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GLOBAL EXISTENCE AND STRONG TRACE PROPERTY OF ENTROPY SOLUTIONS BY THE SOURCE-CONCENTRATION GLIMM SCHEME FOR NONLINEAR HYPERBOLIC BALANCE LAWS*

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Abstract. In this paper, we investigate the initial-boundary value problem for a nonlinear hyperbolic system of balance laws with source terms $a_x g$ and $a_t h$. We assume that the boundary data satisfy a linear or smooth nonlinear relation. The generalized Riemann and boundary Riemann solutions are provided with the variation of a concentrated on a thin T-shaped region in each grid. We generalize Goodman's boundary interaction estimates [7], introduce a new version of Glimm scheme to construct the approximation solutions, and provide their stability by considering two types of functions of $a(x,t)$. The global existence of entropy solutions is established. Under some sampling condition, we find the entropy solutions converge to their boundary values in L^1_{loc} as x approaches the boundary. In addition, such boundary values match the boundary condition almost everywhere in t.

Key words. Nonlinear balance laws, initial-boundary value problem, Riemann problem, generalized Glimm scheme, concentration of source, wave interaction estimates, entropy solutions, boundary regularity.

Mathematics Subject Classification. 35L60, 35L65, 35L67.

1. Introduction.

We are interested with initial-boundary value problem (IBVP for short) of the following $n \times n$ hyperbolic system of balance laws

$$\begin{cases} u_t + f(a,u)_x = a_x g(a,u) + a_t h(a,u), & (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^+ \cup \{0\}, \\ \vartheta(u_B(t)) = \pi(t), & t \in \mathbb{R}^+ \cup \{0\}, \end{cases} \tag{1.1}$$

where $u(x,t) = (u_1(x,t), \cdots, u_n(x,t))$, $a = a(x,t)$ is a given Lipschitz function, $f = (f_1, \cdots, f_n)$, $g = (g_1, \cdots, g_n)$, $h = (h_1, \cdots, h_n)$ are smooth functions of (a,u), u_0 and u_B denote the initial and boundary data, and the third equation gives a nonlinear nonhomogeneous boundary condition. Following the idea of LeFloch [14] and Isaacson-Temple [11], we augment the first equation of problem (1.1) by adding the identical equation $a_t = a_t$ and obtain the following $(n+1) \times (n+1)$ system of balance laws

$$U_t + F(U)_x = a_x G(U) + a_t H(U), \tag{1.2}$$

where $U = (a,u)$, $F(U) = (0, f_1(U), \cdots, f_n(U))$, $G(U) = (0, g_1(U), \cdots, g_n(U))$, and $H(U) = (1, h_1(U), \cdots, h_n(U))$.

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In this paper, we consider the following IBVP that is a little more general than problem (1.1):

$$\begin{cases} U_t + F(U)_x = a_x G(U) + a_t H(U), & (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ U(x,0) = U_0(x) \in \Omega, & x \in \mathbb{R}^+ \cup \{0\}, \\ \Theta(U_B(t), \Pi(t)) = 0, & t \in \mathbb{R}^+ \cup \{0\}, \end{cases} \tag{1.3}$$

where Ω is an open region in $\mathbb{R}^{n+1}$, $\Theta : \mathbb{R}^{n+1} \times \mathbb{R}^{n-k+1} \to \mathbb{R}^{n-k+1}$ is a given linear or smooth nonlinear function, and $\Pi : \mathbb{R}^+ \to \mathbb{R}^{n-k+1}$ is a given function. Here $U_0(x) = (a_0(x), u_0(x)) := (a(x,0), u(x,0))$ and $U_B(t) = (a_B(t), u_B(t)) := (a(0,t), u(0,t))$ denote the initial and boundary data of (1.3) respectively. In addition, we assume that

- (A_1) T. V.$[a(\cdot,t)]$ is sufficiently small, where T. V. denotes the total variation.
- (A_2) The eigenvalues $\{\lambda_0(U), \lambda_1(U), \cdots, \lambda_n(U)\}$ of Jacobian matrix DF satisfy

$$\lambda_1(U) < \cdots < \lambda_k(U) < \lambda_0(U) = 0 < \lambda_{k+1}(U) < \cdots < \lambda_n(U), \quad U \in \Omega.$$

  Let $\{R_0^*(U), R_1(U), \cdots, R_n(U)\}$ denote the corresponding right eigenvectors.
- (A_3) Each i-th characteristic field except $i = 0$ is either genuinely nonlinear ($\nabla\lambda_i(U) \cdot R_i(U) \neq 0$) or linearly degenerate ($\nabla\lambda_i(U) \cdot R_i(U) \equiv 0$) for all $U \in \Omega$. For the genuinely nonlinear fields, we normalize $R_i(U)$ by $\nabla\lambda_i(U) \cdot R_i(U) = 1$.
- (A_4) $\Pi(t) \in \Omega'$ for some open region $\Omega' \in \mathbb{R}^{n-k+1}$ and T. V.$[\Pi]$ is sufficiently small.
- (A_5) Set $R_0(U) := \big(1, (D_u f)^{-1} \cdot (g - f_a)(U)\big)^T$. For $U_B \in \Omega$ and $\Pi \in \Omega'$, the function Θ satisfies the following condition:

$$D_{U_B}\Theta \cdot \begin{pmatrix} | & | & & | \\ R_0 & R_{k+1} & \cdots & R_n \\ | & | & & | \end{pmatrix} \quad \text{has rank } n-k+1,$$

where (A_1)–(A_3) are also needed in Cauchy problem, while (A_5) provides the existence and uniqueness of the solution for our boundary-Riemann problems (see section 2). We note that (A_2) implies the invertibility of $D_u f$, thus R_0 in (A_5) is well-defined. The conditions for the smallness of T. V.$[a(\cdot,t)]$ and T. V.$[\Pi]$ are given in the proof of Theorem 3.3.

An important case of system (1.3) is the 1-dimensional compressible Euler equations describing the dynamics of nozzle flows in general geometry:

$$\begin{aligned} \rho_t + (\rho v)_x &= -\frac{a_x}{a}\rho v - \frac{a_t}{a}\rho, \\ (\rho v)_t + (\rho v^2 + p)_x &= -\frac{a_x}{a}\rho v^2 - \frac{a_t}{a}\rho v, \\ (\rho E)_t + (\rho v E + p v)_x &= -\frac{a_x}{a}(\rho v E + p v) - \frac{a_t}{a}\rho E, \end{aligned}$$

where $a = a(x,t) > 0$ is the area of the cross section of a variable duct at position x and time t, and ρ, v, E, p represent the density, velocity, total energy, and pressure of the compressible gas, respectively.

We review some previous results related to this topic. The entropy solutions to the Riemann problem of

$$U_t + F(U)_x = 0 \tag{1.4}$$

were first obtained by Lax [13]. The solutions are self-similar functions consisting of constant states separated by elementary waves (rarefaction waves, shock waves and contact discontinuities). To the Cauchy problem of (1.4), the global existence of weak solutions was established by Glimm [6] when initial data is uniformly bounded and of small total variation.

Comparing with the Cauchy problem, the study of IBVP for (1.4) gains an extra diffuculty due to the complexities by the appearance of boundaries even if the boundary condition is assumed to be linear. To investigate this kind of problems, there are at least two classical methods: (1) the numerical approximation methods based on the Riemann problems, and (2) the vanishing viscosity method. Using method (1), Nishida and Smoller [16] and Liu [15] studied the piston and double piston problems for particular systems of gas dynamics and obtained the global existence of weak solutions. The general systems case for non-characteristic boundaries was first studied by Goodman [7]. Method (2) is interesting which can be generalized to the multidimensional case, cf. [19]. In 1988, Dubois and LeFloch [5] proved that the two formulations of boundary conditions based on these two methods are equivalent for linear systems and scalar nonlinear equations. Unfortunately, for the nonlinear systems case, the first formulation is more stringent than the second one and only the first one leads to a well-posed problem, cf. [12].

In 2010, Colombo and Guerra [4] considered the IBVP with non-local source and non-characteristic boundary of the form

$$\begin{cases} U_t + F(U)_x = G(U), & x > \gamma(t), \\ U(x,0) = U_0(x), & x \geq \gamma(0), \\ \Theta(U(t,\gamma(t))) = \Pi(t), & t \geq 0. \end{cases} \tag{1.5}$$

By using the wave front tracking method and the general metric space technique in [3], they obtained the well-posedness results for (1.5). But the initial data U_0 is assumed in L^1 (up to a constant) and this setting is not applicable to the problem of nozzle flow.

In recent papers [1, 10], the IBVP of hyperbolic balance laws for Fanno-Rayleigh flows and hydrodynamic escape problem were studied. As in this paper, the authors in [1, 10] also applied Glimm's method to obtain the global existence of the entropy solutions. However, they didn't deal with the boundary regularity of solutions.

In this paper, we wish to establish the global existence of entropy solutions to (1.3) based on the generalized Glimm's method with concentration of source [6], and generalize Goodman's framework [7] to the boundary regularity problem. In [7], the author considered conservation laws with non-characteristic boundaries and homogeneous boundary condition. We extend the results of [7] to the problem of nonlinear balance laws with nonlinear nonhomogeneous boundary condition. It is worthy to mention that, by using a non-staggered generalized Glimm scheme, we simplify the Van der Corput sampling condition in [7] by conditions (4.5)–(4.6). Since a depends on t, we are not able to find the global standing waves as a member of solutions for the Riemann problem. Motivated by [9], to construct the approximate solutions of (1.3), we consider a version of generalized Riemann problem that concentrates the variation of a on a thin T-shaped region in each grid (see sections 2–3). In general, there exist some technical difficulties to employ the concerntration method to construct the approximate solutions due to the low regularity of the solutions in nonlinear balance laws. The results of this paper, together with the ones of [2, 8] indicates that the source-concentration method also holds in the hyperbolic system of balance laws.

We now give the definition of entropy solutions to problem (1.3) and state our main result.

DEFINITION 1.1. *For a given vector-valued function U and a test function $\phi \in C_0^1([0,\infty)\times[0,\infty))$, we define the residual $R_\phi(U)$ of U by*

$$R_\phi(U) := \int_0^\infty \int_0^\infty \{U\phi_t + F(U)\phi_x + a_x G(U)\phi + a_t H(U)\phi\}\,dx\,dt + \int_0^\infty U_0(x)\phi(x,0)\,dx + \int_0^\infty F(U(0,t))\phi(0,t)\,dt.$$

Then a bounded measurable function U is called a weak solution of IBVP (1.3) *if $R_\phi(U)=0$ for all $\phi \in C_0^1([0,\infty)\times[0,\infty))$.*

DEFINITION 1.2. *Let $\mathcal{D} \subset \mathbb{R}^{n+1}$ be a convex set, and let $\mathcal{U} : \mathcal{D} \to \mathbb{R}$ and $\mathcal{F} : \mathcal{D} \to \mathbb{R}$. We say that $(\mathcal{U},\mathcal{F})$ is an entropy pair of* (1.3) *if $\mathcal{U}$ is convex on $\mathcal{D}$ and $(\mathcal{U},\mathcal{F})$ satisfies*

$$D\mathcal{F} = (D\mathcal{U})(DF) \quad \text{on } \mathcal{D}.$$

Furthermore, U is called an entropy solution of (1.3) *if U is a weak solution and satisfies*

$$\int_0^\infty \int_0^\infty \{\mathcal{U}(U)\phi_t + \mathcal{F}(U)\phi_x + D\mathcal{U}[a_x G(U) + a_t H(U)]\phi\}\,dx\,dt + \int_0^\infty \mathcal{U}(U_0(x))\phi(x,0)\,dx + \int_0^\infty \mathcal{F}(U(0,t))\phi(0,t)\,dt \ge 0 \tag{1.6}$$

for every entropy pair $(\mathcal{U},\mathcal{F})$ and positive test function $\phi \in C_0^1([0,\infty)\times[0,\infty))$.

