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SECOND PROOF OF THE GLOBAL REGULARITY OF THE TWO-DIMENSIONAL MHD SYSTEM WITH FULL DIFFUSION AND ARBITRARY WEAK DISSIPATION*

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Abstract. In regards to the mathematical issue of whether a system of equations admits a unique solution for all time or not, given an arbitrary initial data sufficiently smooth, the case of the magnetohydrodynamics system may be arguably more difficult than that of the Navier-Stokes equations. In the last several years, an explosive amount of work by many mathematicians was devoted to make progress toward the global well-posedness of the two-dimensional magnetohydrodynamics system with diffusion in terms of a full Laplacian but with zero dissipation; nevertheless, this problem remains open. The purpose of this manuscript is to provide a second proof of the global well-posedness in case the diffusion is in the form of a full Laplacian, and the dissipation is in the form of a fractional Laplacian with an arbitrary small power. In contrast to the first proof of this result in the literature that took advantage of the property of a heat kernel, the main tools in this manuscript consist of Besov space techniques, in particular fractional chain rule, which has been proven to possess potentials to lead to resolutions of difficult problems, in particular of fluid dynamics partial differential equations.

Key words. Besov space, fractional Laplacians, global regularity, magnetohydrodynamics system, Navier-Stokes equations.

Mathematics Subject Classification. 35B65, 35Q35, 35Q61.

1. Introduction. One of the most difficult outstanding open problems in mathematical analysis questions whether or not, given an initial data u_0 that is sufficiently smooth, the following Navier-Stokes equations (NSE) of fluid mechanics in case $N = 3$ admits a unique solution for all time:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla \pi - \nu \Delta u = 0, \tag{1a}$$

$$\nabla \cdot u = 0, \quad u(x,0) = u_0(x), \tag{1b}$$

where $u \triangleq (u_1, \dots, u_N)(x,t), \pi(x,t)$ represent the velocity, pressure fields respectively, while $\nu \geq 0$ the viscosity coefficient. Due to the scaling property that $u_\lambda(x,t) \triangleq \lambda^2 u(\lambda x, \lambda^2 t), \lambda \in \mathbb{R}^+$, solves the NSE if $u(x,t)$ does, for simplicity hereafter we assume $\nu = 1$; moreover, let us also write $\frac{\partial}{\partial t} = \partial_t, \frac{\partial}{\partial x_i} = \partial_i, i \in \mathbb{N}$, where $x = (x_1, \dots, x_N)$, and $\int f = \int_{\mathbb{R}^N} f(x)dx$.

The magnetohydrodynamics (MHD) system describes the motion of electrically conducting fluids, has broad applications in applied sciences such as astrophysics, geophysics and plasma physics, and has been studied extensively ever since the pioneering work of [1, 6]. We first introduce a fractional Laplacian $\Lambda \triangleq (-\Delta)^{\frac{1}{2}}$ defined through Fourier transform of

$$\widehat{\Lambda^{2r} f}(\xi) \triangleq |\xi|^{2r} \hat{f}(\xi), \quad r \in \mathbb{R},$$

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and now the generalized MHD system, which is a coupling of the NSE (1a)-(1b) with Maxwell's equations of electromagnetism:

$$\partial_t u + (u \cdot \nabla)u + \nabla \pi + \Lambda^{2\alpha} u = (b \cdot \nabla)b, \tag{2a}$$

$$\partial_t b + (u \cdot \nabla)b + \Lambda^{2\beta} b = (b \cdot \nabla)u, \tag{2b}$$

$$\nabla \cdot u = \nabla \cdot b = 0, \quad (u, b)(x, 0) = (u_0, b_0)(x), \tag{2c}$$

where $b = (b_1, \dots, b_N)$ represents the magnetic field, $\alpha, \beta \in [0, 1]$; let us call the system (2a)-(2c) at $\alpha = \beta = 1$ the classical MHD system. We remark that at $\alpha = 1$, the equation (2a) is exactly (1a) forced by the term $(b \cdot \nabla)b$. It is obvious that if one accomplishes in proving that given an arbitrary initial data (u_0, b_0) sufficiently smooth, there exists a unique smooth solution to the MHD system (2a)-(2c), then taking $b_0 \equiv 0$ deduces by uniqueness, the smooth solution to the NSE (1a)-(1b); therefore, it may be argued that the proof of the regularity of the solution is harder for the classical MHD system (2a)-(2c) than the NSE (1a)-(1b).

For the classical MHD system, the existence of the unique strong solution globally in two-dimensional (2d) case and locally in three-dimensional (3d) case, as well as the existence of a weak solution globally in both dimensions were shown in [21]. For the more general case with fractional Laplacians, it was shown in [27] that if

$$\alpha \geq \frac{1}{2} + \frac{N}{4}, \quad \beta \geq \frac{1}{2} + \frac{N}{4},$$

then the global regularity result may be attained in $\mathbb{R}^N, N \geq 2$; in particular, in the case $N = 2$, this requires $\alpha = \beta = 1$. Taking L^2-inner products of (2a)-(2b) with (u, b) respectively, and taking advantage of (2c) lead to

$$\|u(t)\|_{L^2}^2 + \|b(t)\|_{L^2}^2 + \int_0^t \|\Lambda^\alpha u\|_{L^2}^2 d\tau + \int_0^t \|\Lambda^\beta b\|_{L^2}^2 d\tau = \|u_0\|_{L^2}^2 + \|b_0\|_{L^2}^2 \tag{3}$$

for all $t \in [0, T]$, the interval over which the solution exists; this represents the conservation of energy and cumulative energy dissipation and diffusion. This threshold of $\frac{1}{2} + \frac{N}{4}$ may be seen as the endpoint of the energy criticality such that if either power, α or β, lies below, then the dissipation and diffusion are no longer strong enough to suppress the non-linear terms unless the better bound beyond those in (3) are discovered (see logarithmic improvements in [23, 26, 29, 31]).

A remarkable feature of the solution to the 2d NSE due to the incompressibility condition (1b) is that only in this dimension, the better bound in fact exists, even with zero dissipation, the case in which (1a)-(1b) recovers the Euler equations. Indeed, upon applying a curl operator on (1a) without the dissipation, we see that the vorticity $w \triangleq \nabla \times u$ evolves in time over the transport equation of

$$\partial_t w + (u \cdot \nabla)w = 0$$

so that taking L^p-estimate, $p \in [2, \infty)$, using that

$$\int (u \cdot \nabla)w|w|^{p-2}w = \frac{1}{p}\int (u \cdot \nabla)|w|^p = 0$$

due to (1b), deduces

$$\frac{1}{p}\partial_t \|w\|_{L^p}^p = 0; \tag{4}$$

hence, writing $\frac{1}{p}\partial_t \|w\|_{L^p}^p = \|w\|_{L^p}^{p-1}\partial_t \|w\|_{L^p}$, dividing by $\|w\|_{L^p}^{p-1}$ and thereafter taking the limit $p \to \infty$ shows that L^∞-norm of w is bounded, which is almost as good as that of ∇u ([38]). It is a natural question to ask whether such a favorable formulation of the vorticity equation may be utilized to improve the results in [21, 27] that stated that the global well-posedness of the generalized MHD system (2a)-(2c) requires $\alpha \geq 1, \beta \geq 1$. Here, in contrast to (4), the additional difficulty is that upon the L^p-estimate of w where $w \triangleq \nabla \times u$ and u solves (2a), $p \in [2, \infty)$, we are faced with

$$\frac{1}{p}\partial_t \|w\|_{L^p} + \int \Lambda^{2\alpha} w|w|^{p-2}w = \int (b \cdot \nabla) j |w|^{p-2} w \tag{5}$$

(see (12a)) and hence the task of having to estimate $\int (b \cdot \nabla) j |w|^{p-2} w$, $j \triangleq \nabla \times b$, while also making use of the fractional dissipative term $\int \Lambda^{2\alpha} w|w|^{p-2}w$ if necessary. This explains precisely why many realized that if β is sufficiently large so that $(b \cdot \nabla) j$ within $\int (b \cdot \nabla) j |w|^{p-2} w$ has an adequate bound or α is sufficiently large so that $\int \Lambda^{2\alpha} w|w|^{p-2}w$ can provide adequate support to help estimate $\int (b \cdot \nabla) j |w|^{p-2} w$, then the regularity of the solution (u, b) to the system (2a)-(2c) may be attained. We now review prominent results by those who made significant contribution based on this observation.

For the case $\alpha = 0$; i.e. no dissipation term, Tran, Yu and Zhai in [25] showed that $\beta > 2$ suffices to prove the regularity of the solution for all time. Jiu and Zhao in [13] and the author [30] independently improved to $\beta > \frac{3}{2}$. Thereafter, Cao, Wu and Yuan in [5] showed that $\beta > 1$ in fact suffices via Besov space techniques approach; subsequently, Jiu and Zhao in [14] also obtained the same result by a completely different approach from [5]. On the other hand, for the case $\beta = 1$, Tran, Yu and Zhai in [25] showed that $\alpha \geq \frac{1}{2}$ suffices. Subsequently, Yuan and Bai in [37], as well as the author in [32], independently improved this result to $\alpha > \frac{1}{3}$. Thereafter, Ye and Xu in [36] improved to $\alpha \geq \frac{1}{4}$. Finally, making use of the property of a heat kernel, the authors in [11] proved that $\alpha > 0$ suffices.

Therefore, coming from both directions, $\alpha = 0$ fixed and reducing the powers of $\beta \geq 1$, or $\beta = 1$ fixed and reducing the powers of $\alpha \geq 0$, we have come to the crossroad at which the only case left is $\alpha = 0, \beta = 1$, which is the extension of the classical result from [38] (see also [16]). Numerical analysis indicates that the regularity is more likely than the blow-up (e.g. [24]), and various regularity criteria have been obtained (e.g. [12, 35]); however, this problem has remained open since, seemingly asking for a new idea. It is worth emphasizing that the resolution of this problem should lead immediately to analogous results for various related systems of equations such as magnetic Bénard problem, and possibly magneto-micropolar fluid system ([33, 34]). The purpose of this manuscript is to provide a second proof of the global well-posedness of the system (2a)-(2c) in case $\alpha > 0, \beta = 1$ as follows.

Theorem 1.1. *Let $\alpha > 0, \beta = 1$. For every $(u_0, b_0) \in H^s(\mathbb{R}^2), s \geq 3$, there exists a unique solution pair (u, b) to the generalized MHD system (2a)-(2c) such that*

$$u \in C([0, \infty); H^s(\mathbb{R}^2)) \cap L^2([0, \infty); H^{s+\alpha}(\mathbb{R}^2)),$$
$$b \in C([0, \infty); H^s(\mathbb{R}^2)) \cap L^2([0, \infty); H^{s+1}(\mathbb{R}^2)).$$

Remark 1.1. The proof follows the approach of [5] that considered $\beta > 1$ and zero dissipation in (2a)-(2c). Let us already point out the distinctive difference between their iteration schemes and ours. The authors in [5] obtained the L^{q_1}-bound of w and L^{q_1}-bound of j (see Proposition 3.1 [5]), and then $L_T^1 B_{q_1,1}^{\delta}$-bound of j (see Proposition 4.1 [5]), specifically

$$\int_0^T \|j\|_{B^{\delta}_{q_1,1}} d\tau \leq c, \quad \text{where } 2 \leq q_1 \leq \frac{2}{2-\beta}, \ \frac{2}{q_1} < \delta < 2\beta - 1, \tag{6}$$

which consequently leads in particular to the bound of

$$\int_0^T \|\nabla j\|_{L^{q_2}} d\tau, \quad 2 \leq q_2 < \frac{2}{4-3\beta} \ \text{ if } \beta \leq \frac{4}{3} \tag{7}$$

(see Proposition 5.2 [5]). This improvement through the $L_T^1 B_{q_1,1}^{\delta}$-bound allows one to go back and attain L^{q_2}-estimate of w, j and repeat. It is clear that one cannot readily extend this strategy to the case $\beta = 1$, because if $\beta = 1$, then we require $q_1 = 2$ in (6) which disallows us to find any δ such that $\frac{2}{q_1} < \delta < 2\beta - 1$, and similarly we will not be able to find any $r \in [2, \frac{2}{4-3\beta})$ in (7) if $\beta = 1$. Moreover, for the authors in [5] to make the crucial improvement through this $L_T^1 B_{q_1,1}^{\delta}$-bound, it seems $\beta > 1$ is a crucial assumption which is absent in our hypothesis.

