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ON MICROLOCAL SMOOTHNESS OF SOLUTIONS OF FIRST ORDER NONLINEAR PDE*

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Abstract. We study the microlocal smoothness of C^2 solutions u of the first-order nonlinear partial differential equation

$$u_t = f(x, t, u, u_x)$$

where $f(x, t, \zeta_0, \zeta)$ is a complex-valued function which is C^∞ in all the variables (x, t, ζ_0, ζ) and holomorphic in the variables (ζ_0, ζ). If the solution u is C^2, $\sigma \in \text{Char}(L^u)$ and $\frac{1}{\sqrt{-1}}\sigma([L^u, \bar{L^u}]) < 0$, then we show that $\sigma \notin WF(u)$. Here $WF(u)$ denotes the C^∞ wave front set of u and $\text{Char}(L^u)$ denotes the characteristic set of the linearized operator

$$L^u = \frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x, t, u, u_x)\frac{\partial}{\partial x_j}.$$

Key words. C^∞ wave front set, linearized operator.

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1. Introduction. In this paper we study the regularity of C^2 solutions of the first order nonlinear PDE

$$u_t = f(x, t, u, u_x) \tag{1.1}$$

where $f(x, t, \zeta_0, \zeta)$ is complex-valued, C^∞ in all the variables (x, t, ζ_0, ζ), and holomorphic in (ζ_0, ζ). The variable x varies in an open set in $\mathbb{R}^m$, t in an interval of $\mathbb{R}$, and (ζ_0, ζ) in an open set in $\mathbb{C} \times \mathbb{C}^m = \mathbb{C}^{m+1}$. If u is a C^2 solution of (1.1), it was shown in [8] and [2] that the C^∞ wave-front set of u is contained in the characteristic set of the linearized vector field

$$L^u = \frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x, t, u, u_x)\frac{\partial}{\partial x_j} \tag{1.2}$$

In Hanges and Treves [10] it was shown that under the additional hypothesis that f is analytic in the variables (x, t, ζ_0, ζ), the analytic wave front set of u, denoted $WF_a(u)$, is contained in the characteristic set of the linearized operator L^u. In [4] it was proved that when u is a C^2 solution of (1.1), f is real analytic, $\sigma \in \text{Char}(L^u)$ and $\frac{1}{\sqrt{-1}}\sigma([L^u, \bar{L^u}]) < 0$, then $\sigma \notin WF_a(u)$. The work [4] only established a result that involves two brackets.

In this paper we will prove theorem 3.1 . We were motivated by the linear result of Berhanu and Xiao ([7]).

We first recall some of the known results concerning the C^∞ and analytic wave front sets of solutions of first order linear and nonlinear PDEs. The reader can find more results in the articles [1], [3], and [12]. Let

$$L = \sum_{j=1}^{m} a_j(x)\frac{\partial}{\partial x_j}$$

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be a complex vector field.

THEOREM 1.1. *If L is real analytic and $Lu = 0$, then $WF_a(u) \subset Char(L)$(see [11]).*

THEOREM 1.2. *If L is smooth and $Lu = 0$, then $WF(u) \subset Char(L)$(see [11]).*

THEOREM 1.3 (N. Hanges and F. Treves, 1992). *If f is real analytic in (x, t, ζ_0, ζ), holomorphic in (ζ_0, ζ) and $u_t = f(x, t, u, u_x)$, then $WF_a(u) \subset Char(L^u)$.*

THEOREM 1.4 (J. Y. Chemin, 1988). *If f is C^∞ in (x, t, ζ_0, ζ), holomorphic in (ζ_0, ζ) and $u_t = f(x, t, u, u_x)$, then $WF(u) \subset Char(L^u)$.*

Asano gave a simpler proof of the latter result using the standard FBI transform (see [2]).

THEOREM 1.5 (S. Berhanu, 2009). *Suppose f is real analytic in (x, t, ζ_0, ζ), holomorphic in (ζ_0, ζ) , $u_t = f(x, t, u, u_x)$, and $\sigma \in Char(L^u)$. If*

$$\frac{1}{\sqrt{-1}} \left\langle \sigma, \left[L^u, \overline{L^u}\right] \right\rangle < 0,$$

then $\sigma \notin WF_a(u)$.

THEOREM 1.6 (S. Berhanu and Ming Xiao, 2014). *Suppose L is a smooth vector field and u is C^1 solution of $Lu = 0$. If $\sigma \in Char(L)$ and $\frac{1}{\sqrt{-1}}\sigma([L, \overline{L}]) < 0$, then $\sigma \notin WF(u)$.*

2. Some preliminaries on first-order linear PDEs. We will use the following lemma whose proof is found in [2].

LEMMA 2.1. *Let $\Omega \subset \mathbb{R}^N$ be open, $J \subset \mathbb{R}$ be an open interval centered at 0 and let $\mathcal{N} \subset \mathbb{C}^M$ be open. Let*

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{N} a_j(x, t, \zeta) \frac{\partial}{\partial x_j} + \sum_{k=1}^{M} b_k(x, t, \zeta) \frac{\partial}{\partial \zeta_k}$$

where the coefficients a_j and b_k are C^∞ in the variables $(x, t, \zeta) \in \Omega \times J \times \mathcal{N}$ and holomorphic in the variable $\zeta \in \mathcal{N}$. Let $f(x, \zeta)$ be a C^∞ function defined on $\Omega \times \mathcal{N}$, holomorphic in ζ. Then there exists a C^∞ function $u(x, t, \zeta)$ defined on $\Omega \times J \times \mathcal{N}$ holomorphic in ζ which is an approximate solution of $Lu = 0$ in the sense that

$$Lu(x, t, \zeta) = O(t^k), \qquad k = 1, 2, \ldots \tag{2.1}$$

and such that $u(x, 0, \zeta) = f(x, \zeta)$.

Let $\Omega \subset \mathbb{R}^m_x \times \mathbb{R}_t$ be a neighborhood of the origin and consider the complex vector field defined on Ω

$$L = \frac{\partial}{\partial t} + \sum_{j=1}^{m} a_j(x, t) \frac{\partial}{\partial x_j},$$

where $a_j \in C^1(\Omega)$ for $j = 1, 2, \ldots, m$.

To L we associate another vector field

$$L_1 = \frac{\partial}{\partial s} + \sqrt{-1}L$$

where $s \in \mathbb{R}$ is a new variable. Then L_1 is a C^1 complex vector field on $\Omega \times \mathbb{R}$.

Suppose that there exist C^1 functions $\Psi_1(x,t,s), \ldots, \Psi_m(x,t,s)$ defined on $\Omega \times J$ ($J \subset \mathbb{R}$ is an open interval centered at 0) such

$$Z_j(x,t,s) = x_j + s\Psi_j(x,t,s), j = 1, \ldots, m$$

are approximate solutions of $L_1 Z_j(x,t,s) = 0$ in the sense that $L_1 Z_j(x,t,s)$ is s-flat at $s = 0$, i.e

$$\forall k \in \mathbb{N}, \quad \exists C_k > 0 : |L_1 Z_j(x,t,s)| \leq C_k |s|^k, \forall (x,t,s) \in \Omega \times J. \tag{2.2}$$

To get $m+1$ functions of the above type, we let

$$\Psi_{m+1}(x,t,s) = -\sqrt{-1} \quad \text{and} \quad Z_{m+1}(x,t,s) = t - s\sqrt{-1} = t + s\Psi_{m+1}(x,t,s).$$

Then

$$L_1 Z_{m+1} = \left(\frac{\partial}{\partial s} + \sqrt{-1} \left(\frac{\partial}{\partial t} + \sum_{j=1}^{m} a_j(x,t) \frac{\partial}{\partial x_j} \right) \right) (t - s\sqrt{-1}) = 0.$$

Set

$$\Psi = (\Psi_1, \ldots, \Psi_{m+1}) \quad \text{and} \quad Z = (Z_1, \ldots, Z_{m+1}).$$

Then

$$Z(x,t,s) = (x,t) + s\Psi(x,t,s).$$

LEMMA 2.2. *Let $L_1 = \frac{\partial}{\partial s} + \sqrt{-1}L$. Suppose $h(x,t,s)$ is C^1 such that $L_1 h(x,t,s)$ is s-flat at $s = 0$. Assume there exist C^1 functions $\Psi_1(x,t,s), \ldots, \Psi_{m+1}(x,t,s)$ defined on $\Omega \times J$ ($\Omega \subset \mathbb{R}^{m+1}, J \subset \mathbb{R}$ both about the origin) such that $Z = (x,t) + s\Psi(x,t,s)$ is an approximate solution of $L_1 Z = 0$ in the sense that $L_1 Z$ is s-flat at $s = 0$. If $\sigma = (0,0;\xi^0,\tau^0) \in \mathrm{Char}\, L$ and $\frac{1}{\sqrt{-1}}\sigma([L,\overline{L}]) < 0$, then $\sigma \notin WF(w)$ where $w(x,t) = h(x,t,0)$.*

Proof. As in [7], we may assume that

$$L = \frac{\partial}{\partial t} + \sqrt{-1} \sum_{j=1}^{m} b_j(x,t) \frac{\partial}{\partial x_j},$$

where the b_j are C^1 and real valued functions near $(0,0) \in \mathbb{R}^{m+1}$. We then get $\tau^0 = 0$ since $\sigma = (0,0;\xi^0,\tau^0) \in \mathrm{Char}\, L$. A simple calculation shows that

$$[L, \bar{L}] = -2\sqrt{-1} \sum_{k=1}^{m} \frac{\partial b_k}{\partial t}(x,t) \frac{\partial}{\partial x_k}.$$

