omega,E)-(\widetilde{K}_{1,1,\kappa}\psi)(x,\omega,E)|\\&=\Big|\text{p.f.}\int_E^{\kappa E}\frac{f_1(x,\omega,E',E)-f_1(x,\omega,E,E)}{E'-E}dE'\Big|.\end{aligned} \tag{37}$$

Using Lemma 5.1 for $E'\geq E$ (we omit technical details emerging from the factor $\hat{\sigma}_1(x,E',E)$)

$$|f_1(x,\omega,E',E)-f_1(x,\omega,E,E)|\leq C\left\|\psi\right\|_{C(\overline{G},C^1(S\times I))}(E'-E)^{\frac12}.$$

Hence the p.f.-integral in (37) is an ordinary improper integral and

$$\begin{aligned}&|(K_{1,1,\kappa}\psi)(x,\omega,E)-(\widetilde{K}_{1,1,\kappa}\psi)(x,\omega,E)|\leq\int_E^{\kappa E}\Big|\frac{f_1(x,\omega,E',E)-f_1(x,\omega,E,E)}{E'-E}\Big|dE'\\&\leq\int_E^{\kappa E}C_1'\left\|\psi\right\|_{C(\overline{G},C^1(S\times I))}(E'-E)^{\frac12-1}dE'=2C_1'\left\|\psi\right\|_{C(\overline{G},C^1(S\times I))}(\kappa-1)^{\frac12}E^{\frac12}\\&\leq 2C_1'E_m^{\frac12}\left\|\psi\right\|_{C(\overline{G},C^1(S\times I))}(\kappa-1)^{\frac12}.\end{aligned} \tag{38}$$

For $K_{2,1,\kappa}$ we used the approximation

$$f_2(x,\omega,E',E)\approx f_2(x,\omega,E,E)+\frac{\partial f_2}{\partial E'}(x,\omega,E,E)(E'-E) \tag{39}$$

on the interval $[E,\kappa E]$. Due to the formula for the remainder term in Taylor's expansion

$$\begin{aligned}&f_2(x,\omega,E',E)=f_2(x,\omega,E,E)+\int_0^1\frac{\partial f_2}{\partial E'}(x,\omega,E+t(E'-E),E)dt\ (E'-E)\\&=f_2(x,\omega,E,E)+\frac{\partial f_2}{\partial E'}(x,\omega,E,E)(E'-E)\\&+\int_0^1\big(\frac{\partial f_2}{\partial E'}(x,\omega,E+t(E'-E),E)-\frac{\partial f_2}{\partial E'}(x,\omega,E,E)\big)dt\ (E'-E).\end{aligned} \tag{40}$$

Hence the error is

$$\begin{aligned}&|(K_{2,1,\kappa}\psi)(x,\omega,E)-(\widetilde{K}_{2,1,\kappa}\psi)(x,\omega,E)|\\&=\Big|\text{p.f.}\int_E^{\kappa E}\int_0^1\frac{\frac{\partial f_2}{\partial E'}(x,\omega,E+t(E'-E),E)-\frac{\partial f_2}{\partial E'}(x,\omega,E,E)}{E'-E}dE'dt\Big|.\end{aligned} \tag{41}$$

By Theorem 4.1 for $E \neq E'$

$$\frac{\partial f_2}{\partial E'}(x,\omega,E',E) = \frac{\partial \hat{\sigma}_2}{\partial E'}(x,E',E)\int_0^{2\pi} \psi(x,\gamma(E',E,\omega)(s),E')$$
$$\hat{\sigma}_2(x,E',E)\int_0^{2\pi} \Big\langle (\nabla_S\psi)(x,\gamma(E',E,\omega)(s),E'), \frac{\partial\gamma}{\partial E'}(E',E,\omega)(s)\Big\rangle ds$$
$$+ \hat{\sigma}_2(x,E',E)\int_0^{2\pi} \frac{\partial\psi}{\partial E}(x,\gamma(E',E,\omega)(s),E')ds \tag{42}$$

and for $E' = E$

$$\frac{\partial f_2}{\partial E'}(x,\omega,E,E) = 2\pi\ \frac{\partial \hat{\sigma}_2}{\partial E'}(x,E,E)\psi(x,\omega,E)$$
$$+ \hat{\sigma}_2(x,E,E)(A(x,\omega,E,\partial_\omega)\psi)(x,\omega,E) + 2\pi\ \hat{\sigma}_2(x,E,E)\frac{\partial\psi}{\partial E}(x,\omega,E). \tag{43}$$

Due to Lemma 5.2 we find by (41), (42) and (43) similarly as before that

$$|(K_{2,1,\kappa}\psi)(x,\omega,E) - (\widetilde{K}_{2,1,\kappa}\psi)(x,\omega,E)|$$
$$= \int_E^{\kappa E}\int_0^1 \Big| \frac{\frac{\partial f_2}{\partial E'}(x,\omega,E+t(E'-E),E) - \frac{\partial f_2}{\partial E'}(x,\omega,E,E)}{E'-E}\Big| dE'dt$$
$$\leq C_2' \left\|\psi\right\|_{C(\overline{G},C^2(I,C^3(S))}\int_E^{\kappa E}\int_0^1 (t(E'-E))^{\frac{1}{2}-1}dE'dt$$
$$\leq 4C_2' E_m^{\frac{1}{2}} \left\|\psi\right\|_{C(\overline{G},C^2(I,C^3(S))} (\kappa-1)^{\frac{1}{2}}. \tag{44}$$

This completes the proof. □

A similar estimate can be obtained for the formal adjoint operator T^*. The operator T^* can be found e.g. in [36, Section 4.3] and the computation of T_κ^* is routine since, as noticed above, $\Delta_S^* = \Delta_S$. The adjoint operator is relevant in optimal control problems e.g. appearing in dose computations for radiative treatment planning. The computations are similar and therefore omitted here.

Remark 5.4. The assertion of Theorem 5.3 is valid under weaker assumptions. For example, it suffices to assume only that $\hat{\sigma}_1 \in C(\overline{G}, C^\alpha(I'\times I))$, $\hat{\sigma}_2 \in C(\overline{G}, C^{1+\alpha}(I'\times I))$ and $\psi \in C^1(\overline{G}, C^{1+\alpha}(I, C^{2+\alpha}(S))$ for $\alpha > 0$.

6. Existence of solutions for the approximation.

6.1. Assumptions for the restricted collision operator. The existence of solutions for the truncated transport equation (30) that is,

$$T_\kappa\psi := -S_\kappa(x,E)\frac{\partial\psi}{\partial E} - Q_\kappa(x,\omega,E,\partial_\omega)\psi + \omega\cdot\nabla_x\psi + \Sigma_\kappa\psi - K_{r,\kappa}\psi = f \tag{45}$$

can be studied, by applying e.g. the generalized Lax-Milgram Theorem. Other methods could be, for example m-dissipativity and evolution operator theory of unbounded operators and contraction (fixed point) methods. We will use the following statement that can be found e.g. in [38, p. 403] or [14, p. 234].

THEOREM 6.1. *Let X and Y be Hilbert spaces, with Y continuously embedded into X. Assume that $B(\cdot,\cdot) : X \times Y \to \mathbb{R}$ is a bilinear form satisfying the following properties with $M \geq 0,\ c > 0$,*

$$|B(u,v)| \leq M\|u\|_X\|v\|_Y \quad \forall u \in X,\ v \in Y \quad \text{(boundedness)} \tag{46}$$

and

$$B(v,v) \geq c\|v\|_X^2 \quad \forall v \in Y \quad \text{(coercivity)}. \tag{47}$$

Suppose that $F : X \to \mathbb{R}$ is a bounded linear form. Then there exists $u \in X$ (possibly non-unique) such that

$$B(u,v) = F(v) \quad \forall v \in Y. \tag{48}$$

We assume that the restricted collision operator is of the form

$$(K_r\psi)(x,\omega,E) = \int_{I'}\int_0^{2\pi} \hat{\sigma}_r(x,E',E)\psi(x,\gamma(E',E,\omega)(s),E')dsdE' \tag{49}$$

where $\hat{\sigma}_r : G \times I' \times I \to \mathbb{R}$ is a non-negative measurable function such that

$$\begin{aligned} &\int_{I'} \hat{\sigma}_r(x,E',E)dE' \leq M_1,\ \text{a.e. } (x,E) \in G \times I, \\ &\int_{I'} \hat{\sigma}_r(x,E,E')dE' \leq M_2,\ \text{a.e. } (x,E) \in G \times I. \end{aligned} \tag{50}$$

To obtain the desired coercitivity property of the operator $\Sigma - K_r$ we assume that

$$\Sigma(x,\omega,E) - 2\pi\int_{I'} \hat{\sigma}_r(x,E,E')dE' \geq c, \tag{51}$$

and

$$\Sigma(x,\omega,E) - 2\pi\int_{I'} \hat{\sigma}_r(x,E',E)dE' \geq c. \tag{52}$$

for a.e. $(x,\omega,E) \in G \times S \times I$ where $c > 0$.