Throughout this paper, we assume that $a(x,t)$ satisfies one of the following two conditions:

(B) $a(x,t)$ is independent of t if $t \ge T$ for some $T > 0$, and

$$|a_t(x,t)| \le \nu, \quad \mathrm{T.V.}[a(\cdot,t+\Delta t) - a(\cdot,t)] \le \nu\Delta t, \quad \text{for some small constant } \nu;$$

(C) $a(x,t)$ is a smooth function on $\mathbb{R}\times[0,\infty)$, and

$$\int_0^\infty \sup_{x\ge 0} |a_t(x,t)|\,dt \le \nu, \quad \|a_{xt}\|_{L^1([0,\infty)\times[0,\infty))} \le \nu, \quad \text{for some small constant } \nu.$$

THEOREM 1.3. *Consider IBVP* (1.3), *where a, F, G, H, Θ, and Π satisfy conditions (A_1)–(A_5). Assume that $\mathrm{T.V.}[U_0]$ is sufficiently small and $a(x,t)$ satisfies one of conditions* (B) *and* (C) *for sufficiently small ν. Let $\{U^\varepsilon_{\theta,\Delta x}\}$ be the sequence of approximate solutions for* (1.3) *by the generalized Glimm scheme described in section 3. Then there exists a null set $N \in \Phi$ and two sequences $\{\varepsilon_i\}, \{\Delta x_i\} \to 0$ such that if $\theta \in \Phi \setminus N$,*

$$U_\theta(x,t) := \lim_{\varepsilon_i,\,\Delta x_i \to 0} U^{\varepsilon_i}_{\theta,\Delta x_i}(x,t)$$

is an entropy solution to (1.3). *If we further assume that $\mathrm{T.V.}[a(x,\cdot)]$ is continuous near the boundary and the sampling sequence θ satisfies conditions* (4.5)–(4.6), *then $U_\theta(x,t)$ satisfies the boundary condition*

$$\Theta(U_\theta(0,t),\Pi(t)) = 0 \quad \text{a.e. in } t$$

in the sense

$$\lim_{d\to 0^+} U_\theta(d,\cdot) = U_\theta(0,\cdot) \quad in\ L^1_{\text{loc}}.$$

The rest of this paper is organized as follows. In section 2, we consider a new version of generalized Riemann and boundary-Riemann problems, and construct the solutions. In section 3, the approximate solutions of (1.3) are constructed by inventing a version of generalized Glimm scheme (GGS for short). We obtain the wave interaction estimates and the boundary interaction estimates so that the stability of GGS is established, which leads to the weak convergence of a subsequence of the approximate solutions. The weak limits of the approximate solutions are shown to be the global entropy solutions of (1.3) by the entropy inequality. Section 4 is devoted to dealing with the boundary regularity of the entropy solutions. To show that the entropy solutions have strong traces, we show the boundedness for the vertical total variation of the approximate solutions by wave tracing method under the assumption of sampling conditions (4.5)–(4.6). We also prove that the entropy solutions satisfy the boundary condition in (1.3) almost everywhere.

2. Generalized Riemann and boundary-Riemann problems. In this section, we construct the solutions to the generalized Riemann and boundary-Riemann problems of (1.3). For $\kappa,\delta > 0$ and $(x_0,t_0)\in\mathbb{R}\times\mathbb{R}^+$, we consider a small grid $D^{\kappa\delta} = D^{\kappa\delta}(x_0,t_0) := \{(x,t) : |x-x_0| \le \kappa,\ 0 \le t-t_0 \le \delta\}$. Let $\mathring{D}^{\kappa\delta}$ denote the interior of $D^{\kappa\delta}$. The traditional Riemann problem of (1.3) centered at (x_0,t_0) reads

$$\begin{cases} U_t + F(U)_x = a_x G(U) + a_t H(U), & (x,t)\in \mathring{D}^{\kappa\delta},\\ U(x,0) = \begin{cases} U_L, & x<x_0,\\ U_R, & x>x_0,\end{cases} \end{cases}$$

where $F(U)$, $G(U)$, $H(U)$ satisfy the conditions (A_2)–(A_3), $U_L = (a_L,u_L), U_R = (a_R,u_R)\in\Omega$ are two nearby constant states, and

$$a(x,t) = \begin{cases} a_L & \text{if } x<x_0,\\ a_R & \text{if } x>x_0.\end{cases}$$

However, we observe that, since $a(x,t)$ is not continuous, the source terms $a_xG(U)$ and $a_tH(U)$ are not defined in the sense of distributions. The technique of asymptotic expansion to a in [18] also fails in our case. To overcome this difficulty, Chou-Lin [2] choose $0<\varepsilon\ll 1$, decompose $D^{\kappa\delta}$ into the following six sub-regions:

$$\begin{aligned}
D^\varepsilon_L &:= \{(x,t) : -\kappa \le x-x_0 < -\varepsilon\kappa,\ 0\le t-t_0 \le \delta-\varepsilon\delta\},\\
D^\varepsilon_M &:= \{(x,t) : |x-x_0| \le \varepsilon\kappa,\ 0\le t-t_0 \le \delta-\varepsilon\delta\},\\
D^\varepsilon_R &:= \{(x,t) : \varepsilon\kappa < x-x_0 \le \kappa,\ 0\le t-t_0 \le \delta-\varepsilon\delta\},\\
\widetilde{D}^\varepsilon_L &:= \{(x,t) : -\kappa \le x-x_0 < -\varepsilon\kappa,\ \delta-\varepsilon\delta < t-t_0 \le \delta\},\\
\widetilde{D}^\varepsilon_M &:= \{(x,t) : |x-x_0| \le \varepsilon\kappa,\ \delta-\varepsilon\delta < t-t_0 < \delta\},\\
\widetilde{D}^\varepsilon_R &:= \{(x,t) : \varepsilon\kappa < x-x_0 \le \kappa,\ \delta-\varepsilon\delta < t-t_0 \le \delta\},
\end{aligned}$$

and set

$$D^\varepsilon_{\mathcal{B}} := D^\varepsilon_L \cup D^\varepsilon_M \cup D^\varepsilon_R, \qquad D^\varepsilon_{\mathcal{T}} := \widetilde{D}^\varepsilon_L \cup \widetilde{D}^\varepsilon_M \cup \widetilde{D}^\varepsilon_R,$$

see Figure 1.

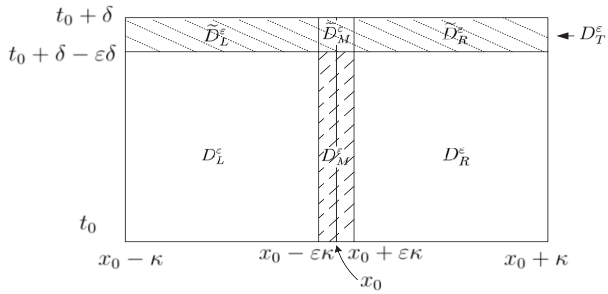

FIG. 1. *T-shaped region* $D_{\mathcal{T}}^\varepsilon \cup D_M^\varepsilon$.

They also reformulate a by

$$a^\varepsilon(x,t) := \begin{cases} \phi^\varepsilon(x), & (x,t)\in D_{\mathcal{B}}^\varepsilon, \\ (1-\frac{t_0+\delta-t}{\varepsilon\delta})\tilde{\phi}^\varepsilon(x) + \frac{t_0+\delta-t}{\varepsilon\delta}\phi^\varepsilon(x), & (x,t)\in D_{\mathcal{T}}^\varepsilon, \end{cases}$$

where $a_L = a(x_0-\kappa, t_0)$, $a_R = a(x_0+\kappa, t_0)$, $a'_L = a(x_0-\kappa, t_0+\delta)$, $a'_R = a(x_0+\kappa, t_0+\delta)$, and

$$\phi^\varepsilon(x) = \begin{cases} a_L, & -\kappa \le x-x_0 < -\varepsilon\kappa, \\ (\frac{1}{2}-\frac{x-x_0}{2\varepsilon\kappa})a_L + (\frac{1}{2}+\frac{x-x_0}{2\varepsilon\kappa})a_R, & |x-x_0|\le \varepsilon\kappa, \\ a_R, & \varepsilon\kappa < x-x_0 \le \kappa, \end{cases}$$

$$\tilde{\phi}^\varepsilon(x) = \begin{cases} a'_L, & -\kappa \le x-x_0 < -\varepsilon\kappa, \\ (\frac{1}{2}-\frac{x-x_0}{2\varepsilon\kappa})a'_L + (\frac{1}{2}+\frac{x-x_0}{2\varepsilon\kappa})a'_R, & |x-x_0|\le \varepsilon\kappa, \\ a'_R, & \varepsilon\kappa < x-x_0 \le \kappa, \end{cases}$$

and consider the generalized Riemann problem on $D^{\kappa\delta}(x_0,t_0)$, denoted by $\mathcal{R}(x_0,t_0)$, as follows:

$$\mathcal{R}(x_0,t_0): \quad \begin{cases} U_t^\varepsilon + F(U^\varepsilon)_x = a_x^\varepsilon G(U^\varepsilon), & (x,t)\in \mathring{D}^{\kappa\delta}\backslash\mathring{D}_{\mathcal{T}}^\varepsilon, \\ U_t^\varepsilon = a_t^\varepsilon H(U^\varepsilon), & (x,t)\in \mathring{D}_{\mathcal{T}}^\varepsilon, \\ U^\varepsilon(x,t_0) = \begin{cases} U_L, & -\kappa \le x-x_0 < -\varepsilon\kappa, \\ \Psi^\varepsilon(x), & |x-x_0|\le\varepsilon\kappa, \\ U_R, & \varepsilon\kappa < x-x_0 \le \kappa, \end{cases} \end{cases} \tag{2.1}$$

where $U^\varepsilon = (a^\varepsilon, u^\varepsilon)$, while $\Psi^\varepsilon(x) = (\phi^\varepsilon(x), \psi^\varepsilon(x))$ is a function connecting $U_L = (a_L, u_L)$ at $x = x_0 - \varepsilon\kappa$ and $U_R = (a_R, u_R)$ at $x = x_0 + \varepsilon\kappa$, and each component of $\psi^\varepsilon(x)$ is monotone. The solvability of $\mathcal{R}(x_0,t_0)$ by the modified Lax method is given as follows.