However, we can make use of $\alpha > 0$ in the L^{q_2}-estimate of j immediately. In other words, our iteration cycle will take place as follows: L^{q_1}-bound of w (see Proposition 3.2), L^{q_2}-bound of j where $q_2 = \frac{q_1}{1-\alpha}$ (see Proposition 3.3), the $L_T^1 B_{q_2,1}^{\delta}$-bound of j (see Proposition 3.4) which leads to L^{q_2}-bound of w (see Proposition 3.5) and repeat thereafter. Here, upon making careful use of $\alpha > 0$, in contrast to the approach of [5], it will be crucial to rely on the fractional chain rule Lemma 2.3 (see the proof of Proposition 3.5, in particular (49), (50), (51)).

In the next section, we set up notations and state key lemmas. The proof of the local existence result is standard; we refer to e.g. [20] where by using mollifiers the local existence proof is shown in the case of the NSE and the Euler equations. The initial regularity space in the statement of Theorem 1.1 may be generalized in many ways; we choose to focus on the *a priori* estimates in this manuscript.

2. Preliminaries. Let us use the notations $A \lesssim_{a,b} B, A \approx_{a,b} B$ to imply that there exists a constant $c(a,b)$ that depends on a, b such that $A \leq cB, A = cB$ respectively. The following lower bound on the fractional Laplacian has found many applications:

Lemma 2.1 ([10] and Lemma 3.3 [15]). *Let $r \in [0,1]$, and $f, \Lambda^{2r} f \in L^p(\mathbb{R}^2), p \geq 2$. Then*

$$2\int |\Lambda^r |f|^{\frac{p}{2}}|^2 \leq p \int |f|^{p-2} f \Lambda^{2r} f.$$

We use the following well-known inequalities:

Lemma 2.2 (e.g. Theorem 3.1.1 [7]). *Suppose f satisfies $\nabla \cdot f = 0$, $\nabla f \in L^p(\mathbb{R}^2), p \in (1,\infty)$. Then*

$$\|\nabla f\|_{L^p} \leq c \frac{p^2}{p-1} \|\nabla \times f\|_{L^p}.$$

LEMMA 2.3 (e.g. Lemma A1 [18], Proposition 3.1 [9]). *Let F be C^1 mapping such that $F(0) = 0$ and*

$$|F'(\tau f + (1-\tau)g)| \le h(\tau)|G(f) + G(g)|$$

where $G \in C, G > 0, h \in L^1([0,1])$. Then for $\delta \in (0,1), p, p_2 \in (1,\infty), p_1 \in (1,\infty]$ satisfying $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$,

$$\|F \circ f\|_{\dot{W}^{\delta,p}} \lesssim \|G \circ f\|_{L^{p_1}} \|f\|_{\dot{W}^{\delta,p_2}}.$$

LEMMA 2.4 (e.g. Lemma A.2 [17], see also [7, 20]). *Let $f \in W^{\delta,p_1}(\mathbb{R}^2) \cap L^{q_2}(\mathbb{R}^2), g \in W^{\delta,p_2}(\mathbb{R}^2) \cap L^{q_1}(\mathbb{R}^2), \delta \ge 0, 1 < p_k < \infty, 1 < q_k \le \infty, \dfrac{1}{p_k} + \dfrac{1}{q_k} = \dfrac{1}{p}, k = 1,2$. Then*

$$\|fg\|_{\dot{W}^{\delta,p}} \lesssim (\|f\|_{\dot{W}^{\delta,p_1}} \|g\|_{L^{q_1}} + \|f\|_{L^{q_2}} \|g\|_{\dot{W}^{\delta,p_2}}).$$

LEMMA 2.5 ([4]; see also the Appendix [30] for proof). *Let $f \in H^s(\mathbb{R}^2), s > 2$, satisfy $\nabla \cdot f = 0, \nabla \times f \in L^\infty(\mathbb{R}^2)$. Then there exists a constant $c \ge 0$ such that*

$$\|\nabla f\|_{L^\infty} \lesssim (\|f\|_{L^2} + \|\nabla \times f\|_{L^\infty} \log_2(2 + \|f\|_{H^s}) + 1).$$

LEMMA 2.6 ([19]). *Let f, g be smooth such that $\nabla f \in L^{p_1}(\mathbb{R}^2), \Lambda^{s-1} g \in L^{p_2}(\mathbb{R}^2)$, $\Lambda^s f \in L^{p_3}(\mathbb{R}^2), g \in L^{p_4}(\mathbb{R}^2), p \in (1,\infty), \dfrac{1}{p} = \dfrac{1}{p_1} + \dfrac{1}{p_2} = \dfrac{1}{p_3} + \dfrac{1}{p_4}$, $p_2, p_3 \in (1,\infty)$, $s > 0$. Then*

$$\|\Lambda^s(fg) - f\Lambda^s g\|_{L^p} \lesssim (\|\nabla f\|_{L^{p_1}} \|\Lambda^{s-1} g\|_{L^{p_2}} + \|\Lambda^s f\|_{L^{p_3}} \|g\|_{L^{p_4}}).$$

Let us recall the notion of Besov spaces (cf. [2, 7]). We denote by $\mathcal{S}(\mathbb{R}^2)$ the Schwartz class functions and $\mathcal{S}'(\mathbb{R}^2)$, its dual. We define $\mathcal{S}_0$ to be the subspace of $\mathcal{S}$ in the following sense:

$$\mathcal{S}_0 \triangleq \{\phi \in \mathcal{S}, \int_{\mathbb{R}^2} \phi(x) x^r dx = 0, |r| = 0, 1, 2, ...\}.$$

Its dual $\mathcal{S}_0'$ is given by $\mathcal{S}_0' \triangleq \mathcal{S}/\mathcal{S}_0^\perp = \mathcal{S}'/\mathcal{P}$ where $\mathcal{P}$ is the space of polynomials. For $k \in \mathbb{Z}$ we define

$$A_k \triangleq \{\xi \in \mathbb{R}^2 : 2^{k-1} < |\xi| < 2^{k+1}\}.$$

It is well-known that there exists a sequence $\{\Phi_k\} \in \mathcal{S}(\mathbb{R}^2)$ such that

$$\text{supp } \hat{\Phi}_k \subset A_k, \quad \hat{\Phi}_k(\xi) = \hat{\Phi}_0(2^{-k}\xi) \quad \text{or} \quad \Phi_k(x) = 2^{2k}\Phi_0(2^k x) \quad \text{and}$$

$$\sum_{k=-\infty}^{\infty} \hat{\Phi}_k(\xi) = \begin{cases} 1 & \text{if } \xi \in \mathbb{R}^2 \setminus \{0\}, \\ 0 & \text{if } \xi = 0. \end{cases}$$

Consequently,

$$\sum_{k=-\infty}^{\infty} \Phi_k * f = f \quad \text{for any } f \in \mathcal{S}_0'.$$

We set $\Psi \in C_0^\infty(\mathbb{R}^2)$ be such that

$$1 = \hat{\Psi}(\xi) + \sum_{k=0}^{\infty} \hat{\Phi}_k(\xi), \qquad \Psi * f + \sum_{k=0}^{\infty} \Phi_k * f = f,$$

for any $f \in \mathcal{S}'$. With that, we set

$$\Delta_k f \triangleq \begin{cases} 0 & \text{if } k \leq -2, \\ \Psi * f & \text{if } k = -1, \\ \Phi_k * f & \text{if } k = 0, 1, 2, ..., \end{cases}$$

and define for any $s \in \mathbb{R}, p, q \in [1, \infty]$, the inhomogeneous Besov space

$$B_{p,q}^s \triangleq \{f \in \mathcal{S}' : \|f\|_{B_{p,q}^s} < \infty\},$$

where

$$\|f\|_{B_{p,q}^s} \triangleq \begin{cases} (\sum_{k=-1}^{\infty} (2^{ks} \|\Delta_k f\|_{L^p})^q)^{\frac{1}{q}}, & \text{if } q < \infty, \\ \sup_{-1 \leq k < \infty} 2^{ks} \|\Delta_k f\|_{L^p} & \text{if } q = \infty. \end{cases}$$

For any $s \in \mathbb{R}, 1 < p < \infty$,

$$B_{p,\min\{p,2\}}^s \subset W^{s,p} \subset B_{p,\max\{p,2\}}^s \tag{8}$$

(pg. 152 [3], Theorems 2.40 and 2.41 in [2]). The following lemmas will be useful in obtaining upper and lower estimates:

LEMMA 2.7 (cf. [7]). *Bernstein's inequality: Let $f \in L^p(\mathbb{R}^2)$ with $1 \leq p \leq q \leq \infty$ and $0 < r < R$. Then for all $k \in \mathbb{Z}^+ \cup \{0\}$, and $\lambda > 0$, there exists a constant $C_k \geq 0$ such that*

$$\begin{cases} \sup_{|r|=k} \|\partial^r f\|_{L^q} \leq C_k \lambda^{k+2(\frac{1}{p}-\frac{1}{q})} \|f\|_{L^p} & \text{if supp } \hat{f} \subset \{\xi : |\xi| \leq \lambda r\}, \\ C_k^{-1} \lambda^k \|f\|_{L^p} \leq \sup_{|r|=k} \|\partial^r f\|_{L^p} \leq C_k \lambda^k \|f\|_{L^p} & \text{if supp } \hat{f} \subset \{\xi : \lambda r \leq |\xi| \leq \lambda R\}, \end{cases}$$

and if we replace derivative ∂^r by the fractional Laplacian, the inequalities remain valid only with trivial modifications.

