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MIXED BOUNDARY VALUE PROBLEMS OF THE SYSTEM FOR STEADY FLOW OF HEAT-CONDUCTING INCOMPRESSIBLE VISCOUS FLUIDS WITH DISSIPATIVE HEATING*

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Abstract. In this paper we are concerned with the equation for steady flow of heat-conducting incompressible viscous Newtonian fluids with dissipative heating under mixed boundary conditions. The boundary conditions for fluid may include Tresca slip, leak condition, one-sided leak conditions, velocity, pressure, rotation, stress together and the conditions for temperature may include Dirichlet, Neumann and Robin conditions together. Relying on the relations among strain, rotation, normal derivative of velocity and shape of boundary surface, we get variational formulations consisted of a variational inequality for velocity and a variational equation for temperature, which are equivalent to the original PDE problems for smooth solutions. Then, we study the existence of solutions to the variational problems. To this end, first we study the existence of solutions to auxiliary problems including a parameter for approximation and two or three parameters concerned with the norms of velocity and temperature. Then we determine the parameters concerned with the norms of velocity and temperature in accordance with the data of problems, and we get the existence of solutions by passing to limits as the parameter for approximation goes to zero.

Key words. Heat-conducting fluids, Dissipative heating, Variational inequality, Mixed boundary conditions, Tresca slip, Leak boundary conditions, One-sided leaks, Pressure boundary condition, Existence.

Mathematics Subject Classification. 35Q35, 35J87, 76D03, 76D05, 49J40, 80A20.

1. Introduction. As a mathematical model for steady motion of heat conducting incompressible viscous fluids with dissipative heating the system

$$\begin{cases} -2\nabla\cdot\mathcal{D}+(v\cdot\nabla)v+\nabla p=(1-\alpha_0\theta)f, \\ \nabla\cdot v=0 \quad \text{in } \Omega, \\ -\nabla\cdot(\kappa(\theta)\nabla\theta)+v\cdot\nabla(\gamma(\theta)\theta)-\alpha_2\mathcal{D}:\mathcal{E}(v)=\alpha_1\theta f\cdot v+g \end{cases} \tag{1.1}$$

is used(cf. [18], [30]). Here v, p and θ are, respectively, velocity, pressure and temperature, and α_0 - parameter for buoyancy effect, α_1 - parameter for dissipation of energy due to expansion, α_2 - a positive real number, f - body force, g - heat source, $\kappa(\theta)$ - thermal conductivity, $\gamma(\theta)$ - specific heat of the fluid. The strain tensor $\mathcal{E}(v)$ is the one with the components $\varepsilon_{ij}(v)=\frac{1}{2}(\partial_{x_i}v_j+\partial_{x_j}v_i)$ and the tensor $\mathcal{D}$ usually depends on v,θ. For two matrices $A=\{a_{ij}\}, B=\{b_{ij}\}$ $A:B=\sum_{ij}a_{ij}b_{ij}$. In the case of Newtonian fluid $\mathcal{D}=\mu(\theta)\mathcal{E}(v)$, where $\mu(\theta)$ is viscosity, and the term $\mathcal{D}:\mathcal{E}(v)$ represents the dissipation of energy due to viscosity (the Joule effect). When $\alpha_1=\alpha_2=0$, we get the well known steady Boussinesq system.

Several papers are concerned with (1.1) and the corresponding non-steady system.

In [9] for non-Newtonian fluid with $\alpha_0=\alpha_1=0$, under homogeneous Dirichlet boundary condition for velocity and mixture of homogeneous Dirichlet and Robin conditions for temperature the existence of a solution is studied and the corresponding

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non-steady problem is studied in [10]. In [14] for the stationary problems of a generalized Newtonian fluid with $\alpha_0 = \alpha_1 = 0$ regularity of weak solutions is proved under homogeneous Dirichlet boundary conditions of velocity and temperature. In [11] for the stationary problems of a non-Newtonian fluid with $\alpha_0 = \alpha_1 = 0$ the existence of a solutions is proved under homogeneous Dirichlet boundary conditions for velocity and temperature on a portion of boundary and a generalized Navier-slip and Robin conditions for velocity and temperature on another portion of boundary. In [31] for the steady problem of a non-Newtonian fluid with heat sources allowed in L^1 and even as measures, under homogeneous Dirichlet boundary conditions for velocity and Robin condition for temperature the existence of a distributional solution is shown for sufficiently small data, and the corresponding non-steady problem is studied in [29]. But all results above exclude Newtonian fluid owing to conditions for nonlinear terms for strein.

In [25] the steady problem for the Newtonian fluid with $\alpha_0 = \alpha_1 = 0$ is studied under homogeneous Dirichlet boundary conditions for velocity and mixture of Dirichlet condition and homogeneous Neumann condition for temperature, and the corresponding non-steady problem is studied in [26] under homogeneous Dirichlet boundary condition. In [27] when $|\alpha_0|, |\alpha_1|$ are small enough or $\alpha_0 \neq 0, \frac{\alpha_1}{\alpha_0} > 1$, existence of a solution to the steady problems for the Newtonian fluid is studied under homogeneous Dirichlet boundary conditions for velocity and mixture of non-homogeneous Dirichlet condition and homogeneous Neumann condition for temperature, and in [28] the corresponding non-steady problem with $|\alpha_0|, |\alpha_1|$ small enough is studied under homogeneous Dirichlet boundary condition for velocity and mixture of homogeneous Dirichlet condition and homogeneous Neumann condition for temperature. All papers above are concerned with homogeneous Dirichlet boundary condition for velocity.

The Problems with mixture of Dirichlet boundary condition of velocity and outlet condition for fluid(cf. Introduction in [19]) are studied in [2], [4]-[7]. In [5] for steady problem of Newtonian fluid on an open channel domain when $\alpha_0 = \alpha_1 = 0$, the local existence of a solution is studied under mixed boundary conditions above for fluid and mixture of non-homogeneous Dirichlet and Neumann boundary conditions for temperature. In [4] when the viscosity, specific heat and thermal conductivity are independent of temperature, the buoyancy term $(1-\alpha_0\theta)f$ is changed by $\rho(\theta)f$, where $0 < \rho(\theta) < \rho_1$, and $\alpha_1 = 0$, for steady problem on an open channel domain under mixed boundary conditions above for fluid and mixture of non-homogeneous Dirichlet and homogeneous Neumann boundary conditions for temperature it is proved that if the data of problem are small enough, then there exists a unique strong solution. In [2] when the viscosity, specific heat and thermal conductivity are independent of temperature, for non-steady problem on an open 2-D channel domain the local-in-time existence of a solution is proved under mixed boundary conditions above for fluid and mixture of non-homogeneous Dirichlet and homogeneous Neumann boundary conditions for temperature. In [7] when specific heat is independent of temperature, the buoyancy term is changed by $\rho(\theta)$, where $0 < \rho(\theta) < \rho_1$, and $\alpha_1 = 0$, for non-steady problem on exterior-like domains, which is surround with a solid surface and a disjoint artificial boundary outside of the solid, local-in-time existence of a solution is proved under mixed boundary conditions above for fluid and mixture of homogeneous Dirichlet and Neumann boundary conditions for temperature. In [6] when viscosity, specific heat and thermal conductivity are independent of temperature and the buoyancy term $(1 - \alpha_0\theta)f$ and $\alpha_1\theta f \cdot v$ are replaced, respectively, by $\rho(\theta)f$ and $\alpha_1\rho(\theta)f \cdot v$, where $0 < \rho_0 < \rho(\theta) < \rho_1$, and the data of problem are small enough, for non-steady problem on 3-D pipes the existence, regularity and uniqueness of a solution are proved under mixed boundary conditions above for fluid and mixture of Robin and homogeneous Neumann boundary conditions for temperature. In the papers with mixed boundary conditions for fluid above the assumptions for shape of domain is essential because in such cases the solutions to the corresponding steady Stokes problem belong to more smooth space than $W^{1,2}(\Omega)$, which is used for estimations of approximate solutions.

In [3] when the viscosity, specific heat and thermal conductivity are independent of temperature, for the steady problem on 2-D bounded domain the existence of a strong solution is proved under mixture of Dirichlet boundary condition of velocity, tangent stress and stress condition for fluid and mixture of non-homogeneous Dirichlet and homogeneous Neumann boundary conditions for temperature.

For the other papers for the non-steady Newtonian fluid with $\alpha_0 = \alpha_1 = 0$ refer to [8], [12]–[13], [16].

On the other hand, for movement of fluid (v, p) different kinds of boundary conditions are used and in practice we deal with mixture of some kinds of boundary conditions. On some portions of boundary we can use boundary conditions with stress or rotation, whereas when there is flux through a portion of boundary, we can deal with the static pressure p or the total pressure (Bernoulli's pressure) $\frac{1}{2}|v|^2 + p$ boundary conditions. There are many literatures for the Navier-Stokes problem with mixed boundary conditions(cf. Introduction of [19], [21] and references therein). Recently, several papers are devoted to problems with Tresca slip boundary condition(condition (8) in (2.2)) or leak boundary condition (condition (9) in (2.2)). Also, in practice we deal with one-sided leak condition for fluid (conditions (10, 11) in (2.2) or (10, 11) in (2.3)). Thus, in [20] and [22], respectively, the steady and non-steady Navier-Stokes problems with mixed boundary conditions including Tresca slip, leak and one-sided leak conditions called the boundary conditions of friction type are studied. For meaning and physical backgrounds of friction boundary conditions refer to Introduction of [20] and references therein.

In the present paper, we are concerned with the system for steady flow of heat-conducting Newtonian fluids with dissipative heating under mixed boundary conditions. Non-steady problem will be studied in another paper. Boundary conditions for velocity may include Tresca slip, leak condition, one-sided leak conditions, velocity, pressure, rotation, stresses together and the conditions for temperature may include Dirichlet, Neumann and Robin conditions together. According to whether the boundary conditions for fluid include static pressure and stress or total pressure and total stress, the problems are distinguished, and we are concerned with two problems. Relying on the relations among strain, rotation, normal derivative of velocity and shape of boundary surface in [19], we get variational formulations consisted of a variational inequality for velocity and a variational equation for temperature. Then, we prove existence of weak solutions to the problems with such boundary conditions.

This paper consists of 5 sections. In the last part of Section 1 we give notations.

In Section 2, the problems and assumptions are stated. According to the static pressure or the total pressure (correspondingly, the stress or the total stress) in the boundary conditions for the fluid, Problem I and Problem II are distinguished.

In Section 3, we first get the variational formulations which consist of six formulae with six unknown functions, that is, using velocity, tangent stress on slip surface, normal stress on leak surface, normal stresses on one-sided leak surfaces and temperature together as unknown functions. Except friction type conditions, other boundary conditions are reflected in variational equations as usual (Problems I-VE, II-VE).

When the solutions are smooth enough, these variational formulations are equivalent to the original PDE problems (Theorems 3.1, 3.2). Then, we get other variational formulations equivalent to the variational formulations above, which are consisted of one variational inequality for velocity and a variational equation for temperature (Problems I-VI, II-VI). In the end of Section 3, the main results of this paper are stated (Theorems 3.3, 3.4). Theorem 3.3 for Problem I involving the static pressure and stress boundary conditions asserts that if the body force and boundary data for fluid are small enough and buoyancy effect (α_0) and energy dissipation effect due to expansion (α_1) are small enough in accordance with the data of problem, then there exists a solution. However, Theorem 3.4 for Problem II involving the total pressure and total stress boundary conditions asserts that without smallness of the body force and boundary data for fluid if buoyancy effect (α_0) and energy dissipation effect due to expansion (α_1) are small enough in accordance with the data of problem, then there exists a solution.

Section 4 is devoted to the proof of Theorem 3.3. First in Subsection 4.1 we consider an auxiliary problem involving two parameters δ, ζ concerned with the norm of velocity (which is useful when there is fluid flux across a portion of boundary), one parameter λ concerned with the norm of temperature (which is useful to deal with buoyancy effect and energy dissipation effect due to expansion) and a parameter ε for approximation. We prove the existence of a solution to the auxiliary problem with the parameters δ, ζ λ, ε (Theorem 4.5). In Subsection 4.2 under a condition $|\alpha_1|\sqrt{\lambda} \leq 1$ we first get an estimate independent of ε, δ, ζ for the negative part of temperature $\theta_\varepsilon^-(x) := \min\{\theta_\varepsilon(x), 0\}$ in the Sobolev space $W^{1,2}(\Omega)$ (Lemma 4.6), which is a key for dealing with the general mixed boundary conditions for temperature. Next, under smallness condition of the body force and boundary data for fluid we determine the parameters δ, ζ (Lemma 4.7). Then, under a condition we get an estimate independent of ε for the the temperature $\theta_\varepsilon(x)$ in the space $W^{1,2}(\Omega)$, and finally determine the parameter λ(Lemma 4.8). Therefore, under smallness of the body force, the boundary data for fluid and $\max\{|\alpha_0|, |\alpha_1|\}$, we get estimates independent of ε for solutions of an auxiliary problem which includes only parameter ε (Theorem 4.9). Passing to the limits as ε goes to zero, in Subsection 4.3 we get the existence and estimates of a solution to the problem.

Section 5 is devoted to the prove of Theorem 3.4 for Problem II. To this end, we consider another auxiliary problem involving parameters ζ, λ, ε. Unlike Problem I, we need not to introduce parameter δ concerned with the norm of velocity. Thus, without smallness of the body force and boundary data for fluid we prove existence of a solution to the problem under smallness condition of $\max\{|\alpha_0|, |\alpha_1|\}$ in accordance with the data of problem.

Throughout this paper we will use the following notation.

Let Ω be a connected bounded open subset of $\mathbb{R}^l$, $l = 2, 3$. $\partial\Omega \in C^{0,1}$,

$$\partial\Omega = \cup_{i=1}^{11} \overline{\Gamma}_i = \overline{\Gamma}_D \cup \overline{\Gamma}_R,$$

$\Gamma_D \cap \Gamma_R = \varnothing$, $\Gamma_i \cap \Gamma_j = \varnothing$ for $i \neq j$, $\Gamma_i = \bigcup_j \Gamma_{ij}$, where Γ_{ij} are connected open subsets of $\partial\Omega$ and $\Gamma_{ij} \in C^2$ for $i = 2, 3, 7$ and $\Gamma_{ij} \in C^1$ for others. When X is a Banach space, $\mathbf{X} = X^l$. Let $W^{k,p}(\Omega)$ be Sobolev spaces, $H^1(\Omega) = W^{1,2}(\Omega)$, and so $\mathbf{H}^1(\Omega) = \{H^1(\Omega)\}^l$.

An inner product and norm in the space $\mathbf{L}^2(\Omega)$ are, respectively, denoted by $(\cdot\,,\cdot)$ and $\|\cdot\|$; and $\langle\cdot\,,\cdot\rangle$ means the duality pairing between a Sobolev space X and its dual one. Also, $(\cdot\,,\cdot)_{\Gamma_i}$ is an inner product in $\mathbf{L}^2(\Gamma_i)$ or $L^2(\Gamma_i)$; and $\langle\cdot\,,\cdot\rangle_{\Gamma_i}$ means the duality pairing between $\mathbf{H}^{\frac{1}{2}}(\Gamma_i)$ and $\mathbf{H}^{-\frac{1}{2}}(\Gamma_i)$ or between $H^{\frac{1}{2}}(\Gamma_i)$ and $H^{-\frac{1}{2}}(\Gamma_i)$. The inner product and norms in $\mathbb{R}^l$, respectively, are denoted by $(\cdot\,,\cdot)_{\mathbb{R}^l}$ and $|\cdot|$. Sometimes the inner product between a and b in $\mathbb{R}^l$ is denoted by $a \cdot b$. For convenience, in the case that $l = 2$, $y = (y_1(x_1, x_2), y_2(x_1, x_2))$ is identified with $\bar{y} = (y_1, y_2, 0)$, and so $\operatorname{rot} y = \operatorname{rot} \bar{y}$. Thus, for $y = (y_1, y_2)$ and $v = (v_1, v_2)$, $\operatorname{rot} y \times v$ is the 2-D vector consisted of the first two components of $\operatorname{rot} \bar{y} \times \bar{v}$.

Let $n(x)$ and $\tau(x)$ be, respectively, outward normal and tangent unit vectors at x in $\partial\Omega$. When $f \in H^{-1/2}(\Gamma_i)$, if $\langle f, w\rangle_{\Gamma_i} \geq 0$ (≤ 0) $\forall w \in C_0^\infty(\Gamma_i)$ with $w \geq 0$, then we denoted by $f \geq 0$ (≤ 0) on Γ_i. Also, $a^- = \min\{a, 0\}$, $a^+ = \max\{a, 0\}$. For convergence in spaces, $\to$ and $\rightharpoonup$ mean, respectively, strong and weak convergence.

2. Problems and assumptions. For temperature we consider the boundary conditions

$$\begin{aligned} &(1) \quad \theta|_{\Gamma_D} = \theta_D|_{\Gamma_D}, \quad \theta_D - \text{a given function on } \Omega, \\ &(2) \quad \Big(\kappa(\theta)\frac{\partial\theta}{\partial n} + \beta(x)\theta\Big)\Big|_{\Gamma_R} = g_R(x), \quad \beta(x),\, g_R(x) - \text{given functions on } \Gamma_R. \end{aligned} \tag{2.1}$$

Let us consider the boundary conditions for fluid. Stress tensor $S(v, p)$ is the one with components $s_{ij} = -p\delta_{ij} + 2\mu(\theta)\varepsilon_{ij}(v)$ and total stress tensor S^t is the one with components $s^t_{ij} = -(p+\frac{1}{2}|v|^2)\delta_{ij}+2\mu(\theta)\varepsilon_{ij}(v)$. Stress vector and total stress vector on the boundary surface, respectively, are $\sigma(\theta, v, p) = S\cdot n$ and $\sigma^t(\theta, v, p) = S^t \cdot n$. Normal stress vector and total normal stress vector on the boundary surface, respectively, are $\sigma_n(\theta, v, p) = \sigma \cdot n$ and $\sigma^t_n(\theta, v, p) = \sigma^t \cdot n$. And $\sigma_\tau(\theta, v, p) = \sigma(\theta, v, p) - \sigma_n(\theta, v, p)n$, $\sigma^t_\tau(\theta, v, p) = \sigma^t(\theta, v, p) - \sigma^t_n(\theta, v, p)n$. Note

$$\sigma_\tau(\theta, v, p) = \sigma^t_\tau(\theta, v, p) = 2\mu(\theta)\varepsilon_{n\tau}(v),$$

that is, these are independent of pressure p, and we use notation $\sigma_\tau(\theta, v) = \sigma^t_\tau(\theta, v)$.

According to boundary conditions for fluid, the problems I and II are distinguished. Problem I is the one with the boundary conditions

$$\begin{aligned} &(1) \quad v|_{\Gamma_1} = 0, \\ &(2) \quad v_\tau|_{\Gamma_2} = 0,\ -p|_{\Gamma_2} = \phi_2, \\ &(3) \quad v_n|_{\Gamma_3} = 0,\ \operatorname{rot} v \times n|_{\Gamma_3} = \phi_3/\mu(\theta), \\ &(4) \quad v_\tau|_{\Gamma_4} = 0,\ (-p + 2\mu(\theta)\varepsilon_{nn}(v))|_{\Gamma_4} = \phi_4, \\ &(5) \quad v_n|_{\Gamma_5} = 0,\ 2(\mu(\theta)\varepsilon_{n\tau}(v) + \alpha v_\tau)|_{\Gamma_5} = \phi_5, \ \ \alpha : \text{a matrix}, \\ &(6) \quad (-pn + 2\mu(\theta)\varepsilon_n(v))|_{\Gamma_6} = \phi_6, \\ &(7) \quad v_\tau|_{\Gamma_7} = 0,\ (-p + \mu(\theta)\frac{\partial v}{\partial n} \cdot n)|_{\Gamma_7} = \phi_7, \\ &(8) \quad v_n = 0,\ |\sigma_\tau(\theta, v)| \leq g_\tau,\ \sigma_\tau(\theta, v) \cdot v_\tau + g_\tau|v_\tau| = 0 \quad \text{on } \Gamma_8, \\ &(9) \quad v_\tau = 0,\ |\sigma_n(\theta, v, p)| \leq g_n,\ \sigma_n(\theta, v, p)v_n + g_n|v_n| = 0 \quad \text{on } \Gamma_9, \\ &(10) \quad v_\tau = 0,\ v_n \geq 0,\ \sigma_n(\theta, v, p) + g_{+n} \geq 0,\ (\sigma_n(\theta, v, p) + g_{+n})v_n = 0 \quad \text{on } \Gamma_{10}, \\ &(11) \quad v_\tau = 0,\ v_n \leq 0,\ \sigma_n(\theta, v, p) - g_{-n} \leq 0,\ (\sigma_n(\theta, v, p) - g_{-n})v_n = 0 \quad \text{on } \Gamma_{11}, \end{aligned} \tag{2.2}$$

and Problem II is the one with the conditions

$$
\begin{aligned}
&(1) \quad v|_{\Gamma_1} = 0,\\
&(2) \quad v_\tau|_{\Gamma_2} = 0,\ -(p + \frac{1}{2}|v|^2)|_{\Gamma_2} = \phi_2,\\
&(3) \quad v_n|_{\Gamma_3} = 0,\ \operatorname{rot} v \times n|_{\Gamma_3} = \phi_3/\mu(\theta),\\
&(4) \quad v_\tau|_{\Gamma_4} = 0,\ (-p - \frac{1}{2}|v|^2 + 2\mu(\theta)\varepsilon_{nn}(v))|_{\Gamma_4} = \phi_4,\\
&(5) \quad v_n|_{\Gamma_5} = 0,\ 2(\mu(\theta)\varepsilon_{n\tau}(v) + \alpha v_\tau)|_{\Gamma_5} = \phi_5,\ \ \alpha : \text{a matrix},\\
&(6) \quad (-pn - \frac{1}{2}|v|^2 n + 2\mu(\theta)\varepsilon_n(v))|_{\Gamma_6} = \phi_6,\\
&(7) \quad v_\tau|_{\Gamma_7} = 0, (-p - \frac{1}{2}|v|^2 + \mu(\theta)\frac{\partial v}{\partial n}\cdot n)|_{\Gamma_7} = \phi_7,\\
&(8) \quad v_n = 0,\ |\sigma^t_\tau(\theta, v)| \le g_\tau,\ \sigma^t_\tau(\theta, v)\cdot v_\tau + g_\tau|v_\tau| = 0 \quad \text{on } \Gamma_8,\\
&(9) \quad v_\tau = 0,\ |\sigma^t_n(\theta, v, p)| \le g_n,\ \sigma^t_n(\theta, v, p)v_n + g_n|v_n| = 0 \quad \text{on } \Gamma_9,\\
&(10) \quad v_\tau = 0,\ v_n \ge 0,\ \sigma^t_n(\theta, v, p) + g_{+n} \ge 0,\ (\sigma^t_n(\theta, v, p) + g_{+n})v_n = 0 \quad \text{on } \Gamma_{10},\\
&(11) \quad v_\tau = 0, v_n \le 0,\ \sigma^t_n(\theta, v, p) - g_{-n} \le 0,\ (\sigma^t_n(\theta, v, p) - g_{-n})v_n = 0 \quad \text{on } \Gamma_{11},
\end{aligned}
\tag{2.3}
$$

where $\varepsilon_n(v) = \varepsilon(v)n$, $\varepsilon_{nn}(v) = (\varepsilon(v)n, n)_{R^l}$, $\varepsilon_{n\tau}(v) = \varepsilon(v)n - \varepsilon_{nn}(v)n$, $v_\tau = v-(v\cdot n)n$, $v_n = v\cdot n$ and h_i, ϕ_i, α_{ij}(components of matrix α) are given functions or vectors of functions. And $g_\tau \in L^2(\Gamma_8)$, $g_n \in L^2(\Gamma_9)$, $g_{+n} \in L^2(\Gamma_{10})$, $g_{-n} \in L^2(\Gamma_{11})$, $g_\tau > 0$, $g_n > 0$, $g_{+n} > 0$, $g_{-n} > 0$, at a.e. x of the portions of boundary.