We recall the following results [35].

THEOREM 6.2. *Suppose that the assumptions (50) are valid. Then K_r is a bounded operator $L^2(G \times S \times I) \to L^2(G \times S \times I)$.*

THEOREM 6.3. *Suppose that $\Sigma \in L^\infty(G\times S\times I),\ \Sigma \geq 0$ and that the assumptions (50), (51) and (52) are valid. Then*

$$\langle(\Sigma - K_r)\psi,\psi\rangle_{L^2(G\times S\times I)} \geq c\|\psi\|^2_{L^2(G\times S\times I)} \ \text{ for all } \psi \in L^2(G \times S \times I). \tag{53}$$

REMARK 6.4. Note that the integrals appearing in the conditions (50), (51), (52) for boundedness and coercitivity (that is, conditions (4.7) and (4.8) imposed in [35]) for the above approximative operator $K_{r,\kappa}$ (that is, (31)) become

$$\int_{I'} \hat{\sigma}_{r,\kappa}(x,E',E)dE'$$
$$= \int_{E}^{E_m} \hat{\sigma}_0(x,E',E)dE' - \int_{\kappa E}^{E_m} \hat{\sigma}_1(x,E',E)\frac{1}{E'-E}dE'$$
$$+ \int_{\kappa E}^{E_m} \hat{\sigma}_2(x,E',E)\frac{1}{(E'-E)^2}dE'$$

and

$$\int_{I} \hat{\sigma}_{r,\kappa}(x,E',E)dE$$
$$= \int_{E_0}^{E'} \hat{\sigma}_0(x,E',E)dE - \int_{E_0}^{E'/\kappa} \hat{\sigma}_1(x,E',E)\frac{1}{E'-E}dE$$
$$+ \int_{E_0}^{E'/\kappa} \hat{\sigma}_2(x,E',E)\frac{1}{(E'-E)^2}dE.$$

6.2. Existence of weak solutions. Consider the problem

$$T\psi = f,\ \psi_{|\Gamma_-} = g,\ \psi(.,.,E_m) = 0 \tag{54}$$

where $f \in L^2(G\times S\times I),\ g \in T^2(\Gamma_-)$.

The approximative operators T_κ are given by the discussion in the previous section, by equation (45). However, for the existence result we can slightly generalize the discussion and assume a more general form for the stopping power S_κ and for the operator Q_κ. Hence, in the following we assume that T is of the form mentioned in the introduction, i.e.,

$$T\psi = a(x,E)\frac{\partial\psi}{\partial E} + b(x,\omega,E,\partial_\omega)\psi + \omega\cdot\nabla_x\psi + \Sigma\psi - K_r\psi \tag{55}$$

where $b(x,\omega,E,\partial_\omega)\psi$ is of the form (here $d=(d_1,d_2)\sim d_1\frac{\partial}{\partial\omega_1}+d_2\frac{\partial}{\partial\omega_2}$ and $d\cdot\nabla_S$ is the Riemannian inner product on $T(S)$)

$$b(x,\omega,E,\partial_\omega)\psi = c(x,E)\Delta_S\psi + d(x,\omega,E)\cdot\nabla_S\psi. \tag{56}$$

The operator T_κ in (45) is of the form (55). We continue therefore with a discussion of the general operator T. Further, denote by

$$d(x,\omega,E,\partial_\omega)\psi := d(x,\omega,E)\cdot\nabla_S\psi. \tag{57}$$

REMARK 6.5. Suppose that $a(x,\omega,E,\partial_\omega) = \sum_{|\alpha|\le 2} a_\alpha(x,\omega,E)\partial_\omega^\alpha$ is any second order partial differential operator with $L^\infty(G\times I, W^{\infty,1}(S))$-coefficients such that

$$|\langle a(x,\omega,E,\partial_\omega)\psi, v\rangle_{L^2(G\times S\times I)}| \le M'\,\|\psi\|_{L^2(G\times I,H^1(S))}\,\|v\|_{L^2(G\times I,H^1(S))} \tag{58}$$

for all $\psi,\ v\in C^1(\overline{G}\times I, C^2(S))$ and

$$\langle a(x,\omega,E,\partial_\omega)v, v\rangle_{L^2(G\times S\times I)} \ge c''\,\|v\|^2_{L^2(G\times I,H^1(S))} \tag{59}$$

for all $v \in C^1(\overline{G} \times I, C^2(S))$ where $c'' > 0$. In the subsequent considerations we could replace $c(x,E)\Delta_S\psi$ (after minor modifications) with any operator $a(x,\omega,E,\partial_\omega)$ satisfying (58) and (59). To keep the treatments concise we restrict ourselves to the case where $b(x,\omega,E,\partial_\omega)$ is of the form (56).

At first we verify formally the corresponding variational equation. Assume that ψ is a solution of (54) in the classical sense. Let $\psi,\ v \in C^1(\overline{G} \times I, C^2(S))$. By the properties of Laplace-Beltrami operator we have

$$\langle \Delta_S\psi(x,.,E), v(x,.,E)\rangle_{L^2(S)} = -\int_S \langle (\nabla_S\psi)(x,\omega,E), (\nabla_S v)(x,\omega,E)\rangle\, d\omega$$

where $\langle \nabla_S\psi, \nabla_S v\rangle$ is the chosen Riemannian inner product on S. Hence,

$$\begin{aligned}
&\langle c(x,E)\Delta_S\psi, v\rangle_{L^2(G\times S\times I)} = -\langle \nabla_S\psi, c(x,E)\nabla_S v\rangle_{L^2(G\times S\times I)} \\
&:= -\int_G\int_S\int_I \langle \nabla_S\psi(x,\omega,E), c(x,E), \nabla_S v(x,\omega,E)\rangle\, dx d\omega dE
\end{aligned} \tag{60}$$

and

$$\langle d(x,\omega,E,\partial_\omega)\psi, v\rangle_{L^2(G\times S\times I)} = \langle \psi, d^*(x,\omega,E,\partial_\omega)v\rangle_{L^2(G\times S\times I)}$$

where $d^*(x,\omega,E,\partial_\omega)$ is the formal adjoint of $d(x,\omega,E,\partial_\omega)$. Note that

$$d^*(x,\omega,E,\partial_\omega)v = -d(x,\omega,E,\partial_\omega)v - \mathrm{div}_S(d)\ v$$

where div_S is the divergence on S. By integration by parts

$$\begin{aligned}
&\left\langle a\frac{\partial\psi}{\partial E}, v\right\rangle_{L^2(G\times S\times I)} = \int_G\int_S a(x,E_m)\psi(x,\omega,E_m)v(x,\omega,E_m)d\omega dx \\
&-\int_G\int_S a(x,E_0)\psi(x,\omega,E_0)v(x,\omega,E_0)d\omega dx - \left\langle \psi, \frac{\partial(av)}{\partial E}\right\rangle_{L^2(G\times S\times I)} \\
&= -\left\langle \psi, a\frac{\partial v}{\partial E}\right\rangle_{L^2(G\times S\times I)} - \left\langle \psi, \frac{\partial a}{\partial E}v\right\rangle_{L^2(G\times S\times I)} \\
&+ \langle a(.,E_m)\psi(.,.,E_m), v(.,.,E_m)\rangle_{L^2(G\times S)} \\
&- \langle a(.,E_0)\psi(.,.,E_0), v(.,.,E_0)\rangle_{L^2(G\times S)}\,.
\end{aligned} \tag{61}$$