Thus, the assumption that

$$\frac{1}{\sqrt{-1}}\left\langle(\xi^0,0),[L,\bar{L}]_0\right\rangle<0$$

implies

$$-\frac{\partial b}{\partial t}(0,0)\cdot\xi^0<0. \tag{2.3}$$

Since $L_1Z_k(x,t,s)=O(s^n),\quad n=1,2,\dots,k=1,\dots,m+1$, we have for any $k=1,\dots,m$

$$\left(\frac{\partial}{\partial s}+\sqrt{-1}\frac{\partial}{\partial t}-\sum_{j=1}^m b_j(x,t)\frac{\partial}{\partial x_j}\right)(x_k+s\Psi_k(x,t,s))=O(s^2)$$

and so

$$\begin{aligned}\Psi_k(x,t,s)+s\frac{\partial\Psi_k}{\partial s}(x,t,s)+\sqrt{-1}s\frac{\partial\Psi_k}{\partial t}(x,t,s)\\ -\sum_{j=1}^m b_j(x,t)\left(\delta_{jk}+s\frac{\partial\Psi_k}{\partial x_j}(x,t,s)\right)=O(s^2).\end{aligned} \tag{2.4}$$

For each $k=1,\dots,m$, let

$$\begin{aligned}A_k(x,t,s)=\Psi_k(x,t,s)+s\frac{\partial\Psi_k}{\partial s}(x,t,s)+\sqrt{-1}s\frac{\partial\Psi_k}{\partial t}(x,t,s)\\ -\sum_{j=1}^m b_j(x,t)\left(\delta_{jk}+s\frac{\partial\Psi_k}{\partial x_j}(x,t,s)\right).\end{aligned} \tag{2.5}$$

Then for $s\neq0$,

$$\frac{A_k(x,t,s)-A_k(x,t,0)}{s}=O(s). \tag{2.6}$$

Since Ψ_k is C^1 letting $s\to0$ in (2.6) gives

$$2\frac{\partial\Psi_k}{\partial s}(x,t,0)+\sqrt{-1}\frac{\partial\Psi_k}{\partial t}(x,t,0)-\sum_{j=1}^m b_j(x,t)\frac{\partial\Psi_k}{\partial x_j}(x,t,0)=0. \tag{2.7}$$

Evaluating (2.4) at $s=0$ we have for each $k=1,\dots,m$

$$\Psi_k(x,t,0)=b_k(x,t). \tag{2.8}$$

Since $\Im b_k(x,t)=0,\forall\ k=1,\dots,m$, we have from (2.7) and (2.8)

$$\Im\Psi_k(x,t,0)=0\quad\text{and}\qquad\frac{\partial\Im\Psi_k}{\partial s}(x,t,0)=-\frac{1}{2}\frac{\partial b_k}{\partial t}(x,t),\quad\forall\ \ k=1,\dots,m. \tag{2.9}$$

Let $x'=(x,t)$. Since $Z_{x'}(x,t,0)=I$, there is a neighborhood Ω of $(0,0,0)$ in $\mathbb{R}^{m+2}$ such that $Z_{x'}(x,t,s)$ is non singular on Ω. Let

$$(\mu_{jk}(x,t,s))_{(m+1)\times(m+1)}=(Z_{x'}(x,t,s))^{-1},\quad(x,t,s)\in\Omega.$$

Then

$$\sum_{k=1}^{m+1} \mu_{kj}(x,t,s)\frac{\partial Z_r}{\partial x'_k}(x,t,s) = \delta_{jr}$$

for all $1 \leq j, r \leq m+1$. Let

$$c(x,t,s) = (\mu_{jk}(x,t,s))^t\,, \quad \big(A^t \text{ denotes transpose of a matrix } A\big)\,.$$

For $j = 1, 2, \ldots, m+1$, set

$$M_j = \sum_{k=1}^{m+1} c_{jk}(x,t,s)\frac{\partial}{\partial x'_k}.$$

Then M_j are continuous vector fields satisfying

$$M_j Z_r = \sum_{k=1}^{m+1} c_{jk}(x,t,s)\frac{\partial Z_r}{\partial x'_k} = \sum_{k=1}^{m+1} \mu_{kj}(x,t,s)\frac{\partial Z_r}{\partial x'_k} = \delta_{jr}.$$

If $\sum_{j=1}^{m+1} A_j M_j + AL_1 = 0$, then evaluating at the functions $s, Z_1, \ldots, Z_{m+1}$ shows that the vector fields $\{L_1, M_1, \ldots, M_{m+1}\}$ are linearly independent on Ω. Thus, $\{L_1, M_1, \ldots, M_{m+1}\}$ is a basis for the complexified tangent space $\mathbb{C}T\mathbb{R}^{m+2}$ on Ω.

Recall that for $x' = (x,t)$, $Z_k(x,t,s) = x'_k + s\Psi_k(x,t,s)$, $k = 1, 2, \ldots, m+1$. Since $dZ_k(x,t,0) = dx'_k$, and

$$\{dx_1, \ldots, dx_m, dt, ds\}$$

are linearly independent, by contracting Ω if necessary, we get that

$$\{dZ_1(x,t,s), \ldots, dZ_{m+1}(x,t,s), ds\}$$

is a basis of $\mathbb{C}T^*\mathbb{R}^{m+2}$ on Ω.

For any C^1 function g,

$$dg = \sum_{j=1}^{m+1} A_j dZ_j + B ds$$

for some continuous coefficients A_j and B. Evaluating at the vector fields $L_1, M_1, \ldots, M_{m+1}$ gives

$$dg = \sum_{k=1}^{m+1} M_k(g) dZ_k + \left(L_1 g - \sum_{k=1}^{m+1} M_k(g) L_1 Z_k\right) ds. \tag{2.10}$$

Using (2.10), we have

$$\begin{aligned} d(g dZ_1 \wedge \ldots \wedge dZ_{m+1}) &= dg \wedge dZ_1 \wedge \ldots \wedge dZ_{m+1} \\ &= \left(L_1 g - \sum_{k=1}^{m+1} M_k(g) L_1 Z_k\right) ds \wedge dZ_1 \wedge \ldots \wedge dZ_{m+1} \end{aligned} \tag{2.11}$$

since $dZ_j \wedge dZ_1 \wedge \ldots \wedge dZ_{m+1} = 0$, $\forall\ j = 1, 2, \ldots, m+1$.

For $(\xi,\tau)\in\mathbb{R}^{m+1}\setminus\{0\},(x',t')\in\mathbb{R}^{m+1}$ and for $K>0$ to be determined later, let

$$E(x',t',\xi,\tau,x,t,s)=\sqrt{-1}(\xi,\tau)\cdot(x'-Z'(x,t,s),t'-Z_{m+1}(x,t,s))$$
$$-K|(\xi,\tau)|\left[\langle x'-Z'(x,t,s)\rangle^2+(t'-Z_{m+1}(x,t,s))^2\right]$$

where $Z'=(Z_1,\dots,Z_m)$ and $\langle x'-Z'(x,t,s)\rangle^2=\sum_{j=1}^m(x'_j-Z'_j(x,t,s))^2$. Let $r>0$ such that $B=\left\{(x,t)\in\mathbb{R}^{m+1}:|x|^2+t^2<2r\right\}\subset\subset\Omega$. Let $\phi\in C_0^\infty(B)$, $\phi\equiv 1$ on $\left\{(x,t)\in\mathbb{R}^{m+1}:|x|^2+t^2\le r\right\}$. Set $dZ=dZ_1\wedge\dots\wedge dZ_{m+1}$. Apply (2.11) to the function $g(x',t',\xi,\tau,x,t,s)=\phi(x,t)h(x,t,s)e^{E(x',t',\xi,\tau,x,t,s)}$ to get

$$d(gdZ)=\left(L_1g-\sum_{k=1}^{m+1}M_k(g)L_1Z_k\right)ds\wedge dZ$$
$$=\left(L_1(\phi he^E)-\sum_{k=1}^{m+1}M_k(\phi he^E)L_1Z_k\right)ds\wedge dZ. \qquad (2.12)$$

Fix $|s_1|$ small. Let $J=[0,s_1],s_1>0$ or $J=[s_1,0],s_1<0$. Set

$$D=\left\{(x,t,s)\in\mathbb{R}^{m+2}:(x,t)\in B,s\in J\right\}.$$

Since $\phi(x,t)=0$ for $(x,t)\in\partial B$, we have by Stokes' theorem

$$\int_B g(x',t',\xi,\tau,x,t,0)dxdt=\int_B g(x',t',\xi,\tau,x,t,s_1)dZ(x,t,s_1)+\int_B\int_J d(gdZ)$$
$$=I_1(x',t',\xi,\tau)+I_2(x',t',\xi,\tau). \qquad (2.13)$$

We will estimate the integrals I_1 and I_2 for (x',t') near $(0,0)$ in $\mathbb{R}^{m+1}$ and (ξ,τ) in some conic neighborhood Γ of $(\xi^0,0)$ in $\mathbb{R}^{m+1}$. We will take $s_1>0$ when $\tau>0$ and $s_1<0$ for $\tau<0$ in (2.13).

