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AN INTRINSIC APPROACH TO STABLE EMBEDDING OF NORMAL SURFACE DEFORMATIONS*

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Abstract. We introduce the notion of *involutive* Kodaira-Spencer deformations of the regular part X_0 of a normal surface singularity, which form a subspace of the analytic cohomology $H^1(X_0, T^{1,0}X_0)$. Examples of involutive deformations for which the Stein completion does not embed in a complex Euclidean space of stable dimension are in fact well-known. Under the assumption that X_0 admits a Kähler metric with L^2-curvature, we show that unstable deformations are avoided if the holomorphic functions which determine an embedding of the central fibre are correspondingly deformed into functions which can be uniformly bounded on compact subsets.

Key words. Singularities, Kodaira-Spencer deformations, holomorphic embedding.

Mathematics Subject Classification. 32G05, 32S30.

1. Introduction. Let $X \subset \mathbf{C}^N$ be a reduced complex analytic space with normal isolated singularity at a point x_0 corresponding to the origin, and let X_0 denote the regular part, i.e., $X \setminus \{x_0\}$. If $[\dot{\psi}] \in H^1(X_0, TX_0)$ is the cohomology class tangent to an integrable Kodaira-Spencer deformation $\psi(t)$ (i.e., $\frac{d}{dt}\,|_{t=0}\ \psi = \dot{\psi}$) of complex structure on the underlying (strictly pseudoconvex) locus of X_0, let the associated complex manifold be denoted $X_0^{\psi(t)}$ for $t \in \Delta \subseteq \mathbf{C}$. When $\dim_{\mathbf{C}}(X) = 2$ the existence of a Stein completion of X_0^{ψ} is guaranteed if and only if the CR-structure induced on the link $X \cap \mathbf{S}^{2N-1}_{\varepsilon}$ is itself an "embeddible" structure in $\mathbf{C}^{N'}$ (cf. Yau [14], cf. also [16] and [15] for more recent advances on the problem of interior regularity). Existence of Stein embeddings of strictly pseudoconvex manifolds was explored in a seminal article of Andreotti-Siu [1], and a more recent refinement of their methods in the case of surfaces was obtained by Marinescu-Dinh [10]. Given separate embeddings of X_0 and X_0^{ψ}, however, it does not follow automatically that the ambient dimensions N and N' are the same, though ψ belongs to a continuously parametrized family of deformations on X_0. For a stably embeddible family of surface-deformations $X(t)$, i.e., one for which the ambient dimension is constant, it is well-known that an essential requirement is the t-independence of the geometric genus $p_g = \dim_{\mathbf{C}} H^1(X'(t), \mathcal{O}_{X'(t)})$, where $X'(t)$ denotes the fibre-wise resolution of Stein surfaces $X(t)$ within a given t-parameter family. If the geometric genus is an invariant of the family this ensures that holomorphic functions on the central fibre extend to neighbouring fibres, which entails both stability of embeddings of the $X(t)$ and "simultaneous blowing-down" of the associated family of their resolutions. Henry Laufer's theory of deformation of resolutions of isolated surface singularities [8] is developed in the context of simultaneous blowing-down.

A famous example outside this context, based on a case of unstable deformation of a CR-structure, is due to Rossi [13]. Another, exposed by Catlin and Lempert [3], is the following. Consider a simple holomorphic deformation of a one-point divisor, $\psi(t) = [kp(t)] \in Pic_k(C)$, where $p(0)$ is an (isolated) Weierstrass-point on the "Fermat curve"

$$C \;:\; Z_0^k = Z_1^k + Z_2^k \subset \mathbf{CP}^2 \quad \text{for } k \geq 5\ .$$

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The line-bundle associated with $\psi(t)$ is very ample for $t = 0$ only, hence the family of surfaces $X'(t)$ determined by the total spaces of the dual bundles $[-kp(t)]$ over C does not permit simultaneous collapse of the exceptional divisor within each fibre, and the resulting family of normal surface singularities is not stably embedded. In this respect, at least for $\dim_{\mathbf{C}}(X) = 2$, there are phenomena within the intrinsic (cf. "analytic") theory of deformation of normal isolated singularities which are not wholly accounted for by the versal family of flat algebraic deformations of X. In fact, Miyajima has shown [11] that there is a one-to-one correspondence between members of the versal space of algebraic deformations and those CR-deformations induced on the link of an isolated singularity which are *stably embeddible.*

