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# MAXIMAL IDEAL CYCLES AND MAXIMAL IDEAL TYPES FOR NORMAL SURFACE SINGULARITIES*

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*Dedicated to Professor Henry Laufer on his seventieth birthday*

**Abstract.** In this paper, we explain several results on the relations between the maximal ideal cycles for normal surface singularities and pencil of curves. Also we report recent results by the author on maximal ideal types for normal surface singularities of some type.

**Key words.** Normal surface singularity, maximal ideal cycle, maximal ideal type.

**Mathematics Subject Classification.** 32: Several complex variables and analytic spaces.

**1. Introduction.** This is a review article on the recent results of the maximal ideal cycles and related topics for normal surface singularities. For a normal complex surface singularity $(X, o)$, let $\pi\colon (\tilde{X}, E) \to (X, o)$ be a resolution. The maximal ideal cycle $M_E$ and the fundamental cycle $Z_E$ are important objects when we study $\mathrm{mul}(X, o)$ (=the multiplicity of $(X, o)$) and $\mathfrak{m}\mathcal{O}_{\tilde{X}}$ (=the pull-back of the maximal ideal $\mathfrak{m}$ of $\mathcal{O}_{X,o}$). Since M. Artin's fundamental work ([Art]) on rational surface singularities, many important results for $M_E$, $Z_E$ and $\mathfrak{m}\mathcal{O}_{\tilde{X}}$ have been obtained by P. Wagreich, H. Laufer, S.S.T. Yau and others ([Wag], [L1] and [Y]). We explain them in Section 2.

When we consider a rough classification of complex manifolds, the notion of the topological equivalence is natural. However, "topological equivalence" is too fine in order of the rough classification of singularities. For example, if we consider two singularities $x^2 + y^2 + z^m$ and $x^2 + y^2 + z^n$ $(m \neq n)$, then they are not topologically equivalent, but those singularities have similar properties to each other. Hence, it is natural to consider the sequences $\{A_n\}$. This point of view (i. e., sequence of singularities) is important for rough classification of singularities. In 1980, S.S.T. Yau introduced the notion of elliptic sequences. Though it was represented as sequences of cycles on the exceptional set of an elliptic singularity, we can regard them as sequences of singularities. For a numerically Gorenstein elliptic surface singularity, he proved that the geometric genus $p_g(X, o)$ is smaller than or equal to the length of the elliptic sequence. Also, he proved that if the both values are equal, then $(X, o)$ is a Gorenstein singularity (he named it a *maxially elliptic singularity*). For example, let $(X_n, o) := \{z^{6n+c} = x^2 + y^3\}$ for $n = 1, 2, \cdots$ and $0 \leqq c < 6$. Then, $\{(X_n, o)\}_{n=1,2,\cdots,N}$ corresponds to the elliptic sequence of $(X_N, o)$ and $(X_N, o)$ is a maximally elliptic singularity for each fixed $N$. It is a minimally elliptic singularity if and only if $n = 1$. For the higher genus case (i. e., the arithmetic genus $p_f(X, o)$ of the fundamental cycle is larger than or equal to 2), the author also studied a similar sequences and called it *Yau sequences* ([To1]). Also, from the point of view of compact complex surface theory, K. Konno studied a similar sequences and obtained several nice results ([Kon]). In those works, the maximal ideal cycle and the fundamental cycle played important roles.

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On the other hand, after K. Kodaira's famous work [Kod], the relation between surface singularities and pencil of curves has been studied by V. Kulikov [Ku], M. Reid [R], U. Karras [Ka], J. Stevens [S] and others. From two decades ago, the author also has been studying such relation ([To2],[To4]-[To6]). Especially, the notion of Kodaira singularity defined by Karras (using pencils of curves) seems to be important. When we consider elliptic or Yau sequences, we often encounter the Kodaira singularities. For example, Yau [Y] described concrete examples of elliptic sequences of three different types; in which each member is a hypersurface maximally elliptic singularity. We can easily check that all members are Kodaira singularities. When we consider the relation between surface singularities and pencil of curves, the maximal ideal cycle $M_E$ and the fundamental cycle $Z_E$ play important roles. For all rational singularities (resp. all Kodaira singularities), it is known that those two cycles coincide for any resolution (resp. for the minimal resolution). In general, we have $M_E \geqq Z_E$. However, for many normal surface singularities, it is unknown what the condition for $M_E = Z_E$ is.

Around 5 years ago, for a hypersurface singularity of Brieskorn type (i. e., $x^a+y^b+z^c = 0$ with $a \leqq b \leqq c$), K. Konno and D. Nagashima [KN] proved the necessary and sufficient numerical condition on $(a, b, c)$ to be $M_E = Z_E$ on the minimal resolution. A few years later, F. Meng and T. Okuma [MO] generalized the result to complete intersections surface singularities of Brieskorn type. Recently, M. Tomari and the author studied the problem for more general normal surface singularities with $\mathbb{C}^*$-action, because complete intersection surface singularities of Brieskorn type form small subclass among normal surface singularities with $\mathbb{C}^*$-action. In Section 4, for the preparation of Section 5, we prepare some facts on normal surface singularities with $\mathbb{C}^*$-action. In Section 5, we explain our results on maximal ideal cycles.

Since last year, for cyclic coverings over normal surface singularities, the author has been considering the relation between $M_E$ and $Z_E$ and $\mathfrak{m}\mathcal{O}_{\tilde{X}}$ ([To7]). Also, we consider a *maximal ideal type* of a normal surface singularity $(X, o)$ (see Definition 6.1). We discuss the maximal ideal type for hypersurface singularities of Brieskorn type.

In Section 7, we consider some examples of sequences of normal surface singularities, and compute the maximal ideal type of each member for the sequences.

**Contents.**



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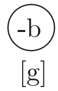
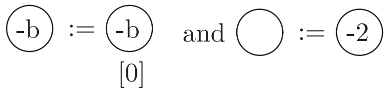
**Notation.** In this paper, the configuration of the exceptional set $E$ of a resolution of a normal surface singularity is represented by the w.d.graph (weighted dual graph). If the irreducible component $E_i$ of $E$ is a non-singular curve of genus $g$ and the self-intersection number $-b$, then $E_i$ is given by the following:




Further, we use the following notations:




**2. Fundamental facts.** Let $\pi\colon (\tilde{X}, E) \to (X, o)$ be a resolution of a normal complex surface singularity. Let $E = \bigcup_{i=1}^{r} E_i$ be the irreducible decomposition of the exceptional set. On the exceptional set $E$, the following two cycles $Z_E$ and $M_E$ are well-known.

Definition 2.1 (M. Artin [Art]). Let

$$Z_E := \min\{D = \sum_{i=1}^{r} a_i E_i \mid a_i > 0 \text{ is a positive integer and } DE_i \leqq 0 \text{ for any } i\},$$

and it is called the *fundamental cycle*.

Definition 2.2 (S. S.-T. Yau [Y]). Let $M_E := \min\{(f \circ \pi)_E \mid f \in \mathfrak{m}, f \neq 0\}$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_{X,o}$ and $(f \circ \pi)_E = \sum_{i=1}^{r} v_{E_i}(f \circ \pi) E_i$ and $v_{E_i}(f \circ \pi)$ is the vanishing order of $f \circ \pi$ on $E_i$. $M_E$ is called *the maximal ideal cycle on $E$*.

The values of $Z_E^2$ and $p_a(Z_E)$ are independent of the choice of a resolution $(\tilde{X}, E)$. Then, in this paper, we represent them $Z^2$ and $p_f(X, o)$. However, $M_E^2$ and $p_a(M_E)$ depend on the choice of a resolution of $(X, o)$.

Facts 2.3. From the definitions, we can easily see the following facts.
(i) $0 < Z_E \leqq M_E$ for any resolution.
(ii) $M_E = Z_E \;\Leftrightarrow\; M_E^2 = Z^2$.

For the cycles $M_E$ and $Z_E$, the following fundamental facts are proven.