THEOREM 2.1 ([2]). *Given* $0<\varepsilon\ll 1$, *consider problem* $\mathcal{R}(x_0,t_0)$ *where* F *and* G *satisfy conditions* (A_2)–(A_3). *Suppose* $U_L\in\Omega$, *then there exists a neighborhood* $N\subset\Omega$ *of* U_L *such that if* $U_R\in N$, $\mathcal{R}(x_0,t_0)$ *has a unique solution* U^ε. *On* $D_{\mathcal{B}}^\varepsilon$, *the solution consists of at most* $n+2$ *constant states separated by shocks, rarefaction waves,* *or contact discontinuities U_i^ε, $i = 1, \cdots, n$, in $D_L^\varepsilon \cup D_R^\varepsilon$, and a standing wave U_0^ε on D_M^ε. The integral curve of U_i^ε, $i = 0, 1, \cdots, n$, starting at $\bar{U}$ can be parameterized as*

$$U_i^\varepsilon(\eta_i; \bar{U}) = \bar{U} + \eta_i R_i(\bar{U}) + \frac{\eta_i^2}{2} R_i \cdot \nabla R_i(\bar{U}) + O(\eta_i^3), \tag{2.2}$$

where $\eta_0 = \phi^\varepsilon - a_L$. All the constant states and waves evolve along with the change of the value of a^ε for $(x,t) \in D_{\mathcal{T}}^\varepsilon$; more precisely, if we denote $U^{\varepsilon,x}(t) = U^\varepsilon(x,t)$ for $(x,t) \in D_{\mathcal{T}}^\varepsilon$ and let $\tilde{U}^\varepsilon(x)$ be the solution on $D_{\mathcal{B}}^\varepsilon$ restricted to the time section $t = t_0 + \delta - \varepsilon\delta$, then the solution curve starting at $\bar{U} = \tilde{U}^\varepsilon(x)$ can be described by

$$\frac{dU^{\varepsilon,x}}{d\xi} = H(U^{\varepsilon,x}), \qquad U^{\varepsilon,x}(0) = \tilde{U}^\varepsilon(x), \tag{2.3}$$

where $\xi = a^\varepsilon(x,t) - a^\varepsilon(x, t_0 + \delta - \varepsilon\delta)$, and its Taylor expansion can be expressed as

$$U^{\varepsilon,x}(\xi) = \bar{U} + \xi H(\bar{U}) + \frac{\xi^2}{2} H \cdot \nabla H(\bar{U}) + O(\xi^3). \tag{2.4}$$

For $i = 0, 1, \cdots, n$, let $T_{\eta_i}^i : \Omega \to \mathbb{R}^{n+1}$ be a one-parameter family of transformations defined by

$$T_{\eta_i}^i(U) = U_i^\varepsilon(\eta_i; U), \quad |\eta_i| < \eta^*, \tag{2.5}$$

for some $\eta^* > 0$, which is C^2 in η_i. Let U_L be any given state in Ω and define $V = \{(\eta_0, \cdots, \eta_n) \in \mathbb{R}^{n+1} : |\eta_i| < \eta^*\}$. Then Theorem 2.1 states that, if $U_R \in \Omega$ and $|U_L - U_R|$ is sufficiently small, then there exists a unique $(\eta_0, \cdots, \eta_n) \in V$ such that $T_{\eta_n}^n \cdots T_{\eta_{k+1}}^{k+1} \cdot T_{\eta_0}^0 \cdot T_{\eta_k}^k \cdots T_{\eta_1}^1(U_L) = U_R$.

We now turn to the boundary-Riemann problem of (1.3). For the same $\kappa, \delta > 0$ as above and some $t_0 > 0$, we consider a small boundary grid

$$\mathcal{D}^{\kappa\delta} = \mathcal{D}^{\kappa\delta}(t_0) := \{(x,t) : 0 \le x \le \kappa,\ 0 \le t - t_0 \le \delta\},$$

which can be divided into the following four sub-regions:

$$\begin{aligned}
\mathcal{D}_M^\varepsilon &:= \{(x,t) : 0 \le x \le \varepsilon\kappa,\ 0 \le t - t_0 \le \delta - \varepsilon\delta\}, \\
\mathcal{D}_R^\varepsilon &:= \{(x,t) : \varepsilon\kappa < x \le \kappa,\ 0 \le t - t_0 \le \delta - \varepsilon\delta\}, \\
\widetilde{\mathcal{D}}_M^\varepsilon &:= \{(x,t) : 0 \le x \le \varepsilon\kappa,\ \delta - \varepsilon\delta < t - t_0 < \delta\}, \\
\widetilde{\mathcal{D}}_R^\varepsilon &:= \{(x,t) : \varepsilon\kappa < x \le \kappa,\ \delta - \varepsilon\delta < t - t_0 \le \delta\},
\end{aligned}$$

see Figure 2. We set

$$\mathcal{D}_{\mathcal{T}}^\varepsilon := \widetilde{\mathcal{D}}_M^\varepsilon \cup \widetilde{\mathcal{D}}_R^\varepsilon$$

and reformulate a by

$$\bar{a}^\varepsilon(x,t) := \begin{cases} \phi_B^\varepsilon(x), & (x,t) \in \mathcal{D}_M^\varepsilon \cup \mathcal{D}_R^\varepsilon, \\ (1 - \frac{t_0+\delta-t}{\varepsilon\delta})\tilde{\phi}_B^\varepsilon(x) + \frac{t_0+\delta-t}{\varepsilon\delta}\phi_B^\varepsilon(x), & (x,t) \in \mathcal{D}_{\mathcal{T}}^\varepsilon, \end{cases}$$

where $a_B = a(0, t_0)$, $a_R = a(\kappa, t_0)$, $a_B' = a(0, t_0 + \delta)$, $a_R' = a(\kappa, t_0 + \delta)$, and

$$\phi_B^\varepsilon(x) = \begin{cases} (1 - \frac{x}{\varepsilon\kappa})a_B + \frac{x}{\varepsilon\kappa}a_R, & 0 \le x \le \varepsilon\kappa, \\ a_R, & \varepsilon\kappa < x \le \kappa, \end{cases}$$

$$\tilde{\phi}_B^\varepsilon(x) = \begin{cases} (1-\frac{x}{\varepsilon\kappa})a'_B + \frac{x}{\varepsilon\kappa}a'_R, & 0 \le x \le \varepsilon\kappa, \\ a'_R, & \varepsilon\kappa < x \le \kappa. \end{cases}$$

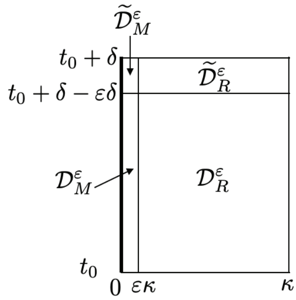

FIG. 2. *Boundary grid.*

Then the corresponding boundary-Riemann problem of (1.3) on $\mathcal{D}^{\kappa\delta}(t_0)$, denoted by $\mathcal{BR}(t_0)$, can be expressed as

$$\mathcal{BR}(t_0): \quad \begin{cases} U_t^\varepsilon + F(U^\varepsilon)_x = \bar{a}_x^\varepsilon G(U^\varepsilon), & (x,t) \in \mathring{\mathcal{D}}^{\kappa\delta} \backslash \mathring{\mathcal{D}}_T^\varepsilon, \\ U_t^\varepsilon = \bar{a}_t^\varepsilon H(U^\varepsilon), & (x,t) \in [0,\kappa] \times (t_0+\delta-\varepsilon\delta, t_0+\delta), \\ U^\varepsilon(0,t) = U_B, & t_0 \le t \le t_0+\delta-\varepsilon\delta, \\ U^\varepsilon(x,t_0) = \begin{cases} \Psi_B^\varepsilon(x), & 0 \le x \le \varepsilon\kappa, \\ U_R, & \varepsilon\kappa < x \le \kappa, \end{cases} \end{cases} \tag{2.6}$$

where $U^\varepsilon = (\bar{a}^\varepsilon, u^\varepsilon)$, the boundary data U_B satisfies the condition

$$\Theta(U_B, \Pi) = 0, \qquad \Pi \in \Omega' \text{ is given,} \tag{2.7}$$

while $\Psi_B^\varepsilon(x) = (\phi_B^\varepsilon(x), \psi_B^\varepsilon(x))$ is a function connecting $U_B = (a_B, u_B)$ at $x = 0$ and $U_R = (a_R, u_R)$ at $x = \varepsilon\kappa$, and each component of $\psi_B^\varepsilon(x)$ is monotone.

To solve problem $\mathcal{BR}(t_0)$, we first focus on the region $\mathcal{D}_M^\varepsilon \cup \mathcal{D}_R^\varepsilon$. By the same argument as problem (2.1) on D_R^ε, the solution of (2.6) on the region $\mathcal{D}_R^\varepsilon$ consists of constant states separated by rarefaction waves, shock waves, and contact discontinuities U_i^ε, $i = k+1, \cdots, n$. Similar to (2.1) on D_M^ε, the solution of (2.6) on the region $\mathcal{D}_M^\varepsilon$ forms a smooth standing wave U_0^ε. For each $i = 0, k+1, \cdots, n$, the integral curve of U_i^ε starting at $\bar{U}$ can also be parameterized as the form in (2.2). Let $T_{\eta_i}^i : \Omega \to \mathbb{R}^{n+1}$, $i = 0, k+1, \cdots, n$, be the one-parameter family of transformations defined as in (2.5) and set $V' = \{(\eta_0, \eta_{k+1}, \cdots, \eta_n) \in \mathbb{R}^{n-k+1} : |\eta_i| < \eta^*\}$. Similar to Theorem 2.1, to get the solvability of (2.6), we need to guarantee that, if $U_B, U_R \in \Omega$ and $|U_B - U_R|$ is sufficiently small, there exists a $(\eta_0, \eta_{k+1}, \cdots, \eta_n) \in V'$ such that $T_{\eta_n}^n \cdots T_{\eta_{k+1}}^{k+1} \cdot T_{\eta_0}^0(U_B) = U_R$. In view of (2.2), that is to say

$$U_B + \eta_0 R_0(U_B) + \sum_{i=k+1}^n \eta_i R_i(U_B) + O\Big(\eta_0^2 + \sum_{i=k+1}^n \eta_i^2\Big) = U_R. \tag{2.8}$$

The relation (2.8) enables us to treat U_B as function of $\eta_0, \eta_{k+1}, \cdots, \eta_n$ and U_R:

$$\begin{aligned} U_B &= U_B(\eta_0, \eta_{k+1}, \cdots, \eta_n; U_R) \\ &= U_R - \eta_0 R_0(U_R) - \sum_{i=k+1}^{n} \eta_i R_i(U_R) + O\Big(\eta_0^2 + \sum_{i=k+1}^{n} \eta_i^2\Big), \end{aligned}$$

which implies that

$$\frac{\partial U_B}{\partial \eta_i}\Big|_{(0,\cdots,0;U_R)} = -R_i(U_R), \qquad i = 0, k+1, \cdots, n. \tag{2.9}$$

Since $\Pi \in \Omega'$ is given, the boundary condition (2.7) gives $n-k+1$ equations with $(n-k+2)$ unknowns $\eta_0, \eta_{k+1}, \cdots, \eta_n$, and U_R:

$$\Theta\big(U_B(\eta_0, \eta_{k+1}, \cdots, \eta_n; U_R), \Pi\big) = 0.$$

If we denote $\eta = (\eta_0, \eta_{k+1}, \cdots, \eta_n)$, then assumption ($A_5$) and (2.9) yield that the matrix $D_\eta \Theta$ is invertible. Applying the implicit function theorem, we will be able to solve for $\eta_0, \eta_{k+1}, \cdots, \eta_n$ and hence U_B in terms of U_R. Therefore, assumption (A_5) gives the existence and uniqueness of the solution to problem $\mathcal{BR}(t_0)$ on $\mathcal{D}_M^\varepsilon \cup \mathcal{D}_R^\varepsilon$. The solution on $\mathcal{D}_\mathcal{T}^\varepsilon$ is just the same as in (2.3).