Recall that $Z=(Z',Z_{m+1})=(x,t)+s\Psi(x,t,s)$, $\Psi=(\Psi',\Psi_{m+1})$ where $Z'=(Z_1,\dots,Z_m)$ and $\Psi'=(\Psi_1,\dots,\Psi_m)$. Then

$$\begin{aligned}
&\Re E(x',t',\xi,\tau,x,t,s)\\
&=\Re\left(\sqrt{-1}(\xi\cdot(x'-x-s\Re\Psi'(x,t,s)-s\sqrt{-1}\Im\Psi'(x,t,s)))\right)\\
&\quad+\Re\left(\sqrt{-1}\tau(t'-s\Re\Psi_{m+1}(x,t,s)-s\sqrt{-1}\Im\Psi_{m+1})\right)\\
&\quad-K|(\xi,\tau)|\Re\left(x'-x-s\Re\Psi'(x,t,s)-s\sqrt{-1}\Im\Psi'(x,t,s)\right)^2\\
&\quad-K|(\xi,\tau)|\Re\left(t'-t-s\Re\Psi_{m+1}(x,t,s)-s\sqrt{-1}\Im\Psi'_{m+1}(x,t,s)\right)^2\\
&=s\xi\cdot\Im\Psi'(x,t,s)+s\tau\Im\Psi_{m+1}(x,t,s)\\
&\quad-K|(\xi,\tau)|\left(\left|x'-x-s\Re\Psi'(x,t,s)\right|^2-|s\Im\Psi'(x,t,s)|^2\right)\\
&\quad-K|(\xi,\tau)|\left(\left|t'-t-s\Re\Psi_{m+1}(x,t,s)\right|^2-|s\Im\Psi_{m+1}(x,t,s)|^2\right).
\end{aligned} \qquad (2.14)$$

Since $Z_{m+1}=t-s\sqrt{-1}$, equation (2.14) becomes

$$\begin{aligned}
\Re E(x',t',\xi,\tau,x,t,s)&=s\xi\cdot\Im\Psi'(x,t,s)-s\tau\\
&\quad-K|(\xi,\tau)|\left(\left|x'-x-s\Re\Psi'(x,t,s)\right|^2-|s\Im\Psi'(x,t,s)|^2\right)\\
&\quad-K|(\xi,\tau)|\left(\left|t'-t\right|^2-s^2\right)
\end{aligned}$$
$$(2.15)$$

From (2.9) we have

$$\Im\Psi'(x,t,0)=0 \text{ and } \frac{\partial\Im\Psi'}{\partial s}(x,t,0)=-\frac{1}{2}\frac{\partial b}{\partial t}(x,t).$$

Therefore, since Ψ' is differentiable at $s=0$ for s near 0 we have

$$\begin{aligned}\Im\Psi'(x,t,s)=&\Im\Psi'(x,t,0)+\frac{\partial\Im\Psi'}{\partial s}(x,t,0)s+o(s)\\ =&\frac{\partial\Im\Psi'}{\partial s}(x,t,0)s+o(s)\\ =&-\frac{1}{2}\frac{\partial b}{\partial t}(x,t)s+o(s)\\ =&-\frac{1}{2}\frac{\partial b}{\partial t}(0,0)s+M(x,t)s+o(s),\\ &(x,t)\text{ near }(0,0)\text{ (since }\frac{\partial b}{\partial t}(x,t)\text{ is continous)}\end{aligned}\tag{2.16}$$

where $\frac{o(s)}{s}\to 0$ as $s\to 0$ and $M(x,t)\to 0$ as $(x,t)\to 0$. Plugging (2.16) into (2.15) results in

$$\begin{aligned}\Re E(x',t',\xi,\tau,x,t,s)=&-\frac{s^2}{2}\xi\cdot\frac{\partial b}{\partial t}(0,0)+s^2\xi\cdot M(x,t)+s\xi\cdot o(s)-s\tau\\ &-K|(\xi,\tau)|\left(|x'-x-s\Re\Psi'(x,t,s)|^2-|s\Im\Psi'(x,t,s)|^2\right)\\ &-K|(\xi,\tau)|\left(|t'-t|^2-s^2\right).\end{aligned}$$

Suppose $\tau>0$ and so take $0\le s\le s_1<1, s_1>0$. If $\tau<0$ we take $s_1<0$. In any case we have $-\tau s\le-\tau s^2$. Then

$$\begin{aligned}\Re E(x',t',\xi,\tau,x,t,s)\le s^2&\left\langle(\xi,\tau),\left(-\frac{1}{2}\frac{\partial b}{\partial t}(0,0),-1\right)\right\rangle+s^2|\xi|M(x,t)+s|\xi|o(s)\\ &-K|(\xi,\tau)|\left(|x'-x-s\Re\Psi'(x,t,s)|^2-|s\Im\Psi'(x,t,s)|^2\right)\\ &-K|(\xi,\tau)|\left(|t'-t|^2-s^2\right).\end{aligned}\tag{2.17}$$

Using (2.3) we have

$$\left\langle\frac{(\xi^0,0)}{|(\xi^0,0|},\left(-\frac{1}{2}\frac{\partial b}{\partial t}(0,0),-1\right)\right\rangle<0.$$

By continuity there is a neighborhood U_0 of $\frac{(\xi^0,0)}{|(\xi^0,0)|}$ in S^m such that for some $A>0$

$$\left\langle(\eta,\sigma),\left(-\frac{1}{2}\frac{\partial b}{\partial t}(0,0),-1\right)\right\rangle<-A,\quad\forall\ (\eta,\sigma)\in U_0.$$

Let

$$\Gamma=\left\{\lambda(\eta,\sigma):\lambda>0,(\eta,\sigma)\in U_0.\right\}.$$

Then Γ is a conic neighborhood of $(\xi^0, 0)$ and

$$\left\langle (\xi,\tau), \left(-\frac{1}{2}\frac{\partial b}{\partial t}(0,0), -1\right)\right\rangle \leq -A|(\xi,\tau)|, \quad \forall\ (\xi,\tau) \in \Gamma. \tag{2.18}$$

Since $M(x,t) \to 0$ as $(x,t) \to 0$ and $\frac{o(s)}{s} \to 0$ as $s \to 0$, taking r and s_1 small we get that

$$|M(x,t)| \leq \frac{A}{4} \quad \text{and} \quad |o(s)| \leq \frac{A}{4}s, \quad \forall\ (x,t) \in B, 0 \leq s \leq s_1. \tag{2.19}$$

Plugging (2.18) and (2.19) into (2.17) and using $|\xi| \leq |(\xi,\tau)|$ yields

$$\begin{aligned} &\Re E(x',t',\xi,\tau,x,t,s) \\ \leq\ &-\frac{s^2}{2}A|(\xi,\tau)| - K|(\xi,\tau)|\left(\left|x'-x-s\Re\Psi'(x,t,s)\right|^2 - |s\Im\Psi'(x,t,s)|^2\right) \\ &- K|(\xi,\tau)|\left(\left|t'-t\right|^2 - s^2\right),\ \forall\ (\xi,\tau) \in \Gamma, (x,t) \in B, 0 \leq s \leq s_1. \end{aligned} \tag{2.20}$$

Set

$$C = \sup_{(x,t)\in\overline{B}, 0\leq s\leq s_1} \left(|\Im\Psi'(x,t,s)|^2 + 1\right).$$

Then (2.20) becomes

$$\begin{aligned} &\Re E(x',t',\xi,\tau,x,t,s) \\ &\leq s^2\left(\frac{-A}{2} + KC + K\right)|(\xi,\tau)| - K|(\xi,\tau)|\left(\left|x'-x-s\Re\Psi'(x,t,s)\right|^2\right) \\ &\quad - K|(\xi,\tau)|\left(\left|t'-t\right|^2\right),\ \forall\ (\xi,\tau) \in \Gamma, (x,t) \in B, 0 \leq s \leq s_1. \end{aligned} \tag{2.21}$$

Choose $K = \frac{A}{4(C+1)}$. Then (2.21) becomes

$$\begin{aligned} &\Re E(x',t',\xi,\tau,x,t,s) \\ \leq\ &-\frac{A}{4}s^2|(\xi,\tau)| - \frac{A}{4(C+1)}|(\xi,\tau)|\left(\left|x'-x-s\Re\Psi'(x,t,s)\right|^2 + (t'-t)^2\right) \\ &,\forall\ (\xi,\tau) \in \Gamma, (x,t) \in B, 0 \leq s \leq s_1, (x',t') \in \mathbb{R}^{m+1}. \end{aligned} \tag{2.22}$$

We now return to the integrals in (2.13).

Consider $I_1(x',t',\xi,\tau)$: For $(x',t',\xi,\tau) \in \mathbb{R}^{m+1} \times \Gamma$ we have using (2.22)

$$\begin{aligned} |I_1(x',t',\xi,\tau)| &= \left|\int_B g(x',t',\xi,\tau,x,t,s_1)dZ(x,t,s_1)\right| \\ &\leq De^{-\frac{A}{4}s_1^2|(\xi,\tau|}, \quad \text{for some} \quad D > 0 \\ &\leq D\frac{k!}{\left(\frac{A}{4}s_1^2|(\xi,\tau|\right)^k},\ k = 0,1,2,\ldots \end{aligned}$$

Therefore, for each $k = 0,1,2\ldots$, there is $C_k^0 > 0$ such that

$$|I_1(x',t',\xi,\tau)| \leq \frac{C_k^0}{|(\xi,\tau)|^k}, \quad \forall\ (x',t',\xi,\tau) \in \mathbb{R}^{m+1} \times \Gamma. \tag{2.23}$$

Consider

$$
\begin{aligned}
I_2(x',t',\xi,\tau) &= \int_B \int_0^{s_1} d(gdZ) \\
&= \int_B \int_0^{s_1} hL_1(\phi)e^E ds \wedge dZ + \int_B \int_0^{s_1} \phi L_1(h) e^E ds \wedge dZ \\
&\quad + \int_B \int_0^{s_1} h\phi L_1(E) e^E ds \wedge dZ - \int_B \int_0^{s_1} \sum_{k=1}^{m+1} h(M_k\phi) L_1 Z_k e^E ds \wedge dZ \\
&\quad - \int_B \int_0^{s_1} \sum_{k=1}^{m+1} \phi(M_k h) L_1 Z_k e^E ds \wedge dZ \\
&\quad - \int_B \int_0^{s_1} \sum_{k=1}^{m+1} h\phi(M_k E) L_1 Z_k e^E ds \wedge dZ \\
&= \sum_{j=1}^{6} J_j(x',t',\xi,\tau). \qquad (2.24)
\end{aligned}
$$

Consider

$$
J_1(x',t',\xi,\tau) = \int_B \int_0^{s_1} hL_1(\phi)e^E ds \wedge dZ :
$$

Since

$$
\begin{aligned}
L_1\phi(x,t) &= \left(\frac{\partial}{\partial s} + \sqrt{-1}L\right)\phi(x,t) \\
&= \sqrt{-1}L\phi = \sqrt{-1}\left(\frac{\partial \phi}{\partial t}(x,t) + \sqrt{-1}\sum_{j=1}^{m} b_j(x,t)\frac{\partial \phi}{\partial x_j}(x,t)\right)
\end{aligned}
$$

and $\phi(x,t) = 1$ for $|x|^2 + t^2 \le r$, we have $L_1\phi(x,t) \equiv 0$ for $|x|^2 + t^2 < r$. In this particular integral we only need to focus on $r \le |x|^2 + t^2 \le 2r$. Let

$$
V = \left\{(x',t') \in \mathbb{R}^{m+1} : |x'|^2 + t'^2 < \frac{r}{4}\right\}.
$$