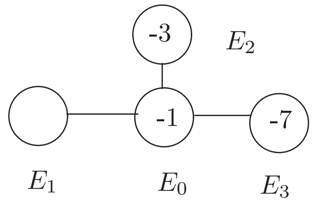
Theorem 2.4 (M. Artin [Art]). *If $(X, o)$ is a rational singularity, then $M_E = Z_E$ for any resolution and mult$(X, o) = -Z^2$.*

Let $\pi\colon (\tilde{X}, E) \to (X, o)$ be a good resolution. Let $\mathfrak{m}\mathcal{O}_{\tilde{X}}$ be the pull-back of the maximal ideal through $\pi$.

Theorem 2.5 (P. Wagreich [Wag]). *Under the condition above, we have*
(i) *mult$(X, o) \geqq -M_E^2 \geqq -Z^2$,*
(ii) *$\mathfrak{m}\mathcal{O}_{\tilde{X}}$ is locally principal $\;\Leftrightarrow\;$ $\mathfrak{m}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-M_E)$.*

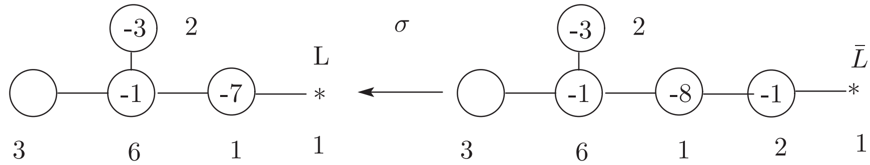
If $\mathfrak{m}\mathcal{O}_{\tilde{X}}$ is not locally principal at $P \in \tilde{X}$, then $P$ is called an *embedded point* of $\mathfrak{m}\mathcal{O}_{\tilde{X}}$. After suitable blowing-ups at embedded points, $\mathfrak{m}\mathcal{O}_{\tilde{X}}$ become to be locally principal. Let us explain this phenomena through an example. The theorem above is very useful to compute the multiplicity of normal surface singularities. For example, T. Okuma [O] proved that the multiplicity of splice quotient singularities are determined topologically.

Example 2.6. Let $\pi\colon (\tilde{X}, E) \to (X, o)$ be the minimal good resolution of $(X, o) := \{(x, y, z) \in \mathbb{C}^3 \mid z^2 = x^3 + y^7\}$. The w.d.graph of $E$ is given as follows:




Let $L$ be the strict transform of the divisor $\{y = 0\}$ in $X$ by $\pi$. Let $\sigma : (\tilde{\tilde{X}}, \tilde{E}) \to (\tilde{X}, E)$ be the blow-up at $P := E_3 \cap L$. Since $P$ is a non-singular point of $\tilde{X}$, $\mathcal{O}_{\tilde{X},P}$ is the convergent power series ring of two variables. Since $\mathrm{Coeff}_{E_0}(x \circ \pi)_E = 14$ and $\mathrm{Coeff}_{E_3}(x \circ \pi)_E = 2$ for the cycle $(x \circ \pi)_E$, we have $(\mathfrak{m}\mathcal{O}_{\tilde{X}})_P = (y \circ \pi, x \circ \pi) = (uv, v^2)$ in $\mathcal{O}_{\tilde{X},P} = \mathbb{C}\{u, v\}$, where $E = \{v = 0\}$ locally near at $P$. We obtain the following figure:

$(y \circ \pi)_{\tilde{X}}$:




Then $(\mathfrak{m}\mathcal{O}_{\tilde{X}})_P$ is not a principal ideal and $P$ is an embedded point of $\mathfrak{m}\mathcal{O}_{\tilde{X}}$. Then, $\mathfrak{m}\mathcal{O}_{\tilde{\tilde{X}}}$ is locally principal at $\bar{E}_3 \cap \bar{L}$, where $\bar{E}_3$ is the proper transform of $E_3$ by $\sigma$.

Because of embedded points, when we compare $M_E$ and $Z_E$, we need to distinguish the kind of resolutions (i. e., the minimal resolution, the minimal good resolution or an arbitrary resolution). In this paper, if $(\tilde{X}, E)$ is the minimal resolution, $M_E$ and $Z_E$ are represented by $M_0$ and $Z_0$. In 1977, H. Laufer proved the following beautiful results.

Theorem 2.7 (H. Laufer [L1]). *Let $\pi: (\tilde{X}, E) \to (X, o)$ be the minimal resolution of a normal surface singularity. Then the following conditions are equivalent:*

(i) $K_{\tilde{X}} E_i = -Z_0 E_i$ *for any* $i$.
(ii) $p_f(X, o) = 1$ *and* $Z_0 =$*minimally elliptic cycle (see Definition 2.9).*
(iii) $(X, o)$ *is Gorenstein and* $p_g(X, o) = 1$.

Singularities satisfying the conditions above are called *minimally elliptic singularities.* He proved the following facts.

Theorem 2.8 (H. Laufer [L1]). *If $\pi: (\tilde{X}, E) \to (X, o)$ is the minimal resolution of a minimally elliptic singularity, then*

(i) $M_0 = Z_0$.
(ii) *If* $Z^2 \leqq -2$, *then* $\mathfrak{m}\mathcal{O}_{\tilde{X}} = \mathcal{O}_{\tilde{X}}(-Z_E)$.
(iii) $\mathrm{mult}(X, o) = \max(2, -Z^2)$.

Here, let us describe the definitions of the minimal cycle and the Yau sequence. If $(X, o)$ is an elliptic singularity (i. e., $p_f(X, o) = 1$), then the minimal cycle (resp. Yau sequence) coincides to the minimally elliptic cycle (resp. elliptic sequence).

Definition 2.9 (minimal cycle). Let $\pi: (\tilde{X}, E) \to (X, o)$ be the minimal good resolution of a normal complex surface singularity with $p_f(X, o) \geqq 1$. Let $Z_{min}$ be a cycle on the exceptional set $E$ satisfying $0 < Z_{min} \leqq Z_E$. Then $Z_{min}$ is said to be a *minimal cycle* on $E$ if $p_a(Z_{min}) = p_f(X, o)$ and $p_a(D) < p_f(X, o)$ for any cycle $D$ with $0 < D < Z_{min}$.

In the case of $p_f(X, o) = 1$, we need not the assumption "$Z_{min} \leqq Z_E$" in Definition 2.9 (see [L]).

Definition 2.10 (Yau sequence). Assume the situation of Definition 2.9. If $Z_E Z_{min} < 0$, then we say that the Yau sequence is $\{Z_E\}$. Suppose $Z_E Z_{min} = 0$. Let $B_1$ be the maximal connected subvariety of $E$ such that $B_1 \supset \operatorname{supp} Z_{min}$ and $Z_E E_i = 0$ for any $E_i \subset B_1$. Since $Z_E^2 < 0$, $B_1$ is properly contained in $E$. Let $Z_{B_1}$ be the fundamental cycle in $B_1$. Suppose $Z_{B_1} Z_{min} = 0$. Let $B_2$ be the maximal connected subvariety of $B_1$ such that $B_2 \supset \operatorname{supp} Z_{min}$ and $Z_{B_1} E_i = 0$ for any $E_i \subset B_2$. By the same argument as above, $B_2$ is properly contained in $B_1$. Continuing this process, if we obtain $B_t$ with $B_t Z_{min} < 0$, we call $\{Z_{B_0} = Z_E, Z_{B_1}, \cdots, Z_{B_t}\}$ *the Yau sequence* of $(X, o)$ and $t + 1$ is called *the length of the Yau sequence.*