THEOREM 2.2. *Given $0 < \varepsilon \ll 1$, consider problem $\mathcal{BR}(t_0)$, where F, G and Θ satisfy conditions* (A_2)–(A_3), (A_5). *Suppose that $\bar{U}_B \in \Omega$ and $\bar{\Pi} \in \Omega'$ satisfy the relation $\Theta(\bar{U}_B, \bar{\Pi}) = 0$. Then there exist neighborhoods $N \subset \Omega$ of $\bar{U}_B$ and $N' \subset \Omega'$ of $\bar{\Pi}$ such that if $U_R \in N$ and $\Pi \in N'$ then condition* (2.7) *gives a unique boundary data $U_B \in \Omega$ and $\mathcal{BR}(t_0)$ admits a unique solution. On $\mathcal{D}_R^\varepsilon$, the solution consists of at most $n-k+1$ constant states separated by shocks, rarefaction waves, or contact discontinuities. On $\mathcal{D}_M^\varepsilon$, the solution forms a standing wave. All the constant states and waves evolve along with the change of the value of a^ε for $(x,t) \in \mathcal{D}_\mathcal{T}^\varepsilon$.*

3. Generalized Glimm scheme; global existence of entropy solutions. In this section, we introduce a non-staggered generalized Glimm scheme for the construction of approximate solutions of (1.3), and investigate its stability. To describe the scheme, we let $x_m = m\Delta x$, $t_p = p\Delta t$, $m, p \in \mathbb{N} \cup \{0\}$, and partition the quarter plane $[0,\infty)\times[0,\infty)$ into the nonoverlapping union of the grids $D^{\kappa\delta}(x_m, t_p) = \{(x,t) : |x - x_m| \le \Delta x,\ 0 \le t - t_p \le \Delta t\}$ and $\mathcal{D}^{\kappa\delta}(t_p) := \{(x,t) : 0 \le x \le \kappa,\ 0 \le t - t_p \le \delta\}$, $m, p \in \mathbb{N} \cup \{0\}$, where $\kappa = \Delta x$ and $\delta = \Delta t$. We assume that Δx and Δt satisfy the Courant-Friedrichs-Lewy (C-F-L) condition

$$\frac{\Delta x}{\Delta t} > \sup\{|\lambda_i(U)| : U \in \Omega,\ i = 1, 2, \cdots, n\},$$

which avoids the interaction of waves in the same time strip if the parameter ε is sufficiently small. Throughout this paper, we assume the ratio $\Delta x/\Delta t = \mu$ is fixed.

Next, we approximate the initial data and boundary data in (1.3) as follows.

$(\mathrm{I})_0$ For $m \in 2\mathbb{N}$, the initial data $U_0^\varepsilon(x)$ is given by

$$U_0^\varepsilon(x) = \begin{cases} U_{m-1}^0 := U_0(x_{m-1}), & x_{m-1} \le x < x_m - \varepsilon\Delta x, \\ \Psi_m^0(x), & |x - x_m| \le \varepsilon\Delta x, \\ U_{m+1}^0 := U_0(x_{m+1}), & x_m + \varepsilon\Delta x < x \le x_{m+1}, \end{cases}$$

where $\Psi_m^0(x) := (\phi_0^\varepsilon(x), \psi_0^\varepsilon(x))$ is a smooth monotone function connecting U_{m-1}^0 and U_{m+1}^0.

(II)$_0$ If $m = 0$, the initial data $U_0^\varepsilon(x)$ is given by

$$U_0^\varepsilon(x) = \begin{cases} \Psi_B^0(x), & 0 \le x \le \varepsilon\Delta x, \\ U_1^0 := U_0(x_1), & \varepsilon\Delta x < x \le x_1, \end{cases}$$

while the boundary data $U_{0,B}^\varepsilon(t) = U_B^0$, $0 \le t \le \Delta t - \varepsilon\Delta t$, is chosen to satisfy

$$\Theta(U_B^0, \Pi(0)) = 0,$$

where $\Psi_B^0(x) := (\phi_{0,B}^\varepsilon(x), \psi_{0,B}^\varepsilon(x))$ is a smooth monotone function connecting U_B^0 and U_1^0.

By solving all the problems $\mathcal{R}(x_m, 0)$ and $\mathcal{BR}(0)$, we get an approximate solution of (1.3) on the zero time strip.

For $p \ge 1$, let $U^{p-1}(x,t)$ denote the approximate solutions constructed by this scheme in the $(p-1)$-th time step. The initial data $U_p^\varepsilon(x, t_p)$ and the boundary data $U_{p,B}^\varepsilon(t)$ in the p-th time step are described as follows:

(I)$_p$ For $m \in 2\mathbb{N}$, $U_p^\varepsilon(x, t_p)$ is given by

$$U_p^\varepsilon(x, t_p) = \begin{cases} U_{m-1}^p := U^{p-1}(x_{m-1} + (1-\varepsilon)\theta_p\Delta x, t_p), & x_{m-1} \le x < x_m - \varepsilon\Delta x, \\ \Psi_m^p(x), & |x - x_m| \le \varepsilon\Delta x, \\ U_{m+1}^p := U^{p-1}(x_{m+1} + (1-\varepsilon)\theta_p\Delta x, t_p), & x_m + \varepsilon\Delta x < x \le x_{m+1}, \end{cases}$$

where $\theta_p \in (-1,1)\backslash\{0\}$ is a random number and $\Psi_m^p(x) = (\phi_p^\varepsilon(x), \psi_p^\varepsilon(x))$ is a smooth monotone function connecting U_{m-1}^p and U_{m+1}^p.

(II)$_p$ If $m = 0$, the initial data $U_p^\varepsilon(x, t_p)$ is given by

$$U_p^\varepsilon(x, t_p) = \begin{cases} \Psi_B^p(x), & 0 \le x \le \varepsilon\Delta x, \\ U_1^p := U_{p-1}(x_1 + \theta_p\Delta x, t_p), & \varepsilon\Delta x < x \le x_1, \end{cases}$$

while the boundary data $U_{p,B}^\varepsilon(t) = U_B^p$, $t_p \le t \le t_p + \Delta t - \varepsilon\Delta t$, is chosen to satisfy

$$\Theta(U_B^p, \Pi(t_p)) = 0,$$

where θ_p is the same as in (I)$_p$ and $\Psi_B^p(x) := (\phi_{p,B}^\varepsilon(x), \psi_{p,B}^\varepsilon(x))$ is a smooth monotone function connecting U_B^p and U_1^p.

Again, by solving all the problems $\mathcal{R}(x_m, t_p)$ and $\mathcal{BR}(t_p)$, we obtain an approximate solution on the p-th time strip.

Let $\theta = \{\theta_1, \theta_2, \cdots\}$ be a sequence of sampling numbers with uniform distribution on $(-1,1) \setminus \{0\}$, where θ_p and θ_q are independent if $|p-q| > M$ for some constant $M > 0$. This M can be chosen the same as in conditions (4.5) and (4.6). For convenience, we denote by Φ the collection of the sampling sequences. For any fixed $\theta \in \Phi$, we repeat the above process for each time step and then obtain an approximate solution $U_{\theta,\Delta x}^\varepsilon(x,t)$ of (1.3). We call this process as the generalized Glimm scheme (GGS for short).

Suppose that $\theta = \{\theta_1, \theta_2, \cdots\} \in \Phi$ is any given sampling sequence. For $m \in 2\mathbb{N}$, $p \in \mathbb{N}$, we define the *mesh points* by $W' = (x_{m-1} + (1-\varepsilon)\theta_p\Delta x, t_p)$, $W = (x_{m-1} + (1-\varepsilon)\theta_p\Delta x, t_p - \varepsilon\Delta t)$, $E' = (x_{m+1} + (1-\varepsilon)\theta_p\Delta x, t_p)$, $E = (x_{m+1} + (1-\varepsilon)\theta_p\Delta x, t_p - \varepsilon\Delta t)$,

$$S = \begin{cases} (x_{m-1} + (1-\varepsilon)\theta_{p-1}\Delta x, t_{p-1}) & \text{if } \theta_p < 0, \\ (x_{m+1} + (1-\varepsilon)\theta_{p-1}\Delta x, t_{p-1}) & \text{if } \theta_p > 0, \end{cases}$$

and

$$N = \begin{cases} (x_{m-1} + (1-\varepsilon)\theta_{p+1}\Delta x, t_{p+1} - \varepsilon\Delta t) & \text{if } \theta_p < 0, \\ (x_{m+1} + (1-\varepsilon)\theta_{p+1}\Delta x, t_{p+1} - \varepsilon\Delta t) & \text{if } \theta_p > 0. \end{cases}$$

If $m = 0$, we set $W'_B, = (0, t_p + \frac{\Delta t}{2})$ and $W_B = (0, t_p - \frac{\Delta t}{2})$. To state the wave interaction estimates and get the stability of GGS, we decompose the (x,t)-quarter-plane by the boundaries of the following three types of polygonal regions:

(1) the interior hexagonal region centered at (x_m, t_p), $m \in 2\mathbb{N}$, $p \in \mathbb{N}$, has vertices W', W, E', E, S, and N;

(2) if $\theta_p > 0$, the boundary hexagonal region centered at $(0, t_p)$, $p \in \mathbb{N}$, has vertices W'_B, W_B, E', E, S, and N with $m = 0$;

(3) if $\theta_p < 0$, the boundary quadrilateral region centered at $(0, t_p)$, $p \in \mathbb{N}$, has vertices W'_B, W_B, E', and E with $m = 0$,

see Figure 3.

An unbounded piecewise linear curve is called a *mesh curve* if it lies on the boundaries of the polygonal regions going from W' to N or W, from W to S or W', from W'_B to N or E', or from W_B to S or E. If I is a mesh curve, then I divides the (x,t)-quarter-plane into I^+ and I^- parts such that I^- contains $t = 0$. We say two mesh curves $I_2 > I_1$ if every point of I_2 is either on I_1 or contained in I_1^+. And I_2 is called an *immediate successor* of I_1 if $I_2 > I_1$ and they differ by a single polygonal region.

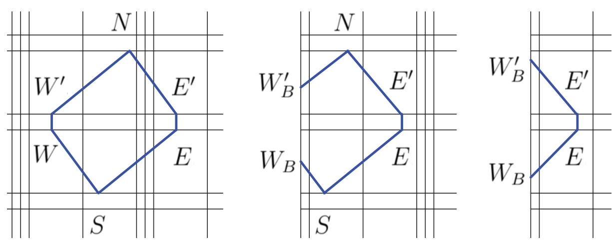

FIG. 3. *The hexagonal region, the boundary hexagonal region, and the boundary quadrilateral region.*

The wave interaction estimates on the single interior hexagonal region Γ centered at (x_m, t_p) were given in [2]. To state it, we let $U'_L = U^\varepsilon_{\theta,\Delta x}(W')$, $U'_R = U^\varepsilon_{\theta,\Delta x}(E')$, $U_L = U^\varepsilon_{\theta,\Delta x}(W)$, $U_R = U^\varepsilon_{\theta,\Delta x}(E)$, and $U_M = U^\varepsilon_{\theta,\Delta x}(S)$. Suppose that (U'_L, U'_R) denotes the approximate solution $U^\varepsilon_{\theta,\Delta x}$ obtained from problem $\mathcal{R}(x_m, t_p)$ and consisting of constant states $U_0 = U'_L, \cdots, U_k, U^\#_k, \cdots, U^\#_n = U'_R$ separated by the waves with the parameterizations $T^0_{\eta_0}(U_k) = U^\#_k$, $T^i_{\eta_i}(U_{i-1}) = U_i$ for $i = 1, \cdots, k$, and $T^i_{\eta_i}(U^\#_{i-1}) = U^\#_i$ for $i = k+1, \cdots, n$. Then (U'_L, U'_R) can be expressed as

$$(U'_L, U'_R) = [(U_0, \cdots, U_k, U^\#_k, \cdots, U^\#_n)/\eta := (\eta_0, \cdots, \eta_n)]. \tag{3.1}$$

We call $|\eta_i|$ the wave strength of the i-wave and write $|\eta| = \sum_{i=0}^n |\eta_i|$. Similarly, we let

$$\begin{aligned} (U_L, U_M) &= [(\bar{U}_0, \cdots, \bar{U}_k, \bar{U}^\#_k, \cdots, \bar{U}^\#_n)/\alpha := (\alpha_0, \cdots, \alpha_n)], \\ (U_M, U_R) &= [(\widetilde{U}_0, \cdots, \widetilde{U}_k, \widetilde{U}^\#_k, \cdots, \widetilde{U}^\#_n)/\beta := (\beta_0, \cdots, \beta_n)], \end{aligned} \tag{3.2}$$

denote the approximation solutions $U^{\varepsilon}_{\theta,\Delta x}$ obtaining from the corresponding generalized Riemann problems connecting U_L to U_M and U_M to U_R respectively, see Figure 4. We say that α and β are *incoming waves* of Γ, η are *outgoing waves* of Γ, and that the i-wave α_i and j-wave β_j are *approaching* if either (i) $\lambda_i > \lambda_j$ or (ii) $i = j$ and at lease one wave is a shock, cf. [17].