From (2.22) we have

$$
\begin{aligned}
\Re E(x',t',\xi,\tau,x,t,s) &\le -\frac{A}{4(C+1)}|(\xi,\tau)|\left(|x'-x-s\Re\Psi'(x,t,s)|^2 + (t'-t)^2\right) \\
&= -\frac{A}{4(C+1)}|(\xi,\tau)|\left(|x'-x|^2 + (t'-t)^2\right) \\
&\quad + \frac{A}{4(C+1)}|(\xi,\tau)|2s(x'-x)\cdot\Re\Psi'(x,t,s) \\
&\quad - \frac{A}{4(C+1)}|(\xi,\tau)|s^2|\Re\Psi'(x,t,s)|^2 \\
&\le -\frac{A}{4(C+1)}|(\xi,\tau)|\left(|x'-x|^2 + (t'-t)^2\right) \\
&\quad + \frac{A}{4(C+1)}|(\xi,\tau)|2s_1|x'-x||\Re\Psi'(x,t,s)|. \qquad (2.25)
\end{aligned}
$$

Let

$$A_1 = \sup_{\substack{|x'|^2 \le r \\ r \le |x|^2+t^2 \le 2r \\ 0 \le s \le s_1}} \left(|x'-x||\Re\Psi'(x,t,s)|\right).$$

For $|x|^2+t^2 \ge r$ and for $(x',t') \in V$ we have

$$|x'-x|^2 + (t'-t)^2 \ge \frac{r}{4}.$$

Then (2.25) becomes

$$\Re E(x',t',\xi,\tau,x,t,s) \le -\frac{rA}{16(C+1)}|(\xi,\tau)| + \frac{2s_1A_1A}{4(C+1)}|(\xi,\tau)|. \tag{2.26}$$

Choose s_1 small such that

$$\frac{2s_1A_1A}{4(C+1)} \le \frac{rA}{32(C+1)} := C_1.$$

Thus

$$\Re E(x',t',\xi,\tau,x,t,s) \le -C_1|(\xi,\tau)|, \quad \forall\ (\xi,\tau) \in \Gamma, (x',t') \in V, |x|^2+t^2 \ge r, 0 \le s \le s_1.$$

Therefore,

$$\begin{aligned} |J_1(x',t',\xi,\tau)| &= \left| \int_B \int_0^{s_1} hL_1(\phi)e^E ds \wedge dZ \right| \\ &\le B'e^{-C_1|(\xi,\tau)|}, \quad \text{for some} \quad B' > 0. \end{aligned}$$

But then for each $k = 0,1,2\ldots$, there is $C_k^1 > 0$ such that

$$|J_1(x',t',\xi,\tau)| \le \frac{C_k^1}{|(\xi,\tau)|^k}, \quad \forall\ (x',t',\xi,\tau) \in V \times \Gamma. \tag{2.27}$$

For the remaining integrals we will use

$$\begin{aligned} \Re E(x',t',\xi,\tau,x,t,s) \le &-\frac{A}{4}s^2|(\xi,\tau)|, \\ &\forall\ (\xi,\tau) \in \Gamma, (x,t) \in B, 0 \le s \le s_1, (x',t') \in \mathbb{R}^{m+1}. \end{aligned}$$

Consider

$$J_2(x',t',\xi,\tau) = \int_B \int_0^{s_1} \phi L_1(h)e^E ds \wedge dZ.$$

By assumption for any $k = 0,1,2\ldots$, there is $A_k > 0$ such that

$$|L_1h(x,t,s)| \le A_k s^{2k}, \forall\ (x,t) \in B.$$

Therefore, for each $k = 0,1,2\ldots$, there is $C_k^2 > 0$ such that

$$|J_2(x'.t',\xi,\tau)| \le \frac{C_k^2}{|(\xi,\tau)|^k}, \quad \forall\ (x',t',\xi,\tau) \in V \times \Gamma. \tag{2.28}$$

Consider

$$J_3(x',t',\xi,\tau)=\int_B\int_0^{s_1}\phi h(L_1E)e^E ds\wedge dZ:$$

Since Z is an approximate solution of $L_1Z=0$, at $s=0$ for each $k=0,1,\ldots,$ there is $B_k>0$ such that

$$|L_1Z(x,t,s)|\le B_k s^{2(k+1)}.$$

Then

$$\begin{aligned}|L_1E|&=\left|L_1\left(\sqrt{-1}(\xi,\tau)\cdot((x',t')-Z(x,t,s))-K|(\xi,\tau)|\langle(x',t')-Z(x,t,s)\rangle^2\right)\right|\\&\le B'_k s^{2(k+1)}|(\xi,\tau)|,\quad\text{some}\quad B'_k>0.\end{aligned}$$

Thus for each $k=0,1,2\ldots,$ there is $C_k^3>0$ such that

$$|J_3(x',t',\xi,\tau)|\le\frac{C_k^3}{|(\xi,\tau)|^k},\quad\forall\ (x',t',\xi,\tau)\in V\times\Gamma. \tag{2.29}$$

Consider

$$J_4(x',t',\xi,\tau)=-\sum_{j=1}^{m+1}\int_B\int_0^{s_1}h(M_j\phi)L_1Z_je^E ds\wedge dZ:$$

By assumption, for each $k=0,1,2,\ldots,$ there is $A_k^j>0$ such that

$$|L_1Z_j|\le A_k^j s^{2k},\quad j=1,2,\ldots,m$$

Therefore, for each $k=0,1,2\ldots,$ there is $C_k^4>0$ such that

$$|J_4(x',t',\xi,\tau)|\le\frac{C_k^4}{|(\xi,\tau)|^k},\quad\forall\ (x',t',\xi,\tau)\in V\times\Gamma. \tag{2.30}$$

Likewise, for each $k=0,1,2\ldots,$ there is $C_k^5>0$ such that

$$\begin{aligned}|J_5(x',t',\xi,\tau)|&=\left|-\sum_{j=1}^{m+1}\int_B\int_0^{s_1}\phi(M_jh)L_1Z_je^E ds\wedge dZ\right|\\&\le\frac{C_k^5}{|(\xi,\tau)|^k},\quad\forall\ (x',t',\xi,\tau)\in V\times\Gamma.\end{aligned} \tag{2.31}$$

Consider

$$J_6(x',t',\xi,\tau)=-\sum_{j=1}^{m+1}\int_B\int_0^{s_1}\phi h(M_jE)L_1Z_je^E ds\wedge dZ:$$

We note that

$$\begin{aligned}|M_jE|&=\left|M_j\left(\sqrt{-1}(\xi,\tau)\cdot((x',t')-Z)-K|(\xi,\tau)|\langle(x',t')-Z\rangle^2\right)\right|\\&\le B|(\xi,\tau)|,\quad\text{for some}\quad B>0\end{aligned}$$

and for each $k = 0, 1, 2, \ldots$, there is $A_k^j > 0$ such that

$$|L_1 Z_j| \leq A_k^j s^{2(1+k)}, \quad j = 1, 2, \ldots, m.$$

Hence for each $k = 0, 1, 2 \ldots$, there is $C_k^6 > 0$ such that

$$|J_6(x', t', \xi, \tau)| \leq \frac{C_k^6}{(|\xi, \tau)|^k}, \quad \forall\ (x', t', \xi, \tau) \in V \times \Gamma. \tag{2.32}$$

Combining equations (2.13), (2.27) – (2.32) we have for each $k = 0, 1, 2 \ldots$, there is $C_k > 0$ such that

$$|\mathcal{F}w(x', t', \xi, \tau)| = \left| \int_B g(x', t', \xi, \tau, x, t, 0) dx dt \right| \leq \frac{C_k}{|(\xi, \tau)|^k}, \quad \forall\ (x', t', \xi, \tau) \in V \times \Gamma$$

where $w(x, t) = h(x, t, 0)$, Γ is a conic nighborhood of $(\xi^0, 0)$ and V is a neighborhood of $(0, 0)$ in $\mathbb{R}^{m+1}$. Thus, by the FBI characterization of the C^∞ wave front set (see[6]),

$$(0, 0; \xi^0, 0) \notin WF(w).$$

□

3. Application to a nonlinear PDE. In this section we will apply the preceding linear results to a nonlinear equation. We will follow very closely section 4 of [2] and [4].

Let $\Omega \subset \mathbb{R}^{m+1}$ be a neighborhood of the origin, $\mathcal{N} \subset \mathbb{C} \times \mathbb{C}^m$ be open and suppose $u(x, t) \in C^2(\Omega)$ is a solution of the first-order nonlinear PDE.

$$u_t = f(x, t, u, u_x) \tag{3.1}$$

where $f(x, t, \zeta_0, \zeta)$ is a C^∞ function in all variables and holomorphic in the variables $(\zeta_0, \zeta) \in \mathcal{N}$ and $(a, w) = (u(0, 0), u_x(0, 0)) \in \mathcal{N}$. Let

$$\mathcal{L} = \frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x, t, \zeta_0, \zeta) \frac{\partial}{\partial x_j}. \tag{3.2}$$

Then $\mathcal{L}$ is a C^∞ vector field on Ω depending on the parameters (ζ_0, ζ).

Let

$$L^u = \frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x, t, u, u_x) \frac{\partial}{\partial x_j}. \tag{3.3}$$

Then L^u is a C^1 vector field on Ω and it is called the **linearization** of the given nonlinear PDE at the solution u. We choose Ω small enough such that

$$(u(x, t), u_x(x, t)) \in \mathcal{N}, \quad \forall\ (x, t) \in \Omega.$$

We now state and prove our main theorem of this paper.