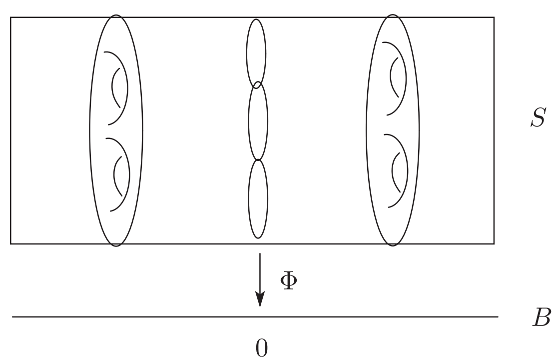
Theorem 2.11 (S. S.-T. Yau [Y]). *Let $(X, o)$ be an elliptic singularity. Then*
(i) *The geometric genus $p_g(X, o) \leqq \ell(X, o)$ (:=the length of the Yau sequence);*
(ii) *If $(X, o)$ is a numerically Gorenstein elliptic singularity satisfying $p_g(X, o) = \ell(X, o)$, then $(X, o)$ is a Gorenstein singularity (such singularities are called maximally elliptic singularities).*

Theorem 2.12 (M. Tomari [Ti]). *If $(X, o)$ is a maximally elliptic singularity, then we have $M_0 = Z_0$.*

**3. Pencils of curves.** In [Arn], V.I.Arnold introduced an invariant (so called "modality") for function germs. He proved that surface singularities with modality 0 coincide to rational double points. Furthermore, he classified surface singularities with modality 1 and 2 and called them unimodal and bimodal singularities respectively. In [Ku], V. Kulikov observed that all unimodal and bimodal singularities are obtained from Kodaira's classification of pencil of elliptic curves in [Ko]. On the other hand, independently from Laufer and Kulikov, M. Reid [R] studied and classified minimally elliptic singularities with small multiplicity; also he observed the relation between those singularities and Kodaira's work on the elliptic pencils above.

After those works, using pencil of curves, U. Karras gave the definition of Kodaira singularities. From the point of view of the rough classification of singularities, we explain the relation between surface singularities and pencils of curves.

Definition 3.1 (K. Kodaira [Ko]). Let $S$ be a smooth complex surface and $\Delta \subset \mathbb{C}$ a small open disc around the origin. If $\Phi\colon S \to \Delta$ is a proper surjective holomorphic map and the generic fiber $S_t\colon = \Phi^{-1}(t)$ $(t \neq 0)$ is a smooth compact connected curve of genus $g$, then it is called a *pencil of curves of genus $g$*.

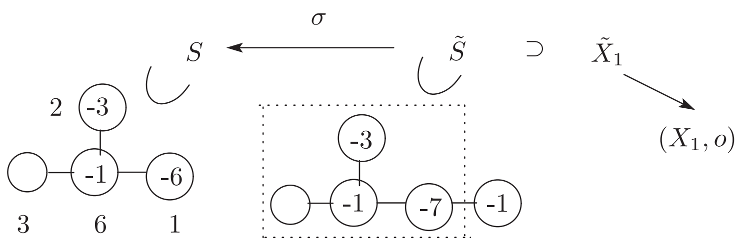


Definition 3.2 (U. Karras [Ka]). Let $\Phi\colon S \to \Delta$ be a pencil of curves of genus $g$ which has reduced components. Let $P_1, \cdots, P_r \in \mathrm{supp}(S_o)$ be non-singular points of $S_o$ (i. e., they are contained in components whose coefficients of $S_o$ equal to one and also smooth points of $\mathrm{red}(S_o)$). Let $S' \xrightarrow{\sigma} S$ be a finite number of blowing-ups with centers $P_1, \cdots, P_r$. Let $\tilde{X}$ be an open neighborhood of the proper transform $E$ $(\subset S')$ of $\mathrm{supp}(S_o)$ by $\sigma$. By contracting $E$ in $\tilde{X}$, we obtain a normal surface singularity $(X, o)$. If a normal surface singularity is isomorphic to a singularity obtained in this way, then it is called a *Kodaira singularity* of genus $g$.

Remark 3.3. From the definition, we can easily see that $M_0 = Z_0$ holds for any Kodaira singularity.

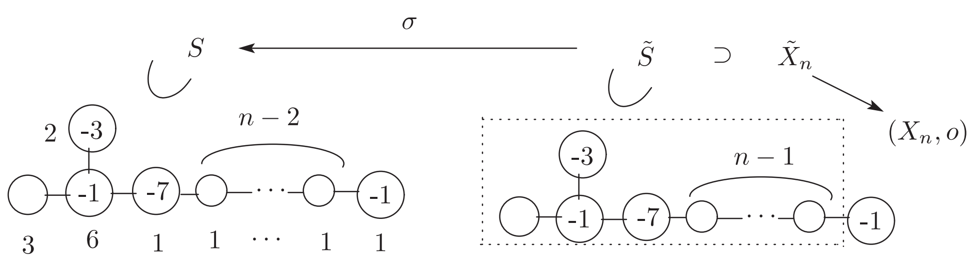
Example 3.4. Let $(X_n, o)$ be a quasi-homogeneous hypersurface singularity defined by $z^{6n+1} = x^2 + y^3$ $(n = 1, 2, \cdots)$. For each $n$, $(X_n, o)$ is an elliptic Kodaira singularity which is constructed from an elliptic $\mathbb{C}^*$-pencil of curves (see [To6]). Let us show them in the following.

Assume $n = 1$. Then, $(X_1, o)$ is a minimally elliptic singularity. It is a Kodaira singularity which is constructed from a relatively minimal elliptic pencil as follows.



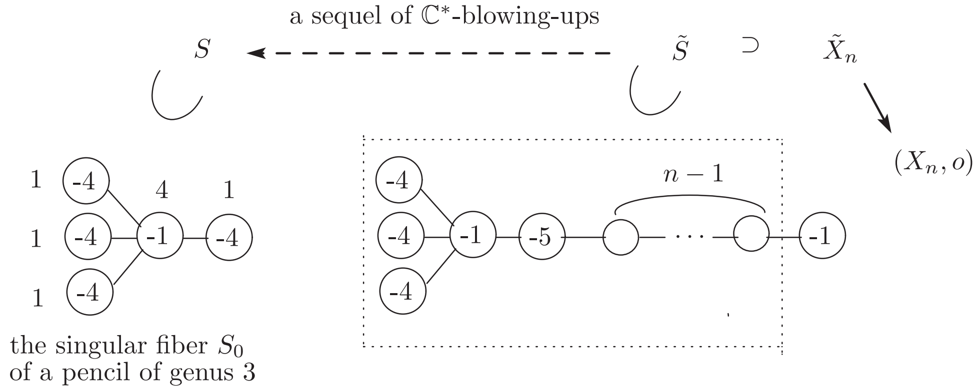

In the above, $\sigma$ is a $\mathbb{C}^*$-blowing-up (i. e., blowing-up at a fixed point of the $\mathbb{C}^*$-action).

Assume $n \geqq 2$. Then, $(X_n, o)$ is a maximally elliptic singularity (but not minimally elliptic). It is also a Kodaira singularity which is constructed from an elliptic $\mathbb{C}^*$-pencil (but not relatively minimal) as follows.




In the above, $\sigma$ is a $\mathbb{C}^*$-blowing-up. For any fixed positive integer $N$, the sequence $\{(X_n, o)\}_{n=1,2,\cdots,N}$ corresponds to the elliptic sequence of $(X_N, o)$ (see [Y]).

Example 3.5. Let $(X_n, o)$ be a quasi-homogeneous hypersurface singularity defined by $z^{12n+3} = x^3 + y^4$ $(n = 1, 2, \cdots)$. Then, for each $n$, $(X_n, o)$ is a Kodaira singularity which is constructed from a relatively minimal $\mathbb{C}^*$-pencil of genus 3 by taking blowing-ups (as in Definition 3.2) several times.




For any fixed positive integer $N$, the sequence $\{(X_n, o)\}_{n=1,2,\cdots,N}$ corresponds to the Yau sequence of $(X_N, o)$.

When we consider sequences of surface singularities, it seems to be interesting to consider the relation between normal surface singularities and pencils of curves.