Note that in the boundary conditions for Problem II the static pressure p and stress in the boundary conditions for Problem I are replaced with the total pressure $p + \frac{1}{2}|v|^2$ and the total stress.

Figure 1 shows a case of the boundary conditions (2.2). On flat portions of the boundary, Γ_3, the rotation boundary condition coincides with the Navier slip boundary condition on Γ_5 (see Remark 3.1 in [19]). For small movements of viscous fluids in an open container, for the open surface the boundary condition on Γ_4 is used instead of the condition on Γ_5 (see Sec. 8.1 in [23]).

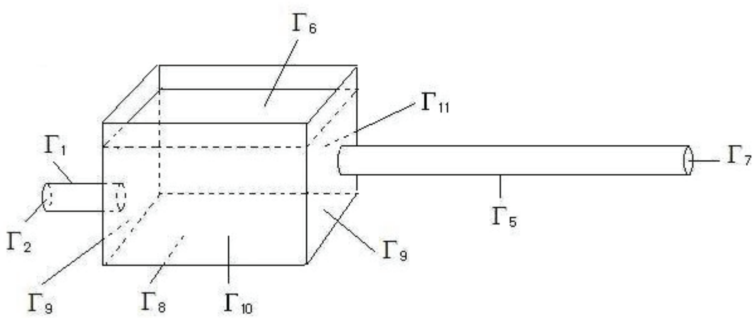

FIG. 1. Γ_1-*stick wall,* Γ_2-*inlet where pressure is measured,* Γ_5-*tube with micro-roughness,* Γ_6-*free surface,* Γ_7-*free outlet,* Γ_8-*surface with Tresca slip,* Γ_9-*sand layer penetrable into and out,* Γ_{10}-*membrane semipermeable out,* Γ_{11}- *membrane semipermeable into.*

We use the following assumption.

ASSUMPTION 2.1. *We assume the followings.*

1) $\Gamma_1 \neq \varnothing$ *and* $\Gamma_D \neq \varnothing$.

2) If Γ_i*, where i is 10 or 11, is nonempty, then at least one of* $\{\Gamma_j : j \in \{\{2,4,7,9-11\}\backslash\{i\}\}$ *is nonempty and there exists a diffeomorphism in* C^1 *between* Γ_i *and* Γ_j.

Also, $\Gamma_{2j}, \Gamma_{3j}, \Gamma_{7j}$ *are convex and*

$$\Gamma_R \subset (\cup_{i=1,3,5,8}\Gamma_i)\,. \tag{2.4}$$

3) For the functions of (1.1) $f \in \mathbf{L}^t(\Omega), t > 3,\ \ g \in L^{6/5}(\Omega)$ *and*

$$\begin{aligned}
&\mu \in C(\mathbb{R}),\ 0 < \mu_0 \le \mu(\xi) \le \mu_1 < \infty \quad \forall \xi \in \mathbb{R};\\
&\kappa \in C(\mathbb{R}),\ 0 < \kappa_0 \le \kappa(\xi) \le \kappa_1 < \infty \quad \forall \xi \in \mathbb{R};\\
&\gamma \in C(\mathbb{R}),\ |\gamma(\xi)| \le \gamma_0 \quad \forall \xi \in \mathbb{R}.
\end{aligned} \tag{2.5}$$

4) For the functions of (2.1), (2.2), (2.3)

$$\begin{aligned}
&\theta_D \in W^{1,2}(\Omega), \nabla\theta_D \in L^\infty(\Omega),\ \theta_D \ge 0, \qquad g_R \in L^{4/3}(\Gamma_R);\\
&\beta_0 \ge \beta(x) \ge 0,\ \beta_0 - \textit{a constant},\ \beta(x) - \textit{measurable};\\
&\phi_i \in H^{-\frac{1}{2}}(\Gamma_i), i = 2,4,7,\ \phi_i \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_i), i = 3,5,6;\\
&\textit{the matrix } \alpha \textit{ is positive},\ \alpha_{ij} \in L_\infty(\Gamma_5).
\end{aligned} \tag{2.6}$$

REMARK 2.1. *On the* $\Gamma_{10}(\Gamma_{11})$ *outflow(inflow) only is possible. Thus, to guarantee* $div\, u = 0$ *the first part of 2) of Assumption 2.1 is used. In Theorems 3.3 and 3.5 of [20] for proof of equivalence of variational formulations to variational inequalities, this assumption was used via Lemma 3.2 of [20]. In this paper this assumption is also necessary to guarantee equivalence between Problem I-VE and Problem I-VI, and between Problem II-VE and Problem II-VI.*

3. Variational formulations and main results. In this section for every problem above we first give a variational formulation which consists of six formulae with six unknown functions, that is, using velocity, tangent stress on slip surface, normal stress (total normal stress) on leak surface, normal stresses (total normal stresses) on one-sided leak surfaces and temperature together as unknown functions. Then, for every problem we get another variational formulation equivalent to the variational formulation above, which consists of one variational inequality for velocity and a variational equation for temperature.

Let

$$\begin{aligned}
\mathbf{V} &= \{u \in \mathbf{H}^1(\Omega) : \operatorname{div} u = 0,\, u|_{\Gamma_1} = 0,\\
&\qquad u_\tau|_{(\Gamma_2\cup\Gamma_4\cup\Gamma_7\cup\Gamma_9\cup\Gamma_{10}\cup\Gamma_{11})} = 0,\, u_n|_{(\Gamma_3\cup\Gamma_5\cup\Gamma_8)} = 0\},\\
K(\Omega) &= \{u \in \mathbf{V} : u_n|_{\Gamma_{10}} \ge 0,\, u_n|_{\Gamma_{11}} \le 0\},\\
W^{1,p}_{\Gamma_D}(\Omega) &= \{y \in W^{1,p}(\Omega) : y|_{\Gamma_D} = 0\}.
\end{aligned}$$

Since $\Gamma_1 \neq \varnothing$ and $\Gamma_D \neq \varnothing$, by Korn's inequality and Poincaré's inequality we use

$$(v,u)_{\mathbf{V}(\Omega)} = (\mathcal{E}(v), \mathcal{E}(u)), \quad (y,z)_{W^{1,2}_{\Gamma_D}(\Omega)} = (\nabla\, y, \nabla\, z).$$

By Theorems 2.1 and 2.2 of [19] for $v \in \mathbf{H}^2(\Omega) \cap \mathbf{V}$, $\theta \in W^{1,2}(\Omega)$ and $u \in \mathbf{V}$

$$\begin{aligned}
&-2\langle \nabla \cdot (\mu(\theta)\mathcal{E}(v)), u\rangle \\
&= 2(\mu(\theta)\mathcal{E}(v), \mathcal{E}(u)) - 2(\mu(\theta)\mathcal{E}(v)n, u)_{\cup_{i=2}^{11}\Gamma_i} \\
&= 2(\mu(\theta)\mathcal{E}(v), \mathcal{E}(u)) + 2(\mu(\theta)k(x)v, u)_{\Gamma_2} - (\mu(\theta)\mathrm{rot}\, v \times n, u)_{\Gamma_3} \\
&\quad + 2(\mu(\theta)S\tilde{v}, \tilde{u})_{\Gamma_3} - 2(\mu(\theta)\varepsilon_{nn}(v), u_n)_{\Gamma_4} \\
&\quad - 2(\mu(\theta)\varepsilon_{n\tau}(v), u)_{\Gamma_5} - 2(\mu(\theta)\varepsilon_n(v), u)_{\Gamma_6} - \left(\mu(\theta)\frac{\partial v}{\partial n}, u\right)_{\Gamma_7} \\
&\quad + (\mu(\theta)k(x)v, u)_{\Gamma_7} - 2(\mu(\theta)\varepsilon_{n\tau}(v), u)_{\Gamma_8} - 2(\mu(\theta)\varepsilon_{nn}(v), u_n)_{\Gamma_9} \\
&\quad - 2(\mu(\theta)\varepsilon_{nn}(v), u_n)_{\Gamma_{10}} - 2(\mu(\theta)\varepsilon_{nn}(v), u_n)_{\Gamma_{11}},
\end{aligned} \tag{3.1}$$

where S is the shape operator of boundary surface and $k(x) = \mathrm{div}\, n(x)$ (cf. (A.1) and Remark 2.1 of [19]).

For $p \in H^1(\Omega)$ and $u \in \mathbf{V}$ we have

$$(\nabla p, u) = (p, u_n)_{\cup_{i=2}^{11}\Gamma_i} = (p, u_n)_{\Gamma_2} + (p, u_n)_{\Gamma_4\cup\Gamma_7\cup\Gamma_9\cup\Gamma_{10}\cup\Gamma_{11}} + (pn, u)_{\Gamma_6}, \tag{3.2}$$

where $u_n |_{\Gamma_3\cup\Gamma_5\cup\Gamma_8} = 0$ was used.
For $\theta \in W^{1,2}(\Omega)$ and $\varphi \in W^{1,2}_{\Gamma_D}(\Omega)$, by (2.1) we have

$$\begin{aligned}
\langle -\nabla \cdot (\kappa(\theta)\nabla\theta), \varphi\rangle &= (\kappa(\theta)\nabla\theta, \nabla\varphi) - \big(\kappa(\theta)\frac{\partial\theta}{\partial n}, \varphi\big)_{\Gamma_R} \\
&= (\kappa(\theta)\nabla\theta, \nabla\varphi) + (\beta\theta - g_R, \varphi)_{\Gamma_R}.
\end{aligned} \tag{3.3}$$

By (2.4) $v_n = 0$ on Γ_R, and so for $v \in \mathbf{V}$, $\theta \in W^{1,2}(\Omega)$ and $\varphi \in W^{1,2}_{\Gamma_D}(\Omega)$ we have

$$\langle v \cdot \nabla(\gamma(\theta)\theta), \varphi\rangle = (v_n\gamma(\theta)\theta, \varphi)_{\Gamma_R} - (\gamma(\theta)\theta v, \nabla\varphi) = -(\gamma(\theta)\theta v, \nabla\varphi). \tag{3.4}$$

3.1. Variational formulations for Problem I. By (3.1)-(3.4), we can see that smooth solutions (v, p, θ) of problem (1.1), (2.1), (2.2) satisfy the following.

$$\left\{\begin{aligned}
&2(\mu(\theta)\mathcal{E}(v), \mathcal{E}(u)) + \langle (v \cdot \nabla)v, u\rangle \\
&\qquad + 2(\mu(\theta)k(x)v, u)_{\Gamma_2} + 2(\mu(\theta)S\tilde{v}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)v, u)_{\Gamma_5} + (\mu(\theta)k(x)v, u)_{\Gamma_7} \\
&\qquad - 2(\mu(\theta)\varepsilon_{n\tau}(v), u)_{\Gamma_8} + (p - 2\mu(\theta)\varepsilon_{nn}(v), u_n)_{\Gamma_9\cup\Gamma_{10}\cup\Gamma_{11}} \\
&\quad = \langle (1 - \alpha_0\theta)f, u\rangle + \sum_{i=2,4,7} \langle \phi_i, u_n\rangle_{\Gamma_i} + \sum_{i=3,5,6} \langle \phi_i, u\rangle_{\Gamma_i} \quad \forall u \in \mathbf{V}, \\
&(\kappa(\theta)\nabla\theta, \nabla\varphi) - (\gamma(\theta)\theta v, \nabla\varphi) - (\alpha_2\mu(\theta)|\mathcal{E}(v)|^2, \varphi) + (\beta\theta, \varphi)_{\Gamma_R} - (\alpha_1\theta f \cdot v, \varphi) \\
&\quad = (g_R, \varphi)_{\Gamma_R} + \langle g, \varphi\rangle \qquad \forall \varphi \in W^{1,\infty}_{\Gamma_D}(\Omega), \\
&|\sigma_\tau(\theta, v)| \le g_\tau,\ \sigma_\tau(\theta, v) \cdot v_\tau + g_\tau|v_\tau| = 0 \quad \text{on } \Gamma_8, \\
&|\sigma_n(\theta, v, p)| \le g_n,\ \sigma_n(\theta, v, p)v_n + g_n|v_n| = 0 \quad \text{on } \Gamma_9, \\
&\sigma_n(\theta, v, p) + g_{+n} \ge 0,\ (\sigma_n(\theta, v, p) + g_{+n})v_n = 0 \quad \text{on } \Gamma_{10}, \\
&\sigma_n(\theta, v, p) - g_{-n} \le 0,\ (\sigma_n(\theta, v, p) - g_{-n})v_n = 0 \quad \text{on } \Gamma_{11}, \\
&\theta|_{\Gamma_D} = \theta_D|_{\Gamma_D} \text{ on } \Gamma_D.
\end{aligned}\right. \tag{3.5}$$

Define $a_0(\theta;\cdot,\cdot), a_1(\cdot,\cdot,\cdot)$ and $f_1 \in \mathbf{V}^*$ by

$$\begin{aligned}a_0(\theta; w, u) &= 2(\mu(\theta)\mathcal{E}(w), \mathcal{E}(u)) + 2(\mu(\theta)k(x)w, u)_{\Gamma_2} + 2(\mu(\theta)S\tilde{w}, \tilde{u})_{\Gamma_3}\\ &\qquad + 2(\alpha(x)w, u)_{\Gamma_5} + (\mu(\theta)k(x)w, u)_{\Gamma_7} \quad \forall w, u \in \mathbf{V},\, \theta \in W^{1,2}(\Omega),\\ a_1(v, w, u) &= \langle (v\cdot\nabla)w, u\rangle \quad \forall v, w, u \in \mathbf{V},\\ \langle f_1, u\rangle &= \sum_{i=2,4,7} \langle \phi_i, u_n\rangle_{\Gamma_i} + \sum_{i=3,5,6} \langle \phi_i, u\rangle_{\Gamma_i} \quad \forall u \in \mathbf{V}.\end{aligned} \tag{3.6}$$

Define $b_0(\theta;\cdot,\cdot)$ and $f_2 \in (W^{1,2}_{\Gamma_D}(\Omega))^*$ by

$$\begin{aligned}b_0(\theta; \tilde{\theta}, \varphi) &= (\kappa(\theta)\nabla\tilde{\theta}, \nabla\varphi) + (\beta(x)\tilde{\theta}, \varphi)_{\Gamma_R} \quad \forall \theta, \tilde{\theta} \in W^{1,2}(\Omega), \varphi \in W^{1,2}_{\Gamma_D}(\Omega),\\ \langle f_2, \varphi\rangle &= (g_R, \varphi)_{\Gamma_R} + \langle g, \varphi\rangle \quad \forall \varphi \in W^{1,2}_{\Gamma_D}(\Omega).\end{aligned} \tag{3.7}$$

Then, taking into account

$$\sigma_\tau(\theta, v) = 2\mu(\theta)\varepsilon_{n\tau}(v), \quad \sigma_n(\theta, v, p) = -p + 2\mu(\theta)\varepsilon_{nn}(v)$$

and (3.3), we introduce the following variational formulation for problem (1.1), (2.1), (2.2).

Problem I-VE. Find $(v, \theta, \sigma_\tau, \sigma_n, \sigma_{+n}, \sigma_{-n}) \in K(\Omega) \times \left(\bigcap_{1\le r<\frac{3}{2}} W^{1,r}(\Omega)\right) \times \mathbf{L}^2_\tau(\Gamma_8) \times L^2(\Gamma_9) \times H^{-1/2}(\Gamma_{10}) \times H^{-1/2}(\Gamma_{11})$ such that $\theta|_{\Gamma_D} = \theta_D|_{\Gamma_D}$ and

$$\begin{cases} a_0(\theta; v, u) + a_1(v, v, u) - (\sigma_\tau, u_\tau)_{\Gamma_8} - (\sigma_n, u_n)_{\Gamma_9}\\ \qquad - \langle \sigma_{+n}, u_n\rangle_{\Gamma_{10}} - \langle \sigma_{-n}, u_n\rangle_{\Gamma_{11}} - \langle f - \alpha_0\theta f, u\rangle = \langle f_1, u\rangle \quad \forall u \in \mathbf{V},\\ b_0(\theta; \theta, \varphi) - \langle \gamma(\theta)\theta v, \nabla\varphi\rangle - \langle \alpha_2\mu(\theta)|\mathcal{E}(v)|^2, \varphi\rangle - \langle \alpha_1\theta f\cdot v, \varphi\rangle\\ \qquad = \langle f_2, \varphi\rangle \quad \forall \varphi \in W^{1,\infty}_{\Gamma_D}(\Omega),\\ |\sigma_\tau| \le g_\tau,\ \sigma_\tau\cdot v_\tau + g_\tau|v_\tau| = 0 \quad \text{on } \Gamma_8,\\ |\sigma_n| \le g_n,\ \sigma_n v_n + g_n|v_n| = 0 \quad \text{on } \Gamma_9,\\ \sigma_{+n} + g_{+n} \ge 0,\ \langle \sigma_{+n} + g_{+n}, v_n\rangle_{\Gamma_{10}} = 0 \quad \text{on } \Gamma_{10},\\ \sigma_{-n} - g_{-n} \le 0,\ \langle \sigma_{-n} - g_{-n}, v_n\rangle_{\Gamma_{11}} = 0 \quad \text{on } \Gamma_{11}, \end{cases} \tag{3.8}$$

where $\mathbf{L}^2_\tau(\Gamma_8)$ is the subspace of $\mathbf{L}^2(\Gamma_8)$ consisting of functions such that $(u, n)_{\mathbf{L}^2(\Gamma_8)} = 0$.

Remark 3.1. *Under 4) of Assumption 2.1 the duality products* $\langle f_1, u\rangle$ *of* (3.6) *has a meaning (cf. Remark 3.1 in [20]).*

Theorem 3.1. *Under Assumption 2.1 if a solution is smooth enough* ($v \in \boldsymbol{H}^2(\Omega), \theta \in W^{2,2}(\Omega), f \in \boldsymbol{L}^2(\Omega)$)*, then Problem I-VE is equivalent to problem* (1.1), (2.1), (2.2). *In addition, if at least one of* $\{\Gamma_i : i = 2, 4, 6, 7, 9-11\}$ *is nonempty, then* p *of problem* (1.1), (2.1), (2.2) *is unique.*

Proof. From problem (1.1), (2.1), (2.2) we deduced Problem I-VE, and it is enough to prove conversion from Problem I-VE to problem (1.1), (2.1), (2.2). By Theorem 3.1 of [20] there exists a p such that (v, p) satisfies (1.1) and the boundary condition (2.2), and p is unique under the condition above. In a routine way(cf. Section 1, ch. 2 of [17]) we can prove that θ satisfies (1.1) and the boundary condition (2.1). □

We will find another variational formulation consisted of a variational inequality and a variational equation, which is equivalent to Problem I-VE.