THEOREM 3.1. *Suppose u is a C^2 solution of the nonlinear equation* (3.1). *If $\sigma \in Char L^u$ and $\frac{1}{\sqrt{-1}}\sigma([L^u, \overline{L^u}]) < 0$, then $\sigma \notin WF(u)$.*

EXAMPLE 3.1. Let $u(x, t)$ be a C^2 solution of the semi-linear equation

$$\frac{\partial u}{\partial t} + a(x, t) \frac{\partial u}{\partial x} = g(x, t, u) \text{ on the rectangle } (a, b) \times (-c, c)$$

where $a(x,t)$ and $g(x,t,\zeta_0)$ are C^∞, and g is holomorphic in ζ_0. Assume that $a(0,0) = 0$ and $\operatorname{Im}\left(\frac{\partial a}{\partial t}(0,0)\right) > 0$. Then by Theorem 3.1, at the origin, $(1,0) \notin WF(u)$.

Proof of Theorem 3.1. Differentiating both sides of (3.1) with respect to x_k for each $k = 1, \ldots, m$, we have

$$\frac{\partial u_t}{\partial x_k} = \frac{\partial f}{\partial x_k}(x,t,u,u_x) + \frac{\partial f}{\partial \zeta_0}(x,t,u,u_x)u_{x_k} + \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,u,u_x)u_{x_j x_k} \tag{3.4}$$

Set $v = (u, u_x)$. Then using (3.1),

$$L^u u = \frac{\partial u}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,u,u_x)\frac{\partial u}{\partial x_j} = f(x,t,v) - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,v)u_{x_j}.$$

Likewise, using (3.4) we have for each $k = 1, \ldots, m$

$$L^u u_{x_k} = \frac{\partial u_{x_k}}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,v)u_{x_j x_k} = \frac{\partial f}{\partial x_k}(x,t,v) + \frac{\partial f}{\partial \zeta_0}(x,t,v)u_{x_k}.$$

Let

$$\begin{aligned} g_0(x,t,\zeta_0,\zeta) &= f(x,t,\zeta_0,\zeta) - \sum_{j=1}^{m} \zeta_j \frac{\partial f}{\partial \zeta_j}(x,t,\zeta_0,\zeta) \\ g_k(x,t,\zeta_0,\zeta) &= \frac{\partial f}{\partial x_k}(x,t,\zeta_0,\zeta) + \zeta_k \frac{\partial f}{\partial \zeta_0}(x,t,\zeta_0,\zeta), \quad k = 1, \ldots, m. \end{aligned} \tag{3.5}$$

Then

$$L^u u = g_0(x,t,v), \quad \text{and} \quad L^u u_{x_k} = g_k(x,t,v), \quad k = 1, \ldots, m. \tag{3.6}$$

Set $g = (g_0, g_1, \ldots, g_m)$. Then g is C^∞ in (x,t) and holomorphic in (ζ_0, ζ). Clearly $v = (u, u_x)$ solves the quasi-linear system

$$L^u v = g(x,t,v). \tag{3.7}$$

Consider now the principal part of the holomorphic Hamiltonian of the system (3.7)

$$H = \mathcal{L} + g_0 \frac{\partial}{\partial \zeta_0} + \sum_{j=1}^{m} g_j \frac{\partial}{\partial \zeta_j}.$$

For $\Psi(x,t,\zeta_0,\zeta)$ a C^∞ function in $(x,t) \in \Omega$ and holomorphic in $(\zeta_0,\zeta) \in \mathcal{N}$ and for any C^1 function $h(x,t)$ with $h(0,0) = (a,\omega)$, we set

$$\Psi^h(x,t) = \Psi(x,t,h(x,t)).$$

For any C^1 function $p(x,t)$, let $\mathcal{L}^p$ denote the vector field in Ω obtained by plugging $p(x,t)$ for (ζ_0,ζ) in the coefficients of $\mathcal{L}$. That is,

$$\mathcal{L}^p = \frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,p(x,t))\frac{\partial}{\partial x_j}.$$

Then

$$\mathcal{L}^v = \frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,v(x,t))\frac{\partial}{\partial x_j} = L^u.$$

Let now $\Psi(x,t,\zeta_0,\zeta)$ be a C^∞ function in $(x,t)\in\Omega$ and holomorphic in $(\zeta_0,\zeta)\in\mathcal{N}$ and let $h(x,t) = (h_0(x,t), h_1(x,t),\dots,h_m(x,t))$ be any C^1 function such that $h(0,0)=(a,\omega)$. Then with the understanding that some of the functions are evaluated at (x,t,h), we have

$$\begin{aligned}
\mathcal{L}^h\Psi^h &= \frac{\partial}{\partial t}\Psi(x,t,h) - \sum_{j=1}^{m}\frac{\partial f}{\partial \zeta_j}(x,t,h)\frac{\partial}{\partial x_j}\Psi(x,t,h)\\
&= \frac{\partial \Psi}{\partial t} + \frac{\partial \Psi}{\partial \zeta_0}\frac{\partial h_0}{\partial t} + \sum_{k=1}^{m}\frac{\partial \Psi}{\partial \zeta_k}\frac{\partial h_k}{\partial t}\\
&\quad - \sum_{j=1}^{m}\frac{\partial f}{\partial \zeta_j}\left(\frac{\partial \Psi}{\partial x_j} + \frac{\partial \Psi}{\partial \zeta_0}\frac{\partial h_0}{\partial x_j} + \sum_{k=1}^{m}\frac{\partial \Psi}{\partial \zeta_k}\frac{\partial h_k}{\partial x_j}\right)\\
&= \frac{\partial \Psi}{\partial t} - \sum_{j=1}^{m}\frac{\partial f}{\partial \zeta_j}\frac{\partial \Psi}{\partial x_j} + \frac{\partial \Psi}{\partial \zeta_0}\left(\frac{\partial h_0}{\partial t} - \sum_{j=1}^{m}\frac{\partial f}{\partial \zeta_j}\frac{\partial h_0}{\partial x_j}\right)\\
&\quad + \sum_{k=1}^{m}\frac{\partial \Psi}{\partial \zeta_k}\left(\frac{\partial h_k}{\partial t} - \sum_{j=1}^{m}\frac{\partial f}{\partial \zeta_j}\frac{\partial h_k}{\partial x_j}\right)\\
&= (\mathcal{L}\Psi)^h + \left(\frac{\partial \Psi}{\partial \zeta_0}\right)^h \mathcal{L}^h h_0 + \sum_{k=1}^{m}\left(\frac{\partial \Psi}{\partial \zeta_k}\right)^h \mathcal{L}^h h_k\\
&= (H\Psi)^h - g_0^h\left(\frac{\partial \Psi}{\partial \zeta_0}\right)^h - \sum_{k=1}^{m} g_k^h\left(\frac{\partial \Psi}{\partial \zeta_k}\right)^h + \left(\frac{\partial \Psi}{\partial \zeta_0}\right)^h \mathcal{L}^h h_0\\
&\quad + \sum_{k=1}^{m}\left(\frac{\partial \Psi}{\partial \zeta_k}\right)^h \mathcal{L}^h h_k \left(\text{since } \ \mathcal{L} = H - g_0\frac{\partial}{\partial \zeta_0} - \sum_{k=1}^{m} g_k \frac{\partial f}{\partial \zeta_k}\right)\\
&= (H\Psi)^h + \left(\frac{\partial \Psi}{\partial \zeta_0}\right)^h (\mathcal{L}^h h_0 - g_0^h) + \sum_{k=1}^{m}\left(\frac{\partial \Psi}{\partial \zeta_k}\right)^h (\mathcal{L}^h h_k - g_k^h). \qquad (3.8)
\end{aligned}$$

But for $h = v = (u, u_x)$, we have using (3.5)

$$g_0^v = g_0(x,t,v) = f(x,t,v) - \sum_{j=1}^{m} u_{x_j}\frac{\partial f}{\partial \zeta_j}(x,t,v)$$

$$\text{and} \quad g_k(x,t,v) = \frac{\partial f}{\partial x_k}(x,t,v) + u_{x_k}\frac{\partial f}{\partial \zeta_0}(x,t,v), \quad k = 1,\dots,m.$$

Plugging this into (3.8) and using equation (3.1) for $h_0 = u$ and $h_k = u_{x_k}$, $k = 1,\dots,m$ we have

$$\begin{aligned}
\mathcal{L}^v u - g_0^v &= \frac{\partial u}{\partial t} - \sum_{j=1}^{m}\frac{\partial f}{\partial \zeta_j}(x,t,v)u_{x_j} - f(x,t,v) + \sum_{j=1}^{m} u_{x_j}\frac{\partial f}{\partial \zeta_j}(x,t,v)\\
&= \frac{\partial u}{\partial t} - f(x,t,v) = 0.
\end{aligned}$$

Similarily, for each $k = 1, \dots, m$, using (3.4) we have

$$\begin{aligned}\mathcal{L}^v u_{x_k} - g_k^v &= \frac{\partial u_{x_k}}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,v) u_{x_j x_k} - \frac{\partial f}{\partial x_k}(x,t,v) - u_{x_k} \frac{\partial f}{\partial \zeta_0}(x,t,v) \\ &= \frac{\partial f}{\partial x_k}(x,t,v) + \frac{\partial f}{\partial \zeta_0}(x,t,v) u_{x_k} + \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,v) u_{x_j x_k} \\ &\quad - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,v) u_{x_j x_k} - \frac{\partial f}{\partial x_k}(x,t,v) - u_{x_k} \frac{\partial f}{\partial \zeta_0}(x,t,v) = 0.\end{aligned}$$