Theorem 3.6 ([To2]). *Let $(X, o) := \{z^n = f(x, y)\}$ be a normal surface singularity. Let $(C, o)$ be a curve singularity defined by $(\{f = 0\}, o) \subset (\mathbb{C}^2, o)$. Let $\sigma : (W, F) \to (\mathbb{C}^2, o)$ be the minimal embedded resolution of $(C, o)$ such that $red(f \circ \sigma)_W$ is normal crossing. Let $N(f) := \max\{ord_{F_i}(f \circ \sigma) \mid F_i$ is a component of $F\}$. Then,*

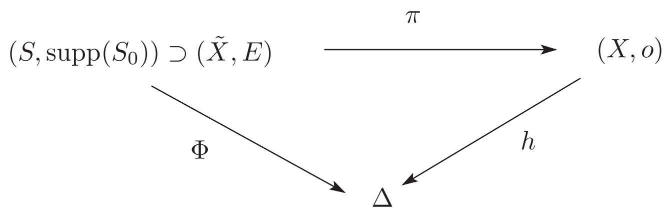
(i) *If $n \mid ord(f)$, then $(X, o)$ is a Kodaira singularity of genus $\dfrac{(n-1)(ord(f)-2)}{2}$ and $Z^2 = -n$.*
(ii) *If $N(f) \leqq n$, then $(X, o)$ is a Kodaira singularity of genus $\dfrac{\mu(f) - r(f) + 1}{2}$ and $Z^2 = -r(f)$, where $\mu(f)$ (resp. $r(f)$) is the Milnor number (resp. the number of irreducible components) of $(C, o)$.*

If $(X, o)$ is a singularity satisfying (i) above, then $\mathrm{mult}(X, o) = -Z^2$. Hence, $\mathrm{mult}(X, o) = -M_0^2 = -Z^2$ and then $M_E = Z_E$ for any resolution. If $n = 2$ and $ord(f)$ is even, then $(X, o)$ is a Kodaira singularity and so $M_E = Z_E$ for any resolution. In [D], D.J. Dixon already proved that if $(X, o) := \{z^2 = f(x, y)\}$ is a normal and $\mathrm{ord}(f)$ is even, $M_0 = Z_0$.

Definition 3.7. For an element $g$ of a ring $R$, if there exists an element $g_0 \in R$ with $g = g_0^\ell$ for a positive integer $\ell \geqq 2$, then $g$ is called *a perfect power element.*

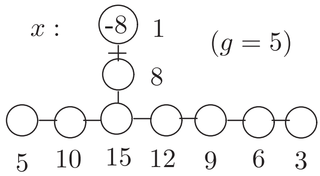
Since any Kodaira singularity has a good resolution which is contained into a pencil of curves, it is natural to ask whether it is always true or not for all normal surface singularities. About this, the author proved the following.

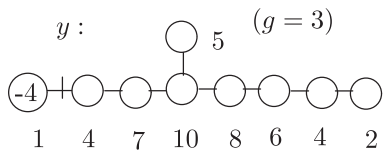
Theorem 3.8 ([T4]). *Let $(X, o)$ be a normal surface singularity. Let $h \in \mathfrak{m}_{X,o}$ be not a perfect power element. If $\pi\colon (\tilde{X}, E) \to (X, o)$ is a good resolution such that $red((h \circ \pi)_{\tilde{X}})$ is a simple normal crossing divisor on $\tilde{X}$, then there exists a pencil of curves $\Phi\colon S \to \Delta$ satisfying $\Phi|_{\tilde{X}} = h \circ \pi$.*

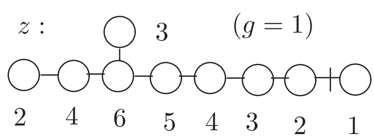



In [To5], the author defined invariants $p_e(X, o, h)$ and $p_e(X, o)$, and call them the *pencil genus* of $(X, o, h)$ and $(X, o)$ respectively.

EXAMPLE 3.9. Let $(X, o) = (\{x^2 + y^3 + z^5 = 0\}, o)$ (a rational double point of type $E_8$). If we choose $x, y$ or $z$ as $h$, the singular fibers of the pencils of curves are given as follows.




For examples above, $p_e(X, o, x) = 5, p_e(X, o, y) = 3, p_e(X, o, z) = 1$ and $p_e(X, o) = 1$.

Though Kodaira singularities were defined as geometric objects, M. Tomari proved the following ring theoretical characterization for Kodaira singularities with some properties.

THEOREM 3.10 (Tomari). *Let $(X, o)$ be a normal surface singularity. The following two conditions are equivalent:*

(i) $\bigoplus_{k=0}^{\infty} \bar{\mathfrak{m}}^k / \bar{\mathfrak{m}}^{k+1}$ *is a reduced ring (*i. e.*, nilpotent free).*

(ii) $(X, o)$ *is a Kodaira singularity of mult$(X, o) = -Z^2$.*

**4. Some facts on normal surface singularities with $\mathbb{C}^*$-action.** In this section, as the preparation of Section 5, we explain some fundamental facts on normal surface singularities with $\mathbb{C}^*$-action.

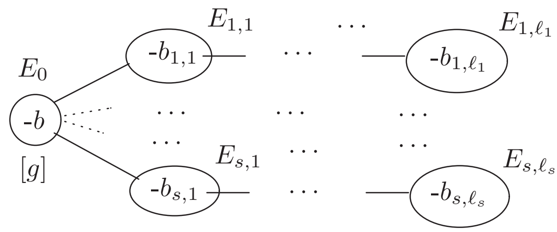
THEOREM 4.1 (Orlik-Wagreich [OW]). *Let $(X, o)$ be a normal surface singularity with $\mathbb{C}^*$-action. There exists a $\mathbb{C}^*$-equivariant resolution $\pi : (\tilde{X}, E) \to (X, o)$ uniquely which satisfies the following:*

(i) *the w.d.graph of $E$ is a star-shaped graph of (4.1),*

(ii) *the $\mathbb{C}^*$-action on $\tilde{X}$ acts trivially on $E_0$,*

(iii) *each $\mathbb{P}^1$-chain $\bigcup_{j=1}^{\ell_i} E_{i,j}$ does not contain a $(-1)$-curve.*

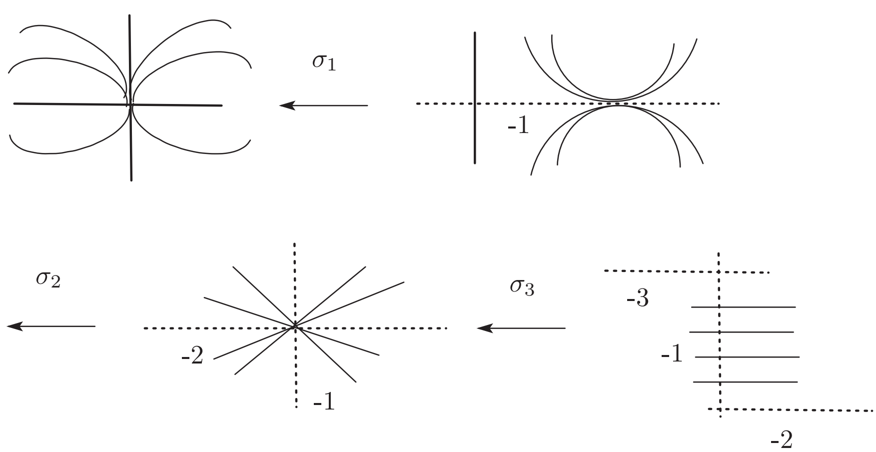
(4.1)



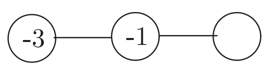

In the above, $E_0$ is called the *central curve* and $\pi$ is called the *minimal* $\mathbb{C}^*$*-good resolution* of $(X, o)$. For cases aside from cyclic quotient singularities, the minimal $\mathbb{C}^*$-good resolution coincides to the minimal good resolution. However, for cyclic quotient singularities, this is not always true (see [To3]).