LEMMA 2.8 (cf. [8, 28]). *Let $r \geq 0$ and $p = 2$ or $0 \leq r \leq 1$ and $2 < p < \infty$. Then for any $k \in \mathbb{Z}, f \in \mathcal{S}'$,*

$$2^{2rk} \|\Delta_k f\|_{L^p}^p \lesssim_{r,p} \int |\Delta_k f|^{p-2} \Delta_k f \Lambda^{2r} \Delta_k f.$$

Moreover, Bony's paraproduct decomposition (cf. [7]) will be used frequently:

$$fg = T_f g + T_g f + R(f, g)$$

where

$$T_f g \triangleq \sum_k S_{k-1} f \Delta_k g, \quad R(f,g) \triangleq \sum_{k,k':|k-k'|\leq 1} \Delta_k f \Delta_{k'} g, \quad S_{k-1} \triangleq \sum_{l:l\leq k-2} \Delta_l. \tag{9}$$

Now let us state the basic energy conservation of the system (2a)-(2c) as in (3):

$$\sup_{t\in[0,T]} \left(\|u\|_{L^2}^2 + \|b\|_{L^2}^2 \right)(t) + \int_0^T \|\Lambda^\alpha u\|_{L^2}^2 + \|\nabla b\|_{L^2}^2 d\tau \lesssim 1. \tag{10}$$

The following proposition was observed in many previous works (e.g. Proposition 3.1 [32]):

PROPOSITION 2.9. *Let $\alpha \geq 0, \beta = 1$. Suppose $(u_0, b_0) \in H^s(\mathbb{R}^2), s \geq 3$. Then its corresponding solution pair (u, b) to the system (2a)-(2c) in $[0,T]$ has the following bounds:*

$$\sup_{t\in[0,T]} \left(\|w\|_{L^2}^2 + \|j\|_{L^2}^2 \right)(t) + \int_0^T \|\Lambda^\alpha w\|_{L^2}^2 + \|\Lambda j\|_{L^2}^2 d\tau \lesssim 1.$$

For completeness we leave the proof in the Appendix. Now in order to clearly explain the plan of the proof of Theorem 1.1, let us discuss the regularity criteria that has been observed in the previous work (e.g. Proposition 3.4 [32]); the proof of Theorem 2.1 would be complete once we attain the following bound:

PROPOSITION 2.10. *Let $\alpha > 0, \beta = 1$. Suppose $(u_0, b_0) \in H^s(\mathbb{R}^2), s \geq 3$ and its corresponding solution pair (u, b) to the system (2a)-(2c) in $[0,T]$ satisfies*

$$\int_0^T \|w\|_{L^{\frac{2}{\alpha}}}^2 d\tau \lesssim 1. \tag{11}$$

Then

$$u \in C([0,T]; H^s(\mathbb{R}^2)) \cap L^2([0,T]; H^{s+\alpha}(\mathbb{R}^2)),$$
$$b \in C([0,T]; H^s(\mathbb{R}^2)) \cap L^2([0,T]; H^{s+1}(\mathbb{R}^2)).$$

For completeness, we sketch its proof in the Appendix.

3. Proof of Theorem 1.1. Let us assume $\alpha \in (0, \frac{1}{3})$; we explain the reason for this restriction at the Remark 3.1. Applying the curl operator on (2a)-(2b), we obtain

$$\partial_t w + \Lambda^{2\alpha} w = -(u\cdot\nabla)w + (b\cdot\nabla)j, \tag{12a}$$
$$\partial_t j + \Lambda^2 j = -(u\cdot\nabla)j + (b\cdot\nabla)w + 2[\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1(\partial_1 b_2 + \partial_2 b_1)]. \tag{12b}$$

The following proposition is similar to but slightly better than Proposition 3.2 [32]:

PROPOSITION 3.1.
Let $\alpha \in (0, \frac{1}{3}), \beta = 1$. Suppose $(u_0, b_0) \in H^s(\mathbb{R}^2), s \geq 3$. Then its corresponding solution pair (u, b) to the system (2a)-(2c) in $[0,T]$ has the following bounds:

$$\sup_{t\in[0,T]} \|\Lambda^\alpha j(t)\|_{L^2}^2 + \int_0^T \|\Lambda^{1+\alpha} j\|_{L^2}^2 d\tau \lesssim 1.$$

For completeness we leave this proof in the Appendix as well. The next proposition is also similar to but slightly better than Proposition 3.3 [32].

PROPOSITION 3.2. *Let $\alpha \in (0, \frac{1}{3}), \beta = 1$. Suppose $(u_0, b_0) \in H^s(\mathbb{R}^2), s \geq 3$. Then its corresponding solution pair (u, b) to the system (2a)-(2c) in $[0, T]$ has the following bounds: for $q_1 \in [2, \frac{2(1+\alpha)}{1-\alpha}]$,*

$$\sup_{t\in[0,T]} \|w(t)\|_{L^{q_1}}^{q_1} + \int_0^T \|w\|_{L^{\frac{q_1}{1-\alpha}}}^{q_1} d\tau \lesssim_{q_1} 1.$$

REMARK 3.1. The reason for the restriction of $\alpha \in (0, \frac{1}{3})$ is because if $\alpha \geq \frac{1}{3}$, then by Proposition 2.10 and Proposition 3.2, the proof of Theorem 1.1 is complete and no iteration scheme is necessary because $\frac{2(1+\alpha)}{(1-\alpha)^2} \geq \frac{2}{\alpha}$ if and only if $\alpha \geq \frac{1}{3}$.

Proof. Let us assume $2(1+\alpha) \leq q_1 \leq \frac{2(1+\alpha)}{1-\alpha}$ below because the case $q_1 \in [2, 2(1+\alpha))$ can be interpolated once we obtain higher L^p-bounds. We multiply (12a) by $|w|^{q_1-2}w$, integrate in space, use divergence-free property of u from (2c) to obtain (5) with p replaced by q_1. We use Lemma 2.1, the Sobolev embedding of $\dot{H}^\alpha(\mathbb{R}^2) \hookrightarrow L^{\frac{2}{1-\alpha}}(\mathbb{R}^2)$ to obtain the lower bound on the dissipation as

$$\tilde{c}(q_1)\|w\|_{L^{\frac{q_1}{1-\alpha}}}^{q_1} \lesssim \frac{2}{q_1}\|\Lambda^\alpha |w|^{\frac{q_1}{2}}\|_{L^2}^2 \leq \int \Lambda^{2\alpha} w |w|^{q_1-2} w$$

for a constant $\tilde{c}(q_1)$ that depends on q_1 so that we can estimate

$$\begin{aligned}
&\frac{1}{q_1}\partial_t \|w\|_{L^{q_1}}^{q_1} + \tilde{c}(q_1)\|w\|_{L^{\frac{q_1}{1-\alpha}}}^{q_1} \\
\lesssim&\|b\|_{L^\infty}\|\nabla j\|_{L^{\frac{q_1}{1+\alpha}}}\|w\|_{L^{q_1}}^{q_1-2}\|w\|_{L^{\frac{q_1}{1-\alpha}}} \\
\lesssim&\|b\|_{L^2}^{\frac{\alpha}{1+\alpha}}\|\Lambda^\alpha j\|_{L^2}^{\frac{1}{1+\alpha}}\|\nabla j\|_{L^{\frac{q_1}{1+\alpha}}}\|w\|_{L^{q_1}}^{q_1-2}\|w\|_{L^{\frac{q_1}{1-\alpha}}} \\
\leq&\frac{\tilde{c}(q_1)}{2}\|w\|_{L^{\frac{q_1}{1-\alpha}}}^{q_1} + c\|\nabla j\|_{L^{\frac{q_1}{1+\alpha}}}^{\frac{q_1}{q_1-1}}\|w\|_{L^{q_1}}^{(\frac{q_1}{q_1-1})(q_1-2)}
\end{aligned} \tag{13}$$

by Hölder's inequalities, Gagliardo-Nirenberg inequality, (10), Proposition 3.1 and Young's inequality. Now for any $q_1 \in \left(2(1+\alpha), \frac{2(1+\alpha)}{1-\alpha}\right)$, we may estimate

$$\|\nabla j\|_{L^{\frac{q_1}{1+\alpha}}} \lesssim \|\nabla j\|_{L^2}^{\frac{2(1+\alpha)-(1-\alpha)q_1}{q_1\alpha}}\|\Lambda^{1+\alpha} j\|_{L^2}^{1-\frac{2(1+\alpha)-(1-\alpha)q_1}{q_1\alpha}} \lesssim_{q_1} 1 + \|\Lambda^{1+\alpha} j\|_{L^2} \tag{14}$$

by Gagliardo-Nirenberg inequality while

$$\|\nabla j\|_{L^{\frac{q_1}{1+\alpha}}} = \begin{cases} \|\nabla j\|_{L^{\frac{2}{1-\alpha}}} \lesssim \|\Lambda^{1+\alpha} j\|_{L^2} & \text{if } q_1 = \frac{2(1+\alpha)}{1-\alpha}, \\ \|\nabla j\|_{L^2} \lesssim 1 + \|\Lambda^{1+\alpha} j\|_{L^2} & \text{if } q_1 = 2(1+\alpha), \end{cases} \tag{15}$$

by Sobolev embedding of $\dot{H}^\alpha(\mathbb{R}^2) \hookrightarrow L^{\frac{2}{1-\alpha}}(\mathbb{R}^2)$ and Proposition 2.9. Therefore, applying (14) and (15) in (13), subtracting $\frac{\tilde{c}(q_1)}{2}\|w\|_{L^{\frac{q_1}{1-\alpha}}}^{q_1}$ from both sides, Young's inequalities lead to

$$\frac{1}{q_1}\partial_t \|w\|_{L^{q_1}}^{q_1} + \frac{\tilde{c}(q_1)}{2}\|w\|_{L^{\frac{q_1}{1-\alpha}}}^{q_1} \lesssim_{q_1} (1 + \|\Lambda^{1+\alpha} j\|_{L^2}^2)(1 + \|w\|_{L^{q_1}}^{q_1}).$$

Applications of Gronwall's inequality and Proposition 3.1 complete the proof of Proposition 3.2. □

We now show that the bound on $\int_0^T \|w\|^{q_1}_{L^{\frac{q_1}{1-\alpha}}} d\tau$ directly leads to an L^{q_2}-bound of j for $q_2 = \frac{q_1}{1-\alpha}$.

PROPOSITION 3.3. *Let* $\alpha \in (0, \frac{1}{3}), \beta = 1$. *Suppose* $(u_0, b_0) \in H^s(\mathbb{R}^2), s \geq 3$. *Then its corresponding solution pair* (u, b) *to the system (2a)-(2c) in* $[0, T]$ *has the following bound: for* $q_2 \in [2, \frac{2(1+\alpha)}{(1-\alpha)^2}]$,

$$\sup_{t\in[0,T]} \|j(t)\|^{q_2}_{L^{q_2}} \lesssim_{q_2} 1.$$

Proof. We consider for simplicity of the proof, $q_2 = \frac{2(1+\alpha)}{(1-\alpha)^2}$ as the lower L^{q_2}-bounds may be attained via interpolation. We multiply (12b) by $|j|^{q_2-2}j$, integrate in space to obtain due to the incompressibility of u from (2c),

$$\begin{aligned}&\frac{1}{q_2}\partial_t \|j\|^{q_2}_{L^{q_2}} + \int |j|^{q_2-2} j \Lambda^2 j \\ = &\int |j|^{q_2-2} j \left((b\cdot\nabla)w + 2[\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - \partial_1 u_1(\partial_2 b_1 + \partial_1 b_2)]\right).\end{aligned} \tag{16}$$