Therefore, equation (3.8) becomes

$$L^u \Psi^v = \mathcal{L}^v \Psi^v = (H\Psi)^v. \tag{3.9}$$

Since

$$(x, \zeta_0, \zeta) \mapsto x_j, \quad j = 1, \dots, m \quad \text{and} \quad (x, \zeta_0, \zeta) \mapsto \zeta_k, \quad k = 0, 1, \dots, m$$

are C^∞ and holomorphic in $(\zeta_0, \zeta) \in \mathcal{N}$, by lemma (2.1) there are C^∞ functions $Z_j(x,t,\zeta_0,\zeta)$, $j = 1, 2, \dots, m$ and $W_k(x,t,\zeta_0,\zeta)$, $k = 0, 1, 2, \dots, m$, holomorphic in $(\zeta_0, \zeta) \in \mathcal{N}$ such that

$$Z_j(x, 0, \zeta_0, \zeta) = x_j, \quad j = 1, \dots, m \quad \text{and} \quad W_k(x, 0, \zeta_0, \zeta) = \zeta_k, \quad k = 0, \dots, m$$

and

$$\begin{aligned}|HZ_j(x,t,\zeta_0,\zeta)| &= O(|t|^n), \quad n = 1, 2, \dots, \quad \forall\, j = 1, \dots, m, \\ |HW_k(x,t,\zeta_0,\zeta)| &= O(|t|^n), \quad n = 1, 2, \dots, \quad \forall\, k = 0, 1, \dots, m.\end{aligned}$$

Set

$$Z = (Z_1, \dots, Z_m) \quad \text{and} \quad W = (W_0, \dots, W_m).$$

Since $Z(x,t,\zeta_0,\zeta)$ and $W(x,t,\zeta_0,\zeta)$ are C^∞ in x, they have almost holomorphic extensions denoted respectively by $\tilde{Z}(z,t,\zeta_0,\zeta)$ and $\tilde{W}(z,t,\zeta_0,\zeta)$ ($z = x + iy \in \mathbb{R}^m \oplus i\mathbb{R}^m$). That is, $\tilde{Z}(x,t,\zeta_0,\zeta) = Z(x,t,\zeta_0,\zeta)$ and $\tilde{W}(x,t,\zeta_0,\zeta) = W(x,t,\zeta_0,\zeta)$ for all $(x,t) \in \Omega$ and for all $k = 1, 2, \dots$, there exists $C_k > 0$ such that for $j = 1, 2, \dots, m$ we have

$$\begin{aligned}\left| \frac{\partial}{\partial \bar{z}_j} \tilde{Z}(z,t,\zeta_0,\zeta) \right| &\le C_k |\Im z|^k, \\ \left| \frac{\partial}{\partial \bar{z}_j} \tilde{W}(z,t,\zeta_0,\zeta) \right| &\le C_k |\Im z|^k.\end{aligned} \tag{3.10}$$

Recall that

$$\tilde{Z}(x, 0, \zeta_0, \zeta) = Z(x, 0, \zeta_0, \zeta) = x \quad \text{and} \quad \tilde{W}(x, 0, \zeta_0, \zeta) = W(x, 0, \zeta_0, \zeta) = (\zeta_0, \zeta).$$

We have

$$\begin{aligned}
&\det \frac{\partial \left(\tilde{Z}(x,0,\zeta_0,\zeta), \tilde{W}(x,0,\zeta_0,\zeta)\right)}{\partial(z,\zeta_0,\zeta)}(0,0,a,\omega) \\
&= \det \begin{pmatrix} \frac{\partial}{\partial z}x & \frac{\partial}{\partial \zeta_0}x & \frac{\partial}{\partial \zeta}x \\ \frac{\partial}{\partial z}(\zeta_0,\zeta) & \frac{\partial}{\partial \zeta_0}(\zeta_0,\zeta) & \frac{\partial}{\partial \zeta}(\zeta_0,\zeta) \end{pmatrix}(0,0,a,\omega) \\
&= \det \begin{pmatrix} \frac{1}{2}I_{m\times m} & 0 \\ 0 & I_{(m+1)\times(m+1)} \end{pmatrix} = \frac{1}{2^m} \neq 0.
\end{aligned}$$

By continuity of the determinant,

$$\frac{\partial(\tilde{Z},\tilde{W})}{\partial(z,\zeta_0,\zeta)}$$

is non-singular near $t=0$. We note that $\tilde{Z}(0,0,a,\omega)=0$ and $\tilde{W}(0,0,a,\omega)=(a,\omega)$. Therefore, by the Implicit Function Theorem, we can solve the system

$$\begin{cases} \tilde{Z}(z,t,\zeta_0,\zeta) &= \tilde{Z}, \\ \tilde{W}(z,t,\zeta_0,\zeta) &= \tilde{W} \end{cases} \tag{3.11}$$

with respect to (z,ζ_0,ζ) in a neighborhood of $(0,a,w)$. That is, there are C^∞ functions $P=(P_1,\dots,P_m)$ and $Q=(Q_0,\dots,Q_m)$ holomorphic in (ζ_0,ζ) such that

$$\begin{cases} z &= P(\tilde{Z},t,\tilde{W}), \\ (\zeta_0,\zeta) &= Q(\tilde{Z},t,\tilde{W}), \end{cases}$$

with $P(0,0,\zeta_0,\zeta)=0$ and $Q(0,0,a,w)=(a,w)$.

Substituting these in to the system (3.11) gives

$$\begin{cases} \tilde{Z}\left(P(\tilde{Z},t,\tilde{W}),t,Q(\tilde{Z},t,\tilde{W})\right) &= \tilde{Z}, \\ \tilde{W}\left(P(\tilde{Z},t,\tilde{W}),t,Q(\tilde{Z},t,\tilde{W})\right) &= \tilde{W}. \end{cases} \tag{3.12}$$

Since $G(\tilde{Z},\tilde{W})=\tilde{Z}$ is holomorphic in $\tilde{Z}$, we get that $\frac{\partial \tilde{Z}}{\partial \bar{\tilde{Z}}}=0$ and $\frac{\partial \tilde{W}}{\partial \bar{\tilde{Z}}}=0$ and so differentiating the system (3.12) with respect to $\bar{\tilde{Z}}$ and using the holomorphic version of the chain rule we obtain

$$\begin{aligned}
&\frac{\partial \tilde{Z}}{\partial(z,\zeta_0,\zeta)}\left(P(\tilde{Z},t,\tilde{W}),t,Q(\tilde{Z},t,\tilde{W})\right)\frac{\partial(P,Q)}{\partial\bar{\tilde{Z}}}(\tilde{Z},t,\tilde{W}) \\
&\quad + \frac{\partial \tilde{Z}}{\partial(\bar{z},\bar{\zeta}_0,\bar{\zeta})}\left(P(\tilde{Z},t,\tilde{W}),t,Q(\tilde{Z},t,\tilde{W})\right)\frac{\partial(\bar{P},\bar{Q})}{\partial\bar{\tilde{Z}}}(\tilde{Z},t,\tilde{W}) = 0.
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}
&\frac{\partial \tilde{W}}{\partial(z,\zeta_0,\zeta)}\left(P(\tilde{Z},t,\tilde{W}),t,Q(\tilde{Z},t,\tilde{W})\right)\frac{\partial(P,Q)}{\partial\bar{\tilde{Z}}}(\tilde{Z},t,\tilde{W}) \\
&\quad + \frac{\partial \tilde{W}}{\partial(\bar{z},\bar{\zeta}_0,\bar{\zeta})}\left(P(\tilde{Z},t,\tilde{W}),t,Q(\tilde{Z},t,\tilde{W})\right)\frac{\partial(\bar{P},\bar{Q})}{\partial\bar{\tilde{Z}}}(\tilde{Z},t,\tilde{W}) = 0.
\end{aligned} \tag{3.14}$$

Combining equations (3.13) and (3.14) gives

$$\begin{aligned}
&\frac{\partial(\tilde{Z},\tilde{W})}{\partial(z,\zeta_0,\zeta)}\left(P(\tilde{Z},t,\tilde{W}),t,Q(\tilde{Z},t,\tilde{W})\right)\frac{\partial(P,Q)}{\partial\overline{\tilde{Z}}}(\tilde{Z},t,\tilde{W})\\
&\quad+\frac{\partial(\tilde{Z},\tilde{W})}{\partial(\bar{z},\bar{\zeta_0},\bar{\zeta})}\left(P(\tilde{Z},t,\tilde{W}),t,Q(\tilde{Z},t,\tilde{W})\right)\frac{\partial(\bar{P},\bar{Q})}{\partial\overline{\tilde{Z}}}(\tilde{Z},t,\tilde{W})=0.
\end{aligned}\tag{3.15}$$

Let $A(z,t,\zeta_0,\zeta)$ denote a generic entry of the matrix

$$\frac{\partial(\tilde{Z},\tilde{W})}{\partial(\bar{z},\bar{\zeta_0},\bar{\zeta})}(z,t,\zeta_0,\zeta).$$

Since $\tilde{Z}(z,t,\zeta_0,\zeta)$ and $\tilde{W}(z,t,\zeta_0,\zeta)$ are holomorphic in (ζ_0,ζ) and using (3.10), for each $k=0,1,\ldots,$ there exists $C_k>0$ such that

$$|A(z,t,\zeta_0,\zeta)|\le C_k|\Im z|^k.$$

Therefore, for each $k=0,1,\ldots,$ there exists $C_k'>0$ such that

$$\left|\frac{\partial(\tilde{Z},\tilde{W})}{\partial(\bar{z},\bar{\zeta_0},\bar{\zeta})}(z,t,\zeta_0,\zeta)\right|\le C_k'|\Im z|^k.\tag{3.16}$$

Let $r>0$ such that

$$\frac{\partial(\tilde{Z},\tilde{W})}{\partial(z,\zeta_0,\zeta)}$$

is nonsingular on

$$B=\{(z,t,\zeta_0,\zeta):|(z,t,\zeta_0,\zeta)|\le r\}.$$

Set

$$A=\left(P\left(\tilde{Z},t,\tilde{W}\right)\right),t,Q\left(\tilde{Z},t,\tilde{W}\right)\right).$$