Example 4.2. Let $(\mathbb{C}^2, o)$ be a point whose $\mathbb{C}^*$-action is given by $t \cdot (x, y) = (t^2 x, t^3 y)$. Let $C_{a,b}$ be the $\mathbb{C}^*$-orbit of $(a, b) \in (\mathbb{C}^*)^2$. Then,

$$\begin{aligned} C_{a,b} = \mathbb{C}^* \cdot (a, b) &= \{(t^2 a, t^3 b) \in \mathbb{C}^2 \mid t \in \mathbb{C}^*\} \\ &= \{(x, y) \in \mathbb{C}^2 \mid (\frac{x}{a})^3 = (\frac{y}{b})^2\} \\ &= \{(x, y) \in \mathbb{C}^2 \mid x^3 = (\frac{a^3}{b^2}) y^2\} \subset (\mathbb{C}^2, o). \end{aligned}$$




Then $\sigma := \sigma_1 \circ \sigma_2 \circ \sigma_3$ is the minimal $\mathbb{C}^*$-good resolution of $(\mathbb{C}^2, o)$ with $\mathbb{C}^*$-action above. Hence the w.d.graph of $E$ is given as follows:



.

Again, let $\pi : (\tilde{X}, E) \to (X, o)$ be the minimal $\mathbb{C}^*$-good resolution of a normal surface singularity with $\mathbb{C}^*$-action such that the w.d.graph of $E$ is given by (4.1).

Theorem 4.3 (A. Fujiki [F]). *The holomorphic structure of* $(X, o)$ *is determined by the holomorphic structures of the following objects:*

(i) $E_0$,

(ii) $N_{E_0/\tilde{X}}$ *(the normal bundle),*

(iii) $P_1, \cdots, P_s$ $(P_i := E_0 \cap E_{i,1})$.

Definition 4.4. Let $E$ be the exceptional set of a star-shaped singularity $(X, o)$ whose w.d.graph is given as (4.1). Let $\dfrac{\alpha_i}{\beta_i} := b_{i,1} - \cfrac{1}{\ddots - \cfrac{1}{b_{i,\ell_i}}}$ and $\gcd(\alpha_i, \beta_i) = 1$ for each $i$. Put $\alpha_0(X, o) := \mathrm{lcm}(\alpha_1, \cdots, \alpha_s)$ if $s > 0$ and $\alpha_0(X, o) := 1$ if $s = 1$.

Theorem 4.5 (H. Pinkham [P]). *Under the condition above, let*

$$D^{(k)} := kN^*_{E_0/\tilde{X}} - \sum_{j=1}^{s} \lceil \frac{\beta_j k}{\alpha_j} \rceil P_j \quad \text{(Pinkham-Demazure's divisor)}.$$

*Then the affine graded ring $R_X$ of $(X, o)$ is given as follows:*

$$R_X \cong \bigoplus_{k=0}^{\infty} H^0(E_o, \mathcal{O}_{E_o}(D^{(k)}))t^k \quad \text{(Pinkham-Demazure's construction)}.$$

Let $\pi : (\tilde{X}, E) \to (X, o)$ be the minimal good resolution. If the w.d.graph of $E$ is given by (4.1) (but has not always $\mathbb{C}^*$-action), then $(X, o)$ is called a *star-shaped singularity*. Then, any normal surface singularity with $\mathbb{C}^*$-action is a special star-shaped singularity.

We can also consider the divisor $D^{(k)}$ for star-shaped surface singularities. The following result was proven by M. Tomari (see [To1; Theorem 3.1]).

Theorem 4.6. Under the condition above, we have

$$\mathrm{Coeff}_{E_0} Z_E = \min\{k \mid deg(D^{(k)}) \geqq 0\}$$

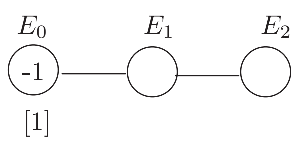
.

Remark 4.7. If $(X, o)$ is a normal surface singularity with $\mathbb{C}^*$-action, then

$$\begin{aligned} \mathrm{Coeff}_{E_0} M_E &= \text{min. deg. of } R_X \text{ (=minimal degree)} \\ &:= \min\{k \mid H^0(E_0, \mathcal{O}_{E_0}(D^{(k)})) \neq 0\}. \end{aligned}$$

**5. Recent results on maximal ideal cycles for normal surface singularities with $\mathbb{C}^*$-action.** In this section, first we explain Konno-Nagashima's result [KN] (resp. Meng-Okuma's result [MO] ) for hypersurface (resp. complete intersection) surface singularities of Brieskorn type. Second we explain the result by Tomari and the author [TT] for normal surface singularities with $\mathbb{C}^*$-action. Before it, we describe the following famous example discovered by H. Laufer.

Example 5.1 (H. Laufer [L2]). Let $(\tilde{X}, E)$ be the minimal resolution space of $(X, o) := \{(x, y, z) \mid z^2 = y(x^4 + y^6)\}$. Then, the w.d.graph of $E$ is given as follows:

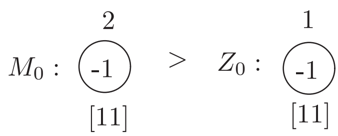



and $M_0 = (x + y)_E = 2E_0 + 2E_1 + E_2 > Z_0 = E_0 + E_1 + E_2$. Let $P \in E_0$ be a point corresponding to $N_{E_0/\tilde{X}}$ (the restriction of the conormal bundle) and let $Q := E_0 \cap E_1$.

Then we have $P \nsim Q$ (not linearly equivalent) and $2P \sim 2Q$. Also, $y$ is a non-reduced element and the pencil of curves given by $y$ (according to the method of Theorem 3.8) is a multiple pencil of curves. From those reasons, the author had been thinking for long time that the phenomenon of $M_0 > Z_0$ does not happen except for some special examples. However, K. Konno and D. Nagashima proved the following.

Theorem 5.2 ([KN]). *Let $(X, o)$ be the Brieskorn hypersurface singularity defined by $z_0^{a_0} + z_1^{a_1} + z_2^{a_2} = 0$. Let $\pi\colon (\tilde{X}, E) \to (X, o)$ be the minimal good resolution. Assume that $a_0 \leqq a_1 \leqq a_2$. Then,*
(i) $M_E = (z_2 \circ \pi)_E$.
(ii) $M_0 = Z_0 \Leftrightarrow \dfrac{\gcd(a_0, a_1)}{\gcd(a_0, a_1, a_2)} \leqq \dfrac{a_2}{\gcd(a_2, \operatorname{lcm}(a_0, a_1))}$.
(iii) $M_E = Z_E$ *for any resolution* $\Leftrightarrow M_0 = Z_0$ *and* $a_0 a_1 > a_0 a_2 - \gcd(a_0, a_1)$.
(iv) $(X, o)$ *is a Kodaira singularity* $\Leftrightarrow \operatorname{lcm}(a_0, a_1) \leqq a_0$.

Example 5.3 ([KN]). If $(X, o) := \{x^6 + y^{10} + z^{15} = 0\}$, then we have the following.




The reason of $M_0 > Z_0$ is simple. We can easily see that $deg(z) = 2$, $deg(y) = 3$ and $deg(x) = 6$. Therefore, the degree of $L := N^*_{E_0/\tilde{X}}$ is 1, but $H^0(E_0, \mathcal{O}_{E_0}(L)) = 0$ and $H^0(E_0, \mathcal{O}_{E_0}(2L)) \neq 0$.

Let $(X, o)$ be a complete intersection surface singularity of Brieskorn type. It is defined by the equations: $z_3^{a_3} = p_3 z_1^{a_1} + q_3 z_2^{a_2}, \cdots, z_m^{a_m} = p_m z_1^{a_1} + q_m z_2^{a_2}$ in $(\mathbb{C}^m, o)$, where $p_i q_j \neq p_j q_i$ for $i \neq j$. This is a normal surface singularity with $\mathbb{C}^*$-action. Meng-Okuma generalized Konno-Nagashima's result to Brieskorn complete intersection singularities.