By Lemma 2.1, we have on the diffusive term,

$$\tilde{c}(q_2)\|\Lambda|j|^{\frac{q_2}{2}}\|^2_{L^2} \leq \int |j|^{q_2-2} j \Lambda^2 j \tag{17}$$

whereas we estimate by integration by parts

$$\begin{aligned}\int |j|^{q_2-2} j (b\cdot\nabla) w =& (q_2 - 1)\int wb\cdot\nabla j |j|^{q_2-2} \\ =& \frac{(q_2-1)2}{q_2}\int_{\{j\neq 0\}} wb\cdot(\nabla|j|^{\frac{q_2}{2}})\frac{1}{\text{sgn}(j)}|j|^{\frac{q_2}{2}-1} \\ \lesssim_{q_2}& \|b\|_{L^\infty}\|w\|_{L^{q_2}}\|\nabla|j|^{\frac{q_2}{2}}\|_{L^2}\|j\|^{\frac{q_2}{2}-1}_{L^{q_2}} \\ \lesssim_{q_2}& \|b\|^{\frac{\alpha}{1+\alpha}}_{L^2}\|\Lambda^\alpha j\|^{\frac{1}{1+\alpha}}_{L^2}\|w\|_{L^{q_2}}\|\nabla|j|^{\frac{q_2}{2}}\|_{L^2}\|j\|^{\frac{q_2}{2}-1}_{L^{q_2}} \\ \leq& \frac{\tilde{c}(q_2)}{4}\|\Lambda|j|^{\frac{q_2}{2}}\|^2_{L^2} + c\|w\|^2_{L^{q_2}}(\|j\|^{q_2}_{L^{q_2}} + 1)\end{aligned} \tag{18}$$

where $\text{sgn}(j) = \frac{j}{|j|}$, by Hölder's inequalities, Gagliardo-Nirenberg inequality, (10), Proposition 3.1, Young's inequalities and Plancherel theorem. Moreover, we estimate

$$\begin{aligned}&\int |j|^{q_2-2} j 2[\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - \partial_1 u_1(\partial_2 b_1 + \partial_1 b_2)] \\ \lesssim& \int |j|^{q_2-1}|\nabla b||\nabla u| \\ \lesssim& \||j|^{\frac{q_2}{2}}\|_{L^{\frac{2q_2}{q_2-2}}}\|\nabla b\|_{L^{q_2}}\|j\|^{\frac{q_2-2}{2}}_{L^{q_2}}\|w\|_{L^{q_2}} \\ \lesssim& \|\Lambda|j|^{\frac{q_2}{2}}\|^{\frac{2}{q_2}}_{L^2}\|j\|^{q_2-1}_{L^{q_2}}\|w\|_{L^{q_2}} \leq \frac{\tilde{c}(q_2)}{4}\|\Lambda|j|^{\frac{q_2}{2}}\|^2_{L^2} + c\|j\|^{q_2}_{L^{q_2}}\|w\|^{\frac{q_2}{q_2-1}}_{L^{q_2}}\end{aligned} \tag{19}$$

by Hölder's inequality, Gagliardo-Nirenberg inequality, Lemma 2.2, and Young's inequality. Applying (17), (18), (19) in (16) and subtracting $\frac{\tilde{c}(q_2)}{2}\|\nabla|j|^{\frac{q_2}{2}}\|_{L^2}^2$ from both sides give

$$\frac{1}{q_2}\partial_t\|j\|_{L^{q_2}}^{q_2} + \frac{\tilde{c}(q_2)}{2}\|\Lambda|j|^{\frac{q_2}{2}}\|_{L^2}^2 \lesssim (\|j\|_{L^{q_2}}^{q_2} + 1)(\|w\|_{L^{q_2}}^2 + \|w\|_{L^{q_2}}^{\frac{q_2}{q_2-1}}).$$

Now Gronwall's inequality and Proposition 3.2 at $q_1 = \frac{2(1+\alpha)}{1-\alpha}$ complete the proof of Proposition 3.3. □

The next proposition follows the work of [5] closely; however, our proof seems slightly more straight-forward (see e.g. (29), (43)).

Proposition 3.4. *Let* $\alpha \in (0, \frac{1}{3}), \beta = 1$. *Suppose* $(u_0, b_0) \in H^s(\mathbb{R}^2), s \geq 3$. *Then its corresponding solution pair* (u, b) *to the system (2a)-(2c) in* $[0, T]$ *has the following bound: for* $\frac{2}{1-\alpha} < q_2 \leq \frac{2(1+\alpha)}{(1-\alpha)^2}, \frac{2}{q_2(1-\alpha)} < \delta < 1$,

$$\int_0^T \|j\|_{B_{q_2,2}^\delta}^2 d\tau \lesssim_{q_2} 1.$$

Proof. We fix $k \geq 3$, apply Δ_k on (12b), multiply by $|\Delta_k j|^{q_2-2}\Delta_k j$ and integrate in space to obtain

$$\begin{aligned}
&\frac{1}{q_2}\partial_t\|\Delta_k j\|_{L^{q_2}}^{q_2} + \int |\Delta_k j|^{q_2-2}\Delta_k j \Lambda^2 \Delta_k j \\
=& \int |\Delta_k j|^{q_2-2}\Delta_k j \Delta_k(-(u\cdot\nabla)j) + \int |\Delta_k j|^{q_2-2}\Delta_k j \Delta_k((b\cdot\nabla)w) \\
&+ \int |\Delta_k j|^{q_2-2}\Delta_k j \Delta_k(2\partial_1 b_1 \partial_1 u_2) + \int |\Delta_k j|^{q_2-2}\Delta_k j \Delta_k(2\partial_1 b_1 \partial_2 u_1) \\
&+ \int |\Delta_k j|^{q_2-2}\Delta_k j \Delta_k(-2\partial_1 u_1 \partial_1 b_2) + \int |\Delta_k j|^{q_2-2}\Delta_k j \Delta_k(-2\partial_1 u_1 \partial_2 b_1) \\
\triangleq& \sum_{i=1}^6 II_i.
\end{aligned} \tag{20}$$

For the diffusive term, by Lemma 2.8 we obtain its lower bound

$$\tilde{c}2^{2k}\|\Delta_k j\|_{L^{q_2}}^{q_2} \leq \int |\Delta_k j|^{q_2-2}\Delta_k j \Lambda^2 \Delta_k j. \tag{21}$$

Now we estimate the nonlinear terms. We apply Bony's product decomposition (9), subtract and add to rewrite II_1 from (20) as

$$\begin{aligned}
II_1 =& - \int |\Delta_k j|^{q_2-2}\Delta_k j \sum_{l:|k-l|\leq 2} [\Delta_k, S_{l-1}u\cdot\nabla]\Delta_l j \\
&- \int |\Delta_k j|^{q_2-2}\Delta_k j \sum_{l:|k-l|\leq 2} (S_{l-1}u - S_k u)\cdot\nabla\Delta_k\Delta_l j \\
&- \int |\Delta_k j|^{q_2-2}\Delta_k j \sum_{l:|k-l|\leq 2} S_k u\cdot\nabla\Delta_k\Delta_l j \\
&- \int |\Delta_k j|^{q_2-2}\Delta_k j \sum_{l:|k-l|\leq 2} \Delta_k(\Delta_l u\cdot S_{l-1}\nabla j) \\
&- \int |\Delta_k j|^{q_2-2}\Delta_k j \sum_{l:l\geq k-1, l':|l-l'|\leq 1} \Delta_k(\Delta_l u\cdot\Delta_{l'}\nabla j) \triangleq \sum_{i=1}^{5} II_{1i}.
\end{aligned} \tag{22}$$

The advantage of this is that $II_{13} = 0$ due to (2c):

$$II_{13} = -\int |\Delta_k j|^{q_2-2}\Delta_k j \sum_{l:|k-l|\leq 2} S_k u\cdot\nabla\Delta_k\Delta_l j = 0. \tag{23}$$

Besides, we first estimate

$$\begin{aligned}
|II_{11}| =& |-\int |\Delta_k j|^{q_2-2}\Delta_k j \sum_{l:|k-l|\leq 2} [\Delta_k, S_{l-1}u\cdot\nabla]\Delta_l j| \\
\lesssim& \sum_{l:|k-l|\leq 2} \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \|[\Delta_k, S_{l-1}u\cdot\nabla]\Delta_l j\|_{L^{q_2}} \\
\lesssim& \|\Delta_k j\|_{L^{q_2}}^{q_2} 2^{k(-1)} \|\nabla S_{k-1}u\|_{L^{q_2}} \|\nabla\Delta_k j\|_{L^\infty} \|x\Phi_0\|_{L^1}
\end{aligned}$$

by (22), Hölder's inequalities, the fact that for all l such that $|k-l|\leq 2$, we may replace l by k modifying constants, and standard commutator estimate (e.g. Lemma 2.1 [29]). We may now continue this estimate as follows:

$$\begin{aligned}
|II_{11}| \lesssim& \|\Delta_k j\|_{L^{q_2}}^{q_2} 2^{k(-1)} \|\nabla S_{k-1}u\|_{L^{q_2}} \|\nabla\Delta_k j\|_{L^\infty} \|x\Phi_0\|_{L^1} \\
\lesssim& \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{-k} \|\nabla u\|_{L^{q_2}} 2^{k(1+2(\frac{1}{q_2}))} \|\Delta_k j\|_{L^{q_2}} \\
\lesssim& \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \|w\|_{L^{q_2}} 2^{k(\frac{2}{q_2})} \|j\|_{L^{q_2}} \lesssim_{q_2} \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{k(\frac{2}{q_2})} \|w\|_{L^{q_2}}
\end{aligned} \tag{24}$$

by Bernstein's inequalities, Lemma 2.2, (8) and Proposition 3.3. Similarly,

$$\begin{aligned}
|II_{12}| \lesssim& \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \sum_{l:|k-l|\leq 2} \|S_{l-1}u - S_k u\|_{L^{q_2}} \|\nabla\Delta_k\Delta_l j\|_{L^\infty} \\
\lesssim& \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \|S_{k-1}u - S_k u\|_{L^{q_2}} \|\nabla\Delta_k j\|_{L^\infty} \\
\lesssim& \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \|\Delta_{k-1}u\|_{L^{q_2}} 2^{k(1+2(\frac{1}{q_2}))} \|\Delta_k j\|_{L^{q_2}} \\
\lesssim& \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \|\Delta_{k-1}\nabla u\|_{L^{q_2}} 2^{k(\frac{2}{q_2})} \|j\|_{L^{q_2}} \lesssim_{q_2} \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{k(\frac{2}{q_2})} \|w\|_{L^{q_2}}
\end{aligned} \tag{25}$$

by (22), Hölder's inequalities, (9), Bernstein's inequalities using that $k\geq 3$, (8) and

Proposition 3.3. Next,

$$
\begin{aligned}
|II_{14}| \lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \sum_{l:|k-l|\leq 2} \|\Delta_l u\|_{L^{q_2}} \|\nabla S_{l-1} j\|_{L^\infty} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \|\Delta_k u\|_{L^{q_2}} \|\nabla S_{k-1} j\|_{L^\infty} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{-k} \|\nabla \Delta_k u\|_{L^{q_2}} \sum_{m:m\leq k-2} \|\nabla \Delta_m j\|_{L^\infty} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \|w\|_{L^{q_2}} 2^{k(\frac{2}{q_2})} \sum_{m:m\leq k-2} 2^{(m-k)(1+\frac{2}{q_2})} \|\Delta_m j\|_{L^{q_2}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \|w\|_{L^{q_2}} 2^{k(\frac{2}{q_2})} \|2^{-|k|(1+\frac{2}{q_2})}\|_{l^1} \|j\|_{B^0_{q_2,\infty}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \|w\|_{L^{q_2}} 2^{k(\frac{2}{q_2})} \|j\|_{B^0_{q_2,q_2}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \|w\|_{L^{q_2}} 2^{k(\frac{2}{q_2})} \|j\|_{L^{q_2}} \lesssim_{q_2} \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{k(\frac{2}{q_2})} \|w\|_{L^{q_2}}
\end{aligned} \tag{26}
$$

by (22), Hölder's inequalities, Bernstein's inequality with the fact that $k \geq 3$, (9), Young's inequality for convolution, (8), and Proposition 3.3. Finally, in particular making use of (2c) we estimate