Then from (3.15) and using (3.16) we have on B

$$\begin{aligned}
&\left|\frac{\partial\,(P,Q)}{\partial\overline{\tilde{Z}}}\left(\tilde{Z},t,\tilde{W}\right)\right|\\
&=\left|\left(\frac{\partial(\tilde{Z},\tilde{W})}{\partial(z,\zeta_0,\zeta)}(A)\right)^{-1}\frac{\partial(\tilde{Z},\tilde{W})}{\partial(\bar{z},\bar{\zeta_0},\bar{\zeta})}(A)\,\frac{\partial(\bar{P},\bar{Q})}{\partial\overline{\tilde{Z}}}\left(\tilde{Z},t,\tilde{W}\right)\right|\\
&=\left|\left(\frac{\partial(\tilde{Z},\tilde{W})}{\partial(z,\zeta_0,\zeta)}(A)\right)^{-1}\right|\left|\frac{\partial(\bar{P},\bar{Q})}{\partial\overline{\tilde{Z}}}\left(\tilde{Z},t,\tilde{W}\right)\right|\left|\frac{\partial(\tilde{Z},\tilde{W})}{\partial(\bar{z},\bar{\zeta_0},\bar{\zeta})}(A)\right|\\
&\le DC_k'\left|\Im P\left(\tilde{Z},t,\tilde{W}\right)\right|^k\left(D=\sup_B\left|\left(\frac{\partial(\tilde{Z},\tilde{W})}{\partial(z,\zeta_0,\zeta)}(A)\right)^{-1}\frac{\partial(\bar{P},\bar{Q})}{\partial\overline{\tilde{Z}}}\left(\tilde{Z},t,\tilde{W}\right)\right|\right).
\end{aligned}$$

In particular, for each $k=0,1,\ldots,$ there is $C_k''>0$ such that

$$\left|\frac{\partial Q_0}{\partial\overline{\tilde{Z}}_j}(\tilde{Z},t,\tilde{W})\right|\le C_k''\left|\Im P\left(\tilde{Z},t,\tilde{W}\right)\right|^k,\quad\forall\,j=1,2,\ldots,m.\tag{3.17}$$

We now define

$$\Psi(z,t,\zeta_0,\zeta) = Q_0\left(\tilde{Z}(z,t,\zeta_0,\zeta),0,\tilde{W}(z,t,\zeta_0,\zeta)\right).$$

Then Ψ is C^∞ in (z,t) and holomorphic in (ζ_0,ζ) since $Q_0, \tilde{Z}$ and $\tilde{W}$ are C^∞ in (z,t) and holomorphic in (ζ_0,ζ).

We observe that

$$\begin{aligned}
\Psi^v(x,0) &= \Psi(x,0,v(x,0)) \\
&= \Psi(x,0,u(x,0),u_x(x,0)) \\
&= Q_0\left(\tilde{Z}\left(x,0,u(x,0),u_x(x,0)\right),0,\tilde{W}\left(x,0,u(x,0),u_x(x,0)\right)\right) \\
&= Q_0\left(Z\left(x,0,u(x,0),u_x(x,0)\right),0,W\left(x,0,u(x,0),u_x(x,0)\right)\right) \\
&= Q_0\left(x,0,u(x,0),u_x(x,0)\right) \\
&= u(x,0).
\end{aligned}$$

We recall that

$$HZ(x,t,\zeta_0,\zeta) \quad \text{and} \quad HW(x,t,\zeta_0,\zeta)$$

are t-flat at $t=0$. Hence

$$H\tilde{Z}(x,t,\zeta_0,\zeta) \quad \text{and} \quad H\tilde{W}(x,t,\zeta_0,\zeta) \tag{3.18}$$

are t-flat at $t=0$. Since

$$\Psi(x,t,\zeta_0,\zeta) = Q_0\left(\tilde{Z}(x,t,\zeta_0,\zeta),0,\tilde{W}(x,t,\zeta_0,\zeta)\right),$$

by the holomorphic version of the chain rule,

$$H\Psi = \sum_{j=1}^{m}\left(\frac{\partial Q_0}{\partial \tilde{Z}_j}H\tilde{Z}_j + \frac{\partial Q_0}{\partial \overline{\tilde{Z}_j}}H\overline{\tilde{Z}_j}\right) + \sum_{k=0}^{m}\left(\frac{\partial Q_0}{\partial \tilde{W}_k}H\tilde{W}_k + \frac{\partial Q_0}{\partial \overline{\tilde{W}_k}}H\overline{\tilde{W}_k}\right). \tag{3.19}$$

We will show that $H\Psi$ is t-flat at $t=0$. Since $P\left(\tilde{Z},t,\tilde{W}\right) = z$, we have

$$\begin{aligned}
P(x,0,\zeta_0,\zeta) &= P\left(Z(x,0,\zeta_0,\zeta),0,W(x,0,\zeta_0,\zeta)\right) \\
&= P\left(\tilde{Z}(x,0,\zeta_0,\zeta),0,\tilde{W}(x,0,\zeta_0,\zeta)\right) = x.
\end{aligned}$$

Hence

$$\Im P\left(\tilde{Z}(x,0,\zeta_0,\zeta),0,\tilde{W}(x,0,\zeta_0,\zeta)\right) = 0.$$

Since $\Im P\left(\tilde{Z}(x,t,\zeta_0,\zeta),0,\tilde{W}(x,t,\zeta_0,\zeta)\right)$ is C^1, by Taylor's theorem for t near zero there is a point $t' = t'(x,t,\zeta_0,\zeta)$ between t and 0 such that

$$\begin{aligned}
\left|\Im P\left(\tilde{Z}(x,t,\zeta_0,\zeta),0,\tilde{W}(x,t,\zeta_0,\zeta)\right)\right| &= \Big|\Im P\left(\tilde{Z}(x,0,\zeta_0,\zeta),0,\tilde{W}(x,0,\zeta_0,\zeta)\right) \\
&\qquad + \partial_t \Im P\left(\tilde{Z}(x,t',\zeta_0,\zeta),0,\tilde{W}(x,t',\zeta_0,\zeta)\right)t\Big| \\
&= \left|\partial_t \Im P\left(\tilde{Z}(x,t',\zeta_0,\zeta),0,\tilde{W}(x,t',\zeta_0,\zeta)\right)t\right| \\
&\le c|t|
\end{aligned}$$

where

$$c = \sup_B \left|\partial_t \Im P\left(\tilde{Z}(x,t,\zeta_0,\zeta),0,\tilde{W}(x,t',\zeta_0,\zeta)\right)\right|.$$

Thus using (3.17) we have for all $\forall\ j = 1,2,\dots,m$,

$$\begin{aligned}\left|\frac{\partial Q_0}{\partial \overline{\tilde{Z}_j}}\left(\tilde{Z}(x,t,\zeta_0,\zeta),0,\tilde{W}(x,t,\zeta_0,\zeta)\right)\right| &\le C''_k \left|\Im P\left(\tilde{Z}(x,t,\zeta_0,\zeta),0,\tilde{W}(x,t,\zeta_0,\zeta)\right)\right|^k \\ &\le C''_k c^k |t|^k.\end{aligned}$$

This shows that

$$\frac{\partial Q_0}{\partial \overline{\tilde{Z}_j}}\left(\tilde{Z}(x,t,\zeta_0,\zeta),0,\tilde{W}(x,t,\zeta_0,\zeta)\right)$$

is t-flat at $t=0$ for all $j=1,\dots,m$. Similarly, we can show that

$$\frac{\partial Q_0}{\partial \overline{\tilde{W}_k}}\left(\tilde{Z}(x,t,\zeta_0,\zeta),0,\tilde{W}(x,t,\zeta_0,\zeta)\right)$$

is t-flat at $t=0$ for all $k=0,1,\dots,m$. Thus going back to equation (3.19) and using (3.10) and (3.18) we have

$$\begin{aligned}|H\Psi(x,t,\zeta_0,\zeta)| &= \left|\sum_{j=1}^m \left(\frac{\partial Q_0}{\partial \tilde{Z}_j} H\tilde{Z}_j + \frac{\partial Q_0}{\partial \overline{\tilde{Z}_j}} H\overline{\tilde{Z}_j}\right) + \sum_{k=0}^m \left(\frac{\partial Q_0}{\partial \tilde{W}_k} H\tilde{W}_k + \frac{\partial Q_0}{\partial \overline{\tilde{W}_k}} H\overline{\tilde{W}_k}\right)\right| \\ &\le \sum_{j=1}^m \left|\frac{\partial Q_0}{\partial \tilde{Z}_j} H\tilde{Z}_j + \frac{\partial Q_0}{\partial \overline{\tilde{Z}_j}} H\overline{\tilde{Z}_j}\right| + \sum_{k=0}^m \left|\frac{\partial Q_0}{\partial \tilde{W}_k} H\tilde{W}_k + \frac{\partial Q_0}{\partial \overline{\tilde{W}_k}} H\overline{\tilde{W}_k}\right| \\ &\le \sum_{j=1}^m \left(A_j |H\tilde{Z}_j| + A'_j \left|\frac{\partial Q_0}{\partial \overline{\tilde{Z}_j}}\right|\right) + \sum_{k=0}^m \left(B_k |H\tilde{W}_k| + B'_k \left|\frac{\partial Q_0}{\partial \overline{\tilde{W}_k}}\right|\right)\end{aligned}$$

which is t-flat at $t=0$, where

$$A_j = \sup_B \left|\frac{\partial Q_0}{\partial \tilde{Z}_j}\right|, \quad A'_j = \sup_B \left|H\overline{\tilde{Z}_j}\right|, \quad B_k = \sup_B \left|\frac{\partial Q_0}{\partial \tilde{W}_k}\right|, \quad B'_k = \sup_B \left|H\overline{\tilde{W}_k}\right|.$$

But then

$$L^u \Psi^v = \mathcal{L}^v \Psi^v = (H\Psi)^v$$

is t-flat at $t=0$.