Let $(v, \theta, \sigma_\tau, \sigma_n, \sigma_{+n}, \sigma_{-n})$ be a solution of Problem I-VE. Subtracting the first formula of (3.8) with $u = v$ from the first formula of (3.8), we get

$$\begin{aligned}
&a_0(\theta; v, u-v) + a_1(v, v, u-v) - (\sigma_\tau, u_\tau - v_\tau)_{\Gamma_8} - (\sigma_n, u_n - v_n)_{\Gamma_9} \\
&\quad - \langle \sigma_{+n}, u_n - v_n \rangle_{\Gamma_{10}} - \langle \sigma_{-n}, u_n - v_n \rangle_{\Gamma_{11}} - \langle f - \alpha_0 \theta f, u - v \rangle \\
&= \langle f_1, u - v \rangle \quad \forall u \in \mathbf{V}.
\end{aligned} \tag{3.9}$$

Define the functionals $\phi_\tau, \phi_n, \phi_+, \phi_-$, respectively, by

$$\begin{aligned}
\phi_\tau(\eta) &= \int_{\Gamma_8} g_\tau |\eta| \, dx \quad \forall \eta \in \mathbf{L}^2_\tau(\Gamma_8), \\
\phi_n(\eta) &= \int_{\Gamma_9} g_n |\eta| \, dx \quad \forall \eta \in L^2(\Gamma_9), \\
\phi_+(\eta) &= \int_{\Gamma_{10}} g_{+n} \eta \, dx \quad \forall \eta \in L^2(\Gamma_{10}), \\
\phi_-(\eta) &= -\int_{\Gamma_{11}} g_{-n} \eta \, dx \quad \forall \eta \in L^2(\Gamma_{11}).
\end{aligned} \tag{3.10}$$

Since if $u \in K(\Omega)$, then $u|_{\Gamma_8} \in \mathbf{L}^2_\tau(\Gamma_8)$, $u_n|_{\Gamma_9} \in L^2(\Gamma_9)$, $u_n|_{\Gamma_{10}} \in L^2(\Gamma_{10})$, $u_n|_{\Gamma_{11}} \in L^2(\Gamma_{11})$, in what follows for convenience we use the notation

$$\begin{aligned}
&\phi_\tau(u) = \phi_\tau(u|_{\Gamma_8}), \ \phi_n(u) = \phi_n(u_n|_{\Gamma_9}), \ \phi_+(u) = \phi_+(u_n|_{\Gamma_{10}}), \\
&\phi_-(u) = \phi_-(u_n|_{\Gamma_{11}}) \quad \forall u \in K(\Omega).
\end{aligned}$$

Define a functional $\Phi(v) \in (\mathbf{V} \to \overline{\mathbb{R}}), \overline{\mathbb{R}} \equiv \mathbb{R} \cup +\infty$, by

$$\Phi(u) = \begin{cases} \phi_\tau(u) + \phi_n(u) + \phi_+(u) + \phi_-(u) & \forall u \in K(\Omega), \\ +\infty & \forall u \notin K(\Omega). \end{cases} \tag{3.11}$$

Then, Φ is proper, convex lower weak semi-continuous. Note $\Phi \geq 0$ since $u_n|_{\Gamma_{10}} \geq 0, u_n|_{\Gamma_{11}} \leq 0 \ \forall u \in K(\Omega)$.

By Theorem 3.3 in [20], under Assumption 2.1 for fixed θ the problem

$$\begin{cases}
a_0(\theta; v, u) + a_1(v, v, u) - (\sigma_\tau, u_\tau)_{\Gamma_8} - (\sigma_n, u_n)_{\Gamma_9} \\
\qquad - \langle \sigma_{+n}, u_n \rangle_{\Gamma_{10}} - \langle \sigma_{-n}, u_n \rangle_{\Gamma_{11}} - \langle f - \alpha_0 \theta f, u \rangle = \langle f_1, u \rangle, \\
|\sigma_\tau| \leq g_\tau, \ \sigma_\tau \cdot v_\tau + g_\tau |v_\tau| = 0 \quad \text{on } \Gamma_8, \\
|\sigma_n| \leq g_n, \ \sigma_n v_n + g_n |v_n| = 0 \quad \text{on } \Gamma_9, \\
\sigma_{+n} + g_{+n} \geq 0, \ \langle \sigma_{+n} + g_{+n}, v_n \rangle_{\Gamma_{10}} = 0 \quad \text{on } \Gamma_{10}, \\
\sigma_{-n} - g_{-n} \leq 0, \ \langle \sigma_{-n} - g_{-n}, v_n \rangle_{\Gamma_{11}} = 0 \quad \text{on } \Gamma_{11}
\end{cases} \tag{3.12}$$

is equivalent to the following variational inequality.

Find $v \in \mathbf{V}$ such that

$$\begin{aligned}
&a_0(\theta; v, u-v) + a_1(v, v, u-v) + \Phi(u) - \Phi(v) - \langle f - \alpha_0 \theta f, u - v \rangle \\
&\geq \langle f_1, u - v \rangle \quad \forall u \in \mathbf{V}.
\end{aligned} \tag{3.13}$$

Therefore, we have the following variational formulation equivalent to Problem I-VE which consists of a variational inequality for velocity and a variational equation for temperature.

Problem I-VI. Find $(v,\theta) \in \mathbf{V} \times \left(\bigcap_{1\le r<\frac{3}{2}} W^{1,r}(\Omega)\right)$ such that $\theta|_{\Gamma_D} = \theta_D|_{\Gamma_D}$ and

$$\begin{cases} a_0(\theta; v, u-v) + a_1(v,v,u-v) + \Phi(u) - \Phi(v) - \langle f - \alpha_0\theta f, u-v\rangle \\ \qquad \geq \langle f_1, u-v\rangle \quad \forall u \in \mathbf{V}, \\ b_0(\theta;\theta,\varphi) - \langle \gamma(\theta)\theta v, \nabla\varphi\rangle - \langle \alpha_2\mu(\theta)|\mathcal{E}(u)|^2, \varphi\rangle - \langle \alpha_1\theta f\cdot v, \varphi\rangle \\ \qquad = \langle f_2, \varphi\rangle \quad \forall \varphi \in W^{1,\infty}_{\Gamma_D}(\Omega). \end{cases} \tag{3.14}$$

3.2. Variational formulations for Problem II. Taking $(v\cdot\nabla)v = \mathrm{rot}\, v \times v + \frac{1}{2}\mathrm{grad}|v|^2$ into account, by (3.1)-(3.4) we can see that smooth solutions (v,p,θ) of problem (1.1), (2.1), (2.3) satisfy the following.

$$\begin{cases} 2(\mu(\theta)\mathcal{E}(v), \mathcal{E}(u)) + \langle \mathrm{rot} v \times v, u\rangle \\ \qquad + 2(\mu(\theta)k(x)v, u)_{\Gamma_2} + 2(\mu(\theta)S\tilde{v}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)v,u)_{\Gamma_5} + (\mu(\theta)k(x)v,u)_{\Gamma_7} \\ \qquad - 2(\mu(\theta)\varepsilon_{n\tau}(v), u)_{\Gamma_8} + (p + \frac{1}{2}|v|^2 - 2\mu(\theta)\varepsilon_{nn}(v), u_n)_{\Gamma_9\cup\Gamma_{10}\cup\Gamma_{11}} \\ \quad = \langle (1-\alpha_0\theta)f, u\rangle + \sum_{i=2,4,7} \langle \phi_i, u_n\rangle_{\Gamma_i} + \sum_{i=3,5,6} \langle \phi_i, u\rangle_{\Gamma_i} \quad \forall u \in \mathbf{V}, \\ (\kappa(\theta)\nabla\theta, \nabla\varphi) - (\gamma(\theta)\theta v, \nabla\varphi) - (\alpha_2\mu(\theta)|\mathcal{E}(v)|^2, \varphi) + (\beta\theta,\varphi)_{\Gamma_R} - (\alpha_1\theta f\cdot v, \varphi) \\ \quad = (g_R, \varphi)_{\Gamma_R} + \langle g, \varphi\rangle \qquad \forall\varphi \in W^{1,\infty}_{\Gamma_D}(\Omega), \\ |\sigma^t_\tau(\theta,v)| \le g_\tau,\ \sigma^t_\tau(\theta,v)\cdot v_\tau + g_\tau|v_\tau| = 0 \quad \text{on } \Gamma_8, \\ |\sigma^t_n(\theta,v,p)| \le g_n,\ \sigma^t_n(\theta,v,p)v_n + g_n|v_n| = 0 \quad \text{on } \Gamma_9, \\ \sigma^t_n(\theta,v,p) + g_{+n} \ge 0,\ (\sigma^t_n(\theta,v,p) + g_{+n})v_n = 0 \quad \text{on } \Gamma_{10}, \\ \sigma^t_n(\theta,v,p) - g_{-n} \le 0,\ (\sigma^t_n(\theta,v,p) - g_{-n})v_n = 0 \quad \text{on } \Gamma_{11}, \\ \theta|_{\Gamma_D} = \theta_D|_{\Gamma_D} \quad \text{on } \Gamma_D. \end{cases} \tag{3.15}$$

Define $a_2(\cdot,\cdot,\cdot)$ by

$$a_2(v,u,w) = \langle \mathrm{rot}\, v \times u, w\rangle \quad \forall v,u,w \in \mathbf{V}. \tag{3.16}$$

Then, taking into account

$$\sigma^t_\tau(\theta,v) = 2\mu(\theta)\varepsilon_{n\tau}(v), \quad \sigma^t_n(\theta,v,p) = -(p + \frac{1}{2}|v|^2) + 2\mu(\theta)\varepsilon_{nn}(v)$$

and (3.15), we introduce the following variational formulation for problem (1.1), (2.1) (2.3).

Problem II-VE. Find $(v,\theta,\sigma^t_\tau,\sigma^t_n,\sigma^t_{+n},\sigma^t_{-n}) \in K(\Omega) \times \left(\bigcap_{1\le r<\frac{3}{2}} W^{1,r}(\Omega)\right) \times$

$\mathbf{L}^2_\tau(\Gamma_8)\times L^2(\Gamma_9)\times H^{-1/2}(\Gamma_{10})\times H^{-1/2}(\Gamma_{11})$ such that $\theta|_{\Gamma_D}=\theta_D|_{\Gamma_D}$ and

$$\begin{cases} a_0(\theta;v,u)+a_2(v,v,u)-(\sigma^t_\tau,u_\tau)_{\Gamma_8}-(\sigma^t_n,u_n)_{\Gamma_9} \\ \qquad -\left\langle\sigma^t_{+n},u_n\right\rangle_{\Gamma_{10}}-\left\langle\sigma^t_{-n},u_n\right\rangle_{\Gamma_{11}}-\langle f-\alpha_0\theta f,u\rangle=\langle f_1,u\rangle \quad \forall u\in\mathbf{V}, \\ b_0(\theta;\theta,\varphi)-\langle\gamma(\theta)\theta v,\nabla\varphi\rangle-\langle\alpha_2\mu(\theta)|\mathcal{E}(v)|^2,\varphi\rangle-\langle\alpha_1\theta f\cdot v,\varphi\rangle \\ \qquad =\langle f_2,\varphi\rangle \quad \forall\varphi\in W^{1,\infty}_{\Gamma_D}(\Omega), \\ |\sigma^t_\tau|\le g_\tau,\ \sigma^t_\tau\cdot v_\tau+g_\tau|v_\tau|=0 \quad \text{on } \Gamma_8, \\ |\sigma^t_n|\le g_n,\ \sigma^t_n v_n+g_n|v_n|=0 \quad \text{on } \Gamma_9, \\ \sigma^t_{+n}+g_{+n}\ge 0,\ \left\langle\sigma^t_{+n}+g_{+n},v_n\right\rangle_{\Gamma_{10}}=0 \quad \text{on } \Gamma_{10}, \\ \sigma^t_{-n}-g_{-n}\le 0,\ \left\langle\sigma^t_{-n}-g_{-n},v_n\right\rangle_{\Gamma_{11}}=0 \quad \text{on } \Gamma_{11}. \end{cases} \tag{3.17}$$

Relying on Theorem 3.4 in [20], in the same way as Theorem 3.1 we have

THEOREM 3.2. *Under Assumption 2.1 if a solution is smooth enough ($v\in \boldsymbol{H}^2(\Omega),\theta\in W^{2,2}(\Omega), f\in \boldsymbol{L}^2(\Omega)$), then Problem II-VE is equivalent to problem* (1.1), (2.1), (2.3). *In addition, if at least one of $\{\Gamma_i : i=2,4,6,7,9-11\}$ is nonempty, then p of problem* (1.1), (2.1), (2.3) *is unique.*

Then, relying on Theorem 3.5 in [20], in the same way as Problem I-VI we get Problem II-VI equivalent to Problem II-VE which consists of a variational inequality for velocity and a variational equation for temperature.

Problem II-VI. Find $(v,\theta)\in\mathbf{V}\times\Big(\bigcap_{1\le r<\frac{3}{2}}W^{1,r}(\Omega)\Big)$ such that $\theta|_{\Gamma_D}=\theta_D|_{\Gamma_D}$ and

$$\begin{cases} a_0(\theta;v,u-v)+a_2(v,v,u-v)+\Phi(u)-\Phi(v)-\langle f-\alpha_0\theta f,u-v\rangle \\ \qquad \ge\langle f_1,u-v\rangle \quad \forall u\in\mathbf{V}, \\ b_0(\theta;\theta,\varphi)-\langle\gamma(\theta)\theta v,\nabla\varphi\rangle-\langle\alpha_2\mu(\theta)|\mathcal{E}(v)|^2,\varphi\rangle-\langle\alpha_1\theta f\cdot v,\varphi\rangle \\ \qquad =\langle f_2,\varphi\rangle \quad \forall\varphi\in W^{1,\infty}_{\Gamma_D}(\Omega), \end{cases} \tag{3.18}$$

where $a_2(\cdot,\cdot,\cdot)$ is the one in (3.16) and Φ is defined by (3.10), (3.11).

3.3. Main results. In view of the results above, we will study Problem I-VI and Problem II-VI. Main results of this paper are the following theorems.

THEOREM 3.3. *Under Assumption 2.1 assume that*

1) $f,\phi_i, i=2-7$, are small enough in the spaces in 3), 4) of Assumption 2.1 (cf. (4.37)*),*

2) $\max\{|\alpha_0|,|\alpha_1|\}$ is small enough in accordance with $f,\phi_i, i=2-7, g, g_R,\theta_D$ (cf. (4.84)*).*

Then, there exists a solution (v,θ) to Problem I-VI such that

$$\begin{aligned} &\|v\|_{\boldsymbol{V}}\le\frac{\mu_0}{K}, \\ &\|\theta^-(x)\|_{W^{1,2}(\Omega)}\le c\Big(\frac{\mu_0}{K}\|f\|_{\boldsymbol{L}^t}+\|g_R\|_{L^{4/3}(\Gamma_R)}+\|g\|_{L^{6/5}(\Omega)}\Big), \\ &\int_\Omega|\nabla\theta|^r\,dx\le cK_\sigma^{r/(1-\sigma)}\big(1+\|\theta_D\|^2_{W^{1,2}}\big) \quad \forall r\ (1<r<3/2), \end{aligned} \tag{3.19}$$

where K is the one in (4.5) *below, K_σ is the one in* (4.77) *and $\sigma=\frac{3-2r}{3-r}$.*

THEOREM 3.4. *Under Assumption 2.1 assume that* $\max\{|\alpha_0|, |\alpha_1|\}$ *is small enough in accordance with* $f, \phi_i, i = 2-7, g, g_R, \theta_D$ *(cf.* (5.13)*).*

Then, there exists a solution (v, θ) *to Problem II-VI such that*

$$\begin{aligned}
&\|v\|_{\boldsymbol{V}} \leq c\big(1 + \|f\|_{\boldsymbol{L}^t} + \|f_1\|_{\boldsymbol{V}^*}\big),\\
&\|\theta^-(x)\|_{W^{1,2}(\Omega)} \leq c\Big(\|f\|_{\boldsymbol{L}^t} + \|f\|^2_{\boldsymbol{L}^t} + \|f_1\|_{\boldsymbol{V}^*}\|f\|_{\boldsymbol{L}^t} + \|g_R\|_{L^{4/3}(\Gamma_R)} + \|g\|_{L^{6/5}(\Omega)}\Big), \qquad (3.20)\\
&\int_\Omega |\nabla\theta|^r\,dx \leq cL_\sigma^{r/(1-\sigma)}\big(1 + \|\theta_D\|^2_{W^{1,2}}\big) \quad \forall r\,(1 < r < 3/2),
\end{aligned}$$

where L_σ *is the one in* (5.10) *below and* $\sigma = \frac{3-2r}{3-r}$.

4. Proof of Theorem 3.3. Let us consider a lemma which is an immediate consequence of Vitali's convergence theorem (Theorem 1.4.12 of [15]) and corollaries necessary later.

LEMMA 4.1. *Let* Ω *be a bounded domain of* $\mathbb{R}^l$. *If* $\{u_n\}$ *is such that*

$$\{u_n\} \text{ is bounded in } L^\infty(\Omega), \quad u_n(x) \to u(x) \text{ a.e. in } \Omega,$$

then

$$u_n \to u \text{ in } L^p(\Omega)\ \forall p, 1 < p < \infty.$$

COROLLARY 4.2. *Let* Ω *be a bounded domain of* $\mathbb{R}^l$ *and* $\mu(\xi), \xi \in \mathbb{R}$, *be a bounded continuous function. If* $\{u_n\}$ *and* $\{v_n\}$ *are such that*

$$u_n(x) \to u(x) \text{ a.e. in } \Omega, \quad v_n \rightharpoonup v \text{ in } L^p(\Omega), 1 < p < \infty,$$

then

$$\mu(u_n)v_n \rightharpoonup \mu(u)v \text{ in } L^p(\Omega).$$

Proof. By Lemma 4.1,

$$\mu(u_n) \to \mu(u) \text{ in } L^q(\Omega),\ \frac{1}{p} + \frac{1}{q} = 1.$$

Thus, $\mu(u_n)v_n \rightharpoonup \mu(u)v$ in $L^1(\Omega)$, that is,

$$\int_\Omega \mu(u_n(x))v_n(x)\phi(x)\,dx \to \int_\Omega \mu(u(x))v(x)\phi(x)\,dx \quad \forall\phi \in C_0^\infty(\Omega).$$

Since $\{\|\mu(u_n)v_n\|_{L^p}\}$ is bounded and $C_0^\infty(\Omega)$ is dense in $L^q(\Omega)$, from above we get the asserted conclusion(cf. Theorem 3, ch. 5, in [32]). □

COROLLARY 4.3. *Let* Ω *be a bounded domain of* $\mathbb{R}^l$ *and* $\mu(\xi), \xi \in \mathbb{R}$, *be a positive bounded continuous function. If* $\{u_n\}$ *and* $\{v_n\}$ *are such that*

$$u_n(x) \to u(x) \text{ a.e. in } \Omega, \quad v_n \rightharpoonup v \text{ in } L^2(\Omega),$$

then

$$\big(\mu(u)v, v\big) \leq \liminf_{n\to\infty} \big(\mu(u_k)v_n, v_n\big).$$

Proof. By Corollary 4.2

$$\sqrt{\mu(u_n)}v_n \rightharpoonup \sqrt{\mu(u)}v \text{ in } L^2(\Omega).$$

Then,

$$\|\sqrt{\mu(u)}v\|_{L^2(\Omega)} \leq \liminf_{n\to\infty} \|\sqrt{\mu(u_n)}v_n\|_{L^2(\Omega)},$$

which implies the asserted conclusion. □

To prove the main results, we use the following

PROPOSITION 4.4. *Let $A : X \to X^*$ be an operator on the real reflexive Banach space X. Let A be coercive and bounded. If for every sequence such that*

$$\begin{aligned} &x_n \rightharpoonup x \text{ in } X, \\ &\limsup_{n\to\infty} \langle Ax_n, x_n - x\rangle \leq 0 \end{aligned} \tag{4.1}$$

there exists a subsequence such that

$$\liminf_{k\to\infty} \langle Ax_k, x_k - v\rangle \geq \langle Ax, x - v\rangle \quad \forall v \in X, \tag{4.2}$$

then for any $f \in X^*$ there exists a solution to

$$Au = f.$$

REMARK 4.1. *If* (4.1) *implies (not for a subsequence)*

$$\liminf_{n\to\infty} \langle Ax_n, x_n - v\rangle \geq \langle Ax, x - v\rangle \quad \forall v \in X,$$

then A is called pseudo-monotone. For the coercive, bounded and pseudo-monotone operators A the asserted conclusion was proved(cf. Theorem 27.A in [33] or Theorem 2.7, ch. 2 in [24]). However, proofs of the facts that A has property (M) (Proposition 2.5, ch.2 in [24]) and A is demicontinuous (footnote of (2.27) of ch.2. in [24]), which guarantee existence of a solution to $Au = f$, hold with (4.2) *for subsequence, and so we have the conclusion.*

For every $\varepsilon > 0$, define Φ_ε by

$$\Phi_\varepsilon(y) = \inf\left\{\frac{\|y-u\|^2_{\mathbf{V}}}{2\varepsilon} + \Phi(u);\ u \in \mathbf{V}\right\}, \quad y \in \mathbf{V},$$

which is called the Moreau regularization of Φ. When $\partial\Phi : \mathbf{V} \to 2^{\mathbf{V}}$ is the subdifferential of Φ, let $J_\varepsilon = (I + \varepsilon\partial\Phi)^{-1}$ and $(\partial\Phi)_\varepsilon := \varepsilon^{-1}(I - J_\varepsilon)$ (the Yosida approximation of $\partial\phi$) for all $\varepsilon > 0$. Then the functional Φ_ε is convex, continuous, Fréchet differentiable and $\nabla\Phi_\varepsilon = (\partial\Phi)_\varepsilon \equiv \varepsilon^{-1}(I - J_\varepsilon)$ for all $\varepsilon > 0$. Moreover

$$\Phi_\varepsilon(y) = \frac{\|y - J_\varepsilon y\|^2_{\mathbf{V}}}{2\varepsilon} + \Phi(J_\varepsilon y) \quad \forall y \in \mathbf{V}, \tag{4.3}$$

$$\lim_{\varepsilon\to 0} \Phi_\varepsilon(y) = \Phi(y), \quad \Phi(J_\varepsilon y) \leq \Phi_\varepsilon(y) \leq \Phi(y) \quad \forall y \in \mathbf{V} \tag{4.4}$$

(cf. Theorem 2.9 in [1]). The operator $\nabla\Phi_\varepsilon$ is Lipschitz continuous with the constant ε^{-1}(cf. Proposition 2.3 in [1]) and monotone(cf. Lemma 4.10 of ch. III in [17]).