Using Green's formula (5) we obtain

$$\begin{aligned}
&\langle \omega\cdot\nabla_x\psi, v\rangle_{L^2(G\times S\times I)} = -\langle \psi, \omega\cdot\nabla_x v\rangle_{L^2(G\times S\times I)} + \int_{\partial G\times S\times I} (\omega\cdot\nu)\psi v d\sigma d\omega dE \\
&= -\langle \psi, \omega\cdot\nabla_x v\rangle_{L^2(G\times S\times I)} + \int_{\Gamma_+} (\omega\cdot\nu)_+\psi v d\sigma d\omega dE - \int_{\Gamma_-} (\omega\cdot\nu)_-\psi v d\sigma d\omega dE
\end{aligned} \tag{62}$$

where $(\omega\cdot\nu)_\pm$ is the positive/negative part of $\omega\cdot\nu$. Finally,

$$\langle K_r\psi, v\rangle_{L^2(G\times S\times I} = \langle \psi, K_r^* v\rangle_{L^2(G\times S\times I}\,,\ \ \langle \Sigma\psi, v\rangle_{L^2(G\times S\times I} = \langle \psi, \Sigma^* v\rangle_{L^2(G\times S\times I}\,. \tag{63}$$

As a conclusion we see that if ψ is a solution of problem (54) then the following weak formulation is fulfilled

$$\begin{aligned} B(\psi,v) := &-\left\langle \psi, a\frac{\partial v}{\partial E}\right\rangle_{L^2(G\times S\times I)} - \left\langle \psi, \frac{\partial a}{\partial E}v\right\rangle_{L^2(G\times S\times I)} \\ &- \langle a(.,E_0)\psi(.,.,E_0), v(.,.,E_0)\rangle_{L^2(G\times S)} \\ &- \langle \nabla_S\psi, c(x,E)\nabla_S v\rangle_{L^2(G\times S\times I)} + \langle \psi, d^*(x,\omega,E,\partial_\omega)v\rangle_{L^2(G\times S\times I)} \\ &- \langle \psi, \omega\cdot\nabla_x v\rangle_{L^2(G\times S\times I)} + \langle \gamma_+(\psi),\gamma_+(v)\rangle_{T^2(\Gamma_+)} \\ &+ \langle \psi, \Sigma^* v - K_r^* v\rangle_{L^2(G\times S\times I)} = \langle f, v\rangle_{L^2(G\times S\times I)} + \langle g, \gamma_-(v)\rangle_{T^2(\Gamma_-)}. \end{aligned} \tag{64}$$

Define in $C^1(\overline{G}\times I, C^2(S))$ inner products

$$\begin{aligned} \langle \psi, v\rangle_{\mathcal{H}} := &\langle \psi, v\rangle_{L^2(G\times S\times I)} + \langle \gamma(\psi),\gamma(v)\rangle_{T^2(\Gamma)} \\ &+ \langle \psi(.,.,E_0), v(.,.,E_0)\rangle_{L^2(G\times S)} + \langle \psi(.,.,E_m), v(.,.,E_m)\rangle_{L^2(G\times S)} \\ &+ \langle \psi, v\rangle_{L^2(G\times I, H^1(S))} \end{aligned} \tag{65}$$

and

$$\langle \psi, v\rangle_{\widehat{\mathcal{H}}} = \langle \psi, v\rangle_{\mathcal{H}} + \langle \omega\cdot\nabla_x\psi, \omega\cdot\nabla_x v\rangle_{L^2(G\times S\times I)} + \left\langle \frac{\partial \psi}{\partial E}, \frac{\partial v}{\partial E}\right\rangle_{L^2(G\times S\times I)}. \tag{66}$$

Let $\mathcal{H}$ and $\widehat{\mathcal{H}}$ be the completions of $C^1(\overline{G}\times I, C^2(S))$ with respect to inner products $\langle .,.\rangle_{\mathcal{H}}$ and $\langle .,.\rangle_{\widehat{\mathcal{H}}}$, respectively.

In the following theorem we yield sufficient criteria under which the bilinear form $B : C^1(\overline{G}\times I, C^2(S))\times C^1(\overline{G}\times I, C^2(S))\to\mathbb{R}$ is *bounded and coercive.* We collect here all needed assumptions of this section together.

THEOREM 6.6. *We assume for the coefficients that*

$$a\in L^\infty(G, W^{\infty,1}(I)), c\in L^\infty(G\times I),\quad d_j\in L^\infty(G\times I, W^{\infty,1}(S)) \tag{67}$$

$$-\Big(\frac{\partial a}{\partial E}(x,E) + (\mathrm{div}_S\ d)(x,\omega,E)\Big)\geq q_1 > 0, a.e., \tag{68}$$

$$-c(x,E)\geq q_2 > 0, a.e., \tag{69}$$

$$-a(x,E_0)\geq q_3>0,\ -a(x,E_m)\geq q_3 > 0, a.e.. \tag{70}$$

Furthermore, we assume that

$$\Sigma\in L^\infty(G\times S\times I),\ \Sigma(x,\omega,E)\geq 0 \text{ a.e.}, \tag{71}$$

and that $\hat{\sigma}_r : G\times I'\times I\to\mathbb{R}$ is a non-negative measurable function such that

$$\begin{aligned} &\int_{I'}\hat{\sigma}_r(x,E',E)dE'\leq M_1,\ \text{a.e. } (x,E)\in G\times I, \\ &\int_{I'}\hat{\sigma}_r(x,E,E')dE'\leq M_2,\ \text{a.e. } (x,E)\in G\times I, \end{aligned} \tag{72}$$

$$\Sigma(x,\omega,E) - 2\pi\int_{I'}\hat{\sigma}_r(x,E,E')dE' \geq c, \tag{73}$$

and

$$\Sigma(x,\omega,E) - 2\pi\int_{I'}\hat{\sigma}_r(x,E',E)dE' \geq c, \tag{74}$$

for a.e. $(x,\omega,E)\in G\times S\times I$ *where* $c>0$*. Then there exists a constant* $M>0$ *such that*

$$|B(\psi,v)| \leq M\,\|\psi\|_{\mathcal{H}}\,\|v\|_{\widehat{\mathcal{H}}} \quad \forall\psi,\ v\in C^1(\overline{G}\times I, C^2(S)) \tag{75}$$

and

$$B(v,v) \geq c'\,\|v\|_{\mathcal{H}}^2 \quad \forall v\in C^1(\overline{G}\times I, C^2(S)) \tag{76}$$

where

$$c' := \min\{\frac{q_1}{2},\frac{q_3}{2},q_2,\frac{1}{2},c\}. \tag{77}$$

Proof. A. Noting that

$$|\left\langle\nabla_S\psi, c(x,E)\nabla_S v\right\rangle_{L^2(G\times S\times I)}| \leq \|c\|_{L^\infty(G\times I)}\,\|\nabla_S\psi\|_{L^2(G\times S\times I)}\,\|\nabla_S v\|_{L^2(G\times S\times I)}$$

and

$$\begin{aligned}&|\left\langle\psi, d^*(x,\omega,E,\partial_\omega)v\right\rangle_{L^2(G\times S\times I)}|\\ &\leq C_1\,\|d\|_{L^\infty(G\times I,W^{\infty,1}(S))}\,\|\psi\|_{L^2(G\times S\times I)}\,\|v\|_{L^2(G\times I,H^1(S))}\end{aligned}$$

the proof of boundedness is analogous to the proof of Theorem 6.4 given in [33].