Theorem 5.4 ([MO]). *Under the situation above, we assume that $a_0 \leqq a_1 \leqq \cdots \leqq a_m$. Let $\pi\colon (\tilde{X}, E) \to (X, o)$ be the minimal good resolution. Then,*
(i) $M_E = (z_m \circ \pi)_E$.
(ii) $M_0 = Z_0 \Leftrightarrow$ *mini.deg*$(R_X) \leqq \alpha_0(X, o)$ *(see Definition 4.4).*

Complete intersection surface singularities of Brieskorn type form a small class among all normal surface with $\mathbb{C}^*$-action. In [TT], Tomari and the author considered same problem to compare $M_0$ and $Z_0$. In the following, let $(X, o)$ be a normal surface singularity with $\mathbb{C}^*$-action.

Definition 5.5. Let $\pi\colon (\tilde{X}, E) \to (X, o)$ be the minimal $\mathbb{C}^*$-good resolution. We define three conditions (1)-(3) as follows:

$$(1)\ M_E = Z_E, \quad (2)\ \operatorname{Coeff}_{E_0} M_E = \operatorname{Coeff}_{E_0} Z_E, \quad (3)\ \text{min.deg}(R_X) \leqq \alpha_0(X, o).$$

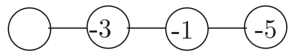
We can easily see that (1) $\Rightarrow$ (2). Moreover, $D^{(\alpha_0(X,o))}$ is an integral cycle with $\deg D^{(\alpha_0(X,o))} > 0$. Hence, from the definition of $Z_E$ and Tomari's formula (Theorem 4.6), we have $\operatorname{Coeff}_{E_0} Z_E \leqq \alpha_0(X, o)$. Hence, $\text{min.deg}(R_X) = \operatorname{Coeff}_{E_0} M_E = \operatorname{Coeff}_{E_0} Z_E \leqq \alpha_0(X, o)$; and then "(2) $\Rightarrow$ (3)" holds. However, the converse "(3) $\Rightarrow$ (2)" is not always true. Because $z^2 = y(x^4 + y^6)$ of Example 5.1 satisfies the following:

$$\text{min.deg}(R_X) = 2 < \alpha_0(X, o) = 3 \text{ (so (3) holds)},$$

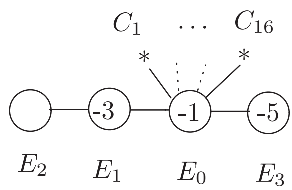
but (2) doesn't hold. Next, let us consider the implication "(2) $\Rightarrow$ (1)". If $E_0 = \mathbb{P}^1$, then (2) always holds from Theorem 3.6.

QUESTION 5.6. Is there an example of a normal surface singularity with $\mathbb{C}^*$-action whose central curve is $\mathbb{P}^1$ and $M_0 > Z_0$ ?

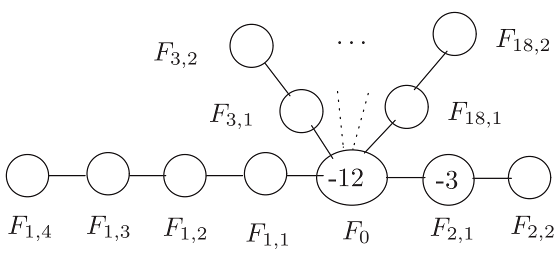
EXAMPLE 5.7 (a counterexample for "(2) $\Rightarrow$ (1)"). Let $(X, o)$ be a cyclic quotient singularity $C_{10,3} := \mathbb{C}^2/\langle (e_{10}, e_{10}^3) \rangle$, where $e_n :=\exp(2\pi\sqrt{-1}/n)$. Consider a $\mathbb{C}^*$-action on $(X, o)$ induced from $t \cdot (x, y) := (tx, ty)$ on $\mathbb{C}^2$. The exceptional set of the minimal $\mathbb{C}^*$-good resolution of $(X, o)$ is given as follows.




Let $h \in R_X$ be a homogeneous element whose divisor $(h \circ \sigma)_E$ is given by $40E_0 + 16E_1 + 8E_2 + 8E_3 + \sum_{i=1}^{16} C_i$ in the next diagram.




Let $(Y, o)$ be the 3-fold cyclic covering over $(X, o)$ defined by $z^3 = h$. Then the exceptional set of the minimal resolution of $(Y, o)$ is given as follows.




Then, $M_F = 6F_0 + \sum_{i=1}^{4}(6-i)F_{1,i} + 3F_{2,1} + 3F_{2,2} + \sum_{i=3}^{18}(4F_{i,1} + 2F_{i,2}) > Z_F = M_F - F_{2,2}$. In the following, we descrive the main results in [TT].

THEOREM 5.8 ([TT]). *Let $\pi \colon (\tilde{X}, E) \to (X, o)$ be the minimal good resolution of a normal surface singularity with $\mathbb{C}^*$-action. Let $E_0$ be the central curve of $E$. Assume that there exists a reduced homogeneous element $f \in R_X$ satisfying $(f \circ \pi)_E = M_E$. Then, $M_E = Z_E$ if and only if $\operatorname{Coeff}_{E_0} M_E = \operatorname{Coeff}_{E_0} Z_E$.*

REMARK 5.9. In the above, from the hypothsis of the existence of a reduced homogeneous element $f$, the form of $M_E$ is subject to strong restriction. This is a key point of the proof. Complete intersection singularities of Brieskorn type satisfies this condition. Moreover, except for complete intersection singularities of Brieskorn type, there are many kind of singularities satisfying the condition of Theorem 5.8.

Let $\pi_X \colon (\tilde{X}, E) \to (X, o)$ be a $\mathbb{C}^*$-good resolution of a normal complex surface singularity with $\mathbb{C}^*$-action and $I_X$ the defining ideal of $(X, o)$ $(\subset \mathbb{C}^N)$ (i.e., $I_X \subset \mathbb{C}[z_1, \cdots, z_N]$). Let $h_1, \cdots, h_m$ be homogeneous reduced elements of $R_X$ and $C_i$ the non-exceptional part of the curve $(h_i \circ \pi)_{\tilde{X}} = 0$ for each $i$. Consider the ideal $I_Y$ in

$\mathbb{C}[z_1, \cdots, z_{N+m}]$ generated by $z_{N+1}^{a_1} - h_1, \cdots, z_{N+m}^{a_m} - h_m$ and $I_X$, where each $a_i$ is an integer ($\geqq 2$). Let $(Y, o)$ be a normal surface singularity with $\mathbb{C}^*$-action defined by $I_Y$ in $\mathbb{C}^{N+m}$. If we assume that $C_i \cap C_j = \emptyset$ for $i \neq j$, then $(Y, o)$ is normal (M. Tomari–K-i.Watanabe [TW]) and it is called a Kummer covering over $(X, o)$.

Let $d_i := deg(h_i)$ for each $i$. Under a suitable exchange of indices, we may assume that $\dfrac{d_1}{a_1} \geqq \cdots \geqq \dfrac{d_m}{a_m}$. Let $\pi_Y : (\tilde{Y}, F) \to (Y, o)$ be the minimal $\mathbb{C}^*$-good resolution.

THEOREM 5.10 ([TT, Theorem 4.2]). *Assume that* $\mathrm{Coeff}_{E_0} M_E \geqq \dfrac{d_m}{a_m}$.

(i) $M_F = (z_m \circ \pi_Y)_F$,
(ii) $M_F = Z_F \;\Leftrightarrow\; \mathrm{Coeff}_{E_0} M_F = \mathrm{Coeff}_{E_0} Z_F$.

REMARK 5.11. Any complete intersection singularity of Brieskorn type is a Kummer covering over $(\mathbb{C}^2, 0)$. We can prove Meng-Okuma's result (Theorem 5.4) as a corollary of Theorem 5.10.

**6. Maximal ideal types of normal surface singularities.** From the point of view on the maximal ideal cycle and the fundamental cycle and the pull-back of the maximal ideal, we consider four types I $\sim$ IV for normal surface singularities. In [To7], the author studied cyclic coverings of normal surface singularities, those types seem to be useful.