$$
\begin{aligned}
|II_{15}| \lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \sum_{l:k\leq l+3} \|\Delta_k(\mathrm{div}(\Delta_l u \Delta_l j))\|_{L^{q_2}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \sum_{l:k\leq l+3} 2^{k-l} \|\Delta_l \nabla u\|_{L^{q_2}} 2^{l2(\frac{1}{q_2})} \|\Delta_l j\|_{L^{q_2}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{k(\frac{2}{q_2})} \sum_{l:k\leq l+3} 2^{(k-l)(1-\frac{2}{q_2})} \|\Delta_l w\|_{L^{q_2}} \|\Delta_l j\|_{L^{q_2}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{k(\frac{2}{q_2})} \sum_{l:k\leq l+3} 2^{(k-l)(1-\frac{2}{q_2})} \|\Delta_l j\|_{L^{q_2}} \|w\|_{B^0_{q_2,\infty}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{k(\frac{2}{q_2})} \sum_{l:k\leq l+3} 2^{(k-l)(1-\frac{2}{q_2})} \|\Delta_l j\|_{L^{q_2}} \|w\|_{B^0_{q_2,q_2}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{k(\frac{2}{q_2})} \sum_{l:k\leq l+3} 2^{(k-l)(1-\frac{2}{q_2})} \|\Delta_l j\|_{L^{q_2}} \|w\|_{L^{q_2}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{k(\frac{2}{q_2})} \|j\|_{L^{q_2}} \|w\|_{L^{q_2}} \lesssim_{q_2} \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{k(\frac{2}{q_2})} \|w\|_{L^{q_2}}
\end{aligned} \tag{27}
$$

by (22), Hölder's inequalities, Bernstein's inequalities using $k \geq 3$, (8), Young's inequalities for convolution, and Proposition 3.3. Therefore, considering (23), (24), (25), (26), (27) in (22), we have

$$
|II_1| \lesssim_{q_2} \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{k(\frac{2}{q_2})} \|w\|_{L^{q_2}}. \tag{28}
$$

Next, we rewrite II_2 from (20) as

$$
\begin{aligned}
II_2 = & \int |\Delta_k j|^{q_2-2} \Delta_k j \Delta_k((b\cdot\nabla)w) \\
= & \int |\Delta_k j|^{q_2-2} \Delta_k j \Delta_k \Bigg(\sum_{l:|k-l|\leq 2} S_{l-1} b \cdot \nabla \Delta_l w + \sum_{l:|k-l|\leq 2} \Delta_l b \cdot S_{l-1} \nabla w \\
& + \sum_{l:l\geq k-1, l':|l-l|\leq 1} \Delta_l b \cdot \Delta_{l'} \nabla w \Bigg) \triangleq \sum_{i=1}^{3} II_{2i}
\end{aligned} \tag{29}
$$

by Bony's paraproduct decomposition. Below we will estimate (29) in a more straightforward manner than the computations in [5] (cf. $L_{21}, L_{22}, L_{23}, L_{24}, L_{25}$ in the proof of Proposition 4.1 [5]). Firstly, we estimate

$$\begin{aligned}
|II_{21}| \lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \sum_{l:|k-l|\leq 2} \|\Delta_k(S_{l-1}b\cdot\nabla\Delta_l w)\|_{L^{q_2}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \|S_{k-1}b\|_{L^\infty} \|\nabla\Delta_k w\|_{L^{q_2}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \|b\|_{L^\infty} 2^k \|\Delta_k w\|_{L^{q_2}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \|b\|_{L^2}^{\frac{\alpha}{1+\alpha}} \|\Lambda^{1+\alpha} b\|_{L^2}^{\frac{1}{1+\alpha}} 2^k \|w\|_{L^{q_2}} \lesssim \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^k \|w\|_{L^{q_2}}
\end{aligned} \tag{30}$$

by (29), Hölder's inequality, Bernstein's inequality and Gagliardo-Nirenberg inequalities, (8), (10) and Proposition 3.1. Next,

$$\begin{aligned}
|II_{22}| \lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \sum_{l:|k-l|\leq 2} \|\Delta_k \mathrm{div}(\Delta_l b S_{l-1} w)\|_{L^{q_2}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \sum_{l:|k-l|\leq 2} 2^k \|\Delta_l b\|_{L^\infty} \|S_{l-1} w\|_{L^{q_2}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{2k(\frac{1}{q_2})} \|\Delta_k \nabla b\|_{L^{q_2}} \|w\|_{L^{q_2}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{k(\frac{2}{q_2})} \|j\|_{L^{q_2}} \|w\|_{L^{q_2}} \lesssim_{q_2} \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{k(\frac{2}{q_2})} \|w\|_{L^{q_2}}
\end{aligned} \tag{31}$$

by (29), Hölder's inequality, (2c), Bernstein's inequalities with the fact that $k \geq 3$, (8) and Proposition 3.3. Finally,

$$\begin{aligned}
|II_{23}| \lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \sum_{l:k\leq l+3} \|\Delta_k \mathrm{div}(\Delta_l b \Delta_l w)\|_{L^{q_2}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \sum_{l:k\leq l+3} 2^k \|\Delta_l b\|_{L^\infty} \|\Delta_l w\|_{L^{q_2}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \sum_{l:k\leq l+3} 2^k \|\Delta_l b\|_{L^\infty} \|w\|_{B^0_{q_2,\infty}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \sum_{l:k\leq l+3} 2^k \|\Delta_l b\|_{L^\infty} \|w\|_{B^0_{q_2,q_2}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{k(\frac{2}{q_2})} \left(\sum_{l:k\leq l+3} 2^{(k-l)(1-\frac{2}{q_2})} \|\Delta_l j\|_{L^{q_2}} \right) \|w\|_{L^{q_2}} \\
\lesssim & \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{k(\frac{2}{q_2})} \|j\|_{L^{q_2}} \|w\|_{L^{q_2}} \lesssim_{q_2} \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{k(\frac{2}{q_2})} \|w\|_{L^{q_2}}
\end{aligned} \tag{32}$$

by (29), Hölder's inequalities, (2c), Bernstein's inequalities with the fact that $k \geq 3$, (8), Young's inequalities for convolution, and Proposition 3.3. Thus, considering (30), (31), (32) in (29) gives

$$|II_2| \lesssim_{q_2} \|\Delta_k j\|_{L^{q_2}}^{q_2-1} (2^{k(\frac{2}{q_2})} + 2^k) \|w\|_{L^{q_2}}. \tag{33}$$

Next, we work on II_3 from (20), noting that identical estimates will work for

II_4, II_5, II_6: we first rewrite

$$\begin{aligned} II_3 =& \int |\Delta_k j|^{q_2-2}\Delta_k j \Delta_k (2\partial_1 b_1 \partial_1 u_2) \\ =& 2\int |\Delta_k j|^{q_2-2}\Delta_k j\Delta_k (\sum_{l:|k-l|\leq 2} S_{l-1}\partial_1 b_1 \Delta_l \partial_1 u_2 \\ &+ \sum_{l:|k-l|\leq 2} \Delta_l \partial_1 b_1 S_{l-1}\partial_1 u_2 \\ &+ \sum_{l:l\geq k-1, l':|l-l'|\leq 1} \Delta_l \partial_1 b_1 \Delta_{l'}\partial_1 u_2) \triangleq \sum_{i=1}^{3} II_{3i} \end{aligned} \tag{34}$$

by (20) and (9). Firstly,

$$\begin{aligned} |II_{31}| \lesssim& \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \sum_{l:|k-l|\leq 2} \|\Delta_k (S_{l-1}\partial_1 b_1 \Delta_l \partial_1 u_2)\|_{L^{q_2}} \\ \lesssim& \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \|S_{k-1} j\|_{L^{q_2}} \|\Delta_k w\|_{L^\infty} \\ \lesssim& \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \|j\|_{L^{q_2}} 2^{2k(\frac{1}{q_2})} \|\Delta_k w\|_{L^{q_2}} \lesssim_{q_2} \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{k(\frac{2}{q_2})}\|w\|_{L^{q_2}} \end{aligned} \tag{35}$$

by (34), Hölder's inequality, Bernstein's inequality, and Proposition 3.3. Next,

$$\begin{aligned} |II_{32}| \lesssim& \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \sum_{l:|k-l|\leq 2} \|\Delta_l \partial_1 b_1\|_{L^\infty} \|S_{l-1}\partial_1 u_2\|_{L^{q_2}} \\ \lesssim& \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \|\Delta_k \partial_1 b_1\|_{L^\infty} \|w\|_{L^{q_2}} \\ \lesssim& \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{k2(\frac{1}{q_2})} \|\Delta_k j\|_{L^{q_2}} \|w\|_{L^{q_2}} \\ \lesssim& \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{k2(\frac{1}{q_2})} \|j\|_{L^{q_2}} \|w\|_{L^{q_2}} \lesssim_{q_2} \|\Delta_k j\|_{L^{q_2}}^{q_2-1} 2^{k(\frac{2}{q_2})}\|w\|_{L^{q_2}} \end{aligned} \tag{36}$$

by Hölder's inequality, Bernstein's inequality, (8) and Proposition 3.3. Finally,

$$\begin{aligned} |II_{33}| \lesssim& \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \sum_{l:k\leq l+3} \|\Delta_l \partial_1 b_1\|_{L^\infty} \|\Delta_l \partial_1 u_2\|_{L^{q_2}} \\ \lesssim& \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \sum_{l:k\leq l+3} 2^{l2(\frac{1}{q_2})} \|\Delta_l \partial_1 b_1\|_{L^{q_2}} 2^{l2(\frac{1}{q_2(1-\alpha)}-\frac{1}{q_2})} \|\Delta_l w\|_{L^{q_2(1-\alpha)}} \\ \lesssim& \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \sum_{l:k\leq l+3} 2^{\frac{2l}{q_2(1-\alpha)}} \|\Delta_l j\|_{L^{q_2}} \|w\|_{L^{q_2(1-\alpha)}} \\ \lesssim& \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \sum_{l:k\leq l+3} 2^{\frac{2l}{q_2(1-\alpha)}} \|\Delta_l j\|_{L^{q_2}} \end{aligned} \tag{37}$$

by (34), Hölder's inequalities, Bernstein's inequality, (8) and Proposition 3.2; we also used the crucial hypothesis that $q_2(1-\alpha) > 2$. Therefore, we obtain from (35), (36), (37) applied to (34),

$$|II_3| \lesssim_{q_2} \|\Delta_k j\|_{L^{q_2}}^{q_2-1} \left(2^{k(\frac{2}{q_2})}\|w\|_{L^{q_2}} + \sum_{l:k\leq l+3} 2^{\frac{2l}{q_2(1-\alpha)}} \|\Delta_l j\|_{L^{q_2}} \right). \tag{38}$$

Considering (21), (28), (33), (38) applied to (20) gives

$$\frac{1}{q_2}\partial_t\|\Delta_k j\|_{L^{q_2}}^{q_2} + \tilde{c}2^{2k}\|\Delta_k j\|_{L^{q_2}}^{q_2}$$
$$\lesssim_{q_2} \|\Delta_k j\|_{L^{q_2}}^{q_2-1}\left((2^{k(\frac{2}{q_2})} + 2^k)\|w\|_{L^{q_2}} + \sum_{l:k\leq l+3} 2^{\frac{2l}{q_2(1-\alpha)}}\|\Delta_l j\|_{L^{q_2}}\right).$$