4.1. Existence of a solution to an auxiliary problem. Since

$$|a_1(v,v,u)| = |((v\cdot\nabla)v,u)| \le K\|v\|_{\mathbf{V}}^2\|u\|_{\mathbf{V}} \quad \forall v,u\in\mathbf{V}, \tag{4.5}$$

define $\overline{a}_1(v)\in\mathbf{V}^*$ by

$$\langle \overline{a}_1(v),u\rangle = a_1(v,v,u) \quad \forall v,u\in\mathbf{V}.$$

Define $\gamma_\varepsilon(t)$ by

$$\gamma_\varepsilon(t) := \frac{\gamma(t)t}{(1+\varepsilon|\gamma(t)|)(1+\varepsilon|t|)} \quad t\in\mathbb{R},\ \varepsilon>0.$$

Then,

$$|\gamma_\varepsilon(t)|\le\frac{1}{\varepsilon^2},\quad |\gamma_\varepsilon(t)|\le|\gamma(t)||t|\le\gamma_0|t|,\quad \gamma_\varepsilon(t)\to\gamma(t)t \text{ as } \varepsilon\to 0. \tag{4.6}$$

We first consider an auxiliary problem:

Problem I-VIA. Let $\delta>0$, $\zeta>0$, $\lambda>0$, $\varepsilon>0$ and $q\in(\frac{12}{5},6)$. Find $(v,\theta)\in\mathbf{V}\times W^{1,2}(\Omega)$ such that $\eta=\theta-\theta_D\in W^{1,2}_{\Gamma_D}(\Omega)$ and

$$\begin{cases} a_0(\theta;v,u)+\dfrac{\delta}{\max\{\delta,\|\overline{a}_1(v)\|_{\mathbf{V}^*}\}}a_1(v,v,u)+\big\langle\nabla\Phi_\varepsilon(v),u\big\rangle \\ \qquad -\Big\langle\Big(1-\dfrac{\alpha_0\lambda}{\max\{\lambda,\|\theta\|^2_{L^q}\}}\theta\Big)f_\varepsilon,u\Big\rangle=\langle f_1,u\rangle \quad \forall u\in\mathbf{V}, \\ b_0(\theta;\theta,\varphi)-\dfrac{\zeta}{\max\{\zeta,\|v\|_{\mathbf{V}}\}}\langle\gamma_\varepsilon(\theta)v,\nabla\varphi\rangle-\Big\langle\alpha_2\mu(\theta)\dfrac{|\mathcal{E}(v)|^2}{1+\varepsilon|\mathcal{E}(v)|^2},\varphi\Big\rangle \\ \qquad -\Big\langle\dfrac{\alpha_1\lambda}{\max\{\lambda,\|\theta\|^2_{L^q}\}}\theta f_\varepsilon\cdot v,\varphi\Big\rangle=\langle f_2,\varphi\rangle \quad \forall\varphi\in W^{1,2}_{\Gamma_D}(\Omega), \end{cases} \tag{4.7}$$

where $f_\varepsilon\in\mathbf{L}^\infty(\Omega)$ is such that $\|f-f_\varepsilon\|_{\mathbf{L}^t}\le\varepsilon$.

THEOREM 4.5. *There exists a solution $(v_\varepsilon,\theta_\varepsilon)\in\boldsymbol{V}\times W^{1,2}(\Omega)$ to Problem I-VIA.*

Proof. Let r be such that

$$\frac{2}{q}+\frac{1}{r}=1. \tag{4.8}$$

Since $q\in(\frac{12}{5},6)$, we know that $\frac{3}{2}<r<6$. Let $\mathscr{H}=\mathbf{V}\times W^{1,2}_{\Gamma_D}(\Omega)$. Define an operator $\mathscr{A}:\mathscr{H}\to\mathscr{H}^*$ by

$$\begin{aligned} &\big\langle\mathscr{A}(v,\eta),(u,\phi)\big\rangle \\ &=a_0(\eta+\theta_D;v,u)+\frac{\delta}{\max\{\delta,\|\overline{a}_1(v)\|_{\mathbf{V}^*}\}}a_1(v,v,u)+\big\langle\nabla\Phi_\varepsilon(v),u\big\rangle \\ &-\Big\langle\Big(1-\frac{\alpha_0\lambda}{\max\{\lambda,\|\eta+\theta_D\|^2_{L^q}\}}(\eta+\theta_D)\Big)f_\varepsilon,u\Big\rangle+b_0(\eta+\theta_D;\eta+\theta_D,\phi) \\ &-\frac{\zeta}{\max\{\zeta,\|v\|_{\mathbf{V}}\}}\langle\gamma_\varepsilon(\eta+\theta_D)v,\nabla\phi\rangle-\Big\langle\alpha_2\mu(\eta+\theta_D)\frac{|\mathcal{E}(v)|^2}{1+\varepsilon|\mathcal{E}(v)|^2},\phi\Big\rangle \\ &-\Big\langle\frac{\alpha_1\lambda}{\max\{\lambda,\|\eta+\theta_D\|^2_{L^q}\}}(\eta+\theta_D)f_\varepsilon\cdot v,\phi\Big\rangle \quad \forall(v,\eta),(u,\phi)\in\mathscr{H}. \end{aligned} \tag{4.9}$$

Let us see this operator is well-defined. By definition of Φ_ε, $\Phi_\varepsilon(0_{\mathbf{V}}) = 0$ and $\nabla\Phi_\varepsilon(0_{\mathbf{V}}) = 0$. Since $\nabla\Phi_\varepsilon$ is Lipschitz continuous with the constant ε^{-1},

$$\big|\big\langle\nabla\Phi_\varepsilon(v), u\big\rangle\big| = \big|\big\langle\nabla\Phi_\varepsilon(v) - \nabla\Phi_\varepsilon(0_{\mathbf{V}}), u\big\rangle\big| \le \varepsilon^{-1}\|v\|_{\mathbf{V}}\|u\|_{\mathbf{V}}. \tag{4.10}$$

By (4.5) we have

$$\Big|\frac{\delta}{\max\{\delta, \|\overline{a}_1(v)\|_{\mathbf{V}^*}\}}a_1(v, v, u)\Big| \le \delta\|u\|_{\mathbf{V}}. \tag{4.11}$$

By (4.6) we have

$$\Big|\frac{\zeta}{\max\{\zeta, \|v\|_{\mathbf{V}}\}}\langle\gamma_\varepsilon(\eta+\theta_D)v, \nabla\phi\rangle\Big| \le \frac{c\zeta}{\varepsilon^2}\|\phi\|_{W^{1,2}}. \tag{4.12}$$

By inequality $ab \le \max\{a^2, b^2\}$,

$$\frac{\sqrt{\lambda}}{\max\{\lambda, \|\eta+\theta_D\|^2_{L^q}\}}\|\eta+\theta_D\|_{L^q} \le 1. \tag{4.13}$$

Taking into account (4.8) and (4.13), we have

$$\begin{aligned}
&\big|\langle\frac{\alpha_0\lambda}{\max\{\lambda, \|\eta+\theta_D\|^2_{L^q}\}}(\eta+\theta_D)f_\varepsilon, u\rangle\big| \\
&\quad\le \frac{\alpha_0\lambda}{\max\{\lambda, \|\eta+\theta_D\|^2_{L^q}\}}\|\eta+\theta_D\|_{L^q}\|f_\varepsilon\|_{\mathbf{L}^q}\|u\|_{\mathbf{L}^r} \\
&\quad\le c\alpha_0\sqrt{\lambda}\|f_\varepsilon\|_{\mathbf{L}^q}\|u\|_{\mathbf{L}^r}, \\
&\Big|\Big\langle\frac{\alpha_1\lambda}{\max\{\lambda, \|\eta+\theta_D\|^2_{L^q}\}}(\eta+\theta_D)f_\varepsilon\cdot v, \phi\Big\rangle\Big| \\
&\quad\le \frac{\alpha_1\lambda}{\max\{\lambda, \|\eta+\theta_D\|^2_{L^q}\}}\|\eta+\theta_D\|_{L^q}\|f_\varepsilon\|_{\mathbf{L}^\infty}\|v\|_{\mathbf{L}^r}\|\phi\|_{L^q} \\
&\quad\le c\alpha_1\sqrt{\lambda}\|f_\varepsilon\|_{\mathbf{L}^\infty}\|v\|_{\mathbf{L}^r}\|\phi\|_{L^q}.
\end{aligned} \tag{4.14}$$

Estimation of other terms is easy, and so operator $\mathscr{A}$ is well defined.

Then, the existence of a solution to Problem I-VIA is equivalent to the existence of a solution to

$$\mathscr{A}(v,\eta) = \mathscr{F}, \quad \mathscr{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Relying to Proposition 4.4, we will prove the existence of a solution to the equation above.

(i) Let us prove that $\mathscr{A}$ is coercive, i.e.,

$$\frac{1}{\|(v,\eta)\|_{\mathscr{H}}}\big\langle\mathscr{A}(v,\eta), (v,\eta)\big\rangle \to \infty \quad \text{as } \|(v,\eta)\|_{\mathscr{H}} \to \infty.$$

Since $\Gamma_{2j}, \Gamma_{3j}, \Gamma_{7j}$ are convex(cf. Lemma A.3 of [19]) and the matrix α is positive, from (3.6) we have

$$a_0(\eta+\theta_D; v, v) \ge 2\mu_0\|v\|^2_{\mathbf{V}}. \tag{4.15}$$

Taking into account (4.11), (4.12), (4.15) and the first formula of (4.14), we have

$$\begin{aligned}
&\big\langle \mathscr{A}(v,\eta),(v,\eta)\big\rangle = a_0(\eta+\theta_D;v,v)+\frac{\delta}{\max\{\delta,\|\overline{a}_1(v)\|_{\mathbf{V}^*}\}}a_1(v,v,v)+\big\langle \nabla\Phi_\varepsilon(v),v\big\rangle\\
&-\Big\langle\Big(1-\frac{\alpha_0\lambda}{\max\{\lambda,\|\eta+\theta_D\|^2_{L^q}\}}(\eta+\theta_D)\Big)f_\varepsilon,v\Big\rangle+b_0(\eta+\theta_D;\eta,\eta)+b_0(\eta+\theta_D;\theta_D,\eta)\\
&-\frac{\zeta}{\max\{\zeta,\|v\|_{\mathbf{V}}\}}\langle\gamma_\varepsilon(\eta+\theta_D)v,\nabla\eta\rangle-\Big\langle\alpha_2\mu(\eta+\theta_D)\frac{|\mathcal{E}(v)|^2}{1+\varepsilon|\mathcal{E}(v)|^2},\eta\Big\rangle\\
&-\Big\langle\frac{\alpha_1\lambda}{\max\{\lambda,\|\eta+\theta_D\|^2_{L^q}\}}(\eta+\theta_D)f_\varepsilon\cdot v,\eta\Big\rangle\\
&\geq\frac{3}{4}\min\{2\mu_0,\kappa_0\}\big(\|v\|^2_{\mathbf{V}}+\|\eta\|^2_{W^{1,2}_{\Gamma_D}}\big)-\delta\|v\|-c\big(\|f_\varepsilon\|^2_{\mathbf{L}^q}+\|\theta_D\|^2_{L^2}+\frac{1}{\varepsilon^2}\big)\\
&+\big\langle\nabla\Phi_\varepsilon(v),v\big\rangle-\frac{c\zeta}{\varepsilon^2}\|\eta\|_{W^{1,2}_{\Gamma_D}}-\Big\langle\frac{\alpha_1\lambda}{\max\{\lambda,\|\eta+\theta_D\|^2_{L^q}\}}(\eta+\theta_D)f_\varepsilon\cdot v,\eta\Big\rangle\quad\forall(v,\eta)\in\mathscr{H}.
\end{aligned}\tag{4.16}$$

Since the operator $\nabla\Phi_\varepsilon$ is monotone and $\nabla\Phi_\varepsilon(0_{\mathbf{V}})=0$, we have

$$\big\langle\nabla\Phi_\varepsilon(v),v\big\rangle=\big\langle\nabla\Phi_\varepsilon(v)-\nabla\Phi_\varepsilon(0_{\mathbf{V}}),v-0_{\mathbf{V}}\big\rangle\geq 0.\tag{4.17}$$

Since $\langle(\eta+\theta_D)f\cdot v,\eta\rangle=\langle(\eta+\theta_D)f\cdot v,(\eta+\theta_D)\rangle-\langle(\eta+\theta_D)f\cdot v,\theta_D\rangle$, by (4.8), (4.13) we have

$$\begin{aligned}
&\Big|\Big\langle\frac{\alpha_1\lambda}{\max\{\lambda,\|\eta+\theta_D\|^2_{L^q}\}}(\eta+\theta_D)f_\varepsilon\cdot v,\eta\Big\rangle\Big|\\
&\leq c\frac{\lambda\|\eta+\theta_D\|^2_{L^q}}{\max\{\lambda,\|\eta+\theta_D\|^2_{L^q}\}}\|f_\varepsilon\|_{\mathbf{L}^\infty}\|v\|_{\mathbf{L}^r}\\
&\quad+c\frac{\lambda\|\eta+\theta_D\|_{L^q}}{\max\{\lambda,\|\eta+\theta_D\|^2_{L^q}\}}\|f_\varepsilon\|_{\mathbf{L}^\infty}\|v\|_{\mathbf{L}^r}\|\theta_D\|_{L^q}\\
&\leq c\lambda\|f_\varepsilon\|_{\mathbf{L}^\infty}\|v\|_{\mathbf{L}^r}+c\sqrt{\lambda}\|f_\varepsilon\|_{\mathbf{L}^\infty}\|v\|_{\mathbf{L}^r}\|\theta_D\|_{L^q}.
\end{aligned}\tag{4.18}$$

By virtue of (4.16)-(4.18), it follows that

$$\begin{aligned}
\big\langle\mathscr{A}(v,\eta),(v,\eta)\big\rangle\geq{}&\frac{1}{4}\min\{2\mu_0,\kappa_0\}\Big(\|v\|^2_{\mathbf{V}}+\|\eta\|^2_{W^{1,2}_{\Gamma_D}}\Big)\\
&-\delta\|v\|-\frac{c\zeta}{\varepsilon^2}\|\eta\|_{W^{1,2}_{\Gamma_D}}-c\lambda\|f_\varepsilon\|_{\mathbf{L}^\infty}\|v\|_{\mathbf{V}}\\
&-c\sqrt{\lambda}\|f_\varepsilon\|_{\mathbf{L}^\infty}\|v\|_{\mathbf{V}}\|\theta_D\|_{L^q}\\
&-c\Big(\|f_\varepsilon\|^2_{\mathbf{L}^t}+\|\theta_D\|^2_{L^2}+\frac{1}{\varepsilon^2}\Big)\quad\forall(v,\eta)\in\mathscr{H},
\end{aligned}$$

which implies coercive property of $\mathscr{A}$.

(ii) Taking into account (4.10)-(4.14), we have from (4.9)

$$\begin{aligned}
&\|\mathscr{A}(v,\eta)\|_{\mathscr{H}^*}\\
&=\sup_{\|(u,\phi)\|_{\mathscr{H}}=1}\big\langle\mathscr{A}(v,\eta),(u,\phi)\big\rangle\\
&\leq c\Big(\|v\|_{\mathbf{V}}+\delta+\frac{c\zeta}{\varepsilon^2}+\frac{1}{\varepsilon}\|v\|_{\mathbf{V}}+\|f_\varepsilon\|_{\mathbf{L}^2}+\sqrt{\lambda}\|f_\varepsilon\|_{\mathbf{L}^q}+\|\eta+\theta_D\|_{W^{1,2}_{\Gamma_D}}\\
&\quad+\|v\|_{\mathbf{L}^2}+\frac{1}{\varepsilon}+\sqrt{\lambda}\|f_\varepsilon\|_{\mathbf{L}^\infty}\|v\|_{\mathbf{V}}\Big)\qquad\forall(v,\eta),(u,\phi)\in\mathscr{H},
\end{aligned}\tag{4.19}$$

which shows that $\mathscr{A}$ maps bounded sets of $\mathscr{H}$ into bounded sets of $\mathscr{H}^*$.

(iii) Let $\{(v_k, \eta_k)\}$ be a sequence such that

$$(v_k, \eta_k) \rightharpoonup (v, \eta) \text{ in } \mathscr{H},$$
$$\limsup_{k\to\infty} \langle \mathscr{A}(v_k, \eta_k), (v_k, \eta_k) - (v, \eta)\rangle \le 0.$$

By taking a subsequence and denoting with same subindex if necessary, we may assume

$$\begin{aligned} v_k \to v, \ \ \eta_k \to \eta \text{ in } \mathbf{L}^s(\Omega), L^s(\Omega)(1 \le s < 6) \text{ respectively,}\\ \text{and a.e. in } \Omega \text{ as } k \to \infty. \end{aligned} \tag{4.20}$$

Since

$$\begin{aligned} a_0(\eta_k + \theta_D; v_k - v, v_k - v) &= a_0(\eta_k + \theta_D; v_k, v_k - v) - a_0(\eta_k + \theta_D; v, v_k - v),\\ b_0(\eta_k + \theta_D; \eta_k - \eta, \eta_k - \eta) &= b_0(\eta_k + \theta_D; \eta_k + \theta_D, \eta_k - \eta)\\ &\quad - b_0(\eta_k + \theta_D; \eta + \theta_D, \eta_k - \eta), \end{aligned}$$

by (4.9) we have

$$\begin{aligned} &\min\{\mu_0, \kappa_0\}\big(\|v_k - v\|^2_{\mathbf{V}} + \|\eta_k - \eta\|_{W^{1,2}_{\Gamma_D}}\big)\\ &\le \langle \mathscr{A}(v_k, \eta_k), (v_k, \eta_k) - (v, \eta)\rangle\\ &\quad - a_0(\eta_k + \theta_D; v, v_k - v) - b_0(\eta_k + \theta_D; \eta + \theta_D, \eta_k - \eta)\\ &\quad - \frac{\delta}{\max\{\delta, \|\overline{a}_1(v_k)\|_{\mathbf{V}^*}\}} a_1(v_k, v_k, v_k - v) - \big\langle \nabla\Phi_\varepsilon(v_k), v_k - v\big\rangle\\ &\quad + \Big\langle \Big(1 - \frac{\alpha_0\lambda}{\max\{\lambda, \|\eta_k + \theta_D\|^2_{L^q}\}}(\eta_k + \theta_D)\Big) f_\varepsilon, v_k - v\Big\rangle\\ &\quad + \frac{\zeta}{\max\{\zeta, \|v_k\|_{\mathbf{V}}\}} \langle \gamma_\varepsilon(\eta_k + \theta_D) v_k, \nabla(\eta_k - \eta)\rangle\\ &\quad + \Big\langle \alpha_2\mu(\theta)\frac{|\mathcal{E}(v_k)|^2}{1 + \varepsilon|\mathcal{E}(v_k)|^2}, \eta_k - \eta\Big\rangle\\ &\quad + \Big\langle \frac{\alpha_1\lambda}{\max\{\lambda, \|\eta_k + \theta_D\|^2_{L^q}\}}(\eta_k + \theta_D) f_\varepsilon \cdot v_k, \eta_k - \eta\Big\rangle. \end{aligned} \tag{4.21}$$

By Corollary 4.2, we have

$$\begin{aligned} &a_0(\eta_k + \theta_D; v, v_k - v)\\ &= 2(\mu(\eta_k + \theta_D)\mathcal{E}(v), \mathcal{E}(v_k - v)) + 2(\mu(\eta_k + \theta_D)k(x)v, v_k - v)_{\Gamma_2}\\ &\quad + 2(\mu(\eta_k + \theta_D)S\tilde{v}, \tilde{v}_k - \tilde{v})_{\Gamma_3} + 2(\alpha(x)v, v_k - v)_{\Gamma_5}\\ &\quad + (\mu(\eta_k + \theta_D)k(x)v, v_k - v)_{\Gamma_7} \to 0,\\ &b_0(\eta_k + \theta_D; \eta + \theta_D, \eta_k - \eta)\\ &= \big(\kappa(\eta_k + \theta_D)\nabla(\eta + \theta_D), \nabla(\eta_k - \eta)\big) + \big(\beta(x)(\eta + \theta_D), (\eta_k - \eta)\big)_{\Gamma_R} \to 0 \end{aligned} \tag{4.22}$$

as $k \to \infty$. Also,

$$\begin{aligned} &\Big|\frac{\delta}{\max\{\delta, \|\overline{a}_1(v_k)\|_{\mathbf{V}^*}\}} a_1(v_k, v_k, v_k - v)\Big|\\ &\le \frac{\delta}{\max\{\delta, \|\overline{a}_1(v_k)\|_{\mathbf{V}^*}\}} \|v_k\|_{\mathbf{L}^4}\|\nabla v_k\|_{\mathbf{L}^2}\|v_k - v\|_{\mathbf{L}^4} \to 0 \quad \text{as } k \to \infty. \end{aligned} \tag{4.23}$$

Since $\nabla\Phi_\varepsilon$ is monotone,

$$\begin{aligned}-\big\langle\nabla\Phi_\varepsilon(v_k),v_k-v\big\rangle&=-\big\langle\nabla\Phi_\varepsilon(v_k)-\nabla\Phi_\varepsilon(v),v_k-v\big\rangle-\big\langle\nabla\Phi_\varepsilon(v),v_k-v\big\rangle\\&\le-\big\langle\nabla\Phi_\varepsilon(v),v_k-v\big\rangle\to0\quad\text{as } k\to\infty.\end{aligned}\tag{4.24}$$

By (4.13) and (4.20), the followings hold.