DEFINITION 6.1. Let $\pi : (\tilde{X}, E) \to (X, o)$ be the minimal good resolution. From Theorem 2.5, we have $\mathrm{mult}(X, o) \geqq -M_0^2 \geqq -Z^2$. Then, we can consider the following four types:

| | $M_0 = Z_0$ | $M_0 > Z_0$ |
|---|---|---|
| $\mathfrak{m}\mathcal{O}_{\tilde{X}}$ is locally principal | type I | type II |
| $\mathfrak{m}\mathcal{O}_{\tilde{X}}$ is not locally principal | type III | type IV |

.

We remark that $\mathrm{mult}(X, o) = -M_0^2$ if and only if $\mathfrak{m}\mathcal{O}_{\tilde{X}}$ is locally principal (i. e., locally free). The type above is called *the maximal ideal type of* $(X, o)$ or the *type of* $(X, o)$ in short.

From the results we told until now, we can easily see the following facts.

(i) Any rational singularity is of type I (Theorem 2.4).
(ii) Any minimally elliptic singularity with $Z^2 \leqq -2$ is of type I (Theorem 2.8 (ii)).
(iii) Any minimally elliptic singularity with $Z^2 = -1$ is of type III (Theorem 2.8 (iii)).
(iv) A normal hypersurface singularity defined by $z^n = f(x, y)$ with $n \mid ord(f)$ is of type I (Theorem 3.6 (i)).
(v) A hypersurface singularity defined by $z^2 = y(x^4 + y^6)$ is of type II (Example 5.1).

If $(X, o)$ is a Kodaira singularity, then it is of type I or III. Tomari's result (Theorem 3.10) is a ring theoretical characterization of Kodaira singularities of type I; hence we propose the following.

Problem 6.2. Is there a similar ring theoretical characterization of Kodaira singularities of type III ?

In the following, we consider the maximal ideal types of hypersurface singularities of Brieskorn type. Let $(X, o)$ be a hypersurface singularity defined by $z_0^{a_0}+z_1^{a_1}+z_2^{a_2}=0$ $(a_0 \leqq a_1 \leqq a_2)$. In this section, we study the maximal ideal type of $(X, o)$.

As a corollary of Konno-Nagashima's result (Theorem 5.2), we can determine the maximal ideal type of $(X, o)$ as follows:

Proposition 6.3. *Let* $(X, o) = \{z_0^{a_0} + z_1^{a_1} + z_2^{a_2} = 0\}$ $(a_0 \leqq a_1 \leqq a_2)$*. The maximal ideal type of* $(X, o)$ *is determined numerically from* $a_0, a_0$ *and* $a_0$ *as follows:*

| | $\alpha \leqq \beta$ | $\alpha > \beta$ |
|---|---|---|
| $a_0a_1 > a_0a_2 - a_2 \cdot \gcd(a_0, a_1)$ | type I | type II |
| $a_0a_1 \leqq a_0a_2 - a_2 \cdot \gcd(a_0, a_1)$ | type III | type IV |

,

*where* $\alpha := \dfrac{\gcd(a_0, a_1)}{\gcd(a_0, a_1, a_2)}$ *and* $\beta := \dfrac{a_2}{\gcd(a_2, \mathrm{lcm}(a_0, a_1))}$.

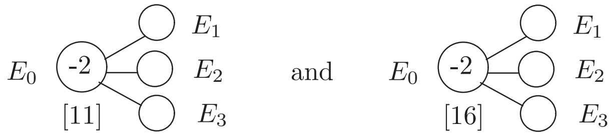
Remark 6.4.If $a_0$ divides $a_1$, then we can easily see that $(X, o)$ is of type I. On the other hand, this fact is also obtained by Theorem 3.6 (i). Because $(X, o)$ is always of type I if $\mathrm{mult}(X, o) = -Z^2$.

Remark 6.5. (i) For complete intersection singularities of Brieskorn type, we can obtain a similar result as Proposition 6.3 from Meng-Okuma [MO].

(ii) Let $(X_\ell, o) = \{z_0^{a_0\ell} + z_1^{a_1\ell} + z_2^{a_2\ell} = 0\}$ $(a_0 \leqq a_1 \leqq a_2$ and $\ell = 1, 2, \cdots)$. Then the maximal ideal type of $(X_\ell, o)$ coincides to one of $(X_1, o)$.

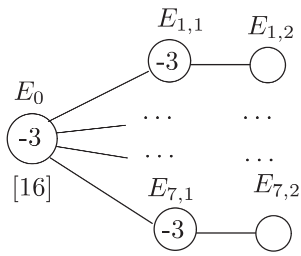
We can easily find many examples of type I and III. Hence we describe some examples of type II and IV in the following.

Example 6.6 (type II). (i) Let $(X_1, o) = \{z_0^6 + z_1^{15} + z_2^{20} = 0\}$ and $(X_2, o) = \{z_0^6 + z_1^{21} + z_2^{28} = 0\}$. Their w.d.graphs associated to the minimal resolutions are respectively given as follows.



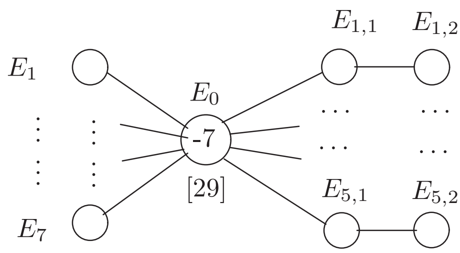

For both cases, we have $M_0 = (z_2 \circ \pi)_E = 3E_0 + 2\sum_{i=1}^{3} E_i$ and $Z_0 = 2E_0 + \sum_{i=1}^{3} E_i$. Then $\mathrm{mult}(X, o) = -M_0^2 = 6 > -Z_0^2 = 2$. Therefore, $(X_1, o)$ and $(X_2, o)$ are of type II.

(ii) Let $(X, o) = \{z_0^{14} + z_1^{21} + z_2^{30} = 0\}$. The w.d.graph associated to the minimal resolution is given as follows.




Then we have $M_0 = (z_2 \circ \pi)_E = 7E_0 + 3\sum_{i=1}^{7} E_{i,1} + 2\sum_{i=1}^{7} E_{i,2}$ and $Z_0 = 5E_0 + 2\sum_{i=1}^{7} E_{i,1} + \sum_{i=1}^{7} E_{i,2}$; hence $\mathrm{mult}(X,o) = -M_0^2 = 14 > -Z_0^2 = 5$. Then, $(X,o)$ is of type II.

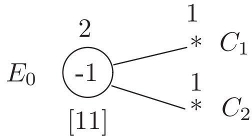
(iii) Let $(X,o) = \{z_0^{20} + z_1^{35} + z_2^{42} = 0\}$. The w.d.graph associated to the exceptional set of the minimal resolution is given as follows.




Then we have $M_0 = (z_2 \circ \pi)_E = 10E_0 + 5\sum_{i=1}^{7} E_i + 7\sum_{i=1}^{5} E_{i,1} + 4\sum_{i=1}^{5} E_{i,2}$ and $Z_0 = 6E_0 + 3\sum_{i=1}^{7} E_i + 4\sum_{i=1}^{5} E_{i,1} + 2\sum_{i=1}^{5} E_{i,2}$. Then $\mathrm{mult}(X,o) = -M_0^2 = 20 > -Z_0^2 = 6$. Therefore, $(X,o)$ is of type II.

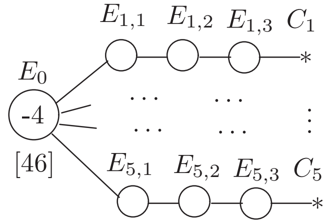
Example 6.7 (type IV). In the following figures, each $C_i$ is a component of the strict transform of a divisor on the singularity, and it is usually represented by "*".

(i) Let $(X,o) = \{z_0^6 + z_1^{10} + z_2^{15} = 0\}$ (i.e., the singularity of Example 5.3). The divisor $(z_2 \circ \pi)_{\tilde{X}}$ on the minimal resolutions is given as follows.