Let

$$h(x,t) = \Psi^v(x,t) = \Psi(x,t,v(x,t)).$$

Then $h(x,t)$ is a C^1 function such that

$$L^u h = L^u \Psi^v = \mathcal{L}^v \Psi^v = (H\Psi)^v$$

is t-flat at $t = 0$ and

$$h(x,0) = \Psi(x,0,v(x,0)) = \Psi^v(x,0) = u(x,0).$$

Therefore, if u is a C^2 solution of the PDE $u_t = f(x,t,u,u_x)$ and if L^u is the associated linearized vector field of this PDE, then we have found a C^1 function $h(x,t)$ such that $h(x,0) = u(x,0)$ and $L^u h$ is t-flat at $t = 0$.

To finish our proof, let $s \in \mathbb{R}$ be a new variable. Since $u(x,t)$ is a solution of $u_t = f(x,t,u,u_x)$ and is independent of the variable s, we observe that $u(x,t)$ is also a solution of

$$u_s = -\sqrt{-1}\,(u_t - f(x,t,u,u_x))\,. \tag{3.20}$$

This equation is of the same kind as equation (3.1). We recall that the vector field associated to the PDE

$$u_t = f(x,t,u,u_x)$$

is

$$\mathcal{L} = \frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,\zeta_0,\zeta)\frac{\partial}{\partial x_j}.$$

Our plan is to apply what we did so far but use s in place of t. So, let $x' = (x,t)$ and let

$$u'\,(x',s) = u(x,t).$$

Then u' is a solution of (3.20). Indeed, equation (3.20) is written as

$$u'_s(x',s) = f'\,(x',s,u',u'_{x'})\,,$$

where

$$f'(x',s,\zeta_0,\zeta,\tau) = -\sqrt{-1}\,(\tau - f(x,t,\zeta_0,\zeta))$$

is C^∞ in (x',s) and holomorphic in

$$(\zeta_0,\zeta,\tau) \in \mathcal{N} \times \mathbb{C} \subset \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}.$$

For a vector field M in (x,t), we write

$$M_1 = \frac{\partial}{\partial s} + \sqrt{-1}M$$

where $s \in \mathbb{R}$ is a new variable. With this notation, if we denote the associated vector field to equation (3.20) by $\mathcal{L}'$ as in (3.3), then

$$\begin{aligned}
\mathcal{L}' &= \frac{\partial}{\partial s} - \sum_{j=1}^{m} \frac{\partial f'}{\partial \zeta_j}(x',s,\zeta_0,\zeta,\tau)\frac{\partial}{\partial x_j} - \frac{\partial f'}{\partial \tau}(x',s,\zeta_0,\zeta,\tau)\frac{\partial}{\partial t} \\
&= \frac{\partial}{\partial s} - \sqrt{-1}\sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,\zeta_0,\zeta)\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial t} \\
&= \frac{\partial}{\partial s} + \sqrt{-1}\left(\frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x,t,\zeta_0,\zeta)\frac{\partial}{\partial x_j}\right) \\
&= \frac{\partial}{\partial s} + \sqrt{-1}\mathcal{L} = \mathcal{L}_1.
\end{aligned}$$

Similarly, if we denote the corresponding linearized vector field of the new PDE by $(L')^{u'}$ then

$$\begin{aligned}(L')^{u'} &= \frac{\partial}{\partial s} - \sum_{j=1}^{m} \frac{\partial f'}{\partial \zeta_j}(x', s, u', u'_x, u'_t)\frac{\partial}{\partial x_j} - \frac{\partial f'}{\partial \tau}(x', s, u', u'_x, u'_t)\frac{\partial}{\partial t} \\ &= \frac{\partial}{\partial s} - \sqrt{-1}\sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x, t, u, u_x)\frac{\partial}{\partial x_j} + \sqrt{-1}\frac{\partial}{\partial t} \\ &= \frac{\partial}{\partial s} + \sqrt{-1}\left(\frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}(x, t, u, u_x)\frac{\partial}{\partial x_j}\right) \\ &= \frac{\partial}{\partial s} + \sqrt{-1}L^u. \\ &= (L^u)_1 .\end{aligned}$$

Therefore, by what we saw, there exists a C^1 function $h'(x, t, s)$ such that

$$(L^u)_1 h' = (L')^{u'} h'$$

is s-flat at $s = 0$ and

$$h'(x, t, 0) = h'(x', 0) = u'(x', 0) = u(x, t).$$

In order to apply lemma (2.2), we need to find C^1 functions

$$\Psi_1(x, t, s), \ldots, \Psi_m(x, t, s), \Psi_{m+1}(x, t, s)$$

such that

$$Z = (Z_1, \ldots, Z_{m+1}) = (x, t) + s\Psi(x, t, s) = (x, t) + s(\Psi_1, \ldots, \Psi_{m+1})$$

is an approximate solution of $(L^u)_1 Z = 0$ in the sense that $(L^u)_1 Z(x, t, s)$ is s- flat at $s = 0$. Take $\Psi_{m+1} = -\sqrt{-1}$ and so $Z_{m+1} = t - s\sqrt{-1}$. Then

$$\begin{aligned}(L^u)_1 Z_{m+1} &= \left(\frac{\partial}{\partial s} + \sqrt{-1}L^u\right) Z_{m+1} \\ &= \left(\frac{\partial}{\partial s} + \sqrt{-1}\left(\frac{\partial}{\partial t} - \sum_{j=1}^{m} \frac{\partial f}{\partial \zeta_j}\frac{\partial}{\partial x_j}\right)\right)(t - s\sqrt{-1}) = 0.\end{aligned}$$

Hence it suffices to find C^1 functions $\Psi_1(x, t, s), \ldots, \Psi_m(x, t, s)$ such that

$$Z_j = x_j + s\Psi_j(x, t, s)$$

is an approximate solution of $(L^u)_1 Z_j = 0$ in the sense that $(L^u)_1 Z_j(x, t, s)$ is s- flat at $s = 0$ for all $j = 1, \ldots, m$.

For $x' = (x, t)$, let $v' = (u', u'_{x'}) = (u, u_x, u_t)$. Then as we saw before v' solves the quasi-linear PDE

$$(L')^{u'} v' = g'(x', s, v') \tag{3.21}$$

where $g' = (g'_0, \dots, g'_{m+1})$ with

$$g'_0(x', s, \zeta_0, \zeta, \tau) = f'(x', s, \zeta_0, \zeta, \tau) - \sum_{j=1}^{m} \zeta_j \frac{\partial f'}{\partial \zeta_j}(x', s, \zeta_0, \zeta, \tau) - \tau \frac{\partial f'}{\partial \tau}(x', s, \zeta_0, \zeta, \tau)$$

$$g'_k(x', s, \zeta_0, \zeta, \tau) = \frac{\partial f'}{\partial x_k}(x', s, \zeta_0, \zeta, \tau) + \zeta_k \frac{\partial f'}{\partial \zeta_0}(x', s, \zeta_0, \zeta, \tau), \quad k = 1, \dots, m$$

$$g'_{m+1}(x', s, \zeta_0, \zeta, \tau) = \frac{\partial f'}{\partial t}(x', s, \zeta_0, \zeta, \tau) + \tau \frac{\partial f'}{\partial \zeta_0}(x', s, \zeta_0, \zeta, \tau).$$

Then the corresponding holomorphic Hamiltonian of the system (3.21) is

$$H' = \mathcal{L}' + g'_0 \frac{\partial}{\partial \zeta_0} + \sum_{j=1}^{m} g'_j \frac{\partial}{\partial \zeta_j} + g'_{m+1} \frac{\partial}{\partial \tau}.$$

By lemma (2.1) for each $j = 1, \dots, m$, there is a C^∞ function $\Phi_j(x, t, s, \zeta_0, \zeta, \tau)$ holomorphic in (ζ_0, ζ, τ) such that

$$W_j(x, t, s, \zeta_0, \zeta, \tau) = x_j + s\Phi_j(x, t, s, \zeta_0, \zeta, \tau)$$

is an approximate solution of $HW_j(x, t, s, \zeta_0, \zeta, \tau) = 0$. That is HW_j is s-flat at $s = 0$.

For each $j = 1, \dots, m$, define

$$Z_j(x, t, s) = W_j^{v'}(x, t, s, \zeta, \zeta, \tau) = W_j(x, t, s, v'(x', s)) = W_j(x, t, s, u, u_x, u_t)$$

and

$$\Psi_j(x, t, s) = \Phi_j^{v'}(x, t, s, \zeta_0, \zeta, \tau) = \Phi_j(x, t, s, u, u_x, u_t).$$

Let $Z' = (Z_1, \dots, Z_m)$ and $\Psi' = (\Psi_1, \dots, \Psi_m)$. Then Z' and Ψ' are C^1 functions such that

$$Z'(x, t, s) = x + s\Psi'(x, t, s).$$

Since $H'W'$ and so $(H'W')^{v'}$ is s-flat at $s = 0$ and since

$$(\mathcal{L}')^{v'} = (L')^{u'},$$

we have using (3.9)

$$\begin{aligned} (L^u)_1 Z' &= (L')^{u'} Z' \\ &= (\mathcal{L}')^{v'} (W')^{v'} \\ &= (H'W')^{v'} \end{aligned} \tag{3.22}$$

is s-flat at $s = 0$.

Therefore, for a C^2 solution $u(x, t)$ of $u_t = f(x, t, u, u_x)$ if L^u denotes its linearized vector field, we have obtained C^1 functions $\Psi_1(x, t, s), \dots, \Psi_{m+1}(x, t, s)$ such that $Z(x, t, s) = (x, t) + s\Psi(x, t, s)$ is an approximate solution of $(L^u)_1 Z = 0$ in the sense that $(L^u)_1 Z$ is s-flat at $s = 0$. We also found a C^1 function $h'(x, t, s)$ such that $h'(x, t, 0) = u(x, t)$ and $(L^u)_1 h$ is s-flat at $s = 0$. Therefore, by lemma (2.2) we conclude that $\sigma \notin WF(u)$ and the proof of theorem 3.1 is complete. ◻

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