B. Secondly, we verify the coercitivity (76). Integrating by parts we have

$$\begin{aligned}-\left\langle v, a\frac{\partial v}{\partial E}\right\rangle_{L^2(G\times S\times I)} &= \left\langle v, \frac{\partial(av)}{\partial E}\right\rangle_{L^2(G\times S\times I)}\\ &-\left\langle v(\cdot,\cdot,E_m), a(\cdot,E_m)v(\cdot,\cdot,E_m)\right\rangle_{L^2(G\times S)} + \left\langle v(\cdot,\cdot,0), a(\cdot,0)v(\cdot,\cdot,0)\right\rangle_{L^2(G\times S)}\end{aligned} \tag{78}$$

and then noting that $\frac{\partial(av)}{\partial E} = \frac{\partial a}{\partial E}v + a\frac{\partial v}{\partial E}$ we get

$$\begin{aligned}-\left\langle v, a\frac{\partial v}{\partial E}\right\rangle_{L^2(G\times S\times I)} = \frac{1}{2}\Big(&\left\langle v, \frac{\partial a}{\partial E}v\right\rangle_{L^2(G\times S\times I)}\\ &-\left\langle v(\cdot,\cdot,E_m), a(\cdot,E_m)v(\cdot,\cdot,E_m)\right\rangle_{L^2(G\times S)}\\ &+\left\langle v(\cdot,\cdot,E_0), a(\cdot,E_0)v(\cdot,\cdot,0)\right\rangle_{L^2(G\times S)}\Big).\end{aligned} \tag{79}$$

Using the Green's formula we have

$$-\left\langle v, \omega\cdot\nabla_x v\right\rangle_{L^2(G\times S\times I)} = \left\langle\omega\cdot\nabla_x v, v\right\rangle_{L^2(G\times S\times I)} - \int_{\partial G\times S\times I}(\omega\cdot\nu)v^2 d\sigma d\omega dE \tag{80}$$

which implies

$$\langle \omega\cdot\nabla_x v, v\rangle_{L^2(G\times S\times I)} = \frac{1}{2}\int_{\partial G\times S\times I}(\omega\cdot\nu)v^2 d\sigma d\omega dE$$
$$= \frac{1}{2}\int_{\partial G\times S\times I}((\omega\cdot\nu)_+ - (\omega\cdot\nu)_-)v^2 d\sigma d\omega dE$$
$$= \frac{1}{2}\big(\,\|\gamma_+(v)\|^2_{T^2(\Gamma_+)} - \|\gamma_-(v)\|^2_{T^2(\Gamma_-)}\,\big). \tag{81}$$

Furthermore, we have

$$\langle d(x,\omega,E,\partial_\omega)v, v\rangle_{L^2(G\times S\times I)} = \langle v, d^*(x,\omega,E,\partial_\omega)v\rangle_{L^2(G\times S\times I)}$$

which implies (by recalling (57) that

$$\langle v, d^*(x,\omega,E,\partial_\omega)v\rangle_{L^2(G\times S\times I)} = -\frac{1}{2}\,\langle(\mathrm{div}_S\ d)v, v\rangle_{L^2(G\times S\times I)}\,. \tag{82}$$

Finally, we have

$$-\,\langle\nabla_S v, c(x,E)\nabla_S v\rangle_{L^2(G\times S\times I)}$$
$$= -\int_{G\times S\times I} c(x,E)\,\langle(\nabla_S v)(x,\omega,E), (\nabla_S v)(x,\omega,E)\rangle\, dx d\omega dE. \tag{83}$$

Inserting (79), (81), (82) and (83) in the expression of $B(.,.)$ (with $\psi = v$) we get in virtue of the assumptions (68), (69), (70) and by Theorem 6.3 the estimate

$$\begin{aligned}
&B(v,v) = -\frac{1}{2}\left\langle v, \frac{\partial a}{\partial E}v\right\rangle_{L^2(G\times S\times I)}\\
&+\frac{1}{2}\Big(-\langle v(\cdot,\cdot,E_m), a(\cdot,E_m)v(\cdot,\cdot,E_m)\rangle_{L^2(G\times S)} + \langle v(\cdot,\cdot,E_0), a(\cdot,E_0)v(\cdot,\cdot,E_0)\rangle_{L^2(G\times S)}\Big)\\
&-\langle a(.,E_0)v(.,.,E_0), v(.,.,E_0)\rangle_{L^2(G\times S}\\
&-\int_{G\times S\times I} c(x,E)\,\langle(\nabla_S v)(x,\omega,E), (\nabla_S v)(x,\omega,E)\rangle\, dx d\omega dE - \frac{1}{2}\,\langle(\mathrm{div}_S\ d)v, v\rangle_{L^2(G\times S\times I)}\\
&-\frac{1}{2}\big(\,\langle\gamma_+(v),\gamma_+(v)\rangle_{T^2(\Gamma_+)} - \langle\gamma_-(v),\gamma_-(v)\rangle_{T^2(\Gamma_-)}\,\big) + \langle\gamma_+(v),\gamma_+(v)\rangle_{T^2(\Gamma_+)}\\
&+\langle v, \Sigma^* v - K_r^* v\rangle_{L^2(G\times S\times I)}\\
&\geq \frac{q_1}{2}\,\|v\|^2_{L^2(G\times S\times I)} + \frac{1}{2}q_3\,\langle v(\cdot,\cdot,E_m), v(\cdot,\cdot,E_m)\rangle_{L^2(G\times S)}\\
&+\frac{1}{2}q_3\,\langle v(\cdot,\cdot,E_0), v(\cdot,\cdot,E_0)\rangle_{L^2(G\times S)} + q_2\,\|\nabla_S v\|^2_{L^2(G\times S\times I)}\\
&+\frac{1}{2}\,\|\gamma(v)\|^2_{T^2(\Gamma)} + c\,\|v\|^2_{L^2(G\times S\times I)}\,.
\end{aligned} \tag{84}$$

This completes the proof. □

Remark 6.7. A. Consider the conditions

$$\Sigma(x,\omega,E) - \frac{1}{2}\Big(\frac{\partial a}{\partial E}(x,E) + (\mathrm{div}_S\ d)(x,\omega,E)\Big) - 2\pi\int_{I'}\hat\sigma_r(x,E,E')dE' \geq c, \tag{85}$$

and

$$\Sigma(x,\omega,E) - \frac{1}{2}\Big(\frac{\partial a}{\partial E}(x,E) + (\mathrm{div}_S\ d)(x,\omega,E)\Big) - 2\pi\int_{I'}\hat\sigma_r(x,E',E)dE' \geq c \tag{86}$$

a.e. $(x,\omega,E)\in G\times S\times I$. In the above and subsequent results we could replace the conditions (73), (74), (68) with the weaker assumptions (85), (86). This can be seen by tracking the proofs of Theorem 4.2, [35] and the above Theorem 6.6 (Part B of the proof).

B. Suppose that $-a(x,E)\geq q_3$. Applying the exponential shift $\phi=e^{-CE}\psi$ as e.g. in [34], [8] we may replace more generally the conditions (73), (74), (68) by

$$\Sigma(x,\omega,E)-Ca(x,E)-\frac{1}{2}\Big(\frac{\partial a}{\partial E}(x,E)+(\mathrm{div}_S\ d)(x,\omega,E)\Big)\\ -2\pi\int_{I'}e^{C(E'-E)}\hat\sigma_r(x,E,E')dE'\geq c, \tag{87}$$

and

$$\Sigma(x,\omega,E)-Ca(x,E)-\frac{1}{2}\Big(\frac{\partial a}{\partial E}(x,E)+(\mathrm{div}_S\ d)(x,\omega,E)\Big)\\ -2\pi\int_{I'}e^{C(E-E')}\hat\sigma_r(x,E',E)dE'.\geq c \tag{88}$$

a.e. $(x,\omega,E)\in G\times S\times I$.