From the intrinsic viewpoint we may then ask under what conditions $[\dot{\psi}]$ induces a stably embedded family of complex deformations of X_0. As explained in section 2, a $\bar{\partial}$-closed form $\psi \in C^\infty(M, T_M^{1,0} \otimes (T_M^{0,1})^*)$ on any complex manifold will be said to represent an *involutive* deformation of the complex structure if the Frölicher-Nijenhuis bracket $[\psi, \psi]$ vanishes identically on M. Deformations of this kind were studied in the context of cone singularities (i.e., $M = X_0$) in [5]. Note that they automatically satisfy the Kodaira-Spencer integrability equation and sit directly inside the cohomology space of infinitesimal deformations $H^1(M, T^{1,0}M)$, hence $t\psi$ defines a one-parameter family of integrable deformations. It was shown in [5] that the space of involutive deformations is in fact an infinite-dimensional subspace of $H^1(X_0, TX_0)$ when X_0 is the regular part of a two-dimensional cone. More generally, we have

Theorem (cf. section 2, Theorem 1). *Let M be a smooth complex surface, and $Z \in H^0(M, T_M)$ a holomorphic vector field. Let*

$$L_{\bar{Z}} : C^\infty(M, T_M^{1,0} \otimes (T_M^{0,1})^*) \to C^\infty(M, T_M^{1,0} \otimes (T_M^{0,1})^*)$$

represent the natural first-order operator defined by Lie differentiation along the anti-holomorphic vector field $\bar{Z}$. Then a sufficient condition for any class $[\psi] \in H^1(M, T_M)$ to be an involutive deformation is that it belong to the quotient

$$\Gamma^1_{\bar{Z}} := \frac{im(L_{\bar{Z}}) \cap ker(\bar{\partial})}{im(\bar{\partial})} \ .$$

In this case we note that for any $f \in C^\infty(M, \mathbf{C})$ we have the additional properties

$$\bar{\partial}(\psi(f)) = -\psi(\bar{\partial} f) \ \text{ and}$$

$$\psi(\psi(f)) \equiv 0.$$

The ancillary properties above, which refer to the natural action of ψ as a derivation on functions, will be of particular importance to the results of section 3, hence for the purposes of this article an "involutive deformation" will be assumed, with minimal loss of generality, to satisfy the condition of the theorem above, i.e., there exist Z and φ such that $\psi = L_{\bar{Z}}\varphi$. Even the unstable examples due to Rossi, or Catlin and Lempert above, fall into this subclass of involutive deformations of the regular part of a normal surface singularity.

The analytic approach to the problem of extension of holomorphic functions from X to $X^{t\psi}$ for $t \neq 0$, and hence to the question of stable embedding of the t-parameter family of deformations, like so many problems in complex analysis, is then a matter of solvability of the Cauchy-Riemann equation on X_0. Let $\rho : X \to [0,\infty)$ be a strongly plurisubharmonic exhaustion, such that $\rho(x) = 0$ if and only if $x = x_0$. A Kähler metric g on X_0 will correspond to the real positive form $\omega = \mathbf{i}\bar{\partial}\partial\rho$ and will be assumed to satisfy the following conditions on

$$X_{0,c} = \{x \in X \mid 0 < \rho(x) < c < \infty\} \ :$$

$$(i) \int_{X_{0,c}} |R_g|^2 < \infty \quad ,$$

where R_g denotes the canonical curvature form associated with g. It will further be assumed that the Sobolev inequality holds with respect to this metric, i.e.,

$$(ii) \left(\int_{X_{0,c}} |f|^4\omega^2\right)^{\frac{1}{2}} \leq C\int_{X_{0,c}} |\nabla f|^2\omega^2$$

for smooth compactly supported functions f. In addition, (iii) let $\delta(x_0, x)$ denote the Riemannian metric distance function on $X_{0,c}$, and let $B_\delta(x_0, r)$ be the associated ball of radius r. For some sufficiently small $0 < c' < c$ it will be assumed that there exists a constant $\Omega > 0$ such that

$$\int_{B_\delta(x_0,r)} \omega^2 \leq \Omega r^4 \quad \text{, for all} \quad 0 < r \leq c'.$$