Making use of the fact that $\frac{1}{q_2}\partial_t\|\Delta_k j\|_{L^{q_2}}^{q_2} = \|\Delta_k j\|_{L^{q_2}}^{q_2-1}\partial_t\|\Delta_k j\|_{L^{q_2}}$, this leads to

$$\partial_t\left(\|\Delta_k j\|_{L^{q_2}} e^{\tilde{c}2^{2k}t}\right) \lesssim_{q_2} e^{\tilde{c}2^{2k}t}\left((2^{k(\frac{2}{q_2})} + 2^k)\|w\|_{L^{q_2}} + \sum_{l:k\leq l+3} 2^{\frac{2l}{q_2(1-\alpha)}}\|\Delta_l j\|_{L^{q_2}}\right).$$

Now integrating in time, we obtain

$$\|\Delta_k j(t)\|_{L^{q_2}} \lesssim_{q_2} e^{-\tilde{c}2^{2k}t}\|\Delta_k j_0\|_{L^{q_2}}$$
$$+ \int_0^t e^{-\tilde{c}2^{2k}(t-\tau)}\left((2^{k(\frac{2}{q_2})} + 2^k)\|w(\tau)\|_{L^{q_2}} + \sum_{l:k\leq l+3} 2^{\frac{2l}{q_2(1-\alpha)}}\|\Delta_l j(\tau)\|_{L^{q_2}}\right) d\tau$$

where $j_0 \triangleq \nabla \times b_0$. Taking L_t^2-norm, using Minkowski's inequality, taking squares, and applying Minkowski's inequality for convolution, we obtain

$$\int_0^t \|\Delta_k j\|_{L^{q_2}}^2 d\tau \lesssim_{q_2} \|e^{-\tilde{c}2^{2k}t}\|_{L_t^2}^2\|\Delta_k j_0\|_{L^{q_2}}^2$$
$$+ \|e^{-\tilde{c}2^{2k}t}\|_{L_t^1}^2\left\|(2^{k(\frac{2}{q_2})} + 2^k)\|w(t)\|_{L^{q_2}} + \sum_{l:k\leq l+3} 2^{\frac{2l}{q_2(1-\alpha)}}\|\Delta_l j(t)\|_{L^{q_2}}\right\|_{L_t^2}^2. \tag{39}$$

Now we may compute

$$\|e^{-\tilde{c}2^{2k}t}\|_{L_t^2}^2 \lesssim 2^{-2k}, \qquad \|e^{-\tilde{c}2^{2k}t}\|_{L_t^1}^2 \lesssim (2^{-2k})^2 \approx 2^{-4k}; \tag{40}$$

that we take this Young's inequality for convolution specifically as in (39) is crucial; we will really need this $-4k$ power (see e.g. (45)). Thus, applying (40) to (39) gives for any $k \geq 3$,

$$\int_0^t \|\Delta_k j\|_{L^{q_2}}^2 d\tau \lesssim_{q_2} 2^{-2k}\|\Delta_k j_0\|_{L^{q_2}}^2$$
$$+2^{-4k}\left(\left\|(2^{k(\frac{2}{q_2})} + 2^k)\|w(t)\|_{L^{q_2}}\right\|_{L_t^2}^2 + \left\|\sum_{l:k\leq l+3} 2^{\frac{2l}{q_2(1-\alpha)}}\|\Delta_l j(t)\|_{L^{q_2}}\right\|_{L_t^2}^2\right). \tag{41}$$

We multiply (41) by $2^{2k\delta}$, bound in particular the case $k = -1, 0, 1, 2$ by

$$\int_0^t \sum_{k=-1}^{2} 2^{2k\delta}\|\Delta_k j\|_{L^{q_2}}^2 d\tau$$
$$= \int_0^t 2^{-2\delta}\|\Delta_{-1} j\|_{L^{q_2}}^2 + \|\Delta_0 j\|_{L^{q_2}}^2 + 2^{2\delta}\|\Delta_1 j\|_{L^{q_2}}^2 + 2^{4\delta}\|\Delta_2 j\|_{L^{q_2}}^2 d\tau \tag{42}$$
$$\lesssim T \sup_{t\in[0,T]}\|j(t)\|_{L^{q_2}}^2 \lesssim_{q_2} T$$

for all $t \in [0, T]$ by Proposition 3.3, and hence we may sum over $k \geq -1$ and estimate

$$\begin{aligned}\int_0^t \|j\|_{B^{\delta}_{q_2,2}}^2 d\tau \lesssim_{q_2} & T + \sum_{k\geq 3} \int_0^t 2^{2k\delta} \|\Delta_k j\|_{L^{q_2}}^2 d\tau \\ \lesssim_{q_2} & T + \sum_{k\geq 3} 2^{2k(\delta-1)} \|\Delta_k j_0\|_{L^{q_2}}^2 \\ & + \sum_{k\geq 3} 2^{2k(\delta-2)} \left\| (2^{k(\frac{2}{q_2})} + 2^k) \|w(t)\|_{L^{q_2}} \right\|_{L_t^2}^2 \\ & + \sum_{k\geq 3} 2^{2k(\delta-2)} \int_0^t \left(\sum_{l:k\leq l+3} 2^{\frac{2l}{q_2(1-\alpha)}} \|\Delta_l j(\tau)\|_{L^{q_2}} \right)^2 d\tau \\ \triangleq & cT + III_1 + III_2 + III_3 \end{aligned} \tag{43}$$

by (42), (41). Again our estimates are more straight-forward here in comparison to the computations in [5] (see $M_1+M_2+M_3+M_4+M_5+M_6$ in the proof of Proposition 4.1 [5]). Firstly, we estimate

$$III_1 \lesssim_{q_2} \sum_{k\geq -1} 2^{2k(\delta-1)} \|\Delta_k j_0\|_{L^{q_2}}^2 \approx_{q_2} \|b_0\|_{B^{\delta}_{q_2,2}}^2 \tag{44}$$

by (43). Secondly, we estimate

$$\begin{aligned} III_2 \lesssim_{q_2} & \sum_{k\geq 3} 2^{2k(\delta-2)} \int_0^t (2^{2k(\frac{2}{q_2})} + 2^{2k}) \|w(\tau)\|_{L^{q_2}}^2 d\tau \\ \approx_{q_2} & \sum_{k\geq 3} (2^{2k(\delta-2+\frac{2}{q_2})} + 2^{2k(\delta-1)}) \int_0^t \|w(\tau)\|_{L^{q_2}}^2 d\tau \lesssim_{q_2} 1 \end{aligned} \tag{45}$$

where we used (43), the elementary inequality of

$$(a+b)^2 \leq 4(a^2+b^2) \quad \forall\, a, b \geq 0,$$

and Proposition 3.2. Finally, we estimate

$$\begin{aligned} III_3 \lesssim_{q_2} & \sum_{k\geq 3} 2^{2k(\delta-2)} \int_0^t \left(\sum_{l:k\leq l+3} 2^{l(\frac{2\alpha}{q_2(1-\alpha)}-\alpha)} \|\Delta_l \Lambda^{1+\alpha} j\|_{L^2} \right)^2 d\tau \\ \lesssim_{q_2} & \sum_{k\geq 3} 2^{2k(\delta-2)} \int_0^t \left(\sum_{l:k\leq l+3} 2^{2l(\frac{2\alpha}{q_2(1-\alpha)}-\alpha)} \right) \left(\sum_{l:k\leq l+3} \|\Delta_l \Lambda^{1+\alpha} j\|_{L^2}^2 \right) d\tau \\ \lesssim_{q_2} & \sum_{k\geq 3} 2^{2k(\delta-2)} \int_0^t \left(\sum_{l\geq -1} \|\Delta_l \Lambda^{1+\alpha} j\|_{L^2}^2 \right) d\tau \\ \lesssim_{q_2} & \int_0^t \|\Lambda^{1+\alpha} j\|_{L^2}^2 d\tau \lesssim_{q_2} 1 \end{aligned} \tag{46}$$

by (43), Bernstein's inequalities as $l \geq k-3 \geq 0$, Hölder's inequality, and Proposition 3.1. Therefore, applying (44), (45), (46) in (43) leads to

$$\int_0^t \|j\|_{B^{\delta}_{q_2,2}}^2 d\tau \lesssim_{q_2} T + III_1 + III_2 + III_3 \lesssim T + \|b_0\|_{B^{\delta}_{q_2,2}}^2 + 1 \lesssim 1 \tag{47}$$

since $b_0 \in H^s, s \geq 3$. This completes the proof of Proposition 3.4. □

We now improve the bound of w in Proposition 3.2.

PROPOSITION 3.5. *Let $\alpha \in (0, \frac{1}{3}), \beta = 1$. Suppose $(u_0, b_0) \in H^s(\mathbb{R}^2), s \geq 3$. Then its corresponding solution pair (u, b) to the system (2a)-(2c) in $[0, T]$ has the following bounds: for $q_2 \in (\frac{2}{1-\alpha}, \frac{2(1+\alpha)}{(1-\alpha)^2}]$,*

$$\sup_{t\in[0,T]} \|w(t)\|_{L^{q_2}}^{q_2} + \int_0^T \|w\|_{L^{\frac{q_2}{1-\alpha}}}^{q_2} d\tau \lesssim_{q_2} 1.$$

Proof. As in the proof of Proposition 3.2, we multiply (12a) with $|w|^{q_2-2}w$, integrate in space to obtain

$$\begin{aligned}
&\frac{1}{q_2}\partial_t \|w\|_{L^{q_2}}^{q_2} + \tilde{c}(q_2)\|\Lambda^\alpha |w|^{\frac{q_2}{2}}\|_{L^2}^2 \\
\leq& \int -(u\cdot\nabla)w|w|^{q_2-2}w + (b\cdot\nabla)j|w|^{q_2-2}w \\
=& \int \operatorname{div}(bj)|w|^{q_2-2}w \\
\leq& 2\|\operatorname{div}(bj)\|_{\dot{W}^{-\alpha,q_2}} \||w|^{q_2-1}\|_{\dot{W}^{\alpha,\frac{q_2}{q_2-1}}}
\end{aligned} \tag{48}$$

by Lemma 2.1, (2c) and Hölder's inequality. We remark here that it is crucial to make careful use of the dissipation strength here; in particular, if one uses Hardy-Littlewood-Sobolev theorem ([22]), and continuity of Riesz transform to deduce

$$\|\operatorname{div}(bj)\|_{\dot{W}^{-\alpha,q_2}} \lesssim \|\operatorname{div}(bj)\|_{L^{\frac{2q_2}{2+q_2\alpha}}} \lesssim \|bj\|_{\dot{W}^{1,\frac{2q_2}{2+q_2\alpha}}}, \tag{49}$$

then we will not be able to estimate this well. The problem is that according to Proposition 2.10, we wish to eventually take $q_2 \nearrow \frac{2}{\alpha}$; however, $\lim_{q_2\nearrow\frac{2}{\alpha}} \frac{2q_2}{2+q_2\alpha} = \frac{1}{\alpha}$ where $\alpha > 0$ is arbitrary small, and the best bound we have on b in terms of regularity is $\|\Lambda^{1+\alpha}j\|_{L^2}$ from Proposition 3.1, which cannot bound $\|\nabla j\|_{L^{\frac{2q_2}{2+q_2\alpha}}}$. Instead we estimate as follows:

$$\begin{aligned}
&\|\operatorname{div}(bj)\|_{\dot{W}^{-\alpha,q_2}} \\
\lesssim& \|\Lambda^{1-\alpha}(bj)\|_{L^{q_2}} \\
\lesssim& \|b\|_{L^\infty}\|\Lambda^{1-\alpha}j\|_{L^{q_2}} + \|\Lambda^{1-\alpha}b\|_{L^{q_2}}\|j\|_{L^\infty} \\
\lesssim& \|b\|_{L^2}^{\frac{\alpha}{1+\alpha}}\|\Lambda^\alpha j\|_{L^2}^{\frac{1}{1+\alpha}}\|\Lambda^{1-\alpha}j\|_{L^{q_2}} + \|b\|_{W^{1,q_2}}\|j\|_{L^2}^{\frac{\alpha}{1+\alpha}}\|\Lambda^{1+\alpha}j\|_{L^2}^{\frac{1}{1+\alpha}} \\
\lesssim& \|\Lambda^{1-\alpha}j\|_{L^{q_2}} + (\|b\|_{L^{q_2}} + \|j\|_{L^{q_2}})\|\Lambda^{1+\alpha}j\|_{L^2}^{\frac{1}{1+\alpha}} \\
\lesssim& \|\Lambda^{1-\alpha}j\|_{L^{q_2}} + \|\Lambda^{1+\alpha}j\|_{L^2}^{\frac{1}{1+\alpha}}
\end{aligned} \tag{50}$$

by continuity of Riesz transform, Lemma 2.4, Gagliardo-Nirenberg inequality, (10), Proposition 3.1, Proposition 3.3, Proposition 2.9, Lemma 2.2. On the other hand, we set $F(x) \triangleq x^{2-\frac{2}{q_2}}$, $G(x) \triangleq |x|^{1-\frac{2}{q_2}} \in C$ and $h(x) \triangleq c \in L^1([0,1])$ so that by Lemma 2.3 we deduce

$$\begin{aligned}
\||w|^{q_2-1}\|_{\dot{W}^{\alpha,\frac{q_2}{q_2-1}}} =& \|F(|w|^{\frac{q_2}{2}})\|_{\dot{W}^{\alpha,\frac{q_2}{q_2-1}}} \\
\lesssim& \|G(|w|^{\frac{q_2}{2}})\|_{L^{\frac{2q_2}{q_2-2}}} \||w|^{\frac{q_2}{2}}\|_{\dot{H}^\alpha} \approx \|w\|_{L^{q_2}}^{\frac{q_2-2}{2}} \|\Lambda^\alpha |w|^{\frac{q_2}{2}}\|_{L^2}.
\end{aligned} \tag{51}$$

Therefore, applying (50), (51) in (48) we have

$$\begin{aligned}
&\frac{1}{q_2}\partial_t \|w\|_{L^{q_2}}^{q_2} + \tilde{c}(q_2)\|\Lambda^\alpha |w|^{\frac{q_2}{2}}\|_{L^2}^2 \\
\lesssim&(\|\Lambda^{1-\alpha} j\|_{L^{q_2}} + \|\Lambda^{1+\alpha} j\|_{L^2}^{\frac{1}{1+\alpha}})\|w\|_{L^{q_2}}^{\frac{q_2-2}{2}} \|\Lambda^\alpha |w|^{\frac{q_2}{2}}\|_{L^2} \\
\le&\frac{\tilde{c}(q_2)}{2}\|\Lambda^\alpha |w|^{\frac{q_2}{2}}\|_{L^2}^2 + c(1+\|\Lambda^{1-\alpha} j\|_{L^{q_2}}^2 + \|\Lambda^{1+\alpha} j\|_{L^2}^2)(\|w\|_{L^{q_2}}^{q_2}+1)
\end{aligned} \tag{52}$$

due to Young's inequality. By Proposition 3.1 we know $\int_0^T \|\Lambda^{1+\alpha} j\|_{L^2}^2 d\tau \lesssim 1$; thus, it suffices to show the time integrability of $\|\Lambda^{1-\alpha} j\|_{L^{q_2}}^2$. Since $\alpha > 0$ and $q_2 > \frac{2}{1-\alpha}$, we may find $\delta \in (\max\{1-\alpha, \frac{2}{q_2(1-\alpha)}\}, 1)$ and estimate by Bernstein's and Hölder's inequalities

$$\begin{aligned}
\int_0^T \|\Lambda^{1-\alpha} j\|_{L^{q_2}}^2 d\tau \lesssim& \int_0^T \left(\sum_{k\ge -1} \|\Lambda^{1-\alpha}\Delta_k j\|_{L^{q_2}}\right)^2 d\tau \\
\lesssim& \int_0^T \left(\sum_{k\ge -1} 2^{k(1-\alpha-\delta)} 2^{k\delta}\|\Delta_k j\|_{L^{q_2}}\right)^2 d\tau \\
\lesssim& \int_0^T \left(\sum_{k\ge -1} 2^{2k(1-\alpha-\delta)}\right)\left(\sum_{k\ge -1} 2^{2k\delta}\|\Delta_k j\|_{L^{q_2}}^2\right) d\tau \\
\lesssim& \int_0^T \|j\|_{B_{q_2,2}^\delta}^2 d\tau \lesssim 1
\end{aligned} \tag{53}$$

due to Proposition 3.4 as $1-\alpha-\delta < 0$. Gronwall's inequality applied on (52), along with (53), completes the proof of Proposition 3.5. □

Proof of Theorem 1.1. Here we explain the iteration scheme. If $\alpha \ge \frac{1}{3}$, as we explained in Remark 3.1, our proof was complete at Proposition 3.2 because Proposition 2.10 stated that only $\int_0^T \|w\|_{L^{\frac{2}{\alpha}}}^2 d\tau \lesssim 1$ is needed to complete the proof of Theorem 1.1; we assumed $\alpha \in (0, \frac{1}{3})$ so that

$$\frac{2(1+\alpha)}{(1-\alpha)^2} < \frac{2}{\alpha}$$

which guaranteed that q_2 in the hypothesis of Proposition 3.5 satisfies $q_2 < \frac{2}{\alpha}$. By Proposition 3.5, we now know that $\int_0^T \|w\|_{L^{\frac{q_2}{1-\alpha}}}^{q_2} d\tau \lesssim 1$ for $q_2 = \frac{2(1+\alpha)}{(1-\alpha)^2}$. Thus, we now consider only $\alpha > 0$ such that

$$\frac{2(1+\alpha)}{(1-\alpha)^3} < \frac{2}{\alpha},$$

go back to Proposition 3.3, Proposition 3.4, Proposition 3.5 to obtain higher L^{q_3}-bound, $q_3 \in [2, \frac{2(1+\alpha)}{(1-\alpha)^3}]$ and repeat to deduce in particular

$$\int_0^T \|w\|_{L^{\frac{q_k}{1-\alpha}}}^{q_k} d\tau \lesssim 1, \quad q_k \in [2, \frac{2(1+\alpha)}{(1-\alpha)^k}], \quad \forall k \in \mathbb{Z}^+ \text{ such that } \frac{2(1+\alpha)}{(1-\alpha)^k} < \frac{2}{\alpha}.$$

This implies that because $\alpha > 0$ has been fixed, for clarity we may denote by λ the first k such that

$$\frac{2(1+\alpha)}{(1-\alpha)^{\lambda}} \triangleq \frac{2(1+\alpha)}{(1-\alpha)^{k}} \geq \frac{2}{\alpha}$$

so that consequently

$$\frac{2(1+\alpha)}{(1-\alpha)^{\lambda}} \geq \frac{2}{\alpha} > \frac{2(1+\alpha)}{(1-\alpha)^{\lambda-1}}$$

and therefore we have the bound of

$$\int_0^T \|w\|_{L^{\frac{2(1+\alpha)}{(1-\alpha)^{\lambda}}}}^{\frac{2(1+\alpha)}{(1-\alpha)^{\lambda-1}}} d\tau \lesssim 1. \tag{54}$$

It is now straight-forward to deduce by interpolation

$$\int_0^T \|w\|_{L^{\frac{2}{\alpha}}}^2 d\tau \lesssim \sup_{t\in[0,T]} \|w(t)\|_{L^2}^{2\epsilon} \int_0^T \|w\|_{L^{\frac{2(1+\alpha)}{(1-\alpha)^{\lambda}}}}^{2(1-\epsilon)} d\tau \lesssim 1$$

where

$$\epsilon \triangleq \left[\frac{\alpha}{2} - \frac{(1-\alpha)^{\lambda}}{2(1+\alpha)}\right] \Big/ \left[\frac{1}{2} + \frac{(1-\alpha)^{\lambda}}{2(1+\alpha)}\right]$$

by Proposition 2.9 and (54); by Proposition 2.10, this completes the proof of Theorem 1.1. □

4. Appendix.

4.1. Proof of Proposition 2.9. Taking L^2-inner products of (12a)-(12b) with (w, j) respectively and using the incompressibility of u and b in (2c), we estimate in sum

$$\begin{aligned}
&\frac{1}{2}\partial_t(\|w\|_{L^2}^2 + \|j\|_{L^2}^2) + \|\Lambda^{\alpha} w\|_{L^2}^2 + \|\Lambda j\|_{L^2}^2 \\
=&2\int [\partial_1 b_1(\partial_1 u_2 + \partial_2 u_1) - \partial_1 u_1(\partial_1 b_2 + \partial_2 b_1)]j \\
\lesssim&\|\nabla b\|_{L^4}\|\nabla u\|_{L^2}\|j\|_{L^4} \lesssim \|j\|_{L^2}\|\nabla j\|_{L^2}\|w\|_{L^2} \leq \frac{1}{2}\|\Lambda j\|_{L^2}^2 + c\|j\|_{L^2}^2\|w\|_{L^2}^2
\end{aligned}$$

where we used Hölder's, Gagliardo-Nirenberg and Young's inequalities and Lemma 2.2. Subtracting $\frac{1}{2}\|\Lambda j\|_{L^2}^2$ from both sides, applying (10) and Gronwall's inequality complete the proof of Proposition 2.9.

4.2. Proof of Proposition 2.10. We take L^2-inner products of (12a)-(12b) with $(-\Delta w, -\Delta j)$ respectively, sum to estimate

$$\begin{aligned}
&\frac{1}{2}\partial_t(\|\nabla w\|_{L^2}^2 + \|\nabla j\|_{L^2}^2) + \|\Lambda^{\alpha}\nabla w\|_{L^2}^2 + \|\Delta j\|_{L^2}^2 \\
\lesssim&\int |\nabla u||\nabla w|^2 + |\nabla b||\nabla j||\nabla w| + |u||\nabla j||\Delta j| + |\nabla b||\nabla u||\Delta j| \\
\lesssim&\|\nabla u\|_{L^{\frac{2}{\alpha}}}\|\nabla w\|_{L^{\frac{2}{1-\alpha}}}\|\nabla w\|_{L^2} + \|\nabla b\|_{L^4}\|\nabla j\|_{L^4}\|\nabla w\|_{L^2} \\
&+ \|u\|_{L^\infty}\|\nabla j\|_{L^2}\|\Delta j\|_{L^2} + \|j\|_{L^4}\|w\|_{L^4}\|\Delta j\|_{L^2}
\end{aligned}$$

where we used integration by parts, Hölder's inequalities. We continue to bound by

$$\begin{aligned}
&\frac{1}{2}\partial_t(\|\nabla w\|_{L^2}^2 + \|\nabla j\|_{L^2}^2) + \|\Lambda^\alpha \nabla w\|_{L^2}^2 + \|\Delta j\|_{L^2}^2 \\
\lesssim &\|w\|_{L^{\frac{2}{\alpha}}} \|\Lambda^\alpha \nabla w\|_{L^2} \|\nabla w\|_{L^2} + \|j\|_{L^2}^{\frac{1}{2}} \|\nabla j\|_{L^2} \|\Delta j\|_{L^2}^{\frac{1}{2}} \|\nabla w\|_{L^2} \\
&+ \|u\|_{L^2}^{\frac{1}{2}} \|\nabla w\|_{L^2}^{\frac{1}{2}} \|\nabla j\|_{L^2} \|\Delta j\|_{L^2} + \|j\|_{L^2}^{\frac{1}{2}} \|\nabla j\|_{L^2}^{\frac{1}{2}} \|w\|_{L^2}^{\frac{1}{2}} \|\nabla w\|_{L^2}^{\frac{1}{2}} \|\Delta j\|_{L^2} \\
\leq &\frac{1}{2}(\|\Lambda^\alpha \nabla w\|_{L^2}^2 + \|\Delta j\|_{L^2}^2) + c(\|\nabla w\|_{L^2}^2 + \|\nabla j\|_{L^2}^2 + 1)(\|w\|_{L^{\frac{2}{\alpha}}}^2 + \|\nabla j\|_{L^2}^2 + 1)
\end{aligned}$$

due to Lemma 2.2, Sobolev embedding of $\dot{H}^\alpha(\mathbb{R}^2) \hookrightarrow L^{\frac{2}{1-\alpha}}(\mathbb{R}^2)$, Gagliardo-Nirenberg inequalities, Proposition 2.9, (10), and Young's inequalities. By Proposition 2.9 and hypothesis, this implies

$$\sup_{t\in[0,T]} \left(\|\nabla w\|_{L^2}^2 + \|\nabla j\|_{L^2}^2\right)(t) + \int_0^T \|\Lambda^\alpha \nabla w\|_{L^2}^2 + \|\Delta j\|_{L^2}^2 d\tau \lesssim 1. \tag{55}$$