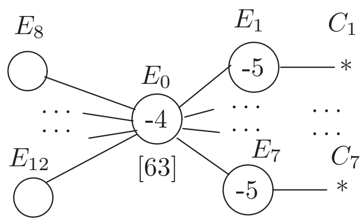

Then $(X,o)$ is of type IV with $\mathrm{mult}(X,o) = 6 > -M_0^2 = 4 > -Z_0^2 = 1$. Since $\mathrm{Coeff}_{E_0}(z_2 \circ \pi)_E = 2$ and $\mathrm{Coeff}_{E_0}(z_1 \circ \pi)_E = 3$, $\mathfrak{m}\mathcal{O}_{\tilde{X}}$ is locally represented by $(u^2v, u^3)$, where $C_i = \{v = 0\}$ in a neighborhood of $E_0 \cap C_i$. Hence, we can easily see that $E_0 \cap C_i$ is an embedded point $(i = 1, 2)$.

(ii) Let $(X,o) = \{z_0^{15} + z_1^{35} + z_2^{84} = 0\}$. The divisor $(z \circ \pi)_{\tilde{X}}$ is given as follows.




Then we have $M_0 = (z_2 \circ \pi)_E = 5E_0 + \sum_{i=1}^{5}\sum_{j=1}^{3}(5-j)E_{i,j}$ and $Z_0 = 4E_0 + \sum_{i=1}^{5}\sum_{j=1}^{3}(4-j)E_{i,j}$. Therefore, $\mathrm{mult}(X,o) = 15 > -M_0^2 = 10 > -Z_0^2 = 4$, and $(X,o)$ is of type IV. Since $\mathrm{Coeff}_{E_0}(z_1 \circ \pi)_E = 12$ and $\mathrm{Coeff}_{E_{i,3}}(z_1 \circ \pi)_E = 3$, we can see that $E_0 \cap C_i$ is an embedded point $(i = 1, \cdots, 5)$ by the same way as (i).

(iii) Let $(X,o) = \{z_0^{35} + z_1^{56} + z_2^{100} = 0\}$. The divisor $(z \circ \pi)_{\tilde{X}}$ is given as follows.




Then we have $M_0 = (z_2 \circ \pi)_E = 14E_0 + 3\sum_{i=1}^{7} E_i + 7\sum_{i=8}^{12} E_i$ and $Z_0 = 10E_0 + 2\sum_{i=1}^{7} E_i + 5\sum_{i=8}^{12} E_i$. Therefore, $\mathrm{mult}(X,o) = 35 > -M_0^2 = 21 > -Z_0^2 = 10$; then $(X,o)$ is of type IV. Since $\mathrm{Coeff}_{E_0}(z_1 \circ \pi)_E = 25$ and $\mathrm{Coeff}_{E_i}(z_1 \circ \pi)_E = 5$ for $i = 1, \cdots, 7$, $E_0 \cap C_i$ is an embedded point $(i = 1, \cdots, 7)$.

**7. Maximal ideal type and sequences of normal surface singularities.** When we consider sequences of singularities, it seems to be interesting to consider the change of the maximal ideal type of each member of the sequence. We show this through some examples.

EXAMPLE 7.1. Let $\{(X_n, o)\}_{n=2,3,\dots}$ be a sequence constructed by hypersurface singularities, where $(X_n, o) = \{z^n = x^6 + y^{10}\}$ $(n = 2, 3, \cdots)$. Then we have the following.

(1) If $n = 2, 3, 4, 6, 8$ or $9$, then $(X_n, o)$ is of type I and $\mathrm{mult}(X_n, o) = -Z^2 = \min\{6, n\}$.
(2) If $n = 5, 7$, then $(X_n, o)$ is of type III; also we have $\mathrm{mult}(X_5, o) = 5 > -M_0^2 = -Z_0^2 = 3$ and $\mathrm{mult}(X_7, o) = 6 > -M_0^2 = -Z_0^2 = 5$.
(3) If $10 \leqq n \leqq 14$, then $(X_n, o)$ is of type I and $\mathrm{mult}(X_n, o) = -Z^2 = 6$.
(4) If $n = 15$, then $(X_{15}, o)$ is of type IV and $\mathrm{mult}(X_{15}, o) = 6 > -M_0^2 = 4 > -Z^2 = 1$ (see Example 5.3 and 6.7 (i)).
(5) If $16 \leqq n \leqq 29$, then $(X_n, o)$ is of type III and $\mathrm{mult}(X_n, o) = 6 > -M_0^2 = -Z^2 = 4$.
(6) If $30 \leqq n$, then $(X_n, o)$ is a Kodaira singularity of type III from Theorem 3.6 (ii) and $\mathrm{mult}(X_n, o) = 6 > -M_0^2 = -Z^2 = 2$.

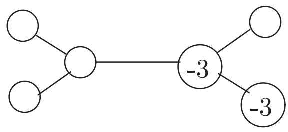
EXAMPLE 7.2. Let $\{(X_n, o)\}_{n=2,3,\dots}$ be a sequence constructed by hypersurface singularities, where $(X_n, o) = \{z^n = (y^2 + x^3)(x^4 + y^6)\}$ $(n = 2, 3, \cdots)$. Then we have the following.

(1) If $2 \leqq n \leqq 13$, then $(X_n, o)$ is of type I and $\mathrm{mult}(X_n, o) = -Z^2 = \min\{6, n\}$. Also, if $n = 2, 3$ or $6$, then $(X_n, o)$ is a Kodaira singularity of type III from Theorem 3.6 (i).

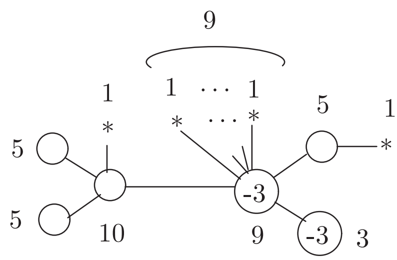
(2) If $14 \leqq n$, then $(X_n, o)$ is of type III and $\mathrm{mult}(X_n, o) = 6$. Also, if $14 \leqq n \leqq 15$ (resp. $16 \leqq n$), then $-M_0^2 = -Z_0^2 = 5$ (resp. $-M_0^2 = -Z_0^2 = 3$). Furthermore, if $16 \leqq n$, then $(X_n, o)$ is a Kodaira singularity of type III.

We remark that the sequence $\{(X_n, o)\}_{n=2,3,\cdots}$ does not contain any singularity with $M_0 > Z_0$.

Example 7.3. Let $(X, o)$ be a rational surface singularity with $\mathrm{mult}(X, o) = -Z^2 = 5$. Let $(\tilde{X}, E)$ be the minimal resolution whose w.d.graph is given as follows.




Let us consider a reduced element $h$ of $\mathfrak{m}$ whose divisor is given as follows.




Let $(X_n, o)$ be a surface singularity defined by $z^n = h$ over $(X, o)$. From Tomari-Watanabe's result [TW], $(X_n, o)$ is a normal singularity. For a sequence $\{(X_n, o)\}_{n=2,3,\cdots}$, we have the following:

(1) If $2 \leqq n \leqq 8$, then $(X_n, o)$ is of type I. Moreover, $\mathrm{mult}(X_2, o) = -Z^2 = 10$, $\mathrm{mult}(X_3, o) = -Z^2 = 15$, $\mathrm{mult}(X_4, o) = -Z^2 = 19$ and $\mathrm{mult}(X_n, o) = -Z^2 = 21$ for $n = 5, 6, 7$ and $8$.

(2) If $9 \leqq n$, then $(X_n, o)$ is of type III and $\mathrm{mult}(X_n, o) = 21$. For $(X_9, o)$, we have $-M_0^2 = -Z^2 = 12$. Also, if $10 \leqq n$, then $(X_n, o)$ is a Kodaira singularity of type III from Theorem 4.14 in [To5] and $-M_0^2 = -Z^2 = 11$.

Therefore, the sequence $\{(X_n, o)\}_{n=2,3,\cdots}$ does not contain any singularity with $M_0 > Z_0$.

In [To7], the author studied the multiplicity of cyclic coverings over normal surface singularities.

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