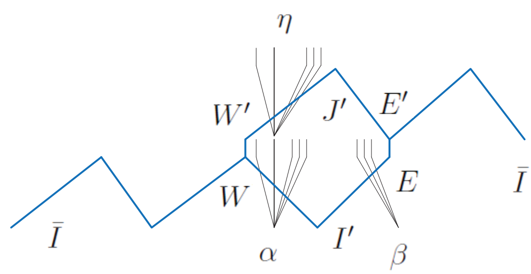

FIG. 4. *Interior wave interaction.*

THEOREM 3.1 ([2]). *Let $\tilde{a}_L = a^{\varepsilon}(W')$, $\tilde{a}_R = a^{\varepsilon}(E')$, $\bar{a}_L = a^{\varepsilon}(W)$, and $\bar{a}_R = a^{\varepsilon}(E)$, where a^{ε} is defined in section 2. If (U'_L, U'_R), (U_L, U_M), (U_M, U_R) are the solutions described in* (3.1)–(3.2)*, then*

$$\eta_i = \alpha_i + \beta_i + O(|\alpha||\beta|) + O(|\zeta_R - \zeta_L|) + O((|\zeta_L| + |\zeta_R|)(|\alpha| + |\beta|)),$$

where $\zeta_L := \tilde{a}_L - \bar{a}_L$ and $\zeta_R := \tilde{a}_R - \bar{a}_R$. As $|\alpha| + |\beta| \to 0$,

$$\eta_i = \alpha_i + \beta_i + C_0 D(\alpha, \beta) + O(|\zeta_R - \zeta_L|) + O((|\zeta_L| + |\zeta_R|)(|\alpha| + |\beta|)),$$

where $D(\alpha, \beta) = \sum |\alpha_i|\,|\beta_j|$, the sum is over all pairs for which the i-wave α_i and the j-wave from β_j are approaching, and $C_0 > 0$ is a constant independent of the point (x_m, t_p).

Next, we provide the wave interaction estimates on the single boundary hexagonal or quadrilateral region Γ_B centered at $(0, t_p)$. Since the latter case is included in the former one, we now present the former case only. Let $U'_B = U^{\varepsilon}_{\theta,\Delta x}(W'_B)$, $U'_R = U^{\varepsilon}_{\theta,\Delta x}(E')$, $U_B = U^{\varepsilon}_{\theta,\Delta x}(W_B)$, $U_R = U^{\varepsilon}_{\theta,\Delta x}(E)$, $U_M = U^{\varepsilon}_{\theta,\Delta x}(S)$, and

$$\Pi_{p-1} = \Pi(t_{p-1}), \quad \Pi_p = \Pi(t_p). \tag{3.3}$$

Suppose that (U'_B, U'_R) denotes the approximate solution $U^{\varepsilon}_{\theta,\Delta x}$ obtained from problem $\mathcal{BR}(0, t_p)$ and consisting of constant states $U'_B, U_k, \cdots, U_n = U'_R$ separated by the waves with the parameterizations $T^0_{\eta_0}(U'_B) = U_k$, $T^i_{\eta_i}(U_{i-1}) = U_i$ for $i = k+1, \cdots, n$. Then (U'_B, U'_R) can be expressed as

$$(U'_B, U'_R) = [(U'_B, U_k, \cdots, U_n)/\eta := (\eta_0, \eta_{k+1}, \cdots, \eta_n)], \tag{3.4}$$

where U'_B satisfies the boundary condition $\Theta(U'_B, \Pi_p) = 0$. On the other hand, we let

$$(U_B, U_M) = [(U_B, \bar{U}_k, \cdots, \bar{U}_n)/\alpha := (\alpha_0, \alpha_{k+1}, \cdots, \alpha_n)] \tag{3.5}$$

be the solution of problem $\mathcal{BR}(0, t_{p-1})$ connecting U_B to U_M and

$$(U_M, U_R) = [(\widetilde{U}_0, \cdots, \widetilde{U}_k)/\beta := (\beta_1, \cdots, \beta_k)] \tag{3.6}$$

be the solution of problem $\mathcal{R}(x_2, t_{p-1})$ connecting U_M to U_R, where U_B satisfies the boundary condition $\Theta(U_B, \Pi_{p-1}) = 0$. We note that $\beta = 0$ if Γ_B is a quadrilateral region. Similar to the interior case, we say that α and β are *incoming waves* of Γ_B and η are *outgoing waves* of Γ_B. According to [7], the following theorem is proved by applying the generalized version of Goodman's wave interaction estimates near the boundary.

THEOREM 3.2 (Boundary interaction estimate). *Let $\tilde{a}_R = \bar{a}^\varepsilon(E')$ and $\bar{a}_R = \bar{a}^\varepsilon(E)$, where $\bar{a}^\varepsilon$ is defined in section 2. If (U'_B, U'_R), (U_B, U_M), (U_M, U_R) are the solutions described in* (3.4)–(3.6)*, then*

$$|\eta_i - \alpha_i| \le \bar{C}_0 \sum_{j=1}^{k} |\beta_j| + O(|\bar{\zeta}_R| + |\Pi_p - \Pi_{p-1}|), \qquad i = 0, k+1, \cdots, n,$$

where $\bar{\zeta}_R := \tilde{a}_R - \bar{a}_R$, Π_{p-1} and Π_p are given in (3.3)*, and $\bar{C}_0 \ge 0$ is a constant that is independent of $(0, t_p)$ and is chosen to be* 0 *if $\beta = 0$.*

Proof. Suppose first that the incoming waves α and β interact and generate outgoing waves $\eta^* := (\eta_0^*, \eta_{k+1}^*, \cdots, \eta_n^*)$ connecting the right state U_R with the boundary state U_B^* satifying the boundary condition $\Theta(U_B^*, \Pi_{p-1}) = 0$, see Figure 5. Since

$$U_B = U_B(\alpha_0, \alpha_{k+1}, \cdots, \alpha_n; U_M) \quad \text{and} \quad U_B^* = U_B(\eta_0^*, \eta_{k+1}^*, \cdots, \eta_n^*; U_R)$$

satisfy the same boundary condition and

$$U_R = U_M + \sum_{j=1}^{k} \beta_j R_j(U_M) + O\Big(\sum_{j=1}^{k} \beta_j^2\Big),$$

the implicit function theorem gives that

$$|\eta_i^* - \alpha_i| \le C|U_R - U_M| \le C \sum_{j=1}^{k} |\beta_j|, \qquad i = 0, k+1, \cdots, n. \tag{3.7}$$

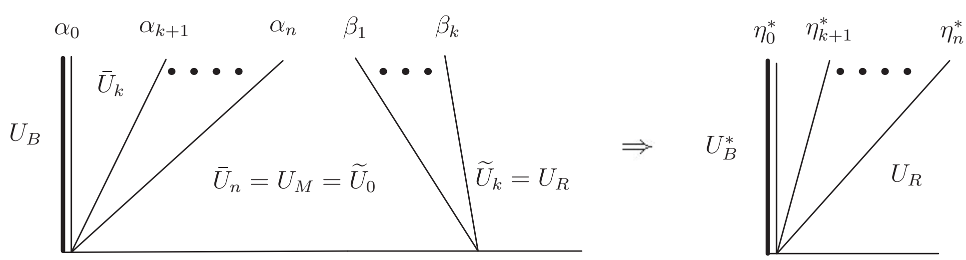

FIG. 5. *α and β interact and generate η^*.*

Next, let $\eta^{**} := (\eta_0^{**}, \eta_{k+1}^{**}, \cdots, \eta_n^{**})$ be waves connecting the right state U'_R with the boundary state U_B^{**} satisfying the same boundary condition as U_B^*. In view of (2.4), we have

$$U'_R = U_R + \zeta_R H(U_R) + O(\zeta_R^2).$$

Applying the implicit function theorem again, we obtain

$$|\eta_i^{**} - \eta_i^*| \le C|U_R' - U_R| \le C|\zeta_R|, \qquad i = 0, k+1, \cdots, n. \tag{3.8}$$

We observe that η have the same right state U_R' as η^{**} but the boundary state U_B' satisfies the different boundary condition $\Theta(U_B', \Pi') = 0$. Thus, the implicit function theorem implies that

$$|\eta_i - \eta_i^{**}| \le C|\Pi_p - \Pi_{p-1}|, \qquad i = 0, k+1, \cdots, n. \tag{3.9}$$

Therefore, the triangle inequality together with (3.7)–(3.9) yields that

$$|\eta_i - \alpha_i| \le C\Big(|\zeta_R| + |\Pi_p - \Pi_{p-1}| + \sum_{j=1}^{k} |\beta_j|\Big), \qquad i = 0, k+1, \cdots, n,$$

which completes the proof. ☐

We now show the stability of GGS. To this end, we control the total variations of the approximate solutions by introducing a version of Glimm functional. For given ε, θ, and Δx, we let $\mathcal{W}$ be the collection of all waves in the approximate solution $U_{\theta,\Delta x}^{\varepsilon}$ and set

$$\alpha_j \in \mathcal{W}^+ \text{ if } \lambda_j \ge 0, \qquad \alpha_j \in \mathcal{W}^- \text{ if } \lambda_j < 0,$$

where α_j denotes a j-wave in $\mathcal{W}$. Then $\mathcal{W} = \mathcal{W}^+ \cup \mathcal{W}^-$. For such approximate solution $U_{\theta,\Delta x}^{\varepsilon}$ and any mesh curve I, we define functionals $\mathcal{L}$ and $\mathcal{Q}$ on each subcurve I' of I by

$$\begin{aligned}
\mathcal{L}(I') &= \sum\{|\alpha| : \alpha \in \mathcal{W}^+ \text{ crosses } I'\} + (1 + \bar{C}_0)\sum\{|\alpha| : \alpha \in \mathcal{W}^- \text{ crosses } I'\}, \\
\mathcal{Q}(I') &= \sum\{|\alpha||\beta| : \alpha, \beta \in \mathcal{W} \text{ cross } I' \text{ and approach}\},
\end{aligned}$$

where $\bar{C}_0$ is given in Theorem 3.2. The Glimm functional $\mathcal{F}(I)$ of $U_{\theta,\Delta x}^{\varepsilon}$ for I is defined by

$$\mathcal{F}(I) = \mathcal{L}(I) + 2C_0\mathcal{Q}(I),$$

where C_0 is the constant in Theorem 3.1. It is easy to see that $\mathcal{L}(I)$ is equivalent to the total variation of $U_{\theta,\Delta x}^{\varepsilon}$ on I and $\mathcal{Q}(I) \le \mathcal{L}(I)^2$.