Because $C^1(\overline{G}\times I,C^2(S))\times C^1(\overline{G}\times I,C^2(S))$ is dense in $\mathcal{H}\times\widehat{\mathcal{H}}$ and since (75) holds, the bilinear form $B(\cdot,\cdot):C^1(\overline{G}\times I,C^2(S))\times C^1(\overline{G}\times I,C^2(S))\to\mathbb{R}$ has a unique extension $\widetilde{B}(\cdot,\cdot):\mathcal{H}\times\widehat{\mathcal{H}}\to\mathbb{R}$ which satisfies

$$|\widetilde{B}(\psi,v)|\leq M\left\|\psi\right\|_{\mathcal{H}}\left\|v\right\|_{\widehat{\mathcal{H}}}\quad\forall\ \psi\in\mathcal{H},\ v\in\widehat{\mathcal{H}} \tag{89}$$

and

$$\widetilde{B}(v,v)\geq c'\left\|v\right\|_{\mathcal{H}}^2\quad\forall v\in\widehat{\mathcal{H}}. \tag{90}$$

We see that actually

$$\widetilde{B}(\psi,v)=-\Big\langle\psi,a\frac{\partial v}{\partial E}\Big\rangle_{L^2(G\times S\times I)}-\Big\langle\psi,\frac{\partial a}{\partial E}v\Big\rangle_{L^2(G\times S\times I)}\\ -\langle a(.,E_0)p_0,v(.,.,E_0)\rangle_{L^2(G\times S)}\\ -\langle\nabla_S\psi,c(x,E)\nabla_S v\rangle_{L^2(G\times S\times I)}+\langle\psi,d^*(x,\omega,E,\partial_\omega)v\rangle_{L^2(G\times S\times I)}\\ -\langle\psi,\omega\cdot\nabla_x v\rangle_{L^2(G\times S\times I)}+\big\langle q_{|\Gamma_+},\gamma_+(v)\big\rangle_{T^2(\Gamma_+)}\\ +\langle\psi,\Sigma^*v-K_r^*v\rangle_{L^2(G\times S\times I)}, \tag{91}$$

where $q\in T^2(\Gamma)$ and $p_0\in L^2(G\times S)$ are explained in [33], p. 14 (here we have E_0 instead of 0 therein).

In addition, since for $\psi\in C^1(\overline{G}\times I,C^2(S))$

$$|F(\psi)|\leq|\langle f,\psi\rangle_{L^2(G\times S\times I)}|+|\langle g,\gamma_-(\psi)\rangle_{T^2(\Gamma_-)}|\\ \leq\left\|f\right\|_{L^2(G\times S\times I)}\left\|\psi\right\|_{L^2(G\times S\times I)}+\left\|g\right\|_{T^2(\Gamma_-)}\left\|\gamma(\psi)\right\|_{T^2(\Gamma_-)}, \tag{92}$$

the linear form $F:C^1(\overline{G}\times I,C^2(S))\to\mathbb{R}$ has a unique bounded extension, which we still denote by F,

$$F:\mathcal{H}\to\mathbb{R};\quad F(\psi)=\langle f,\psi\rangle_{L^2(G\times S\times I)}+\langle g,q\rangle_{T^2(\Gamma_-)}. \tag{93}$$

Note also that the embedding $\widehat{\mathcal{H}} \subset \mathcal{H}$ is continuous.

Let

$$P(x,\omega,E,D)\psi := a(x,E)\frac{\partial\psi}{\partial E} + b(x,\omega,E,\partial_\omega)\psi + \omega\cdot\nabla_x\psi.$$

The space

$$\mathcal{H}_P(G\times S\times I^\circ) := \{\psi \in L^2(G\times S\times I) \mid P(x,\omega,E,D)\psi \in L^2(G\times S\times I) \text{ in the weak sense}\} \quad (94)$$

is a Hilbert space when equipped with the inner product

$$\langle\psi,v\rangle_{\mathcal{H}_P(G\times S\times I^\circ)} = \langle\psi,v\rangle_{L^2(G\times S\times I)} + \langle P(x,\omega,E,D)\psi, P(x,\omega,E,D)v\rangle_{L^2(G\times S\times I)}.$$

With this notation, the equation (54) can be written as

$$P(x,\omega,E,D)\psi + \Sigma\psi - K_r\psi = f.$$

We show the existence of weak solutions *without boundary and initial conditions*

THEOREM 6.8. *Suppose that the assumptions of Theorem 6.6 are valid. Let $f\in L^2(G\times S\times I)$. Then the variational equation*

$$\widetilde{B}(\psi,v) = F(v) \quad \forall v\in\widehat{\mathcal{H}} \quad (95)$$

has a solution $\psi\in\mathcal{H}$. Furthermore, $\psi\in\mathcal{H}_P(G\times S\times I^\circ)$ and it is a weak (distributional) solution of the equation (55) that is,

$$T\psi := a(x,E)\frac{\partial\psi}{\partial E} + b(x,\omega,E,\partial_\omega)\psi + \omega\cdot\nabla_x\psi + \Sigma\psi - K_r\psi = f. \quad (96)$$

Proof. We apply Theorem 6.1 with $X=\mathcal{H}$, $Y=\widehat{\mathcal{H}}$, and with $\widetilde{B}(\cdot,\cdot)$ and F given by (91) and (93), respectively. As explained above $\widetilde{B}(\cdot,\cdot)$ satisfies (46) and (47), while $F: X\to\mathbb{R}$ is a bounded linear functional, hence Theorem 6.1 guarantees the existence of a solution $\psi\in\mathcal{H}$ such that (95) holds.

We verify that $\psi\in L^2(G\times S\times I)$ is a weak solution of the equation (96). From (95) it follows that

$$\widetilde{B}(\psi,v) = F(v), \quad \forall v\in C_0^\infty(G\times S\times I^\circ). \quad (97)$$

Since for $v\in C_0^\infty(G\times S\times I^\circ)$ we have $v(\cdot,\cdot,E_0)=0$ and $v_{|\Gamma_-} = v_{|\Gamma_+} = 0$, we see from (91) that

$$\begin{aligned}\widetilde{B}(\psi,v) = & -\left\langle\psi, a\frac{\partial v}{\partial E}\right\rangle_{L^2(G\times S\times I)} - \left\langle\psi, \frac{\partial a}{\partial E}v\right\rangle_{L^2(G\times S\times I)} \\ & - \langle\nabla_S\psi, c(x,E)\nabla_S v\rangle_{L^2(G\times S\times I)} + \langle\psi, d^*(x,\omega,E,\partial_\omega)v\rangle_{L^2(G\times S\times I)} \quad (98)\\ & - \langle\psi,\omega\cdot\nabla_x v\rangle_{L^2(G\times S\times I)} + \langle\psi,\Sigma^* v - K_r^* v\rangle_{L^2(G\times S\times I)} \\ & = F(v) = \langle f,v\rangle_{L^2(G\times S\times I)}. \quad (99)\end{aligned}$$

Moreover,

$$-\langle\nabla_S\psi, c(x,E)\nabla_S v\rangle_{L^2(G\times S\times I)} = \langle\psi, c(x,E)\Delta_S v\rangle_{L^2(G\times S\times I)}.$$

The formal adjoint T^* of T is

$$T^*v = -a\frac{\partial v}{\partial E} - \frac{\partial a}{\partial E}v + c(x,E)\Delta_S v + d^*(x,\omega,E,\partial_\omega)v - \omega\cdot\nabla_x v + \Sigma^* v - K_r^* v.$$

Hence by (98)

$$\langle \psi, T^*v\rangle_{L^2(G\times S\times I)} = \langle f, v\rangle_{L^2(G\times S\times I)}\,,\ \forall\ v\in C_0^\infty(G\times S\times I^\circ)$$

which completes the proof. □

Remark 6.9. Since for $f\in L^2(G\times I, H^{-1}(S)),\ v\in L^2(G\times I, H^1(S))$

$$\langle f, v\rangle \leq \|f\|_{L^2(G\times I,H^{-1}(S))}\,\|v\|_{L^2(G\times I,H^1(S))}$$

we find that we are able to replace the assumption $f\in L^2(G\times S\times I)$ in the above Theorem 6.8 with $f\in L^2(G\times I, H^{-1}(S))$. Here $\langle .,.\rangle$ is the canonical duality.