$$\begin{aligned}&\Big|\Big\langle\frac{\alpha_0\lambda}{\max\{\lambda,\|\eta_k+\theta_D\|^2_{L^q}\}}(\eta_k+\theta_D)f_\varepsilon,v_k-v\Big\rangle\Big|\le c\sqrt{\lambda}\|f_\varepsilon\|_{\mathbf{L}^\infty}\|v_k-v\|_{\mathbf{L}^3}\to0,\\&\Big|\Big\langle\frac{\alpha_1\lambda}{\max\{\lambda,\|\eta_k+\theta_D\|^2_{L^q}\}}(\eta_k+\theta_D)f_\varepsilon\cdot v_k,\eta_k-\eta\Big\rangle\Big|\\&\le c\sqrt{\lambda}\|f_\varepsilon\|_{\mathbf{L}^\infty}\|v_k\|_{\mathbf{L}^6}\|\eta_k-\eta\|_{L^r}\to0,\\&|\frac{\zeta}{\max\{\zeta,\|v_k\|_{\mathbf{V}}\}}\langle\gamma_\varepsilon(\eta_k+\theta_D)v_k,\nabla(\eta_k-\eta)\rangle|\\&\le|\langle\big(\gamma_\varepsilon(\eta_k+\theta_D)-\gamma_\varepsilon(\eta+\theta_D)\big)v_k,\nabla(\eta_k-\eta)\rangle|\\&\quad+|\langle\gamma_\varepsilon(\eta+\theta_D)(v_k-v),\nabla(\eta_k-\eta)\rangle|+|\langle\gamma_\varepsilon(\eta+\theta_D)v,\nabla(\eta_k-\eta)\rangle|\\&\le c\|\gamma_\varepsilon(\eta_k+\theta_D)-\gamma_\varepsilon(\eta+\theta_D)\|_{L^3}\|v_k\|_{\mathbf{V}}\|\eta_k-\eta\|_{W^{1,2}_{\Gamma_D}}\\&\quad+c\|\gamma_\varepsilon(\eta+\theta_D)\|_{L^6}\|(v_k-v)\|_{\mathbf{L}^3}\|\eta_k-\eta\|_{W^{1,2}_{\Gamma_D}}\\&\quad+\langle\gamma_\varepsilon(\eta+\theta_D)v,\nabla(\eta_k-\eta)\rangle\to0\end{aligned}\tag{4.25}$$

as $k\to\infty$, where the fact that by Lemma 4.1 $\gamma_\varepsilon(\eta_k+\theta_D)\to\gamma_\varepsilon(\eta+\theta_D)$ in $L^3(\Omega)$ as $k\to\infty$ was used. It is easy to prove convergence of other terms on the right hand side of (4.21). Thus, by (4.20)-(4.25) we have

$$\begin{aligned}&\limsup_{k\to\infty}\min\{\mu_0,\kappa_0\}|\big(\|v_k-v\|^2_{\mathbf{V}}+\|\eta_k-\eta\|_{W^{1,2}_{\Gamma_D}}\big)\\&\le\limsup_{k\to\infty}\langle\mathscr{A}(v_k,\eta_k),(v_k,\eta_k)-(v,\eta)\rangle\le0,\end{aligned}$$

which implies

$$\begin{aligned}&(v_k,\eta_k)\to(v,\eta)\text{ in }\mathscr{H}\ \text{ as } k\to\infty,\\&v_k\to v,\ \ \eta_k\to\eta\text{ a.e. in }\Omega\ \text{ as } k\to\infty.\end{aligned}\tag{4.26}$$

By the definition of $\mathscr{A}$, for $(u,\phi)\in\mathscr{H}$

$$\begin{aligned}&\langle\mathscr{A}(v_k,\eta_k),\ (v_k,\eta_k)-(u,\phi)\rangle\\&=a_0(\eta_k+\theta_D;v_k,v_k-u)+\frac{\delta}{\max\{\delta,\|\overline{a}_1(v_k)\|_{\mathbf{V}^*}\}}a_1(v_k,v_k,v_k-u)\\&\quad+\big\langle\nabla\Phi_\varepsilon(v_k),v_k-u\big\rangle-\Big\langle\Big(1-\frac{\alpha_0\lambda}{\max\{\lambda,\|\eta_k+\theta_D\|^2_{L^q}\}}(\eta_k+\theta_D)\Big)f,v_k-u\Big\rangle\\&\quad+b_0(\eta_k+\theta_D,\eta_k-\phi)+\langle\frac{\zeta}{\max\{\zeta,\|v_k\|_{\mathbf{V}}\}}\gamma(\eta_k+\theta_D)v_k,\nabla(\eta_k-\phi)\rangle\\&\quad-\Big\langle\alpha_2\mu(\theta_k)\frac{|\mathcal{E}(v_k)|^2}{1+\varepsilon|\mathcal{E}(v_k)|^2},\eta_k-\phi\Big\rangle\\&\quad-\Big\langle\frac{\alpha_1\lambda}{\max\{\lambda,\|\eta_k+\theta_D\|^2_{L^q}\}}(\eta_k+\theta_D)f_\varepsilon\cdot v_k,\eta_k-\phi\Big\rangle.\end{aligned}\tag{4.27}$$

Taking into account (4.26), by Corollary 4.3 we have

$$\begin{aligned}&\liminf_{k\to\infty} a_0(\eta_k+\theta_D; v_k, v_k) \geq a_0(\eta+\theta_D; v, v),\\ &\liminf_{k\to\infty} b_0(\eta_k+\theta_D; \eta_k, \eta_k) \geq b_0(\eta+\theta_D; \eta, \eta).\end{aligned} \tag{4.28}$$

The sequence $\{\frac{\alpha_0\lambda}{\max\{\lambda,\|\eta_k+\theta_D\|^2_{L^q}\}}(\eta_k+\theta_D)\}$ converges a.e. in Ω to $\frac{\alpha_0\lambda}{\max\{\lambda,\|\eta+\theta_D\|^2_{L^q}\}}(\eta+\theta_D)$ and $\big\|\frac{\alpha_0\lambda}{\max\{\lambda,\|\eta_k+\theta_D\|^2_{L^q}\}}(\eta_k+\theta_D)\big\|_{L^q} \leq \alpha_0\sqrt{\lambda}$, and so this sequence weakly converges in L^q(cf. Lemma 1.3, ch. 1 in [24]). Then by virtue of this fact and the first formula of (4.25), we have

$$\begin{aligned}&\lim_{k\to\infty}\Big\langle\frac{\alpha_0\lambda}{\max\{\lambda,\|\eta_k+\theta_D\|^2_{L^q}\}}(\eta_k+\theta_D)f_\varepsilon, v_k-u\Big\rangle\\ &=\lim_{k\to\infty}\Big\langle\frac{\alpha_0\lambda}{\max\{\lambda,\|\eta_k+\theta_D\|^2_{L^q}\}}(\eta_k+\theta_D)f_\varepsilon, v_k-v\Big\rangle\\ &\quad+\lim_{k\to\infty}\Big\langle\frac{\alpha_0\lambda}{\max\{\lambda,\|\eta_k+\theta_D\|^2_{L^q}\}}(\eta_k+\theta_D)f_\varepsilon, v-u\Big\rangle\\ &=\Big\langle\frac{\alpha_0\lambda}{\max\{\lambda,\|\eta+\theta_D\|^2_{L^q}\}}(\eta+\theta_D)f_\varepsilon, v-u\Big\rangle,\end{aligned} \tag{4.29}$$

where the fact that owing to (4.8) $f_\varepsilon\cdot(v-u)\in L^3\subset L^{q'}$, $\frac{1}{q}+\frac{1}{q'}=1$ was used. Using the second formula in (4.25), in the same way we get

$$\begin{aligned}&\lim_{k\to\infty}\Big\langle\frac{\alpha_1\lambda}{\max\{\lambda,\|\eta_k+\theta_D\|^2_{L^q}\}}(\eta_k+\theta_D)f_\varepsilon\cdot v_k, \eta_k-\phi\Big\rangle\\ &=\Big\langle\frac{\alpha_1\lambda}{\max\{\lambda,\|\eta+\theta_D\|^2_{L^q}\}}(\eta+\theta_D)f_\varepsilon\cdot v, \eta-\phi\Big\rangle.\end{aligned} \tag{4.30}$$

It is easy to prove convergence of other terms in the right hand side of (4.27). Thus, by (4.28)-(4.30) we have existence of a subsequence $\{(v_k,\eta_k)\}$ such that

$$\liminf_{k\to\infty}\langle\mathscr{A}(v_k,\eta_k),(v_k,\eta_k)-(u,\phi)\rangle \geq \langle\mathscr{A}(v,\eta),(v,\eta)-(u,\phi)\rangle\,.$$

Therefore, by virtue of Proposition 4.4 we come to the conclusion. □

4.2. A priori estimates of solutions to the auxiliary problem. Let us choose q_0 such that

$$\frac{1}{q_0}+\frac{1}{t}+\frac{1}{3}\leq 1,\quad \frac{12}{5}<q_0<3. \tag{4.31}$$

Since $t>3$(cf. 3) of Assumption 2.1), such a choice is possible. Then, q_0 satisfies the condition for q in Problem I-VIA.

LEMMA 4.6. *If $v\in\boldsymbol{V}$ and $|\alpha_1|\sqrt{\lambda}\leq 1$, then for all θ_ε satisfying the second formula of* (4.7) *and the condition $\theta_\varepsilon-\theta_D\in W^{1,2}_{\Gamma_D}(\Omega)$ the following estimate holds.*

$$\|\theta^-_\varepsilon(x)\|_{W^{1,2}_{\Gamma_D}(\Omega)}\leq c\Big(\|f_\varepsilon\|_{\boldsymbol{L}^t}\|v\|_{\boldsymbol{V}}+\|g_R\|_{L^{4/3}(\Gamma_R)}+\|g\|_{L^{6/5}(\Omega)}\Big), \tag{4.32}$$

where constant c is independent of ε.

Proof. Since $\theta_\varepsilon|_{\Gamma_D} = \theta_D|_{\Gamma_D} \geq 0$, a function $\varphi = \min\{\theta_\varepsilon, 0\}$ is admissible in the second formula of (4.7) and we have

$$\begin{aligned}&(\kappa(\theta_\varepsilon)\nabla\theta_\varepsilon, \nabla\theta_\varepsilon^-) + (\beta(x)\theta_\varepsilon, \theta_\varepsilon^-)_{\Gamma_R}\\&- \frac{\zeta}{\max\{\zeta, \|v\|_{\mathbf{V}}\}}\langle\gamma_\varepsilon(\theta_\varepsilon)v, \nabla\theta_\varepsilon^-\rangle - \Big\langle \alpha_2\mu(\theta)\frac{|\mathcal{E}(v)|^2}{1+\varepsilon|\mathcal{E}(v)|^2}, \theta_\varepsilon^-\Big\rangle\\&- \Big\langle \frac{\alpha_1\lambda}{\max\{\lambda, \|\theta_\varepsilon\|^2_{L^{q_0}}\}}\theta_\varepsilon f_\varepsilon\cdot v, \theta_\varepsilon^-\Big\rangle = \langle f_2, \theta_\varepsilon^-\rangle.\end{aligned} \tag{4.33}$$

Let us prove

$$\langle\gamma_\varepsilon(\theta_\varepsilon)v, \nabla\theta_\varepsilon^-\rangle = 0. \tag{4.34}$$

To this end, define

$$\Psi(t) := \int_0^t \gamma_\varepsilon(s)\,ds, \quad t\in\mathbb{R}.$$

Then, $\Psi \in C^1(\mathbb{R})$ and

$$\begin{aligned}&\nabla\Psi(\theta) = \gamma_\varepsilon(\theta)\nabla\theta,\ \ \Psi(\theta)\in W^{1,2}(\Omega) \quad \forall\theta\in W^{1,2}(\Omega),\\&\Psi(\theta)|_{\Gamma_D} = 0 \quad \forall\theta\in W^{1,2}_{\Gamma_D}(\Omega).\end{aligned} \tag{4.35}$$

Taking into account $\nabla\theta_\varepsilon^- = 0$ on $\Omega^+ = \{x : \theta(x)\geq 0\}$ and $v\cdot n|_{\Gamma_R} = 0$, by (4.35) we have

$$\langle\gamma_\varepsilon(\theta)v, \nabla\theta_\varepsilon^-\rangle = \int_\Omega \gamma_\varepsilon(\theta_\varepsilon^-)v\cdot\nabla\theta_\varepsilon^-\,dx = \int_\Omega v\cdot\nabla\Psi(\theta_\varepsilon^-) = 0,$$

which means (4.34).

Also,

$$\begin{aligned}&-\Big\langle \alpha_2\mu(\theta)\frac{|\mathcal{E}(v)|^2}{1+\varepsilon|\mathcal{E}(v)|^2}, \theta_\varepsilon^-\Big\rangle \geq 0,\\&\Big|\Big\langle\frac{\alpha_1\lambda}{\max\{\lambda, \|\theta_\varepsilon\|^2_{L^{q_0}}\}}\theta_\varepsilon f_\varepsilon\cdot v, \theta_\varepsilon^-\Big\rangle\Big|\\&\quad \leq c\|f_\varepsilon\|_{\mathbf{L}^t}\|v\|_{\mathbf{V}}\|\theta_\varepsilon^-\|_{W^{1,2}_{\Gamma_D}} \leq \frac{\kappa_0}{4}\|\theta_\varepsilon^-\|^2_{W^{1,2}_{\Gamma_D}} + c\|f_\varepsilon\|^2_{\mathbf{L}^t}\|v\|^2_{\mathbf{V}},\\&\big|\langle f_2, \theta_\varepsilon^-\rangle\big| \leq \frac{\kappa_0}{4}\|\theta_\varepsilon^-\|^2_{H^1} + c\|f_2\|^2_{(W^{1,2}_{\Gamma_D})^*},\\&(\beta(x)\theta_\varepsilon, \theta_\varepsilon^-)_{\Gamma_R}\geq 0,\end{aligned} \tag{4.36}$$

where to get the second inequality (4.13) and (4.31) were used. By (4.33)-(4.36) we have

$$\|\theta_\varepsilon^-\|^2_{W^{1,2}_{\Gamma_D}} \leq \frac{2c}{\kappa_0}\Big(\|f_\varepsilon\|^2_{\mathbf{L}^t}\|v\|^2_{\mathbf{V}} + \|f_2\|^2_{(W^{1,2}_{\Gamma_D})^*}\Big),$$

which implies (4.32). □

LEMMA 4.7. *If* $|\alpha_0|\sqrt{\lambda}\leq 1$ *and*

$$(\|f\|_{\boldsymbol{L}^t} + \|f_1\|_{\boldsymbol{V}^*}) < \frac{\mu_0^2}{Kc_0}, \tag{4.37}$$

where K is the one in (4.5) *and c_0 is the one in* (4.42) *below, then there exist parameters δ and ζ such that*

$$\frac{\delta}{\max\{\delta, \|\overline{a}_1(v_\varepsilon)\|_{\boldsymbol{V}^*}\}} = 1, \quad \frac{\zeta}{\max\{\zeta, \|v_\varepsilon\|_{\boldsymbol{V}}\}} = 1 \tag{4.38}$$

for all small ε and solutions of (4.7) *v_ε. In addition,*

$$\|v_\varepsilon\|_{\boldsymbol{V}} \le \frac{\mu_0}{K}. \tag{4.39}$$

Proof. Since $|\alpha_0|\sqrt{\lambda} \le 1$, we have

$$\begin{aligned} &\Big|\langle \frac{\alpha_0\lambda}{\max\{\lambda, \|\theta_\varepsilon\|^2_{L^{q_0}}\}}\theta_\varepsilon f_\varepsilon, u\rangle\Big| \\ &\le \frac{c\sqrt{\lambda}}{\max\{\lambda, \|\theta_\varepsilon\|^2_{L^{q_0}}\}}\|\theta_\varepsilon\|_{L^{q_0}}\|f_\varepsilon\|_{\mathbf{L}^t}\|u\|_{\mathbf{L}^6} \le c\|f_\varepsilon\|_{\mathbf{L}^t}\|u\|_{\mathbf{V}}. \end{aligned} \tag{4.40}$$

Putting $u = v_\varepsilon$ in the first equation of (4.7), we have

$$\begin{aligned} &a_0(\theta_\varepsilon; v_\varepsilon, v_\varepsilon) + \frac{\delta}{\max\{\delta, \|\overline{a}_1(v_\varepsilon)\|_{\mathbf{V}^*}\}}a_1(v_\varepsilon, v_\varepsilon, v_\varepsilon) + \big\langle \nabla\Phi_\varepsilon(v_\varepsilon), v_\varepsilon\big\rangle \\ &- \Big\langle\Big(1 - \frac{\alpha_0\lambda}{\max\{\lambda, \|\theta_\varepsilon\|^2_{L^{q_0}}\}}\theta_\varepsilon\Big) f_\varepsilon, v_\varepsilon\Big\rangle = \langle f_1, v_\varepsilon\rangle. \end{aligned} \tag{4.41}$$

There exists a constant c_0' such that

$$\|f_\varepsilon\|_{\mathbf{L}^{\frac{6}{5}}} + c\|f_\varepsilon\|_{\mathbf{L}^t} \le c_0'\|f_\varepsilon\|_{\mathbf{L}^t},$$

where c is the one in (4.40). Thus taking into account (4.5), (4.15), (4.17) and (4.40), we have from (4.41)

$$\begin{aligned} 2\mu_0\|v_\varepsilon\|^2_{\mathbf{V}} &\le a_0(\theta_\varepsilon; v_\varepsilon, v_\varepsilon) \\ &\le \frac{\delta}{\max\{\delta, \|\overline{a}_1(v_\varepsilon)\|_{\mathbf{V}^*}\}}|a_1(v_\varepsilon, v_\varepsilon, v_\varepsilon)| \\ &\quad + \Big|\Big\langle\Big(1 - \frac{\alpha_0\lambda}{\max\{\lambda, \|\theta_\varepsilon\|^2_{L^{q_0}}\}}\theta_\varepsilon\Big) f_\varepsilon, v_\varepsilon\Big\rangle\Big| + |\langle f_1, v_\varepsilon\rangle| \\ &\le K\|v_\varepsilon\|^3_{\mathbf{V}} + \big(c_0'\|f_\varepsilon\|_{\mathbf{L}^t} + \|f_1\|_{\mathbf{V}^*}\big)\|v_\varepsilon\|_{\mathbf{V}} \\ &\le K\|v_\varepsilon\|^3_{\mathbf{V}} + c_0(\|f_\varepsilon\|_{\mathbf{L}^t} + \|f_1\|_{\mathbf{V}^*})\|v_\varepsilon\|_{\mathbf{V}}, \end{aligned}$$

where

$$c_0 = \max\{c_0', 1\}. \tag{4.42}$$

Note that the estimate above is independent of δ. This implies

$$0 \le K\|v_\varepsilon\|^2_{\mathbf{V}} - 2\mu_0\|v_\varepsilon\|_{\mathbf{V}} + c_0(\|f_\varepsilon\|_{\mathbf{L}^t} + \|f_1\|_{\mathbf{V}^*}).$$

Let us consider a quadric equation for $x > 0$ concerned with the inequality above

$$Kx^2 - 2\mu_0 x + a = 0.$$

If $0 \le Ka \le \mu_0^2$, then there exists a positive minimum root $x_1 (\le \frac{\mu_0}{K})$ and a maximum root x_2. Thus, we can know that if

$$(\|f_\varepsilon\|_{\mathbf{L}^t} + \|f_1\|_{\mathbf{V}^*}) \le \frac{\mu_0^2}{Kc_0},$$

then

$$\|v_\varepsilon\|_{\mathbf{V}} \le \frac{\mu_0}{K} \quad \text{or} \quad \|v_\varepsilon\|_{\mathbf{V}} \ge x_2. \tag{4.43}$$

On the other hand, we have from (4.41) another estimate under consideration of δ

$$2\mu_0 \|v_\varepsilon\|_{\mathbf{V}}^2 \le a_0(\theta_\varepsilon; v_\varepsilon, v_\varepsilon) \le \delta \|v_\varepsilon\|_{\mathbf{V}} + c_0(\|f_\varepsilon\|_{\mathbf{L}^t} + \|f_1\|_{\mathbf{V}^*})\|v_\varepsilon\|_{\mathbf{V}},$$

which implies

$$\|v_\varepsilon\|_{\mathbf{V}} \le \frac{1}{2\mu_0}\big(\delta + c_0(\|f_\varepsilon\|_{\mathbf{L}^t} + \|f_1\|_{\mathbf{V}^*})\big). \tag{4.44}$$

In view of (4.43), let us take $\delta = K\big(\frac{\mu_0}{K}\big)^2 = \frac{\mu_0^2}{K}$. If

$$(\|f\|_{\mathbf{L}^t} + \|f_1\|_{\mathbf{V}^*}) < \frac{\mu_0^2}{Kc_0},$$

then for all ε small enough

$$(\|f_\varepsilon\|_{\mathbf{L}^t} + \|f_1\|_{\mathbf{V}^*}) \le \frac{\mu_0^2}{Kc_0}.$$

Thus, without loss of generality from (4.44) we have

$$\|v_\varepsilon\|_{\mathbf{V}} \le \frac{\delta}{2\mu_0} + \frac{1}{2\mu_0}\frac{\mu_0^2}{K} = \frac{\mu_0}{K}. \tag{4.45}$$

By (4.45) under the condition (4.37) $\|\overline{a}_1(v_\varepsilon)\|_{\mathbf{V}^*} \le K\|v_\varepsilon\|_{\mathbf{V}}^2 \le \frac{\mu_0^2}{K}$(cf. (4.5)), and so we get the first one in (4.38). Taking $\zeta = \frac{\mu_0}{K}$, we get the second one in (4.38). □

Lemma 4.8. *If* $\max\{|\alpha_0|, |\alpha_1|\}\sqrt{\lambda} \le 1$ *and* (4.37) *holds, then under the parameter* ζ *by Lemma 4.7 there exists a* λ_1 *independent of* ε *such that*

$$\|\theta_\varepsilon\|_{L^{q_0}} \le \sqrt{\lambda_1}. \tag{4.46}$$

In addition, if $1 < r < \frac{3}{2}$, *then*

$$\int_\Omega |\nabla\theta_\varepsilon|^r \, dx \le cK_\sigma^{r/(1-\sigma)}\big(1 + \|\theta_D\|_{W^{1,2}}^2\big), \quad \sigma = \frac{3-2r}{3-r}, \tag{4.47}$$

where K_σ *is the one in* (4.77) *below.*

Proof. Using Lemmas 4.6, 4.7 and following the idea in [27], we will obtain the conclusion. For simplicity of notation, from now on in this lemma we write $v_\varepsilon, \theta_\varepsilon$ by v, θ.

Put

$$d_0 := \|\theta_D\|_{L^\infty(\Omega)} \ge \|\theta_D\|_{L^\infty(\partial\Omega)}. \tag{4.48}$$

Then, since $(\theta - d_0)^+ \equiv \max\{0, \theta - d_0\} = 0$ on Γ_D, the function

$$\varphi = 1 - \frac{1}{(1 + (\theta - d_0)^+)^\sigma}, \quad \sigma > 0, \tag{4.49}$$

belongs to $W^{1,2}_{\Gamma_D}(\Omega)$ and $0 \le \varphi \le 1$ a.e. in Ω.