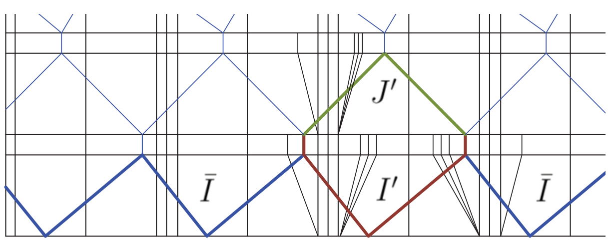

FIG. 6. *Mesh curves I and J.*

THEOREM 3.3. *Suppose that I and J are two mesh curves with $J > I$ and assume that I is in the domain of definition of $U_{\theta,\Delta x}^{\varepsilon}$. If $\mathcal{L}(I)$ is sufficiently small, then J*

is also in the domain of definition of $U^{\varepsilon}_{\theta,\Delta x}$. *Moreover, if* T. V.$[U_0]$ *and* T. V.$[\Pi]$ *are sufficiently small and* $a(x,t)$ *satisfies one of conditions* (B) *and* (C) *for sufficiently small* ν, *then* $U^{\varepsilon}_{\theta,\Delta x}$ *can be defined for* $t \geq 0$.

Proof. Suppose first that J is an immediate successor of I so that they differ by a single polygonal region Γ centered at some (x_m, t_p) or Γ_B centered at some $(0, t_p)$. Let $I = \bar{I} \cup I'$ and $J = \bar{I} \cup J'$, see Figure 6. If I' and J' lie on the boundary of Γ, then Theorem 3.1 gives that

$$\mathcal{L}(J) - \mathcal{L}(I) \leq C_0 \mathcal{Q}(I') + C|\zeta_R - \zeta_L| + C\mathcal{L}(I')(|\zeta_L| + |\zeta_R|), \tag{3.10}$$

$$\mathcal{Q}(J) - \mathcal{Q}(I) \leq -\mathcal{Q}(I') + \mathcal{L}(\bar{I})\{C_0 \mathcal{Q}(I') + C|\zeta_R - \zeta_L| + C\mathcal{L}(I')(|\zeta_L| + |\zeta_R|)\}, \tag{3.11}$$

where ζ_L, ζ_R are given in Theorem 3.1 and the constant C is independent of m and p. If I' and J' lie on the boundary of Γ_B, then by Theorem 3.2,

$$\mathcal{L}(J) - \mathcal{L}(I) \leq C|\bar{\zeta}_R| + C|\Pi_p - \Pi_{p-1}|, \tag{3.12}$$

$$\mathcal{Q}(J) - \mathcal{Q}(I) \leq -\mathcal{Q}(I') + C\mathcal{L}(\bar{I})\{|\bar{\zeta}_R| + |\Pi_p - \Pi_{p-1}|, \}, \tag{3.13}$$

where $\bar{\zeta}_R$ is defined as in Theorem 3.2, and Π_p, Π_{p-1} are given in (3.3).

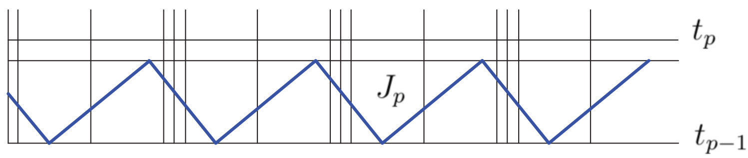

FIG. 7. *The I curve* J_p.

Let J_p denote the mesh curve located on the p-th time strip $t_{p-1} \leq t \leq t_p$ for $p \in \mathbb{N}$, see Figure 7. If $a(x,t)$ satisfies condition (B), then

$$\begin{aligned} |\zeta_L|,\ |\zeta_R|,\ |\bar{\zeta}_R| &\leq \sup_{(x,t)\in\mathbb{R}\times\mathbb{R}^+} |a_t(x,t)|\Delta t \leq \nu\Delta t, \\ |\zeta_R - \zeta_L| &\leq |b(x_{m+1}, t_p) - b(x_{m-1}, t_p)|, \end{aligned} \tag{3.14}$$

where $b(x,t) := a(x,t) - a(x, t-\Delta t)$. Suming up the estimates (3.10)–(3.13) for all $m \in 2\mathbb{N} \cup \{0\}$ together with condition (B) and (3.14), we have

$$\begin{aligned} \mathcal{F}(J_{p+1}) \leq{}& \mathcal{F}(J_p) + C_0\big(2C_0\mathcal{L}(J_p) - 1\big)\mathcal{Q}(J_p) \\ &+ C\big(1 + 2C_0\mathcal{L}(J_p)\big)\,\mathrm{T.\,V.}[a(\cdot, t_p + \Delta t) - a(\cdot, t_p)] \\ &+ C\nu\Delta t\big(1 + 2C_0\mathcal{L}(J_p)\big)\mathcal{L}(J_p) + C\big(1 + 2C_0\mathcal{L}(J_p)\big)|\Pi_p - \Pi_{p-1}| \\ \leq{}& (1 + C\nu\Delta t)\mathcal{F}(J_p) + C\nu\Delta t + C|\Pi_p - \Pi_{p-1}| \end{aligned} \tag{3.15}$$

provided that $\mathcal{L}(J_p) \leq (2C_0)^{-1}$. Let $N = [\frac{T}{\Delta t}] + 1$. Since $a(x,t)$ is independent of t if $t \geq T$, we then have

$$\mathcal{F}(J_n) \leq \mathcal{F}(J_N) + C\,\mathrm{T.\,V.}[\Pi] \quad \text{if } n > N \tag{3.16}$$

and, using (3.15) repeatedly,

$$\begin{aligned} \mathcal{F}(J_n) \leq (1 + C\nu\Delta t)^{n-1}\mathcal{F}(J_1) + C\nu\Delta t \sum_{i=0}^{n-2}(1 + C\nu\Delta t)^i \\ + C(1 + C\nu\Delta t)^{n-2}\,\mathrm{T.\,V.}[\Pi] \quad \text{if } n \leq N. \end{aligned} \tag{3.17}$$

If $\mathcal{L}(J_1) \leq (2C_0)^{-1}$, then

$$\mathcal{F}(J_1) = \mathcal{L}(J_1) + 2C_0\mathcal{Q}(J_1) \leq \mathcal{L}(J_1) + 2C_0(\mathcal{L}(J_1))^2 \leq 2\mathcal{L}(J_1) \leq C\,\mathrm{T.V.}[U_0]. \quad (3.18)$$

In view of (3.16)–(3.18),

$$\begin{aligned}\mathcal{L}(J_n) &\leq \mathcal{F}(J_n) \leq \mathcal{F}(J_N) + C\,\mathrm{T.V.}[\Pi] \\ &\leq C(1+C\nu\Delta t)^{N-1}\big(\mathrm{T.V.}[U_0] + \mathrm{T.V.}[\Pi]\big) + C\nu\Delta t\sum_{i=0}^{N-2}(1+C\nu\Delta t)^i \\ &\leq Ce^{C\nu T}\big(\mathrm{T.V.}[U_0] + \mathrm{T.V.}[\Pi]\big) + C(e^{C\nu T}-1) \qquad \text{for all } n\in\mathbb{N}.\end{aligned}$$

Thus, we can choose ν, $\mathrm{T.V.}[U_0]$, and $\mathrm{T.V.}[\Pi]$ small enough such that $\mathcal{L}(J_n) \leq (2C_0)^{-1}$ for all n.

If $a(x,t)$ satisfies condition (C), then, by the Taylor expansion, we get

$$\begin{aligned}\zeta_L &= (\Delta t)a_t(x_{m-1},t_p) + O((\Delta t)^2), \\ \zeta_R &= (\Delta t)a_t(x_{m+1},t_p) + O((\Delta t)^2), \\ \bar{\zeta}_R &= (\Delta t)a_t(x_1,t_p) + O((\Delta t)^2), \\ \zeta_R - \zeta_L &= 2(\Delta x)(\Delta t)a_{xt}(x_{m-1},t_p) + O((\Delta x)^2(\Delta t)).\end{aligned} \quad (3.19)$$

Then, similar to (3.15) with condition (C) and (3.19) instead of condition (B) and (3.14), we obtain

$$\begin{aligned}\mathcal{F}(J_{p+1}) &\leq \mathcal{F}(J_p) + C(\Delta x)(\Delta t)\sum_m \|a_{xt}\|_{L^\infty(D^{\kappa\delta}_{m,p})} \\ &\quad + C(\Delta t)\sup_{x\in\mathbb{R}}|a_t(x,t_p)|\mathcal{L}(J_p) + C|\Pi_p - \Pi_{p-1}| + O((\Delta x)(\Delta t))\end{aligned} \quad (3.20)$$

provided that $\mathcal{L}(J_p) \leq (2C_0)^{-1}$, where $D^{\kappa\delta}_{m,p} = D^{\kappa\delta}(x_m,t_p) = [x_{m-1},x_{m+1}] \times [t_p,t_{p+1}]$. By (3.18) and (3.20),

$$\begin{aligned}\mathcal{F}(J_{n+1}) &\leq \mathcal{F}(J_1) + C(\Delta x)(\Delta t)\sum_{p=1}^{n}\sum_m \|a_{xt}\|_{L^\infty(D^{\kappa\delta}_{mp})} \\ &\quad + C(\Delta t)\sum_{p=1}^{n}\sup_{x\in\mathbb{R}}|a_t(x,t_p)|\mathcal{L}(J_{p+1}) + C\,\mathrm{T.V.}[\Pi] + O(\Delta x)\end{aligned}$$

for all $n\in\mathbb{N}$. Since

$$\lim_{\Delta x\to 0}\sum_{p=1}^{\infty}\sum_m \|a_{xt}\|_{L^\infty(D^{\kappa\delta}_{mp})}2(\Delta x)(\Delta t) = \|a_{xt}\|_{L^1(\mathbb{R}\times[0,\infty))},$$

$$\lim_{\Delta x\to 0}\sum_{p=1}^{\infty}\sup_{x\in\mathbb{R}}|a_t(x,t_p)|(\Delta t) = \int_0^\infty \sup_{x\in\mathbb{R}}|a_t(x,t)|\,dt,$$

it follows from condition (C) and (3.18) that

$$\begin{aligned}\mathcal{L}(J_{n+1}) &\leq \mathcal{F}(J_{n+1}) \leq \mathcal{F}(J_1) + C\big(\nu + \mathrm{T.V.}[\Pi]\big) + O(\Delta x) \\ &\leq C\big(\mathrm{T.V.}[U_0] + \mathrm{T.V.}[\Pi] + \nu\big) + O(\Delta x).\end{aligned}$$

Hence, we are able to choose ν, T. V.$[U_0]$, and T. V.$[\Pi]$ small enough such that $\mathcal{L}(J_n) \leq (2C_0)^{-1}$ for all n as Δx tends to zero, which completes the proof. □

According to Theorem 3.3 and results in [9, 17], the following compactness results for the approximate solutions $\{U^{\varepsilon}_{\theta,\Delta x}\}$ hold.