6.3. Existence of solutions of the initial and boundary value problem. Let

$$Q(x,\omega,E,D)\psi = a(x,E)\frac{\partial\psi}{\partial E} + \omega\cdot\nabla_x\psi.$$

Define the space $\mathbf{H}_Q(G\times S\times I^\circ)$

$$\mathbf{H}_Q(G\times S\times I^\circ) := \{\psi\in L^2(G\times I, H^1(S))|\ Q(x,\omega,E,D)\psi\in L^2(G\times I, H^{-1}(S))\}$$

equipped with the inner product

$$\begin{aligned}\langle\psi, v\rangle_{\mathbf{H}_Q(G\times S\times I^\circ)} = {} & \langle\psi, v\rangle_{L^2(G\times I,H^1(S))} \\ & + \langle Q(x,\omega,E,D)\psi, Q(x,\omega,E,D)v\rangle_{L^2(G\times I,H^{-1}(S))}\,.\end{aligned}$$

Define the traces $\gamma_\pm(\psi) := \psi_{|\Gamma_\pm}$ and $\gamma_{\rm m}(\psi) := \psi(\cdot,\cdot,E_{\rm m}),\ \gamma_0(\psi) := \psi(\cdot,\cdot,E_0)$. The traces $\gamma_\pm : W^2(G\times S\times I)\to L^2_{\rm loc}(\Gamma_\pm, |\omega\cdot\nu| d\sigma d\omega dE)$ can be shown to be continuous ([4, 33]). Moreover, we have

Theorem 6.10. *Suppose that* $a\in W^{\infty,1}(G\times I^\circ)$ *such that*

$$|a(x,E)|\geq c>0 \text{ a.e. in } G\times I. \tag{100}$$

Then the trace operators

$$\begin{aligned}\gamma_\pm &: \mathbf{H}_Q(G\times S\times I^\circ)\to L^2_{\rm loc}(\Gamma_\pm, |\omega\cdot\nu| d\sigma d\omega dE),\\ \gamma_{\rm m} &: \mathbf{H}_Q(G\times S\times I^\circ)\to L^2_{\rm loc}(G\times S),\\ \gamma_0 &: \mathbf{H}_Q(G\times S\times I^\circ)\to L^2_{\rm loc}(G\times S),\end{aligned}$$

are well-defined and continuous.

Proof. In virtue of density results like Friedrich [13], [29] the space $C^1(\overline{G}\times S\times I)$ is dense in $\mathbf{H}_Q(G\times S\times I^\circ)$ and so it suffices to show the below boundedness estimate for $\psi\in C^1(\overline{G}\times S\times I)$.

Let $\psi\in C^1(\overline{G}\times S\times I)$. We have

$$\frac{\partial(a\psi)}{\partial E} = \frac{\partial a}{\partial E}\psi + a\frac{\partial\psi}{\partial E}$$

and so by partial integration for $\psi\in C^1(\overline{G}\times S\times I)$

$$2\int_{G\times S\times I} a\frac{\partial\psi}{\partial E}\psi\ dxd\omega dE=-\int_{G\times S\times I}\frac{\partial a}{\partial E}\psi^2 dxd\omega dE+\int_{G\times S}\Big|_{E_0}^{E_m} a\psi^2(x,\omega,.)dxd\omega. \tag{101}$$

Similarly by the Green's formula for $\psi\in C^1(\overline{G}\times S\times I)$

$$\begin{aligned}&2\int_{G\times S\times I}\frac{1}{a}(\omega\cdot\nabla_x\psi)\psi dxd\omega dE\\ &=-\int_{G\times S\times I}\omega\cdot\nabla_x\Big(\frac{1}{a}\Big)\psi^2 dxd\omega dE+\int_\Gamma\frac{1}{a}\psi^2(\omega\cdot\nu)d\sigma d\omega dE\end{aligned} \tag{102}$$

where we used that

$$\omega\cdot\nabla_x\Big(\frac{1}{a}\psi\Big)=\frac{1}{a}\omega\cdot\nabla_x\psi+\omega\cdot\nabla_x\Big(\frac{1}{a}\Big)\ \psi.$$

Note that

$$\frac{\partial\psi}{\partial E}+\frac{1}{a}\omega\cdot\nabla_x\psi=\frac{1}{a}Q(x,\omega,E,D)\psi.$$

A. Consider the operator $\gamma_{\rm m}$. Let $\eta\in C_0^\infty(G\times S\times\mathbb{R})$ such that $\eta(.,.,E_0)=0$. Then we get by (102)

$$\begin{aligned}&\|(\eta\psi)(.,.,E_m)\|^2_{L^2(G\times S)}=\int_{G\times S}(\eta\psi)^2(x,\omega,E_m)d\omega dx\\
&=\int_{G\times S}\int_{E_0}^{E_m}\frac{\partial}{\partial E}\Big((\eta\psi)^2(x,\omega,E)\Big)dEd\omega dx\\
&=\int_{G\times S\times I}2(\eta\psi)(x,\omega,E)\frac{\partial(\eta\psi)}{\partial E}(x,\omega,E)dEd\omega dx\\
&=2\int_{G\times S\times I}(\eta\psi)(x,\omega,E)\big(\frac{\partial(\eta\psi)}{\partial E}+\frac{1}{a}\omega\cdot\nabla_x(\eta\psi)\big)(x,\omega,E)dEd\omega dx\\
&+\int_{G\times S\times I}\omega\cdot\nabla_x\Big(\frac{1}{a}\Big)(\eta\psi)^2(x,\omega,E)dxd\omega dE\\
&=2\int_{G\times S\times I}(\eta\psi)(x,\omega,E)\frac{1}{a}(Q(x,\omega,E,D)(\eta\psi))(x,\omega,E)dEd\omega dx\\
&+\int_{G\times S\times I}\omega\cdot\nabla_x\Big(\frac{1}{a}\Big)(\eta\psi)^2(x,\omega,E)dxd\omega dE\\
&\leq 2\left\|\eta\psi\right\|_{L^2(G\times I,H^1(S))}\left\|\frac{1}{a}Q(x,\omega,E,D)(\eta\psi)\right\|_{L^2(G\times I,H^{-1}(S))}\\
&+\left\|\omega\cdot\nabla_x\Big(\frac{1}{a}\Big)\right\|_{L^\infty(G\times S\times I)}\|\eta\psi\|^2_{L^2(G\times S\times I)}\\
&\leq\Big(\ \|\eta\psi\|^2_{L^2(G\times I,H^1(S)}+\left\|\frac{1}{a}Q(x,\omega,E,D)(\eta\psi)\right\|^2_{L^2(G\times I,H^{-1}(S))}\Big)\\
&+\left\|\omega\cdot\nabla_x\Big(\frac{1}{a}\Big)\right\|_{L^\infty(G\times S\times I)}\|\eta\psi\|^2_{L^2(G\times S\times I)}\,.\end{aligned} \tag{103}$$

Since

$$Q(x,\omega,E,D)(\eta\psi)=\eta Q(x,\omega,E,D)\psi+(Q(x,\omega,E,D)\eta)\psi$$

and for $q\in L^\infty(G\times I,W^{\infty,1}(S))$

$$\|q\psi\|_{L^2(G\times I,H^1(S)}\leq\|q\|_{L^\infty(G\times I,W^{\infty,1}(S))}\|\psi\|_{L^2(G\times I,H^1(S)}$$

and

$$\|qU\|_{L^2(G\times I,H^{-1}1(S)}\leq\|q\|_{L^\infty(G\times I,W^{\infty,1}(S))}\|U\|_{L^2(G\times I,H^{-1}(S)}$$

we conclude by (103)

$$\begin{aligned}&\|(\eta\psi)(.,.,E_m)\|^2_{L^2(G\times S)}\\&\leq C'\big(\|\psi\|^2_{L^2(G\times I,H^1(S))}+\|Q(x,\omega,E,D)\psi\|^2_{L^2(G\times I,H^{-1}(S))}\big)\end{aligned}\tag{104}$$

and so the assertion holds for γ_m. The assertion for γ_0 is similarly proved.