Taking φ of (4.49) and ζ satisfying (4.38), we have from the second equation of (4.7)

$$\begin{aligned}
&\sigma \int_\Omega \kappa(\theta) \frac{|\nabla(\theta - d_0)^+|^2}{(1 + (\theta - d_0)^+)^{1+\sigma}}\, dx + (\beta\theta, \varphi)_{\Gamma_R} - \sigma \int_\Omega \gamma_\varepsilon(\theta) v \cdot \frac{\nabla(\theta - d_0)^+}{(1 + (\theta - d_0)^+)^{1+\sigma}}\, dx \\
&= \int_\Omega \alpha_2 \mu(\theta) \frac{|\mathcal{E}(v)|^2}{1 + \varepsilon|\mathcal{E}(v)|^2} \Big(1 - \frac{1}{(1 + (\theta - d_0)^+)^\sigma}\Big)\, dx \\
&\quad + \frac{\alpha_1 \lambda}{\max\{\lambda, \|\theta\|^2_{L^{q_0}}\}} \int_\Omega \theta f_\varepsilon \cdot v \Big(1 - \frac{1}{(1 + (\theta - d_0)^+)^\sigma}\Big)\, dx \\
&\quad + \Big\langle f_2, \Big(1 - \frac{1}{(1 + (\theta - d_0)^+)^\sigma}\Big)\Big\rangle \\
&\equiv I_1 + I_2 + I_3.
\end{aligned} \tag{4.50}$$

Note that $\varphi(x) = 0$ at x such that $\theta(x) \le d_0$, and $(\beta(x)\theta, \varphi)_{\Gamma_R} \ge 0$. Let us show the third term on the left hand side in (4.50) vanishes. To this end, define

$$\Psi_\sigma(t) := \int_0^t \frac{\gamma_\varepsilon(s + d_0)}{(1 + s)^{1+\sigma}}\, ds, \quad t \ge 0.$$

Then, $\Psi_\sigma \in C^1(\mathbb{R})$ and

$$\begin{aligned}
&\Psi_\sigma((\theta - d_0)^+)|_{\Gamma_D} = 0, \\
&\nabla \Psi_\sigma((\theta - d_0)^+) = \gamma_\varepsilon(\theta) \frac{\nabla(\theta - d_0)^+}{(1 + (\theta - d_0)^+)^{1+\sigma}},
\end{aligned} \tag{4.51}$$

where the fact that if $\theta(x) - d_0 \ge 0$, then $(\theta(x) - d_0)^+ + d_0 = \theta(x)$ was used. Taking into account that $v_n = 0$ on Γ_R(cf. (2.4)) and the first equality of (4.51), we have

$$\int_\Omega \gamma_\varepsilon(\theta) v \cdot \frac{\nabla(\theta - d_0)^+}{(1 + (\theta - d_0)^+)^{1+\sigma}}\, dx = \int_\Omega v \cdot \nabla \Psi_\sigma((\theta - d_0)^+) = 0. \tag{4.52}$$

It is easily seen that

$$|I_1| \le c\|v\|^2_{\mathbf{V}}. \tag{4.53}$$

Since $\max\{|\alpha_0|, |\alpha_1|\}\sqrt{\lambda} \le 1$, by (4.13) we have

$$|I_2| \le \frac{\sqrt{\lambda}}{\max\{\lambda, \|\theta\|^2_{L^{q_0}}\}} \|\theta\|_{q_0} \|f_\varepsilon\|_{\mathbf{L}^t} \|v\|_{L^3} \le c\|f_\varepsilon\|_{\mathbf{L}^t} \|v\|_{\mathbf{V}}. \tag{4.54}$$

Also,

$$|I_3| \le c\sqrt{\mathrm{mes}\Omega}\|f_2\|_{(W^{1,2}_{\Gamma_D})^*} \le c\big(\|g_R\|_{L^{4/3}(\Gamma_R)} + \|g\|_{L^{6/5}(\Omega)}\big). \tag{4.55}$$

By (4.52)-(4.55), we have from (4.50)

$$\sigma \int_{\{x;\theta\geq d_0\}} \frac{|\nabla\theta|^2}{(1+(\theta-d_0)^+)^{1+\sigma}}\,dx \leq c\Big(\|v\|_{\mathbf{V}}^2 + \|f_\varepsilon\|_{\mathbf{L}^t}\|v\|_{\mathbf{V}} + \|g_R\|_{L^{4/3}(\Gamma_R)} + \|g\|_{L^{6/5}(\Omega)}\Big). \tag{4.56}$$

Next, taking $\varphi = \min\{\theta-\theta_D, d_0\}$ admissible in the second formula of (4.7), we have

$$\begin{aligned}
&\kappa_0 \int_{\{x;\theta-\theta_D<d_0\}} |\nabla\theta|^2 dx \\
&\leq \int_{\{x;\theta-\theta_D<d_0\}} \kappa(x)\nabla\theta\cdot\nabla\theta_D\,dx - \int_{\Gamma_R}\beta(x)\theta(x)\min\{\theta(x)-\theta_D(x),d_0\}\,dx \\
&\quad + \int_\Omega \gamma_\varepsilon(\theta)v\cdot\nabla(\min\{\theta-\theta_D,d_0\})\,dx \\
&\quad + \int_\Omega \alpha_2\mu(\theta)\frac{|\mathcal{E}(v)|^2}{1+\varepsilon|\mathcal{E}(v)|^2}\min\{\theta-\theta_D,d_0\}\,dx \\
&\quad + \frac{\alpha_1\lambda}{\max\{\lambda,\|\theta\|_{L^{q_0}}^2\}}\int_\Omega \theta f_\varepsilon\cdot v\min\{\theta-\theta_D,d_0\}\,dx + \langle f_2, \min\{\theta-\theta_D,d_0\}\rangle \\
&\equiv \overline{I}_1+\overline{I}_2+\overline{I}_3+\overline{I}_4+\overline{I}_5+\overline{I}_6.
\end{aligned} \tag{4.57}$$

By Young's inequality,

$$|\overline{I}_1| \leq \frac{\kappa_0}{4}\int_{\{x;\theta-\theta_D<d_0\}}|\nabla\theta|^2\,dx + \frac{\kappa_1^2}{\kappa_0}\int_\Omega|\nabla\theta_D|^2\,dx. \tag{4.58}$$

Let us estimate $\overline{I}_2 = -\int_{\Gamma_R}\beta(x)\theta(x)\min\{\theta(x)-\theta_D(x),d_0\}\,dx$.
If $\theta(x)-\theta_D(x)\geq d_0$, then $\varphi(x)=d_0$ and $\theta(x)\geq\theta_D(x)+d_0\geq 0$. Thus

$$\int_{\Gamma_R\cap\{x;\theta(x)-\theta_D(x)\geq d_0\}}\beta(x)\theta(x)\min\{\theta(x)-\theta_D(x),d_0\}\,dx \geq 0. \tag{4.59}$$

If $\theta(x)-\theta_D(x)\leq d_0$, then $\varphi(x)=\theta(x)-\theta_D(x)$ and $\theta(x)\leq\theta_D(x)+d_0$. Thus

$$\begin{aligned}
&\int_{\Gamma_R\cap\{x;\theta(x)-\theta_D(x)\leq d_0\}}\beta(x)\theta(x)\min\{\theta(x)-\theta_D(x),d_0\}\,dx \\
&= \int_{\Gamma_R\cap\{x;\theta(x)\leq\theta_D(x)+d_0\}}\beta(x)\theta(x)\theta(x)\,dx \\
&\quad - \int_{\Gamma_R\cap\{x;\theta(x)\leq\theta_D(x)+d_0\}}\beta(x)\theta(x)\theta_D(x)\,dx \\
&\geq -\int_{\Gamma_R\cap\{x;0\leq\theta(x)\leq\theta_D(x)+d_0\}}\beta(x)\theta(x)\theta_D(x)\,dx \\
&\quad - \int_{\Gamma_R\cap\{x;\theta(x)\leq 0\}}\beta(x)\theta(x)\theta_D(x)\,dx \\
&\geq -\beta_0 2d_0^2\int_{\Gamma_R}dx - \beta_0 d_0\int_{\Gamma_R}|\theta^-(x)|\,dx \\
&\geq -c\big(1+\|\theta^-(x)\|_{W^{1,2}_{\Gamma_D}(\Omega)}\big).
\end{aligned} \tag{4.60}$$

By (4.59) and (4.60), we get

$$\overline{I}_2 \le c\big(1 + \|\theta^-(x)\|_{W^{1,2}_{\Gamma_D}(\Omega)}\big). \tag{4.61}$$

Taking into account (4.6), we have

$$\begin{aligned}\overline{I}_3 &= \int_{\{x;\theta-\theta_D\le d_0\}} \gamma_\varepsilon(\theta) v\cdot\nabla\theta\,dx - \int_{\{x;\theta-\theta_D\le d_0\}} \gamma_\varepsilon(\theta) v\cdot\nabla\theta_D\,dx \\ &\le \frac{\kappa_0}{4}\int_{\{x;\theta-\theta_D\le d_0\}} |\nabla\theta|^2\,dx + c\gamma_0^2\int_{\{x;0\le\theta\le d_0+\theta_D\}} |\theta|^2|v|^2\,dx \\ &\quad + c\gamma_0^2\int_{\{x;\theta\le 0\}} |\theta|^2|v|^2\,d + \gamma_0\int_{\{x;0\le\theta\le d_0+\theta_D\}} |\theta||v\cdot\nabla\theta_D|\,dx \\ &\quad + \gamma_0\int_{\{x;\theta\le 0\}} |\theta||v\cdot\nabla\theta_D|\,dx.\end{aligned} \tag{4.62}$$

Since $|\theta| \le 2d_0$ on $\{x; 0\le\theta\le d_0+\theta_D\}$,

$$\int_{\{x;0\le\theta\le d_0+\theta_D\}} \theta^2|v|^2\,dx \le 4d_0^2\int_\Omega |v|^2\,dx \le c\|v\|^2_{\mathbf{V}}. \tag{4.63}$$

And, by Hölder's inequality we obtain

$$\begin{aligned}&\int_{\{x;\theta\le 0\}} \theta^2|v|^2\,dx \\ &\le \Big(\int_{\{x;\theta\le 0\}} \theta^4\,dx\Big)^{1/2}\Big(\int_{\{x;\theta\le 0\}} |v|^4\,dx\Big)^{1/2} \le c\|\theta^-\|^2_{W^{1,2}_{\Gamma_D}}\|v\|^2_{\mathbf{V}},\end{aligned} \tag{4.64}$$

$$\int_{\{x;0\le\theta\le d_0+\theta_D\}} |\theta||v\cdot\nabla\theta_D|\,dx \le 2d_0\|v\|_{L^2}\|\theta_D\|_{W^{1,2}} \le c\|v\|_{\mathbf{V}}\|\theta_D\|_{W^{1,2}}, \tag{4.65}$$

$$\int_{\{x;\theta\le 0\}} |\theta||v\cdot\nabla\theta_D|\,dx \le c\|\theta^-\|_{W^{1,2}_{\Gamma_D}}\|v\|_{\mathbf{V}}\|\theta_D\|_{W^{1,2}}. \tag{4.66}$$

By (4.62)-(4.66), we see that

$$\begin{aligned}\overline{I}_3 &\le \frac{\kappa_0}{4}\int_{\{x;\theta-\theta_D\le d_0\}} |\nabla\theta|^2\,dx \\ &+c\big(\|v\|^2_{\mathbf{V}} + \|\theta^-\|^2_{W^{1,2}_{\Gamma_D}}\|v\|^2_{\mathbf{V}} + \|v\|_{\mathbf{V}}\|\theta_D\|_{W^{1,2}} + \|\theta^-\|_{W^{1,2}_{\Gamma_D}}\|v\|_{\mathbf{V}}\|\theta_D\|_{W^{1,2}}\big).\end{aligned} \tag{4.67}$$

Next, we have

$$\begin{aligned}\overline{I}_4 &\le \int_{\{x;\theta-\theta_D\ge d_0\}} \alpha_2\mu(\theta)\frac{|\mathcal{E}(v)|^2}{1+\varepsilon|\mathcal{E}(v)|^2} d_0\,dx \\ &\quad + \int_{\{x;0\le\theta\le d_0+\theta_D\}} \alpha_2\mu(\theta)\frac{|\mathcal{E}(v)|^2}{1+\varepsilon|\mathcal{E}(v)|^2} 2d_0\,dx \\ &\quad + \int_{\{x;\theta\le 0\}} \alpha_2\mu(\theta)\frac{|\mathcal{E}(v)|^2}{1+\varepsilon|\mathcal{E}(v)|^2}\theta\,dx + \int_{\{x;\theta\le 0\}} \alpha_2\mu(\theta)\frac{|\mathcal{E}(v)|^2}{1+\varepsilon|\mathcal{E}(v)|^2} d_0\,dx \\ &\le \int_\Omega \alpha_2\mu(\theta)|\mathcal{E}(v)|^2 2d_0\,dx \le c\|v\|^2_{\mathbf{V}}.\end{aligned} \tag{4.68}$$

Taking into account (4.31) and applying Hölder's inequality, we have

$$
\begin{aligned}
\overline{I}_5 \le\; & \frac{|\alpha_1|\lambda}{\max\{\lambda, \|\theta\|^2_{L^{q_0}}\}} \int_{\{x;\theta-\theta_D\ge d_0\}} |\theta f_\varepsilon \cdot v| d_0\, dx \\
& + \frac{|\alpha_1|\lambda}{\max\{\lambda, \|\theta\|^2_{L^{q_0}}\}} \int_{\{x;0\le\theta\le d_0+\theta_D\}} |\theta f_\varepsilon \cdot v| 2d_0\, dx \\
& + \frac{\alpha_1\lambda}{\max\{\lambda, \|\theta\|^2_{L^{q_0}}\}} \int_{\{x;\theta\le 0\}} \theta f_\varepsilon \cdot v \theta\, dx \\
& + \frac{|\alpha_1|\lambda}{\max\{\lambda, \|\theta\|^2_{L^{q_0}}\}} \int_{\{x;\theta\le 0\}} |\theta f_\varepsilon \cdot v| d_0\, dx \\
\le\; & 2d_0 \frac{|\alpha_1|\lambda}{\max\{\lambda, \|\theta\|^2_{L^{q_0}}\}} \int_\Omega |\theta f_\varepsilon \cdot v|\, dx + \frac{|\alpha_1|\lambda}{\max\{\lambda, \|\theta\|^2_{L^{q_0}}\}} \int_\Omega |\theta f_\varepsilon \cdot v||\theta^-|\, dx \\
\le\; & c\|f_\varepsilon\|_{\mathbf{L}^t} \|v\|_{\mathbf{V}} + c\|f_\varepsilon\|_{\mathbf{L}^t} \|v\|_{\mathbf{V}} \|\theta^-\|_{W^{1,2}} \\
\le\; & c\|f_\varepsilon\|_{\mathbf{L}^t} \|v\|_{\mathbf{V}} (1 + \|\theta^-\|_{W^{1,2}}).
\end{aligned} \tag{4.69}
$$

Let us estimate $\overline{I}_6 = (g_R, \varphi)_{\Gamma_R} + (g, \varphi)$, where $\varphi = \min\{\theta(x) - \theta_D(x), d_0\}$. If $\theta(x) - \theta_D(x) \ge d_0$, then $\varphi(x) = d_0$ and

$$
\int_{\Gamma_R \cap \{x;\theta(x)-\theta_D(x)\ge d_0\}} g_R \varphi\, dx \le d_0 \int_{\Gamma_R \cap \{x;\theta(x)-\theta_D(x)\ge d_0\}} |g_R|\, dx. \tag{4.70}
$$

If $0 \le \theta(x) \le d_0 + \theta_D(x)$, then $|\varphi(x)| = |\theta(x) - \theta_D(x)| \le 2d_0$ and

$$
\int_{\Gamma_R \cap \{x;0\le\theta(x)\le d_0+\theta_D(x)\}} g_R \varphi(x)\, dx \le 2d_0 \int_{\Gamma_R \cap \{x;0\le\theta(x)\le d_0+\theta_D(x)\}} |g_R|\, dx. \tag{4.71}
$$

And

$$
\begin{aligned}
\int_{\Gamma_R \cap \{x;\theta(x)\le 0\}} g_R \varphi(x)\, dx &= \int_{\Gamma_R \cap \{x;\theta(x)\le 0\}} g_R(\theta^-(x) - d_0)\, dx \\
&\le d_0 \int_{\Gamma_R \cap \{x;\theta(x)\le 0\}} |g_R|\, dx + \|g_R\|_{L^{4/3}(\Gamma_R)} \|\theta^-\|_{W^{1,2}_{\Gamma_D}(\Omega)}.
\end{aligned} \tag{4.72}
$$

Also, we have

$$
\begin{aligned}
\langle g, \varphi \rangle \le\; & \int_{\{x;\theta-\theta_D\ge d_0\}} |g| d_0\, dx + \int_{\{x;0\le\theta\le d_0+\theta_D\}} |g| 2d_0\, dx + \int_{\{x;\theta\le 0\}} g(\theta^- - d_0)\, dx \\
\le\; & 2d_0 \|g\|_{L^1(\Omega)} + \|g\|_{L^{6/5}(\Omega)} \|\theta^-\|_{W^{1,2}_{\Gamma_D}(\Omega)} \\
\le\; & c\|g\|_{L^{6/5}(\Omega)} (1 + \|\theta^-\|_{W^{1,2}_{\Gamma_D}(\Omega)}).
\end{aligned} \tag{4.73}
$$

By (4.70)-(4.73), we get

$$
\overline{I}_6 \le c(\|g_R\|_{L^{4/3}(\Gamma_R)} + \|g\|_{L^{6/5}(\Omega)})(1 + \|\theta^-\|_{W^{1,2}_{\Gamma_D}(\Omega)}). \tag{4.74}
$$

Therefore, by virtue of (4.57), (4.58), (4.61), (4.67)-(4.69), (4.74) we have

$$
\begin{aligned}
& \int_{\{x;\theta-\theta_D<d_0\}} |\nabla\theta|^2 dx \\
& \le c\Big(\|\theta_D\|^2_{W^{1,2}} + \|v\|^2_{\mathbf{V}} + \|\theta^-\|^2_{W^{1,2}_{\Gamma_D}} \|v\|^2_{\mathbf{V}} \Big) \\
& \quad + c\Big(1 + \|f_\varepsilon\|_{\mathbf{L}^t} \|v\|_{\mathbf{V}} + \|g_R\|_{L^{4/3}(\Gamma_R)} + \|g\|_{L^{6/5}(\Omega)} \Big) \big(1 + \|\theta^-\|_{W^{1,2}_{\Gamma_D}} \big).
\end{aligned} \tag{4.75}
$$

Combining (4.56), (4.75) and taking into account (4.32), we get

$$
\begin{aligned}
&\int_\Omega \frac{|\nabla\theta|^2}{(1+(\theta-d_0)^+)^{1+\sigma}}\,dx\\
&\le \int_{\{x;\theta\ge d_0\}} \frac{|\nabla\theta|^2}{(1+(\theta-d_0)^+)^{1+\sigma}}\,dx + \int_{\{x;\theta\le d_0\}} |\nabla\theta|^2\,dx\\
&\le c\big(1+\frac{1}{\sigma}\big)\Big(1+\|v\|_{\mathbf{V}}^2+\|f_\varepsilon\|_{\mathbf{L}^t}\|v\|_{\mathbf{V}}+\|g_R\|_{L^{4/3}(\Gamma_R)}+\|g\|_{L^{6/5}(\Omega)}\Big)\\
&\quad\times\big(1+\|\theta^-\|_{W^{1,2}_{\Gamma_D}}\big)+c\Big(\|\theta_D\|^2_{W^{1,2}}+\|v\|^2_{\mathbf{V}}+\|\theta^-\|^2_{W^{1,2}_{\Gamma_D}}\|v\|^2_{\mathbf{V}}\Big)\\
&\le c\Bigg[\big(1+\frac{1}{\sigma}\big)\Big(1+\|v\|_{\mathbf{V}}^2+\|f_\varepsilon\|_{\mathbf{L}^t}\|v\|_{\mathbf{V}}+\|g_R\|_{L^{4/3}(\Gamma_R)}+\|g\|_{L^{6/5}(\Omega)}\Big)^2\\
&\quad+\|\theta_D\|^2_{W^{1,2}}+\|v\|^2_{\mathbf{V}}+\Big(\|f_\varepsilon\|_{\mathbf{L}^t}\|v\|_{\mathbf{V}}+\|g_R\|_{L^{4/3}(\Gamma_R)}+\|g\|_{L^{6/5}(\Omega)}\Big)^2\|v\|^2_{\mathbf{V}}\Bigg].
\end{aligned}
\tag{4.76}
$$

Since $f_\varepsilon\to f$ in $\mathbf{L}^t(\Omega)$, we may assume that $\|f_\varepsilon\|_{\mathbf{L}^t}\le 1+\|f\|_{\mathbf{L}^t}$ for all ε. Thus taking into account (4.45), we have from (4.76) that under (4.37)

$$
\begin{aligned}
&\int_\Omega \frac{|\nabla\theta|^2}{(1+(\theta-d_0)^+)^{1+\sigma}}\,dx\\
&\le c\Bigg[\big(1+\frac{1}{\sigma}\big)\Big(1+\big(\frac{\mu_0}{K}\big)^2+(1+\|f\|_{\mathbf{L}^t})\frac{\mu_0}{K}+\|g_R\|_{L^{4/3}(\Gamma_R)}+\|g\|_{L^{6/5}(\Omega)}\Big)^2\\
&\quad+c\Big(\|\theta_D\|^2_{W^{1,2}}+\big(\frac{\mu_0}{K}\big)^2+\big((1+\|f\|_{\mathbf{L}^t})\frac{\mu_0}{K}+\|g_R\|_{L^{4/3}(\Gamma_R)}+\|g\|_{L^{6/5}(\Omega)}\big)^2\big(\frac{\mu_0}{K}\big)^2\Bigg]\\
&\equiv K_\sigma.
\end{aligned}
\tag{4.77}
$$