THEOREM 3.4. *Suppose that* T. V.$[U_0]$ *and* T. V.$[\Pi]$ *are sufficiently small and* $a(x,t)$ *satisfies one of conditions* (B) *and* (C) *for sufficiently small* ν. *Then the approximate solutions* $\{U^{\varepsilon}_{\theta,\Delta x}\}$ *of* (1.3) *constructed by the GGS satisfy*

(i) T. V.$[U^{\varepsilon}_{\theta,\Delta x}(\cdot,t)] \leq C_1\big($T. V.$[U_0]$ + T. V.$[\Pi] + \nu\big)$, *where* C_1 *is a constant independent of* t, ε, θ, *and* Δx;

(ii) T. V.$[U^{\varepsilon}_{\theta,\Delta x}(\cdot,p\Delta t)] + \sup_x U^{\varepsilon}_{\theta,\Delta x}(x,p\Delta t) \leq C_2\big($T. V.$[U_0]$ + T. V.$[\Pi] + \nu\big)$, *where* C_2 *is a constant independent of* p, ε, θ, *and* Δx;

(iii) $\displaystyle\int_0^\infty |U^{\varepsilon}_{\theta,\Delta x}(x,t_2)-U^{\varepsilon}_{\theta,\Delta x}(x,t_1)|\,dx \leq C_3(|t_2-t_1|+\Delta t)$, *where* C_3 *is a constant independent of* ε, θ, *and* Δx.

By Theorem 3.4 and Helly's selection principle, we get the following result immediately.

THEOREM 3.5. *Let* $\{U^{\varepsilon}_{\theta,\Delta x}\}$ *be as in Theorem 3.4. Then there exist a subsequence* $\{U^{\varepsilon_i}_{\theta,\Delta x_i}\}$ *of* $\{U^{\varepsilon}_{\theta,\Delta x}\}$ *and a measurable function* $U_\theta(x,t)$ *such that*

(i) $U^{\varepsilon_i}_{\theta,\Delta x_i}(\cdot,t) \to U_\theta(\cdot,t)$ *in* L^1_{loc} *as* $\varepsilon_i, \Delta x_i \to 0$;

(ii) *for every continuous function* f, $f(U^{\varepsilon_i}_{\theta,\Delta x_i}(\cdot,t)) \to f(U_\theta(\cdot,t))$ *in* L^1_{loc} *as* $\varepsilon_i, \Delta x_i \to 0$.

Following a similar argument as section 4 in [2], one can easily find that $U_\theta(x,t)$ is an entropy solution to (1.3) for almost every sampling sequence θ. We leave the detail to the readers.

THEOREM 3.6. *Let* $U_\theta(x,t)$ *be the limits stated in Theorem 3.5. Then there exists a null set* $N \in \Phi$ *such that if* $\theta \in \Phi \setminus N$, $U_\theta(x,t)$ *satisfies the equality* $R_\phi(U_\theta) = 0$ *and inequality* (1.6) *for all* $\phi \in C^1([0,\infty)\times[0,\infty))$; *that is,* $U_\theta(x,t)$ *is an entropy solution to* (1.3).

4. Boundary regularity of entropy solutions. In section 3, we constructed the approximate solutions $\{U^{\varepsilon}_{\theta,\Delta x}\}$ by GGS and obtain their compactness in L^1_{loc}. Theorem 3.5 indicates the convergence of a subsequence $\{U^{\varepsilon_i}_{\theta,\Delta x_i}\}$ of $\{U^{\varepsilon}_{\theta,\Delta x}\}$ as $\varepsilon_i, \Delta x_i \to 0$. We denote the limit by $U_\theta(x,t)$ and, for almost every sampling sequence θ, $U_\theta(x,t)$ is the desired entropy solution. Under some sampling conditions, we now prove that those entropy solutions in section 3 indeed satisfy the boundary condition in (1.3). Moreover, those entropy solutions satisfy the strong trace property.

To get the boundary regularity, we state the existence of wave traces in every approximate solution $U^{\varepsilon}_{\theta,\Delta x}$ and find the boundedness of the vertical total variation. The wave tracing means to keep track of the strengths of waves. For any given $p_0 \in \mathbb{N}$, let $\mathcal{W}$ denote the collection of all waves in $U^{\varepsilon}_{\theta,\Delta x}$ on $[0,\infty)\times[0,p_0\Delta t)$. We observe from Theorems 3.1 and 3.2 that the strengths of waves begin at the initial time $t = 0$ or are produced from any wave interaction or boundary interaction. On the other hand, those strengths end when the waves are cancelled with each other or when they meet the boundary. To trace the waves completely, we will partition every j-wave α_j

in $\mathcal{W}$ into wavelets $\bar{\alpha}_j^l$, $l = 1, \cdots, h_j$, such that

$$\operatorname{sign} \bar{\alpha}_j^l = \operatorname{sign} \alpha_j \quad \text{and} \quad \sum_{l=1}^{h_j} \bar{\alpha}_j^l = \alpha_j. \tag{4.1}$$

We use the notations $\alpha_j(m,p)$ and $\bar{\alpha}_j^l(m,p)$ to indicate that they originate at (x_m, t_p). If $\bar{\alpha}_j^l(m,p)$ is a portion of rarefaction wave, we let $\lambda_j^m(\bar{\alpha}_j^l(m,p))$ and $\lambda_j^M(\bar{\alpha}_j^l(m,p))$ denote its minimum and maximum wave speed respectively. Let $\mathcal{T}$ denote the family of all wave traces τ for $U^\varepsilon_{\theta,\Delta x}$ on $[0,\infty)\times[0,p_0\Delta t)$. Suppose τ begins when $t = p_m(\tau)\Delta t$ and ends when $t = p_M(\tau)\Delta t$. For $p_m(\tau) \le p \le p_M(\tau)$, the wave trace τ maps p to $(j(\tau), l_p(\tau), m_p(\tau))$ if we track the strength of the wave to the wavelet $\bar{\alpha}_{j(\tau)}^{l_p(\tau)}(m_p(\tau), p)$ when $t = p\Delta t$. The traced wavelet can either stays, moves to the left, or moves to the right, which depends on its eigenspace and whether the sampling point falls to its right or left. We summarize the rule as follows: For $j = 1, \cdots k$,

$$m_{p+1}(\tau) = \begin{cases} m_p(\tau) - 2 & \text{if } \Delta x + \lambda_j^M(\bar{\alpha}_j^l(m,p))\Delta t \le \theta_{p+1}\Delta x, \\ m_p(\tau) & \text{if } \Delta x + \lambda_j^m(\bar{\alpha}_j^l(m,p))\Delta t \ge \theta_{p+1}\Delta x, \end{cases} \tag{4.2}$$

and, for $j = k+1, \cdots, n$,

$$m_{p+1}(\tau) = \begin{cases} m_p(\tau) & \text{if } -\Delta x + \lambda_j^M(\bar{\alpha}_j^l(m,p))\Delta t \le \theta_{p+1}\Delta x, \\ m_p(\tau) + 2 & \text{if } -\Delta x + \lambda_j^m(\bar{\alpha}_j^l(m,p))\Delta t \ge \theta_{p+1}\Delta x, \end{cases} \tag{4.3}$$

where $\lambda_j^m(\bar{\alpha}_j^l(m,p)) = \lambda_j^M(\bar{\alpha}_j^l(m,p))$ if $\bar{\alpha}_j^l(m,p)$ is a portion of shock wave. One can partition the wave $\alpha_j(m,p)$ appropriately such that no wavelet is divided by the sampling point. In addition, we require that

$$|\bar{\alpha}_j^{l'}(m', p+1)| \le |\bar{\alpha}_j^l(m,p)| \qquad \text{for } p \ge p_m(\tau) \tag{4.4}$$

when the two wavelets are on the same trace τ. Then we define $|\tau|$ by

$$|\tau| := |\bar{\alpha}_j^l(m, p_m(\tau))| = \max_{p_m(\tau)\le p\le p_M(\tau)} |\bar{\alpha}_j^{l'}(m', p)|.$$

By a similar argument as in [7, Theorem 3.1], we can extend Goodman's result to our case. We leave the proof to the readers.

Theorem 4.1. *For any sampling sequence $\theta \in \Phi$, consider any approximate solution $U^\varepsilon_{\theta,\Delta x}$ of* (1.3) *constructed in Section 3. Then, for any given $p_0 \in \mathbb{N}$, there exists a family $\mathcal{T}$ of wave traces for $U^\varepsilon_{\theta,\Delta x}$ on $[0,\infty)\times[0,p_0\Delta t)$ satisfying* (4.1)–(4.4). *Every wavelet of a wave in $\mathcal{W}$ lies in some wave trace $\tau \in \mathcal{T}$. Moreover, the total traced wave strength satisfies*

$$\sum_{\tau\in\mathcal{T}} |\tau| \le C\big(\mathrm{T.V.}[U_0] + \mathrm{T.V.}[\Pi] + \nu\big).$$

We now find a bound of the total variation to $U^\varepsilon_{\theta,\Delta x}\big|_{x=x_0} = U^\varepsilon_{\theta,\Delta x}(x_0,\cdot)$. We divide all the cases into two categories: (1) $x_{2m} + \varepsilon\Delta x \le x_0 \le x_{2m+2} - \varepsilon\Delta x$ for some $m \in \mathbb{N}\cup\{0\}$, and (2) $x_{2m} - \varepsilon\Delta x < x_0 < x_{2m} + \varepsilon\Delta x$ for some $m \in \mathbb{N}\cup\{0\}$. For category (1), the vertical total variation is determined by the wavelets that cross $x = x_0$ and the variations of $U^\varepsilon_{\theta,\Delta x}(x_0,\cdot)$ whenever the line $x = x_0$ crosses the top part of each T-shaped region. Since every wavelet lies in some wave trace, it depends on how many times that each wave trace crosses $x = x_0$. If the wave trace τ crosses $x = x_0$ in the time strip $t_p \le t < t_{p+1}$, then it contributes at most $C(|\tau| + |a(x_{2m+1}, t_{p+1}) - a(x_{2m+1}, t_p)|)$ to $\mathrm{T.V.}[U^\varepsilon_{\theta,\Delta x}\big|_{x=x_0}]$. Suppose that τ traces a right going wave. It might cross $x = x_0$ in the p-th time strip only if $m_p(\tau) = 2m$. In view of (4.3), τ will stay or move to the right in the next time strip. Once it moves to the right, it will not cross $x = x_0$ anymore. Thus, the crossing number depends on how many time strips that τ stays on the region $[x_{2m} + \varepsilon\Delta x, x_{2m+2} - \varepsilon\Delta x]$. The argument is similar for the case that τ traces a left going wave. Let

$$\lambda^-_{\max} := \sup_{U\in\Omega} \lambda_k(U) \quad \text{and} \quad \lambda^+_{\min} := \inf_{U\in\Omega} \lambda_{k+1}(U).$$

The crossing numbers for all $\tau \in \mathcal{T}$ are uniformly bounded if there exists an $M > 0$ such that

$$\max\Big\{k : \max\{\theta_p, \theta_{p+1}, \cdots, \theta_{p+k}\} < 1 + \lambda^-_{\max}\frac{\Delta t}{\Delta x}\Big\} \le M, \tag{4.5}$$

and

$$\max\Big\{k : \min\{\theta_p, \theta_{p+1}, \cdots, \theta_{p+k}\} > -1 + \lambda^+_{\min}\frac{\Delta t}{\Delta x}\Big\} \le M \tag{4.6}$$

for all $p \in \mathbb{N} \cup \{0\}$. Therefore, for category (1),

$$\begin{aligned}\mathrm{T.V.}[U^\varepsilon_{\theta,\Delta x}\big|_{x=x_0}] &\le C(M+1)\sum_{\tau\in\mathcal{T}} |\tau| + C\,\mathrm{T.V.}[a(x_{2m+1},\cdot)]\\ &\le C(M+1)\big(\mathrm{T.V.}[U_0] + \mathrm{T.V.}[\Pi] + \nu\big) + C\,\mathrm{T.V.}[a(x_{2m+1},\cdot)].\end{aligned}$$

By (2.2) for $i = 0$, it is easy to see that, for $\Delta x > 0$ is sufficiently small, the vertical total variation for category (2) is controlled as follows:

$$\begin{aligned}\mathrm{T.V.}[U^\varepsilon_{\theta,\Delta x}\big|_{x=x_0}] &\le C\max\big\{\mathrm{T.V.}[U^\varepsilon_{\theta,\Delta x}\big|_{x=x_{2m}-\varepsilon\Delta x}], \mathrm{T.V.}[U^\varepsilon_{\theta,\Delta x}\big|_{x=x_{2m}+\varepsilon\Delta x}]\big\}\\ &\le C(M+1)\big(\mathrm{T.V.}[U_0] + \mathrm{T.V.}[\Pi] + \nu\big)\\ &\quad + C\max\big\{\mathrm{T.V.}[a(x_{2m-1},\cdot)], \mathrm{T.V.}[a(x_{2m+1},\cdot)]\big\}.\end{aligned}$$

We observe that conditions (B) and (C) both imply that $\mathrm{T.V.}[a(x,\cdot)] < \infty$ for all $x \ge 0$. Thus, we have the following vertical total variation estimate.