B. Consider the operator γ_-. Let $\theta\in C_0^\infty(\mathbb{R}^3\times S\times I^\circ)$ such that $\text{supp}(\theta)\cap\Gamma_+$ is empty. Since $(\theta\psi)(x,\omega,.)\in C_0^1(I^\circ)$ we have by the Green's formula (5) and by (101)

$$\begin{aligned}&\int_{\Gamma_-}(\omega\cdot\nu)(\theta\psi)^2d\sigma d\omega dE=2\int_{G\times S\times I}\omega\cdot\nabla_x(\theta\psi)\ (\theta\psi)dEd\omega dx\\&=2\int_{G\times S\times I}\Big(\omega\cdot\nabla_x(\theta\psi)\ (\theta\psi)+a\frac{\partial(\theta\psi)}{\partial E}(\theta\psi)\Big)dEd\omega dx\\&+\int_{G\times S\times I}\frac{\partial a}{\partial E}(\theta\psi)^2dxd\omega dE\\&=2\int_{G\times S\times I}(Q(x,\omega,E,D)(\theta\psi))(\theta\psi)dEd\omega dx\\&+\int_{G\times S\times I}\frac{\partial a}{\partial E}(\theta\psi)^2dxd\omega dE\end{aligned}\tag{105}$$

and so as in (103) we obtain

$$\begin{aligned}&\int_{\Gamma_-}|(\omega\cdot\nu)|(\theta\psi)^2d\sigma d\omega dE\\&\leq C''\big(\|\psi\|^2_{L^2(G\times I,H^1(S))}+\|Q(x,\omega,E,D)\psi\|^2_{L^2(G\times I,H^{-1}(S))}\big).\end{aligned}\tag{106}$$

This completes the claim for γ_-. The boundedness of γ_+ is similarly shown. This finishes the proof. □

Let P^* be the formal adjoint operator of $P(x,\omega,E,D)$

$$P^*(x,\omega,E,D)v=-a\frac{\partial v}{\partial E}-\frac{\partial a}{\partial E}v+b^*(x,\omega,E,\partial_\omega)v-\omega\cdot\nabla_xv$$

where

$$b^*(x,\omega,E,\partial_\omega)v=c(x,E)\Delta_Sv+d^*(x,\omega,E,\partial_\omega)v.$$

For $U\in L^2(G\times I,H^{-1}(S)),\ v\in L^2(G\times I,H^1(S))$ we define

$$\langle U,v\rangle:=\int_{G\times I}(U(x,.,E),v(x,.,E))dxdE$$

where $(U(x,.,E),v(x,.,E))$ is the canonical duality between $H^{-1}(S)$ and $H^1(S)$. We have the following extended Green's formula

LEMMA 6.11. *Suppose that (67), (100) hold and that* $\psi\in\mathcal{H}_P(G\times S\times I^\circ)\cap L^2(G\times I,H^1(S))$ *and* $v\in\widehat{\mathcal{H}}$ *for which* $(\mathrm{supp}(v))\cap\partial(G\times S\times I)$ *is a compact subset of* $\Gamma_-\cup\Gamma_+\cup(G\times S\times\{E_m\})\cup(G\times S\times\{E_0\})$. *Then*

$$\langle P(x,\omega,E,D)\psi,v\rangle-\langle P^*(x,\omega,E,D)v,\psi\rangle=\int_{\partial G\times S\times I}(\omega\cdot\nu)\psi v\ \ d\sigma d\omega dE$$
$$+\int_{G\times S}\big(a(\cdot,E_m)\psi(\cdot,\cdot,E_m)v(\cdot,\cdot,E_m)-a(\cdot,E_0)\psi(\cdot,\cdot,E_0)v(\cdot,\cdot,E_0)\big)dxd\omega. \qquad (107)$$

The proof follows from a density argument and applying standard Green's formula. Theorem 6.10 on the continuity of the trace operator then guarantee that the Green's formula also holds in the limit.

REMARK 6.12. The Green formula has some additional generalizations. Especially, (107) holds for $\psi=v$ in the case when $\psi\ \in\mathcal{H}_P(G\times S\times I^\circ)\cap L^2(G\times I,H^1(S))$ such that $\gamma_\pm(\psi)\in T^2(\Gamma_\pm)$ and $\gamma_{\mathrm{m}}(\psi),\ \gamma_0(\psi)\in L^2(G\times S)$ (cf. [7], p. 225).

Under the assumption (100) the weak solution ψ of the equation (96) obtained in Theorem 6.8 can be shown to be a solution of the initial and inflow boundary value problem. We have

THEOREM 6.13. *Suppose that the assumptions of Theorem 6.6 are valid and that (100) holds that is,*

$$|a(x,E)|\geq c>0 \text{ in } \overline{G}\times I. \qquad (108)$$

Let $f\in L^2(G\times S\times I)$ *and* $g\in T^2(\Gamma_-)$. *Then the transport problem*

$$a(x,E)\frac{\partial\psi}{\partial E}+b(x,\omega,E,\partial_\omega)\psi+\omega\cdot\nabla_x\psi+\Sigma\psi-K_r\psi=f$$
$$\psi_{|\Gamma_-}=g,\ \ \psi(.,.,E_m)=0 \qquad (109)$$

has a unique solution $\psi\in\mathcal{H}\cap\mathcal{H}_P(G\times S\times I^\circ)$.

Proof. The proof is analogous to the proof of Theorem 5.7, [33] (items (ii)-(iii) of the proof) and we omit detailed treatments. □

In addition, the solution ψ obeys the apriori estimate

$$\|\psi\|_{\mathcal{H}}\leq C\big(\,\|f\|_{L^2(G\times S\times I)}+\|g\|_{T^2(\Gamma_-)}\,\big). \qquad (110)$$

REMARK 6.14. Let T be the above transport operator $T=P(x,\omega,E,D)+\Sigma-K_r$ and let $T^*=P^*(x,\omega,E,D)+\Sigma^*-K_r^*$ be the formal transpose of T. Furthermore, let the assumptions of Theorem 6.8 be valid, and let $\psi\in\mathcal{H}$ be the weak solution of (95) guaranteed by the Theorem 6.8. By (91), we find that for any $v\in C^1(\overline{G}\times S\times I)$ for which the *homogeneous adjoint boundary conditions* $v(\cdot,\cdot,E_0)=0$ and $\gamma_+(v)=0$ hold, we have

$$\langle\psi,T^*v\rangle=\widetilde{B}(\psi,v)=\langle f,v\rangle_{L^2(G\times S\times I)}+\langle g,\gamma_-(v)\rangle_{T^2(\Gamma_-)}\,, \qquad (111)$$

that is

$$\langle \psi, T^* v\rangle - \langle T\psi, v\rangle_{L^2(G\times S\times I)} = \langle g, \gamma_-(v)\rangle_{T^2(\Gamma_-)}\,. \tag{112}$$

This means that $\psi \in \mathcal{H}$ is a *weak solution* of the inflow boundary and initial value problem

$$T\psi = f,\ \psi_{|\Gamma_-} = g,\ \psi(.,.,E_m) = 0, \tag{113}$$

a terminology which goes back to [19]. The validity of the trace theorems and the Green formula (107) with $\gamma(\psi) \in T^2(\Gamma)$, $\psi(\cdot,\cdot,E_0)$, $\psi(\cdot,\cdot,E_m) \in L^2(G\times S)$ are keys for obtaining well-defined solutions that is, solutions for which the boundary and initial values hold *strongly*, as in Theorem 6.13. Trace theorems always demand geometrical treatments where the smoothness of the boundary ∂G is essential.

7. Summary and Conclusion. The paper considers an approximation of an exact linear Boltzmann transport operator related to the charged particle transport. The hyper-singularities in the differential cross-sections for certain charged particle collisions lead to partial hyper-singular integral terms in kinematic equations. Hyper-singular integral terms can be approximated by partial differential and non-singular integral operators. These approximations are essentially founded on Taylor's formulas and angular approximations of (new) primary particles. The resulting approximation contains the first-order partial derivatives with respect to energy E. With respect to angle ω also second-order partial derivatives appear that is, the operator contains the term $b(x,\omega,E,\partial_\omega)\psi$ which is corresponding to the second order partial differential term (e.g. Laplace-Beltrami operator, as in the case treated in this paper) with respect to angular variables appearing in various variants of Fokker-Planck equations. In many cases this term turns out to be an elliptic operator (on S) which helps the analysis and which should also improve the numerical treatments due to stabilization properties.