To get boundedness of $\theta\in L^{q_0}(\Omega)$ independent of ε, we will use the property $W^{1,r_0}(\Omega)\subset L^{q_0}$, where r_0 is such that $\frac{1}{r_0}-\frac{1}{3}=\frac{1}{q_0}$. Then, $\frac{4}{3}<r_0<\frac{3}{2}$. To this end, take $\sigma_0>0$ such that

$$
\frac{r_0(1+\sigma_0)}{2-r_0}=q_0.
$$

Then, $\sigma_0=\frac{3-2r_0}{3-r_0}$, and $0<\sigma_0<\frac{1}{5}$. Putting $\sigma=\sigma_0$ in (4.77), by virtue of Hölder's inequality with exponents $\frac{2}{r_0}$, $\frac{2}{2-r_0}$ and an inequality $|a+b|^p\le 2^p(|a|^p+|b|^p)$, $|a|+|b|\le(|a|^{\frac{1}{p}}+|b|^{\frac{1}{p}})^p$, $p\in(1,\infty)$ we have

$$
\begin{aligned}
\int_\Omega|\nabla\theta|^{r_0}\,dx&\le\Big(\int_\Omega\frac{|\nabla\theta|^2}{(1+(\theta-d_0)^+)^{(1+\sigma_0)}}\,dx\Big)^{r_0/2}\\
&\quad\times\Big(\int_\Omega\big(1+(\theta-d_0)^+\big)^{(1+\sigma_0)\frac{r_0}{2-r_0}}\,dx\Big)^{(2-r_0)/2}\\
&\le cK_{\sigma_0}^{r_0/2}\Big(1+\Big(\int_\Omega|\theta|^{3r_0/(3-r_0)}\,dx\Big)^{(2-r_0)/2}\Big),
\end{aligned}
\tag{4.78}
$$

where $(1+\sigma_0)\frac{r_0}{2-r_0}=3r_0/(3-r_0)$ was used and K_{σ_0} is the one with $\sigma=\sigma_0$ in K_σ of

(4.77). By Sobolev's embedding theorem and Friedrichs' inequality,

$$\begin{aligned}&\Big(\int_\Omega |\theta|^{3r_0/(3-r_0)}\,dx\Big)^{(2-r_0)/2}\\&\le c\|\theta\|_{W^{1,r_0}}^{(1+\sigma_0)\frac{r_0}{2}} \le c\Big(\int_\Omega |\nabla\theta|^{r_0}\,dx + \|\theta_D\|_{W^{1,r_0}}^{r_0}\Big)^{(1+\sigma_0)/2}\\&\le c\left(\Big(\int_\Omega |\nabla\theta|^{r_0}\,dx\Big)^{(1+\sigma_0)/2} + \|\theta_D\|_{W^{1,r_0}}^{r_0(1+\sigma_0)/2}\right).\end{aligned} \tag{4.79}$$

Substituting (4.79) into (4.78) and using Young's inequality and Hölder's inequality with exponents $\frac{2}{1-\sigma_0}, \frac{2}{1+\sigma_0}$ on the right hand side, we have

$$\begin{aligned}\int_\Omega |\nabla\theta|^{r_0}\,dx &\le cK_{\sigma_0}^{r_0/2}\Big(\int_\Omega |\nabla\theta|^{r_0}\,dx\Big)^{(1+\sigma_0)/2} + cK_{\sigma_0}^{r_0/2}\big(1+\|\theta_D\|_{W^{1,r_0}}^{r_0(1+\sigma_0)/2}\big)\\&\le \frac{1}{2}\int_\Omega |\nabla\theta|^{r_0}\,dx + cK_{\sigma_0}^{r_0/(1-\sigma_0)} + cK_{\sigma_0}^{r_0/(1-\sigma_0)}\big(1+\|\theta_D\|_{W^{1,r_0}}^{r_0}\big).\end{aligned}$$

Thus, we get

$$\int_\Omega |\nabla\theta|^{r_0}\,dx \le cK_{\sigma_0}^{r_0/(1-\sigma_0)}\big(1+\|\theta_D\|_{W^{1,r_0}}^{r_0}\big). \tag{4.80}$$

By virtue of (4.80), Sobolev's embedding theorem and Friedrichs' inequality, we have

$$\begin{aligned}\|\theta\|_{L^{q_0}} &\le c\|\theta\|_{W^{1,r_0}} \le c\Big(\int_\Omega |\nabla\theta|^{r_0}\,dx + \|\theta_D\|_{W^{1,r_0}}^{r_0}\Big)^{1/r_0}\\&\le c\left(K_{\sigma_0}^{r_0/(1-\sigma_0)}(1+\|\theta_D\|_{W^{1,r_0}}^{r_0}) + \|\theta_D\|_{W^{1,r_0}}^{r_0}\right)^{1/r_0}\\&\le c\left(K_{\sigma_0}^{1/(1-\sigma_0)}(1+\|\theta_D\|_{W^{1,r_0}}) + \|\theta_D\|_{W^{1,r_0}}\right)\\&\le c\left(K_{\sigma_0}^{1/(1-\sigma_0)}(1+\|\theta_D\|_{W^{1,2}}) + \|\theta_D\|_{W^{1,2}}\right) \equiv \overline{K}_{\sigma_0}.\end{aligned} \tag{4.81}$$

$\overline{K}_{\sigma_0}$ is independent of λ, ε and depends on $\|f\|_{\mathbf{L}^t}, \|\theta_D\|_{W^{1,2}}, \|g_R\|_{L^{4/3}(\Gamma_R)}, \|g\|_{L^{6/5}(\Omega)}$ and $\phi_i, i = 2-7$, via f_1 in (3.6).

Putting

$$\sqrt{\lambda_1} = \overline{K}_{\sigma_0}, \tag{4.82}$$

we have (4.46).

Now, for all r such that $1 < r < \frac{3}{2}$ and all $\varepsilon > 0$ let us get an estimate of $\int_\Omega |\nabla\theta|^r\,dx$ independent of ε. Putting $\sigma = \frac{3-2r}{3-r}$, we have that $0 < \sigma < \frac{1}{2}$. Repeating the argument of (4.78)-(4.80), we have

$$\int_\Omega |\nabla\theta|^r\,dx \le cK_\sigma^{r/(1-\sigma)}\big(1+\|\theta_D\|_{W^{1,r}}^r\big),$$

which implies (4.47). □

When $\lambda = \lambda_1$ and

$$\max\{|\alpha_0|, |\alpha_1|\}\sqrt{\lambda_1} \le 1,$$

by (4.81), (4.82) we have

$$\frac{\lambda_1}{\max\{\lambda_1, \|\theta_\varepsilon\|^2_{L^{q_0}}\}} = 1 \quad \forall \varepsilon > 0. \tag{4.83}$$

Therefore, summarizing Lemmas 4.6-4.8 we have

THEOREM 4.9. *If*

$$\begin{aligned} &\max\{|\alpha_0|, |\alpha_1|\}\sqrt{\lambda_1} \le 1, \\ &(\|f\|^2_{\boldsymbol{L}^t} + \|f_1\|^2_{\boldsymbol{V}^*}) < \frac{\mu_0^2}{Kc_0}, \end{aligned} \tag{4.84}$$

then there exists a solution $(v_\varepsilon, \theta_\varepsilon) \in \boldsymbol{V} \times W^{1,2}(\Omega)$ *to the following problem*

$$\begin{cases} a_0(\theta_\varepsilon; v_\varepsilon, u) + a_1(v_\varepsilon, v_\varepsilon, u) + \big\langle \nabla\Phi_\varepsilon(v_\varepsilon), u\big\rangle - \big\langle (1-\alpha_0\theta_\varepsilon) f_\varepsilon, u\big\rangle \\ \quad = \langle f_1, u\rangle \quad \forall u \in \mathbf{V}, \\ b_0(\theta_\varepsilon; \theta_\varepsilon, \varphi) - \langle \gamma_\varepsilon(\theta_\varepsilon) v_\varepsilon, \nabla\varphi\rangle - \Big\langle \alpha_2\mu(\theta_\varepsilon)\dfrac{|\mathcal{E}(v_\varepsilon)|^2}{1+\varepsilon|\mathcal{E}(v_\varepsilon)|^2}, \varphi\Big\rangle \\ \quad - \langle \alpha_1\theta_\varepsilon f_\varepsilon \cdot v_\varepsilon, \varphi\rangle = \langle f_2, \varphi\rangle \qquad \forall \varphi \in W^{1,2}_{\Gamma_D}(\Omega), \end{cases} \tag{4.85}$$

$$\theta_\varepsilon - \theta_D \in W^{1,2}_{\Gamma_D}(\Omega), \tag{4.86}$$

and the solution satisfies:

$$\begin{aligned} &\|v_\varepsilon\|_{\boldsymbol{V}} \le \frac{\mu_0}{K}, \\ &\|\theta_\varepsilon^-\|_{W^{1,2}_{\Gamma_D}(\Omega)} \le c\Big(\|f\|_{\boldsymbol{L}^t}\|v_\varepsilon\|_{\boldsymbol{V}} + \|g_R\|_{L^{4/3}(\Gamma_R)} + \|g\|_{L^{6/5}(\Omega)}\Big), \\ &\int_\Omega |\nabla\theta_\varepsilon|^r\, dx \le cK_\sigma^{r/(1-\sigma)}\big(1 + \|\theta_D\|^2_{W^{1,2}}\big) \quad \forall r, 1 < r < \frac{3}{2}, \end{aligned} \tag{4.87}$$

where $\sigma = \frac{3-2r}{3-r}$.

4.3. Passing to the limit as $\varepsilon \to 0$. By passing to the limit of solutions in Theorem 4.9, we will prove Theorem 3.3. Owing to (4.87) we can extract subsequences, which are denoted as before, such that

$$\begin{aligned} &v_\varepsilon \rightharpoonup v \quad \text{in } \mathbf{V}, \\ &v_\varepsilon \to v \quad \text{in } \mathbf{L}^q, 1 \le q < 6, \quad \text{and a.e. in}\, \Omega, \\ &\theta_\varepsilon \rightharpoonup \theta \quad \text{in } W^{1,r}(\Omega) \quad \forall r,\, 1 \le r < \frac{3}{2}, \\ &\theta_\varepsilon \to \theta \quad \text{in } L^s(\Omega) \quad \forall s, 1 \le s < 3, \quad \text{and a.e. in}\, \Omega, \end{aligned} \tag{4.88}$$

as $\varepsilon \to 0$.

By (4.86), $\theta|_{\Gamma_D} = \theta_D|_{\Gamma_D}$.

Subtracting the formula obtained by putting $u = v_\varepsilon$ from the first formula of (4.85), we have

$$\begin{aligned} &a_0(\theta_\varepsilon; v_\varepsilon, u - v_\varepsilon) + a_1(v_\varepsilon, v_\varepsilon, u - v_\varepsilon) + \big\langle \nabla\Phi_\varepsilon(v_\varepsilon), u - v_\varepsilon\big\rangle \\ &- \big\langle (1-\alpha_0\theta_\varepsilon) f, u - v_\varepsilon\big\rangle = \langle f_1, u - v_\varepsilon\rangle \qquad \forall u \in \mathbf{V}. \end{aligned} \tag{4.89}$$

By corollaries 4.2, 4.3

$$\begin{aligned}&a_0(\theta_\varepsilon; v_\varepsilon, u) \to a_0(\theta; v, u) \quad \text{as } \varepsilon \to 0,\\ &\liminf_{\varepsilon\to 0} a_0(\theta_\varepsilon; v_\varepsilon, v_\varepsilon) \ge a_0(\theta; v, v),\end{aligned}$$

which imply that

$$\limsup_{\varepsilon\to 0} a_0(\theta_\varepsilon; v_\varepsilon, u - v_\varepsilon) \le a_0(\theta; v, u - v). \tag{4.90}$$

It is easy to prove

$$a_1(v_\varepsilon, v_\varepsilon, u - v_\varepsilon) \to a_1(v, v, u - v) \quad \text{as } \varepsilon \to 0. \tag{4.91}$$

Since Φ_ε is convex, continuous and Fréchet differentiable, we have

$$\Phi_\varepsilon(u) - \Phi_\varepsilon(v_\varepsilon) \ge \langle \nabla\Phi_\varepsilon(v_\varepsilon), u - v_\varepsilon\rangle \quad \forall u \in \mathbf{V}, \tag{4.92}$$

which owing to (4.4) implies

$$\Phi_\varepsilon(u) - \Phi(J_\varepsilon v_\varepsilon) \ge \langle \nabla\Phi_\varepsilon(v_\varepsilon), u - v_\varepsilon\rangle \quad \forall u \in \mathbf{V}. \tag{4.93}$$

Since $\Phi(0_{\mathbf{V}}) = 0$, by (4.4) $\Phi_\varepsilon(0_{\mathbf{V}}) = 0$, and so from(4.92) we have

$$\Phi_\varepsilon(v_\varepsilon) \le \langle \nabla\Phi_\varepsilon(v_\varepsilon), v_\varepsilon\rangle . \tag{4.94}$$

On the other hand, putting $u = v_\varepsilon$ in the first formula of (4.85), we have

$$\begin{aligned}&a_0(\theta_\varepsilon; v_\varepsilon, v_\varepsilon) + a_1(v_\varepsilon, v_\varepsilon, v_\varepsilon) + \big\langle \nabla\Phi_\varepsilon(v_\varepsilon), v_\varepsilon\big\rangle\\ &= \big\langle (1 - \alpha_0\theta_\varepsilon) f_\varepsilon, v_\varepsilon\big\rangle + \langle f_1, v_\varepsilon\rangle.\end{aligned} \tag{4.95}$$

From (4.94) and (4.95) we have

$$a_0(\theta_\varepsilon; v_\varepsilon, v_\varepsilon) + a_1(v_\varepsilon, v_\varepsilon, v_\varepsilon) + \Phi_\varepsilon(v_\varepsilon) \le \big\langle (1 - \alpha_0\theta_\varepsilon) f_\varepsilon, v_\varepsilon\big\rangle + \langle f_1, v_\varepsilon\rangle,$$

from which we get

$$|\Phi_\varepsilon(v_\varepsilon)| \le c\big((1 + \overline{K}_{\sigma_0})\|f_\varepsilon\|_{\mathbf{L}^t} + \|f_1\|_{\mathbf{V}^*}\big)\|v_\varepsilon\|_{\mathbf{V}} + |a_1(v_\varepsilon, v_\varepsilon, v_\varepsilon)|, \tag{4.96}$$

where (4.81) was used. By virtue of (4.3), (4.5), (4.45) and (4.96), we have

$$\|v_\varepsilon - J_\varepsilon v_\varepsilon\|^2_{\mathbf{V}} \le \big[c\big((1 + \overline{K}_{\sigma_0})\|f_\varepsilon\|_{\mathbf{L}^t} + \|f_1\|_{\mathbf{V}^*}\big)\frac{\mu_0}{K} + \frac{\mu_0^3}{K^2}\big]2\varepsilon,$$

which shows that

$$J_\varepsilon v_\varepsilon \rightharpoonup v \text{ in } \mathbf{V} \quad \text{as } \varepsilon \to 0.$$

Then, by virtue of lower weak semi-continuity of $\Phi(v)$

$$\liminf_{\varepsilon\to 0} \Phi(J_\varepsilon v_\varepsilon) \ge \Phi(v). \tag{4.97}$$

By (4.4) we have

$$\Phi_\varepsilon(u) \to \Phi(u) \quad \text{as } \varepsilon \to 0. \tag{4.98}$$

Taking into account (4.97) and (4.98), we have from (4.93)

$$\Phi(u) - \Phi(v) \geq \limsup_{\varepsilon \to 0} \langle \nabla \Phi_\varepsilon(v_\varepsilon), u - v_\varepsilon \rangle \quad \forall u \in \mathbf{V}. \tag{4.99}$$

Using

$$\begin{aligned} &|\big\langle \theta_\varepsilon f_\varepsilon, v_\varepsilon \big\rangle - \big\langle \theta f, v \big\rangle| \\ &\leq |\big\langle \theta_\varepsilon f_\varepsilon, v_\varepsilon \big\rangle - \big\langle \theta f_\varepsilon, v_\varepsilon \big\rangle| + |\big\langle \theta f_\varepsilon, v_\varepsilon \big\rangle - \big\langle \theta f, v_\varepsilon \big\rangle| + |\big\langle \theta f, v_\varepsilon \big\rangle - \big\langle \theta f, v \big\rangle| \\ &\leq \|\theta_\varepsilon - \theta\|_{L^2} \|f_\varepsilon\|_{\mathbf{L}^3} \|v_\varepsilon\|_{\mathbf{L}^6} + \|\theta\|_{L^2} \|f_\varepsilon - f\|_{\mathbf{L}^3} \|v_\varepsilon\|_{\mathbf{L}^6} \\ &\quad + \|\theta\|_{L^{15/7}} \|f\|_{\mathbf{L}^3} \|v_\varepsilon - v\|_{\mathbf{L}^5}, \end{aligned} \tag{4.100}$$

we can prove

$$\big\langle (1 - \alpha_0 \theta_\varepsilon) f, u - v_\varepsilon \big\rangle \to \big\langle (1 - \alpha_0 \theta) f, u - v \big\rangle \quad \text{as } \varepsilon \to 0. \tag{4.101}$$

It is easy to prove

$$\langle f_1, u - v_\varepsilon \rangle \to \langle f_1, u - v \rangle \quad \text{as } \varepsilon \to 0. \tag{4.102}$$

By virtue of (4.90), (4.91), (4.99), (4.101) and (4.102), from (4.89) we get

$$\begin{aligned} &a_0(\theta; v, u - v) + a_1(v, v, u - v) + \Phi(u) - \Phi(v) \\ &- \big\langle (1 - \alpha_0 \theta) f, u - v \big\rangle \geq \langle f_1, u - v \rangle \qquad \forall u \in \mathbf{V}, \end{aligned}$$

which is the first formula in (3.14). From above we know that $v \in K(\Omega)$, i.e. $\Phi(v) < +\infty$. Thus putting $u = v$, from (4.99) we have

$$0 \geq \limsup_{\varepsilon \to 0} \langle \nabla \Phi_\varepsilon(v_\varepsilon), v - v_\varepsilon \rangle . \tag{4.103}$$

We will get the second equation in (3.14). By Corollary 4.2, we have

$$b_0(\theta_\varepsilon; \theta_\varepsilon, \varphi) \to b_0(\theta; \theta, \varphi) \quad \forall \varphi \in W^{1,\infty}_{\Gamma_D}(\Omega) \quad \text{as } \varepsilon \to 0. \tag{4.104}$$

Let us prove

$$\langle \gamma_\varepsilon(\theta_\varepsilon) v_\varepsilon, \nabla \phi \rangle \to \langle \gamma(\theta) \theta v, \nabla \varphi \rangle \quad \forall \varphi \in W^{1,\infty}_{\Gamma_D}(\Omega) \quad \text{as } \varepsilon \to 0. \tag{4.105}$$

By Hölder's inequality

$$\begin{aligned} &|\langle \gamma_\varepsilon(\theta_\varepsilon) v_\varepsilon, \nabla \varphi \rangle - \langle \gamma(\theta) \theta v, \nabla \varphi \rangle| \\ &\leq |\langle \gamma_\varepsilon(\theta_\varepsilon) v_\varepsilon, \nabla \varphi \rangle - \langle \gamma(\theta) \theta v_\varepsilon, \nabla \varphi \rangle| + |\langle \gamma(\theta) \theta v_\varepsilon, \nabla \varphi \rangle - \langle \gamma(\theta) \theta v, \nabla \varphi \rangle| \\ &\leq \|\gamma_\varepsilon(\theta_\varepsilon) - \gamma(\theta)\theta\|_{L^{6/5}} \|v_\varepsilon\|_{\mathbf{L}^6} \|\nabla \varphi\|_{\mathbf{L}^\infty} + \|\gamma(\theta)\theta\|_{L^4} \|v_\varepsilon - v\|_{\mathbf{L}^4} \|\nabla \varphi\|_{\mathbf{L}^\infty}. \end{aligned} \tag{4.106}$$

By the definition of $\gamma_\varepsilon(t)$

$$\begin{aligned} &\|\gamma_\varepsilon(\theta_\varepsilon) - \gamma(\theta)\theta\|_{L^{6/5}} \\ &\leq \left\| \frac{\gamma(\theta_\varepsilon)\theta_\varepsilon}{(1 + \varepsilon|\gamma(\theta_\varepsilon)|)(1 + \varepsilon|\theta_\varepsilon|)} - \gamma(\theta)\theta \right\|_{L^{6/5}} \\ &\leq \|\gamma(\theta_\varepsilon)\theta_\varepsilon - \gamma(\theta)\theta\|_{L^{6/5}} + \varepsilon \left\| \gamma(\theta)\theta \big(|\gamma(\theta_\varepsilon)| + |\theta_\varepsilon| + \varepsilon |\gamma(\theta_\varepsilon)||\theta_\varepsilon| \big) \right\|_{L^{6/5}} \end{aligned} \tag{4.107}$$

Then, by virtue of Lemma 4.1, $\gamma(\theta_\varepsilon)$ converges to $\gamma(\theta)$ in space $L^p(\Omega)$ ($\forall p$, $1 < p < \infty$,) as ε goes to zero. Thus, from (4.106), (4.107) we get (4.105).