THEOREM 4.2. *Let $U^\varepsilon_{\theta,\Delta x}$ be any approximate solution constructed in Section 3. Suppose that $a(x,t)$ satisfies one of conditions* (B) *and* (C) *for sufficiently small ν and that $\mathrm{T.V.}[a(x,\cdot)]$ is continuous at $x = x_0$. If $\Delta x > 0$ is small enough and the sampling sequence $\theta \in \Phi$ satisfies conditions* (4.5)–(4.6), *then there is a constant $C > 0$ such that*

$$\mathrm{T.V.}[U^\varepsilon_{\theta,\Delta x}\big|_{x=x_0}] \le C\big(\mathrm{T.V.}[U_0] + \mathrm{T.V.}[\Pi] + \mathrm{T.V.}[a(x_0,\cdot)] + \nu\big).$$

To get the boundary regularity of U_θ, we also need the following vertical local L^1 estimate.

THEOREM 4.3. *Let $U^\varepsilon_{\theta,\Delta x}$ and $a(x,t)$ be the same as in Theorem 4.2. If the sampling sequence $\theta \in \Phi$ satisfies conditions* (4.5)–(4.6)*, then, for $0 \le a < b < \infty$, there is a constant $C > 0$ such that*

$$\int_a^b |U^\varepsilon_{\theta,\Delta x}(x_1,t) - U^\varepsilon_{\theta,\Delta x}(x_2,t)|\, dt \le C(|x_1 - x_2| + \Delta x). \tag{4.7}$$

Proof. It suffices to consider the two cases: (1) $x_{2m} - \varepsilon\Delta x \le x_1, x_2 \le x_{2m} + \varepsilon\Delta x$ for some $m \in \mathbb{N}\cup\{0\}$, and (2) $x_{2m}+\varepsilon\Delta x \le x_1, x_2 \le x_{2m+2}-\varepsilon\Delta x$ for some $m \in \mathbb{N}\cup\{0\}$. For case (1), we use (2.2) for $i = 0$ and the condition that a is Lipschitz to get

$$\begin{aligned}\int_a^b |U^\varepsilon_{\theta,\Delta x}(x_1,t) - U^\varepsilon_{\theta,\Delta x}(x_2,t)|\, dt &\le C\int_a^b |a^\varepsilon(x_{2m} + \varepsilon\Delta x, t) - a^\varepsilon(x_{2m} - \varepsilon\Delta x, t)|\, dt \\ &\le C(b-a)\Delta x.\end{aligned}$$

For case (2), it is sufficient to consider $a = t_{N_1}$ and $b = t_{N_2}$ for some $0 \le N_1 < N_2 < \infty$. We divide the integral on the left hand side of (4.7) by

$$\begin{aligned}&\int_a^b |U^\varepsilon_{\theta,\Delta x}(x_1,t) - U^\varepsilon_{\theta,\Delta x}(x_2,t)|\, dt \\ &\quad= \sum_{p=N_1}^{N_2-1}\left(\int_{t_p}^{t_{p+1}-\varepsilon\Delta t} + \int_{t_{p+1}-\varepsilon\Delta t}^{t_{p+1}}\right)|U^\varepsilon_{\theta,\Delta x}(x_1,t) - U^\varepsilon_{\theta,\Delta x}(x_2,t)|\, dt.\end{aligned} \tag{4.8}$$

If the line $x = x_1$ crosses waves α_i and the line $x = x_2$ crosses waves β_j, then

$$|U^\varepsilon_{\theta,\Delta x}(x_1, t_{p+1} - \varepsilon\Delta t) - U^\varepsilon_{\theta,\Delta x}(x_2, t_{p+1} - \varepsilon\Delta t)| \le C\sum_{i,j}(|\alpha_i| + |\beta_j|) \tag{4.9}$$

and

$$\int_{t_p}^{t_{p+1}-\varepsilon\Delta t} |U^\varepsilon_{\theta,\Delta x}(x_1,t) - U^\varepsilon_{\theta,\Delta x}(x_2,t)|\, dt \le C|x_1 - x_2|\sum_{i,j}(|\alpha_i| + |\beta_j|). \tag{4.10}$$

In view of (2.4) and (4.9), we have

$$\begin{aligned}&\int_{t_{p+1}-\varepsilon\Delta t}^{t_{p+1}} |U^\varepsilon_{\theta,\Delta x}(x_1,t) - U^\varepsilon_{\theta,\Delta x}(x_2,t)|\, dt \\ &\quad\le C\varepsilon\Delta t|U^\varepsilon_{\theta,\Delta x}(x_1, t_{p+1} - \varepsilon\Delta t) - U^\varepsilon_{\theta,\Delta x}(x_2, t_{p+1} - \varepsilon\Delta t)| \\ &\quad\le C\varepsilon\Delta x\sum_{i,j}(|\alpha_i| + |\beta_j|).\end{aligned} \tag{4.11}$$

Combining (4.8), (4.10)-(4.11) and applying Theorem 4.2, we obtain

$$\begin{aligned}&\int_a^b |U^\varepsilon_{\theta,\Delta x}(x_1,t) - U^\varepsilon_{\theta,\Delta x}(x_2,t)|\, dt \\ &\quad\le C(|x_1 - x_2| + \varepsilon\Delta x)\big(\mathrm{T.V.}[U^\varepsilon_{\theta,\Delta x}\big|_{x=x_1}] + \mathrm{T.V.}[U^\varepsilon_{\theta,\Delta x}\big|_{x=x_2}]\big) \\ &\quad\le C(|x_1 - x_2| + \varepsilon\Delta x)\big(\mathrm{T.V.}[U_0] + \mathrm{T.V.}[\Pi] + \mathrm{T.V.}[a(x_0,\cdot)] + \nu\big),\end{aligned}$$

which completes the proof of case (2). □

We now prove that $U_\theta(x,t)$ satisfies the boundary condition

$$\Theta(U_\theta(0,t),\Pi(t)) = 0 \quad \text{a.e. in } t. \tag{4.12}$$

The construction of $U^\varepsilon_{\theta,\Delta x}$ gives that

$$\Theta(U^\varepsilon_{\theta,\Delta x}(0,t),\Pi(t_p)) = 0 \qquad \text{for } t_p \le t < t_{p+1} - \varepsilon\Delta t.$$

Since Θ is Lipschitz continuous, we have

$$\int_{t_p}^{t_{p+1}-\varepsilon\Delta t} |\Theta(U^\varepsilon_{\theta,\Delta x}(0,t),\Pi(t))|\,dt \le C\Delta t \sup_{t_p\le t\le t_{p+1}-\varepsilon\Delta t} |\Pi(t)-\Pi(t_p)| \tag{4.13}$$

and

$$\begin{aligned}\int_{t_{p+1}-\varepsilon\Delta t}^{t_{p+1}} &|\Theta(U^\varepsilon_{\theta,\Delta x}(0,t),\Pi(t))|\,dt\\ &\le C\varepsilon\Delta t\Big(\sup_{t_{p+1}-\varepsilon\Delta t\le t\le t_{p+1}} |\Pi(t)-\Pi(t_p)|\\ &\qquad + \sup_{t_{p+1}-\varepsilon\Delta t\le t\le t_{p+1}} |U^\varepsilon_{\theta,\Delta x}(0,t) - U^\varepsilon_{\theta,\Delta x}(0,t_p)|\Big).\end{aligned} \tag{4.14}$$

For $0 \le a < b < \infty$, we employ Theorem 4.2 together with (4.13)–(4.14) and conclude that

$$\begin{aligned}\int_a^b &|\Theta(U^\varepsilon_{\theta,\Delta x}(0,t),\Pi(t))|\,dt\\ &\le C\Delta t\big(\,\mathrm{T.V.}[U^\varepsilon_{\theta,\Delta x}\big|_{x=0}] + \mathrm{T.V.}[\Pi]\big)\\ &\le C\Delta x\big(\,\mathrm{T.V.}[U_0] + \mathrm{T.V.}[\Pi] + \mathrm{T.V.}[a(x_0,\cdot)] + \nu\big).\end{aligned} \tag{4.15}$$

On the other hand, Theorems 4.2 and 4.3 imply that the subsequence $\{U^{\varepsilon_i}_{\theta,\Delta x_i}\}$ in Theorem 3.5 has a further subsequence (we also denote it by $\{U^{\varepsilon_i}_{\theta,\Delta x_i}\}$) satisfying, for all $x \ge 0$,

$$U^{\varepsilon_i}_{\theta,\Delta x_i}(x,\cdot) \to U_\theta(x,\cdot) \quad \text{in } L^1_{\mathrm{loc}} \text{ as } \ \varepsilon_i, \Delta x_i \to 0. \tag{4.16}$$

Therefore, boundary condition (4.12) follows from (4.15) and (4.16).

Finally, we want to show that

$$U_\theta(d,\cdot) \to U_\theta(0,\cdot) \quad \text{in } L^1_{\mathrm{loc}} \text{ as } d \to 0^+. \tag{4.17}$$

For $0 \le a < b < \infty$, by Theorem 4.3,

$$\int_a^b |U^\varepsilon_{\theta,\Delta x}(d,t) - U^\varepsilon_{\theta,\Delta x}(0,t)|\,dt \le C(d+\Delta x).$$

Convergence (4.16) states that, for any given $\eta > 0$, we can choose sufficiently small ε_i and Δx_i such that

$$\int_a^b |U^{\varepsilon_i}_{\theta,\Delta x_i}(0,t) - U_\theta(0,t)|\,dt < \eta.$$

The triangle inequality gives

$$\int_a^b |U^{\varepsilon_i}_{\theta,\Delta x_i}(d,t) - U_\theta(0,t)|\, dt \le C(d+\Delta x_i) + \eta.$$

Letting $\varepsilon_i, \Delta x_i \to 0$, we have

$$\int_a^b |U_\theta(d,t) - U_\theta(0,t)|\, dt \le Cd + \eta,$$

and hence (4.17) follows since $\eta > 0$ is arbitrary.

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