We derived and studied an approximation of an exact transport operator which is more simple at least from numerical point of view than the exact one. In [36] we considered details of the exact equation and the related variational problem. In principle, the obtained variational problem therein can be solved numerically by applying e.g. Galerkin finite element methods (FEM). In fact, numerical computations and approximations of (hyper)-singular integrals have been studied by various methods for various needs. However, these methods may be too time-consuming and so approximations (as here) which have solid foundations are necessary.

The well-posedness of the transport problems that is, existence and uniqueness of solutions together with pertinent a priori estimates is a central importance. In this paper we applied a Lax-Milgram type Theorem. Other successful methods might be e.g. semigroup-dissipativity-perturbation methods and fixed-point (contraction) methods. Higher regularity of solutions remains so far an open problem. It is known that the transport problems have a limited Sobolev regularity. We remark that the initial inflow boundary value problems related to transport problems have the so called *variable boundary value multiplicity* that is, the dimension of the kernel of the boundary operator is not constant (e.g. [26], [30]). This makes the inflow boundary value transport problems more subtle independently of the applied methods.

Acknowledgement. The authors thank Dr. Petri Kokkonen for useful discussions and help in differential geometry while preparing the paper.

REFERENCES

[1] V. Agoshkov, *Boundary Value Problems for Transport Equations*, Springer Science +Business Media, 1998.

[2] N. Bellomo, A. Palczewsky and G. Toscani, *Mathematical Topics in Nonlinear Kinetic Theory*, World Scientific, 1988.

[3] C. Cercignani, *The Boltzmann Equation and Its Applications*, Springer, 1988.

[4] M. Cessenat, *Théorémes de trace L^p pour des espaces de fonctions de la neutronique*, C. R. Acad. Sc. Paris, t. 299, Serie 1, number 16, 1984.

[5] Y.-S. Chan, et.al, *Finite part integrals and hypersingular kernels*, Advances in Dynamical Systems, 14:S2 (2007), pp. 264–269.

[6] L. Chupin, *Fokker-Planck equation in bounded domain*, Annales de L'Institut Fourier, 60:1 (2010), pp. 217–255.

[7] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology*, Vol. 6. Evolution Problems II, Springer, 1999.

[8] P. Degond, *Global existence of smooth solutions for the Vlasov-Fokker-Planck equation in* 1 *and* 2 *space dimensions*, Ann. scient. Ec. Norm. Sup. 4^e serie, 19:4 (1986), pp. 519–542.

[9] J. Duderstadt and W. Martin, *Transport Theory*, Wiley, 1979.

[10] H. Egger and M. Schlottbom, *An L^p-theory for stationary radiative transfer*, Appl. Anal., 93:6 (2014), pp. 1283–1296.

[11] M. Frank, M. Herty and A. N. Sandjo, *Optimal radiotherapy treatment planning governed by kinetic equations*, Math. Mod. Meth. Appl. Sci., 20:4 (2010), pp. 661–678.

[12] M. Frank and T. Goudon, *On a generalized Boltzmann equation for non-classical particle transport*, Kinet. Relat. Models, 3 (2010), pp. 395–407.

[13] K. O. Friedrichs, *The identity of weak and strong extensions of differential operators*, Trans. Amer. Math. Soc., 55 (1944), pp. 132–151.

[14] P. Grisvard, *Elliptic problems in nonsmooth domains*, vol. 24 of Monographs and Studies (Mathematics), Pitman, 1985.

[15] P. R. Halmos and V. S. Sunder, *Bounded Integral Operators on L^2 Spaces*, Springer 1978.

[16] M. Herty, C. Jörres and A. N. Sandjo, *Optimization of a model Fokker-Planck equation*, Kinetic and Related Models, 5:3 (2012), pp. 485–503.

[17] M. Herty and A. N. Sandjo, *On optimal treatment planning in radiotherapy governed by transport equations*, Math. Models Methods Appl. Sci., 21:2 (2011), pp. 345–359.

[18] G. C. Hsiao and W. L. Wendland, *Boundary Integral Equations*, Springer, 2008.

[19] P. D. Lax and R. S. Phillips, *Local boundary conditions for dissipative symmetric linear differential operators*, Comm. Pure Appl. Math., 13 (1960), pp. 427–455.

[20] C. Le Bris and P.-L. Lions, *Existence and uniqueness of solutions to Fokker-Planck type equations with irregular coefficients*, Comm. Partial Diff. Equations, 33 (2008), pp. 1272–1317.

[21] J. Lee, *Introduction to Smooth Manifolds*, Springer, 2002.

[22] O. Madelung, *Introduction to Solid-State Theory*, Springer-Verlag, 1978.

[23] P. A. Martin and F. J. Rizzo, *Hypersingular integrals: How smooth must be the density be?*, Int. Journal for Num. Meth. in Engineering, 39 (1996), pp. 687–704.

[24] M. Mokhtar-Kharroubi, *Mathematical Topics in Neutron Transport Theory*, World Scientific, 1997.

[25] M. F. Modest, *Radiative Heat Transfer*, Academic Press, 3rd Edition, 2013.

[26] T. Nishitani and M. Takayama, *Characteristic initial boundary value problems for symmetric hyperbolic systems*, Osaka J. Math., 35 (1988), pp. 629–657.

[27] G. C. Pomraning, *The Equations of Radiation Hydrodynamics*, Pergamon Press, 1973.

[28] G. C. Pomraning, *The Fokker-Planck operator as an asymptotic limit*, Math. Models Methods Appl. Sci., 2 (1992), pp. 21–36.

[29] J. Rauch, *Symmetric positive systems with boundary characteristic of constant multiplicity*, Trans. Amer. Math. Soc., 291:1 (1985), pp. 167–187.

[30] J. Rauch, *Boundary value problems with nonuniformly characteristic boundary*, J. Maths. Pure et. Appliquées, 73 (1994), pp. 347–353.

[31] L. Schwarz, *Mathematics for Physical Sciences*, Herman Paris, 1966.

[32] J. Tervo and P. Kokkonen, *On existence of L^1-solutions for coupled Boltzmann transport equation and radiation therapy treatment planning*, ArXiv: 1406.3228v2 [math.OC] 1 Feb 2017.

[33] J. Tervo, P. Kokkonen, M. Frank and M. Herty, *On Existence of L^2-solutions of Coupled Boltzmann Continuous Slowing Down Transport Equation System*, arXiv:1603.05534v3 [math.OC] 5 Apr 2018.

[34] J. Tervo, P. Kokkonen, M. Frank and M. Herty, *On existence of solutions for Boltzmann Continuous Slowing Down transport Equation*, J. Math. Anal. Appl., 460:1 (2018), pp. 271–301.

[35] J. Tervo, P. Kokkonen, M. Frank and M. Herty, *On approximative linear Boltzmann transport equation for charged particle transport*, Math. Models Methods Appl. Sci., 14:28 (2018), pp. 2905–2939.

[36] J. Tervo, *On linear hypersingular Boltzmann transport equation and its variational formulation*, arXiv:1808.09631v3 [math.AP], 2019.

[37] R. Tian, *Existence, uniqueness and regularity property of solutions to Fokker-Planck type equations*, J. Math. Anal. Appl., 2:1 (2014), pp. 53–63.

[38] F. Treves, *Basic Linear Partial Differential Equations*, Academic Press, 1975.

[39] S. Ukai and T. Yang, *Mathematical Theory of Boltzmann Equation*, Reprint, City University of Hong Kong, 2010.

[40] C. Villani, *A review of mathematical topics in collision kinetic theory*, in Handbook of Mathematical Fluid Dynamics (Vol. 1), edited by S. Friedlander and D. Serre, Elsevier Science, 2002.

[41] J. W. Wilson, *Transport Methods and Interactions for Space Radiations*, NASA RP-1257, pp. 14–17, 1991.