Let us consider

$$\begin{aligned}\mu_0\|v-v_\varepsilon\|_{\mathbf{V}}^2 &\le a_0(\theta_\varepsilon; v-v_\varepsilon, v-v_\varepsilon)\\ &= a_0(\theta_\varepsilon; v, v-v_\varepsilon)+a_1(v_\varepsilon, v_\varepsilon, v-v_\varepsilon)+\nabla\Phi_\varepsilon(v_\varepsilon)(v-v_\varepsilon)\\ &\quad -\big\langle(1-\alpha_0\theta_\varepsilon)f, v-v_\varepsilon\big\rangle-\langle f_1, v-v_\varepsilon\rangle,\end{aligned}\tag{4.108}$$

which is obtained from the first formula in (4.85) by putting $u=v-v_\varepsilon$. By virtue of (4.90), (4.91), (4.101) and (4.103), the right hand side of (4.108) converges to 0 as ε goes to 0. Thus, we have

$$v_\varepsilon\to v\quad\text{in }\mathbf{V}\quad\text{as }\varepsilon\to 0.\tag{4.109}$$

On the other hand,

$$\begin{aligned}&\alpha_2\mu(\theta_\varepsilon)\frac{|\mathcal{E}(v_\varepsilon)|^2}{1+\varepsilon|\mathcal{E}(v_\varepsilon)|^2}-\alpha_2\mu(\theta)|\mathcal{E}(v)|^2\\ &=\left(\alpha_2\mu(\theta_\varepsilon)\frac{|\mathcal{E}(v_\varepsilon)|^2}{1+\varepsilon|\mathcal{E}(v_\varepsilon)|^2}-\alpha_2\mu(\theta_\varepsilon)|\mathcal{E}(v)|^2\right)+\alpha_2\big(\mu(\theta_\varepsilon)-\mu(\theta)\big)|\mathcal{E}(v)|^2\\ &=\alpha_2\mu(\theta_\varepsilon)\frac{|\mathcal{E}(v_\varepsilon)|^2-|\mathcal{E}(v)|^2}{1+\varepsilon|\mathcal{E}(v_\varepsilon)|^2}-\alpha_2\mu(\theta_\varepsilon)\frac{\varepsilon|\mathcal{E}(v_\varepsilon)|^2|\mathcal{E}(v)|^2}{1+\varepsilon|\mathcal{E}(v_\varepsilon)|^2}+\alpha_2\big(\mu(\theta_\varepsilon)-\mu(\theta)\big)|\mathcal{E}(v)|^2\\ &\equiv I_{\varepsilon1}+I_{\varepsilon2}+\alpha_2\big(\mu(\theta_\varepsilon)-\mu(\theta)\big)|\mathcal{E}(v)|^2.\end{aligned}$$

By (4.109)

$$\|I_{\varepsilon1}\|_{L^1(\Omega)}\to 0\text{ as }\varepsilon\to 0.$$

Since $|I_{\varepsilon2}|\le c|\mathcal{E}(v)|^2$ and (passing to a subsequence if necessary) $I_{\varepsilon2}\to 0$ a.e. in Ω, by virtue of the dominated convergence theorem we have

$$\|I_{\varepsilon2}\|_{L^1(\Omega)}\to 0\text{ as }\varepsilon\to 0.$$

By Corollary 4.2,

$$\mu(\theta_\varepsilon)\varepsilon_{ij}(v)\rightharpoonup\mu(\theta)\varepsilon_{ij}(v)\text{ in }\ L^2(\Omega)\quad\text{as }\varepsilon\to 0,$$

and so

$$\big\langle\alpha_2\big(\mu(\theta_\varepsilon)-\mu(\theta)\big)|\mathcal{E}(v)|^2,\varphi\big\rangle\to 0\quad\forall\varphi\in W^{1,\infty}_{\Gamma_D}(\Omega)\quad\text{as }\varepsilon\to 0.$$

Thus, we have that

$$\begin{aligned}&\left\langle\alpha_2\mu(\theta_\varepsilon)\frac{|\mathcal{E}(v_\varepsilon)|^2}{1+\varepsilon|\mathcal{E}(v_\varepsilon)|^2}-\alpha_2\mu(\theta)|\mathcal{E}(v)|^2,\varphi\right\rangle\to 0\\ &\qquad\qquad\forall\varphi\in W^{1,\infty}_{\Gamma_D}(\Omega)\quad\text{as }\varepsilon\to 0.\end{aligned}\tag{4.110}$$

Taking into account

$$\begin{aligned}&|\langle\alpha_1\theta_\varepsilon f_\varepsilon\cdot v_\varepsilon,\varphi\rangle-\langle\alpha_1\theta f\cdot v,\varphi\rangle|\\ &\le|\langle\alpha_1\theta_\varepsilon f_\varepsilon\cdot v_\varepsilon,\varphi\rangle-\langle\alpha_1\theta f_\varepsilon\cdot v_\varepsilon,\varphi\rangle|+|\langle\alpha_1\theta f_\varepsilon\cdot v_\varepsilon,\varphi\rangle-\langle\alpha_1\theta f\cdot v_\varepsilon,\varphi\rangle|\\ &\quad+|\langle\alpha_1\theta f\cdot v_\varepsilon,\varphi\rangle-\langle\alpha_1\theta f\cdot v,\varphi\rangle|\\ &\le c\|\theta_\varepsilon-\theta\|_{L^2}\|f_\varepsilon\|_{\mathbf{L}^3}\|v_\varepsilon\|_{\mathbf{L}^6}\|\varphi\|_{L^\infty}+c\|\theta\|_{L^2}\|f_\varepsilon-f\|_{\mathbf{L}^3}\|v\|_{\mathbf{L}^6}\|\varphi\|_{L^\infty}\\ &\quad+c\|\theta\|_{L^2}\|f\|_{\mathbf{L}^3}\|v_\varepsilon-v\|_{\mathbf{L}^6}\|\varphi\|_{L^\infty},\end{aligned}$$

we can prove

$$\langle \alpha_1 \theta_\varepsilon f \cdot v_\varepsilon, \varphi \rangle \to \langle \alpha_1 \theta f \cdot v, \varphi \rangle \quad \forall \varphi \in W^{1,\infty}_{\Gamma_D}(\Omega) \quad \text{as } \varepsilon \to 0. \tag{4.111}$$

By virtue of (4.104), (4.105), (4.110) and (4.111), from the second formula in (4.85) we get the second formula in (3.14).

Estimates (3.19) follow from (4.87) by (4.88). □

5. Proof of Theorem 3.4. First, we look for solutions to the auxiliary problem:

Problem II-VIA. Let $\zeta > 0$, $\lambda > 0$, $\varepsilon > 0$ and $q \in (\frac{12}{5}, 6)$. Find $(v, \theta) \in \mathbf{V} \times W^{1,2}(\Omega)$ such that $\theta - \theta_D \in W^{1,2}_{\Gamma_D}(\Omega)$ and

$$\begin{cases} a_0(\theta; v, u) + a_2(v, v, u) + \big\langle \nabla \Phi_\varepsilon(v), u \big\rangle \\ \quad - \Big\langle \Big(1 - \dfrac{\alpha_0 \lambda}{\max\{\lambda, \|\theta\|^2_{L^q}\}} \theta\Big) f_\varepsilon, u \Big\rangle = \langle f_1, u \rangle \quad \forall u \in \mathbf{V}, \\ b_0(\theta; \theta, \varphi) - \dfrac{\zeta}{\max\{\zeta, \|v_\varepsilon\|_{\mathbf{V}}\}} \langle \gamma_\varepsilon(\theta) v, \nabla \varphi \rangle - \Big\langle \alpha_2 \mu(\theta) \dfrac{|\mathcal{E}(v)|^2}{1 + \varepsilon |\mathcal{E}(v)|^2}, \varphi \Big\rangle \\ \quad - \Big\langle \dfrac{\alpha_1 \lambda}{\max\{\lambda, \|\theta\|^2_{L^q}\}} \theta f_\varepsilon \cdot v, \varphi \Big\rangle = \langle f_2, \varphi \rangle \quad \forall \varphi \in W^{1,2}_{\Gamma_D}(\Omega), \end{cases} \tag{5.1}$$

where $f_\varepsilon \in \mathbf{L}^\infty(\Omega)$ is such that $\|f - f_\varepsilon\|_{\mathbf{L}^t} \le \varepsilon$.

THEOREM 5.1. *There exists a solution $(v_\varepsilon, \theta_\varepsilon) \in \boldsymbol{V} \times W^{1,2}(\Omega)$ to Problem II-VIA.*

Proof. Let $\mathscr{H} = V \times W^{1,2}_{\Gamma_D}(\Omega)$. Define an operator $\mathscr{A} : \mathscr{H} \to \mathscr{H}^*$ by

$$\begin{aligned} &\big\langle \mathscr{A}(v, \eta), (u, \phi) \big\rangle \\ &= a_0(\eta + \theta_D; v, u) + a_2(v, v, u) + \big\langle \nabla \Phi_\varepsilon(v), u \big\rangle \\ &\quad - \Big\langle \Big(1 - \frac{\alpha_0 \lambda}{\max\{\lambda, \|\eta + \theta_D\|^2_{L^q}\}} (\eta + \theta_D)\Big) f_\varepsilon, u \Big\rangle + b_0(\eta + \theta_D; \eta + \theta_D, \varphi) \\ &\quad - \frac{\zeta}{\max\{\zeta, \|v_\varepsilon\|_{\mathbf{V}}\}} \langle \gamma_\varepsilon(\eta + \theta_D) v, \nabla \varphi \rangle - \Big\langle \alpha_2 \mu(\eta + \theta_D) \frac{|\mathcal{E}(v)|^2}{1 + \varepsilon |\mathcal{E}(v)|^2}, \varphi \Big\rangle \\ &\quad - \Big\langle \frac{\alpha_1 \lambda}{\max\{\lambda, \|\eta + \theta_D\|^2_{L^q}\}} (\eta + \theta_D) f_\varepsilon \cdot v, \varphi \Big\rangle \qquad \forall (v, \eta), (u, \phi) \in \mathscr{H}. \end{aligned}$$

Note that instead of $\frac{\delta}{\max\{\delta, \|\overline{a}_1(v)\|_{\mathbf{V}^*}\}} a_1(v, v, u)$ in Problem I-VIA the term $a_2(v, v, u)$ is used for Problem II-VIA.

Using

$$\begin{aligned} &a_2(v, v, v) = 0, \\ &|a_2(v, v, u)| \le K \|v\|^2_{\mathbf{V}} \|u\|_{\mathbf{V}}, \\ &|a_2(v_\varepsilon, v_\varepsilon, v_\varepsilon - u)| \le c \|\nabla v_\varepsilon\|_{\mathbf{L}^2} \|v_\varepsilon\|_{\mathbf{L}^4} \|v_\varepsilon - u\|_{\mathbf{L}^4}, \end{aligned}$$

respectively, in (4.16), (4.19) and (4.23), we can verify that proof of Theorem 4.5 for Problem I-VIA is valid for for Problem II-VIA. Thus, we come to the asserted conclusion. □

As (4.31) let us choose q_0 such that

$$\frac{1}{q_0} + \frac{1}{t} + \frac{1}{3} \le 1, \quad \frac{12}{5} < q_0 < 3.$$

Lemma 5.2. *If* $|\alpha_0|\sqrt{\lambda} \le 1$*, then there exists parameter* ζ *such that*

$$\frac{\zeta}{\max\{\zeta, \|v_\varepsilon\|_{\boldsymbol{V}}\}} = 1 \tag{5.2}$$

for all small ε *and solutions of* (5.1) v_ε*.*

Proof. Since $|\alpha_0|\sqrt{\lambda} \le 1$, we have

$$\begin{aligned}&\Big|\langle \frac{\alpha_0\lambda}{\max\{\lambda, \|\theta_\varepsilon\|^2_{L^{q_0}}\}}\theta_\varepsilon f_\varepsilon, u\rangle\Big| \\ &\le \frac{c\sqrt{\lambda}}{\max\{\lambda, \|\theta_\varepsilon\|^2_{L^{q_0}}\}}\|\theta_\varepsilon\|_{L^{q_0}}\|f_\varepsilon\|_{\mathbf{L}^t}\|u\|_{\mathbf{L}^6} \le c\|f_\varepsilon\|_{\mathbf{L}^t}\|u\|_{\mathbf{V}^*}.\end{aligned} \tag{5.3}$$

Putting $u = v_\varepsilon$ in the first equation of (5.1), we have

$$\begin{aligned}&a_0(\theta_\varepsilon; v_\varepsilon, v_\varepsilon) + a_2(v_\varepsilon, v_\varepsilon, v_\varepsilon) + \big\langle \nabla\Phi_\varepsilon(v_\varepsilon), v_\varepsilon\big\rangle \\ &- \Big\langle \Big(1 - \frac{\alpha_0\lambda}{\max\{\lambda, \|\theta_\varepsilon\|^2_{L^{q_0}}\}}\theta_\varepsilon\Big) f_\varepsilon, v_\varepsilon\Big\rangle = \langle f_1, v_\varepsilon\rangle.\end{aligned} \tag{5.4}$$

Taking into account $a_2(v_\varepsilon, v_\varepsilon, v_\varepsilon) = 0$, (4.17) and (5.3), from (5.4) we have

$$\begin{aligned}2\mu_0\|v_\varepsilon\|^2_{\mathbf{V}} \le a_0(\theta_\varepsilon; v_\varepsilon, v_\varepsilon) &\le \Big\langle \Big(1 - \frac{\alpha_0\lambda}{\max\{\lambda, \|\theta_\varepsilon\|^2_{L^{q_0}}\}}\theta_\varepsilon\Big) f_\varepsilon, v\Big\rangle + \langle f_1, v_\varepsilon\rangle \\ &\le c(\|f_\varepsilon\|_{\mathbf{L}^{\frac{6}{5}}} + \|f_\varepsilon\|_{\mathbf{L}^t} + \|f_1\|_{\mathbf{V}^*})\|v_\varepsilon\|_{\mathbf{V}},\end{aligned}$$

which implies

$$\|v_\varepsilon\|_{\mathbf{V}} \le \frac{c}{2\mu_0}(\|f_\varepsilon\|_{\mathbf{L}^t} + \|f_1\|_{\mathbf{V}^*}).$$

Since $f_\varepsilon \to f$ in $\mathbf{L}^t(\Omega)$, again we may assume that $\|f_\varepsilon\|_{\mathbf{L}^t} \le 1 + \|f\|_{\mathbf{L}^t}$ for all ε. Therefore,

$$\|v_\varepsilon\|_{\mathbf{V}} \le \frac{c}{2\mu_0}(1 + \|f\|_{\mathbf{L}^t} + \|f_1\|_{\mathbf{V}^*}). \tag{5.5}$$

Putting $\zeta = \frac{c}{2\mu_0}(1 + \|f\|_{\mathbf{L}^t} + \|f_1\|_{\mathbf{V}^*})$, we come to the asserted conclusion. □

Lemma 5.3. *If* $\max\{|\alpha_0|, |\alpha_1|\}\sqrt{\lambda} \le 1$*, then under the parameter* ζ *by Lemma 5.2 there exists a* λ_2 *independent of* ε *such that*

$$\|\theta_\varepsilon\|_{L^{q_0}} \le \sqrt{\lambda_2}. \tag{5.6}$$

In addition, if $1 < r < \frac{3}{2}$*, then*

$$\int_\Omega |\nabla\theta_\varepsilon|^r\,dx \le cL_\sigma^{r/(1-\sigma)}\big(1 + \|\theta_D\|^2_{W^{1,2}}\big), \quad \sigma = \frac{3-2r}{3-r}, \tag{5.7}$$

where L_σ *is the one in* (5.10) *below.*

Proof. By Lemma 4.6 (which is valid for the second formula of (5.1)), we have

$$\|\theta_\varepsilon^-\|^2_{W^{1,2}} \le c\big(\|f_\varepsilon\|_{\mathbf{L}^t}\|v_\varepsilon\|_{\mathbf{V}} + \|g_R\|_{L^{4/3}(\Gamma_R)} + \|g\|_{L^{6/5}(\Omega)}\big). \tag{5.8}$$

Using (5.8) and arguing as (4.48)-(4.76), we have

$$\begin{aligned}&\int_\Omega \frac{|\nabla\theta_\varepsilon|^2}{(1+(\theta_\varepsilon-d_0)^+)^{1+\sigma}}\,dx\\ &\le c\Big[(1+\frac{1}{\sigma})\Big(1+\|v_\varepsilon\|^2_{\mathbf{V}}+\|f_\varepsilon\|_{\mathbf{L}^t}\|v_\varepsilon\|_{\mathbf{V}}+\|g_R\|_{L^{4/3}(\Gamma_R)}+\|g\|_{L^{6/5}(\Omega)}\Big)^2\\ &+\|\theta_D\|^2_{W^{1,2}}+\|v_\varepsilon\|^2_{\mathbf{V}}+\Big(\|f_\varepsilon\|_{\mathbf{L}^t}\|v_\varepsilon\|_{\mathbf{V}}+\|g_R\|_{L^{4/3}(\Gamma_R)}+\|g\|_{L^{6/5}(\Omega)}\Big)^2\|v_\varepsilon\|^2_{\mathbf{V}}\Big].\end{aligned}\tag{5.9}$$

Using (5.5), from (5.9) we have

$$\begin{aligned}&\int_\Omega \frac{|\nabla\theta_\varepsilon|^2}{(1+(\theta_\varepsilon-d_0)^+)^{1+\sigma}}\,dx\\ &\le c\Big[(1+\frac{1}{\sigma})\Big(1+(1+\|f\|_{\mathbf{L}^t}+\|f_1\|_{\mathbf{V}^*})^2+\|f\|_{\mathbf{L}^t}(1+\|f\|_{\mathbf{L}^t}+\|f_1\|_{\mathbf{V}^*})\\ &\quad+\|g_R\|_{L^{4/3}(\Gamma_R)}+\|g\|_{L^{6/5}(\Omega)}\Big)^2+\|\theta_D\|^2_{W^{1,2}}+(1+\|f\|_{\mathbf{L}^t}+\|f_1\|_{\mathbf{V}^*})^2\\ &\quad+\Big(\|f\|_{\mathbf{L}^t}(1+\|f\|_{\mathbf{L}^t}+\|f_1\|_{\mathbf{V}^*})+\|g_R\|_{L^{4/3}(\Gamma_R)}+\|g\|_{L^{6/5}(\Omega)}\Big)^2\\ &\quad\times\left(1+\|f\|_{\mathbf{L}^t}+\|f_1\|_{\mathbf{V}^*}\right)^2\Big]\\ &\equiv L_\sigma.\end{aligned}\tag{5.10}$$

Using (5.10), in the same way as (4.81) we have

$$\|\theta_\varepsilon\|_{L^{q_0}}\le c\left(L_{\sigma_0}^{1/(1-\sigma_0)}(1+\|\theta_D\|_{W^{1,2}})+\|\theta_D\|_{W^{1,2}}\right)\equiv\overline{L}_{\sigma_0},\tag{5.11}$$

where L_{σ_0} is the one with σ_0 instead of σ in L_σ of (5.10). Putting

$$\sqrt{\lambda_2}=\overline{L}_{\sigma_0},\tag{5.12}$$

we get (5.6).

Now, for $1<r<\frac{3}{2}$ putting $\sigma=\frac{3-2r}{3-r}$ and repeating the arguments of (4.78)-(4.80), we have (5.7). □

Fixing $\lambda=\lambda_2$, under the condition

$$\max\{|\alpha_0|,|\alpha_1|\}\sqrt{\lambda_2}\le 1,\tag{5.13}$$

by (5.11), (5.12) we have

$$\frac{\lambda_2}{\max\{\lambda_2,\|\theta_\varepsilon\|^2_{L^{q_0}}\}}=1\quad\forall\varepsilon>0.$$

Therefore, by virtue of Lemmas 5.2, 5.3 we have

Theorem 5.4. *If*

$$\max\{|\alpha_0|,|\alpha_1|\}\sqrt{\lambda_2}\le 1,$$

then there exists a solution $(v_\varepsilon, \theta_\varepsilon) \in \boldsymbol{V} \times W^{1,2}(\Omega)$ *to the following problem*

$$\begin{cases} a_0(\theta_\varepsilon; v_\varepsilon, u) + a_2(v_\varepsilon, v_\varepsilon, u) + \big\langle \nabla\Phi_\varepsilon(v_\varepsilon), u\big\rangle - \big\langle (1-\alpha_0\theta_\varepsilon) f_\varepsilon, u\big\rangle = \langle f_1, u\rangle \quad \forall u \in \mathbf{V}, \\ b_0(\theta_\varepsilon; \theta_\varepsilon, \varphi) - \langle \gamma_\varepsilon(\theta_\varepsilon) v_\varepsilon, \nabla\varphi\rangle - \Big\langle \alpha_2\mu(\theta_\varepsilon)\dfrac{|\mathcal{E}(v_\varepsilon)|^2}{1+\varepsilon|\mathcal{E}(v_\varepsilon)|^2}, \varphi\Big\rangle \\ \quad - \langle \alpha_1\theta_\varepsilon f_\varepsilon \cdot v_\varepsilon, \varphi\rangle = \langle f_2, \varphi\rangle \qquad \forall\varphi \in W^{1,2}_{\Gamma_D}(\Omega), \end{cases}$$

$$\theta_\varepsilon - \theta_D \in W^{1,2}_{\Gamma_D}(\Omega),$$

and the solution satisfies:

$$\begin{aligned} &\|v_\varepsilon\|_{\boldsymbol{V}} \le \frac{c}{2\mu_0}(1 + \|f\|_{\boldsymbol{L}^t} + \|f_1\|_{\boldsymbol{V}^*}), \\ &\|\theta_\varepsilon^-\|_{W^{1,2}_{\Gamma_D}(\Omega)} \le c\Big(\|f\|_{\boldsymbol{L}^t}\|v_\varepsilon\|_{\boldsymbol{V}} + \|g_R\|_{L^{4/3}(\Gamma_R)} + \|g\|_{L^{6/5}(\Omega)}\Big), \\ &\int_\Omega |\nabla\theta|^r\, dx \le cL_\sigma^{r/(1-\sigma)}\big(1 + \|\theta_D\|^2_{W^{1,2}}\big) \quad \forall r, 1 < r < \frac{3}{2}, \end{aligned}$$

where $\sigma = \frac{3-2r}{3-r}$.

Now repeating the arguments in subsection 4.3 with the solutions of Theorem 5.4, we complete proof of Theorem 3.4. □

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