This implies by Gagliardo Nirenberg inequalities that

$$\begin{aligned}
&\int_0^T \|w\|_{L^\infty} + \|j\|_{L^\infty} d\tau \\
\lesssim &\sup_{t\in[0,T]} \|w\|_{L^2}^{\frac{\alpha}{1+\alpha}} \int_0^T \|\Lambda^\alpha \nabla w\|_{L^2}^{\frac{1}{1+\alpha}} d\tau + \sup_{t\in[0,T]} \|j\|_{L^2}^{\frac{1}{2}} \int_0^T \|\Delta j\|_{L^2}^{\frac{1}{2}} d\tau \lesssim 1,
\end{aligned} \tag{56}$$

by Proposition 2.9 and (55). This bound immediately leads to higher regularity. We denote for convenience $X(t) \triangleq \|\Lambda^s u(t)\|_{L^2}^2 + \|\Lambda^s b(t)\|_{L^2}^2$, apply Λ^s to (2a)-(2b), take L^2-inner products with $(\Lambda^s u, \Lambda^s b)$ respectively, use (2c) so that

$$\begin{aligned}
&\int u \cdot \nabla \Lambda^s u \cdot \Lambda^s u = 0, \quad \int u \cdot \nabla \Lambda^s b \cdot \Lambda^s b = 0, \\
&\int b \cdot \nabla \Lambda^s b \cdot \Lambda^s u + \int b \cdot \nabla \Lambda^s u \cdot \Lambda^s b = 0,
\end{aligned}$$

to estimate

$$\begin{aligned}
&\frac{1}{2}\partial_t X(t) + \|\Lambda^{s+\alpha} u\|_{L^2}^2 + \|\Lambda^{s+1} b\|_{L^2}^2 \\
= &-\int [\Lambda^s((u\cdot\nabla)u) - u\cdot\nabla\Lambda^s u]\cdot\Lambda^s u + \int [\Lambda^s((b\cdot\nabla)b) - b\cdot\nabla\Lambda^s b]\cdot\Lambda^s u \\
&- \int [\Lambda^s((u\cdot\nabla)b) - u\cdot\nabla\Lambda^s b]\cdot\Lambda^s b + \int [\Lambda^s((b\cdot\nabla)u) - b\cdot\nabla\Lambda^s u]\cdot\Lambda^s b \\
\lesssim &(\|\nabla u\|_{L^\infty} + \|\nabla b\|_{L^\infty}) X(t) \\
\lesssim &(1 + \|w\|_{L^\infty} + \|j\|_{L^\infty}) \log_2(2 + X(t))(2 + X(t))
\end{aligned}$$

by Lemma 2.5, Lemma 2.6 and (10). Dividing by $2 + X(t)$, Gronwall's inequality and (56) completes the proof of Proposition 2.9.

4.3. Proof of Proposition 3.1. *Proof.* We take L^2-inner products of (12b) with $\Lambda^{2\alpha} j$ to obtain

$$\begin{aligned}
&\frac{1}{2}\partial_t \|\Lambda^\alpha j\|_{L^2}^2 + \|\Lambda^{1+\alpha} j\|_{L^2}^2 \\
=& \int \left(-\text{div}(uj) + \text{div}(bw)\right) \Lambda^{2\alpha} j \\
&+ 2\int [\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - \partial_1 u_1(\partial_2 b_1 + \partial_1 b_2)]\Lambda^{2\alpha} j \\
\lesssim& \left(\|\Lambda^{\alpha-1}\text{div}(uj)\|_{L^2} + \|\Lambda^{\alpha-1}\text{div}(bw)\|_{L^2}\right)\|\Lambda^{1+\alpha} j\|_{L^2} \\
&+ \|\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - \partial_1 u_1(\partial_2 b_1 + \partial_1 b_2)\|_{L^2}\|\Lambda^{2\alpha} j\|_{L^2} \triangleq I_1 + I_2
\end{aligned} \tag{57}$$

where we used (2c), Hölder's inequalities. We estimate them separately: firstly,

$$\begin{aligned}
I_1 \approx& \left(\|\Lambda^{\alpha-1}\text{div}(uj)\|_{L^2} + \|\Lambda^{\alpha-1}\text{div}(bw)\|_{L^2}\right)\|\Lambda^{1+\alpha} j\|_{L^2} \\
\leq& \frac{1}{8}\|\Lambda^{1+\alpha} j\|_{L^2}^2 + c\left(\|\Lambda^\alpha (uj)\|_{L^2}^2 + \|\Lambda^\alpha (bw)\|_{L^2}^2\right) \\
\leq& \frac{1}{8}\|\Lambda^{1+\alpha} j\|_{L^2}^2 + c(\|\Lambda^\alpha u\|_{L^2}^2 \|j\|_{L^\infty}^2 + \|u\|_{L^\infty}^2 \|\Lambda^\alpha j\|_{L^2}^2 \\
&\qquad + \|\Lambda^\alpha b\|_{L^{\frac{2}{\alpha}}}^2 \|w\|_{L^{\frac{2}{1-\alpha}}}^2 + \|b\|_{L^\infty}^2 \|\Lambda^\alpha w\|_{L^2}^2) \\
\leq& \frac{1}{8}\|\Lambda^{1+\alpha} j\|_{L^2}^2 + c(\|u\|_{L^2}^{2(1-\alpha)} \|\nabla u\|_{L^2}^{2\alpha} \|j\|_{L^2}^{2(\frac{\alpha}{1+\alpha})} \|\Lambda^{1+\alpha} j\|_{L^2}^{2(\frac{1}{1+\alpha})} \\
&\qquad + \|u\|_{L^2}^{2(\frac{\alpha}{1+\alpha})} \|\Lambda^{1+\alpha} u\|_{L^2}^{2(\frac{1}{1+\alpha})} \|\Lambda^\alpha j\|_{L^2}^2 \\
&\qquad + \|j\|_{L^2}^2 \|\Lambda^\alpha w\|_{L^2}^2 + \|b\|_{L^2}^{2(\frac{\alpha}{1+\alpha})} \|\Lambda^{1+\alpha} b\|_{L^2}^{2(\frac{1}{1+\alpha})} \|\Lambda^\alpha w\|_{L^2}^2)
\end{aligned}$$

where we used (57), Young's inequalities, Lemma 2.4, the Gagliardo-Nirenberg inequalities, and Sobolev embeddings of $\dot{H}^{1-\alpha}(\mathbb{R}^2) \hookrightarrow L^{\frac{2}{\alpha}}(\mathbb{R}^2)$, $\dot{H}^\alpha(\mathbb{R}^2) \hookrightarrow L^{\frac{2}{1-\alpha}}(\mathbb{R}^2)$. By (10), Proposition 2.9, Lemma 2.4 and Young's inequalities we continue to bound by

$$\begin{aligned}
I_1 \leq& \frac{1}{8}\|\Lambda^{1+\alpha} j\|_{L^2}^2 \\
&+ c(\|\Lambda^{1+\alpha} j\|_{L^2}^{\frac{2}{1+\alpha}} + \|\Lambda^\alpha w\|_{L^2}^{\frac{2}{1+\alpha}} \|\Lambda^\alpha j\|_{L^2}^2 \\
&\qquad + \|\Lambda^\alpha w\|_{L^2}^2 + \|\Lambda^\alpha j\|_{L^2}^{\frac{2}{1+\alpha}} \|\Lambda^\alpha w\|_{L^2}^2) \\
\leq& \frac{1}{4}\|\Lambda^{1+\alpha} j\|_{L^2}^2 + c(1 + \|\Lambda^\alpha w\|_{L^2}^2)(1 + \|\Lambda^\alpha j\|_{L^2}^2).
\end{aligned} \tag{58}$$

Next, we work on I_2 from (57):

$$\begin{aligned}
I_2 \approx& \|\partial_1 b_1(\partial_2 u_1 + \partial_1 u_2) - \partial_1 u_1(\partial_2 b_1 + \partial_1 b_2)\|_{L^2}\|\Lambda^{2\alpha} j\|_{L^2} \\
\lesssim& \|\nabla b\|_{L^{\frac{2}{\alpha}}} \|\nabla u\|_{L^{\frac{2}{1-\alpha}}} \|\Lambda^\alpha j\|_{L^2}^{1-\alpha} \|\Lambda^{1+\alpha} j\|_{L^2}^\alpha \\
\lesssim& \|\Lambda j\|_{L^2}^{1-\alpha} \|\Lambda^\alpha w\|_{L^2} \|\Lambda^\alpha j\|_{L^2}^{1-\alpha} \|\Lambda^{1+\alpha} j\|_{L^2}^\alpha \\
\leq& \frac{1}{4}\|\Lambda^{1+\alpha} j\|_{L^2}^2 + c\left(\|\Lambda j\|_{L^2}^2 + \|\Lambda^\alpha w\|_{L^2}^2\right)\left(1 + \|\Lambda^\alpha j\|_{L^2}^2\right)
\end{aligned} \tag{59}$$

by Hölder's inequalities, Gagliardo-Nirenberg inequalities, Sobolev embedding of $\dot{H}^\alpha(\mathbb{R}^2) \hookrightarrow L^{\frac{2}{1-\alpha}}(\mathbb{R}^2)$ and Young's inequalities. Considering (58) and (59) in (57), we obtain

$$\begin{aligned}&\frac{1}{2}\partial_t\|\Lambda^\alpha j\|_{L^2}^2+\|\Lambda^{1+\alpha}j\|_{L^2}^2\\ \leq&\frac{1}{2}\|\Lambda^{1+\alpha}j\|_{L^2}^2+c\left(1+\|\Lambda j\|_{L^2}^2+\|\Lambda^\alpha w\|_{L^2}^2\right)\left(1+\|\Lambda^\alpha j\|_{L^2}^2\right).\end{aligned}$$

Subtracting $\frac{1}{2}\|\Lambda^{1+\alpha}j\|_{L^2}^2$ from both sides, Gronwall's inequality with Proposition 2.9 completes the proof of Proposition 3.1. □

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