As noted in [6], a strongly plurisubharmonic exhaustion ρ corresponding to the Euclidean norm-squared, or its restriction to an embedded Stein surface always satisfies conditions (ii) and (iii). We note also that condition (i) is then equivalent to having the second fundamental form of the embedding belong to $C^\infty \cap L^4(X_0)$. Under the assumptions $(i) - (iii)$ we recall,

Theorem (cf. [6]). *For any* $\eta \in L^2 \cap C^\infty(X_{0,c}, E \otimes (T^{0,1}_{X_0})^*)$ *such that* $\bar{\partial}\eta = 0$, *there exists* $u \in L^2 \cap C^\infty(X_{0,c}, E)$ *such that* $\bar{\partial}u = \eta$, *and* $\|u\|_{L^2} \leq \|\eta\|_{L^2}$.

While the holomorphic vector bundle $E \to X_0$ in the original statement above is governed by quite general assumptions, for the purposes of this article E can simply be taken to be either $T^{1,0}_{X_0}$ or the trivial line bundle corresponding to the structure sheaf $\mathcal{O}_{X_0}$. The proof is derived from the theory of existence, uniqueness and regularity of solutions to the $\bar{\partial}$-Neumann problem for the Laplace-Beltrami equation on complete strongly pseudoconvex domains [7], and its adaptation by Bando [2] to punctured domains. Specifically, unique solution of the equation $\Box\mu_\varepsilon = \eta$ on

$$X_{\varepsilon,c} = \{x \in X \mid 0 < \varepsilon < \rho(x) < c\} \ ,$$

such that the solution μ_ε satisfies the $\bar{\partial}$-Neumann condition on $\{\rho(x) = c\}$ and the Dirichlet condition (i.e., μ_ε vanishes) on $\{\rho(x) = \varepsilon\}$, is a consequence of [6], Lemma 1. As shown initially by Bando, the uniform bound $\|\mu_\varepsilon\|_{L^2} \leq \|\eta\|_{L^2}$ entails a uniform limit μ_0 on $X_{0,c}$, such that $\bar{\partial}\eta = 0$ implies $\bar{\partial}^*\bar{\partial}\mu_0 = 0$, and hence $u = \bar{\partial}^*\mu_0$. If f is now a smooth function on X_0 (or any domain containing $X_{\varepsilon,c}$), the natural action of ψ as a complex derivation allows us to choose $\eta = \psi(f)$, and express the solution $\mu_\varepsilon = G(\psi(f))$, by way of the Green operator. The operator $T^{t\psi}_\varepsilon(f) := t\bar{\partial}^*G\psi(f)$ may then be applied iteratively to smooth functions defined on any domain containing

$X_{\varepsilon,c}$. Together with an inductive application of Bando's method for the $\bar{\partial}$-Neumann-Dirichlet problem on $X_{\varepsilon,c}$, this operator is the key to extension of holomorphic functions f from $X_{0,c}$ to fibres $X^{t\psi}_{0,c}$ when ψ is involutive. We remark that the Theorem cited from [6] above implies moreover that $|\psi|$ cannot itself belong to $L^2(X_{0,c})$ if it is to represent a non-trivial deformation. An iterative approach to holomorphic extension therefore requires a more explicit type of uniform L^2-bound, as follows. Let h be an arbitrary holomorphic function on $X \times \Delta$, with a uniformly convergent power series expansion $\Sigma_{k=0}^{\infty}(-1)^k h_k \cdot t^k$ in the complex parameter $t \in \Delta$, where the functions h_k are all holomorphic on X, and $h_0 = f$. We will make the formal identification

$$\frac{1}{1+T^{t\psi}_{\varepsilon}} := \Sigma_{k=0}^{\infty}(-1)^k (T^{t\psi}_{\varepsilon})^k ,$$

and hence define the action of the *Neumann-Dirichlet deformation operator*

$$\frac{1}{1+T^{t\psi}_{\varepsilon}}(h) = \Sigma_{n=0}^{\infty}(-1)^n f_{\varepsilon,n} \cdot t^n \quad (*)$$

such that for all $n \geq 0$

$$f_{\varepsilon,n} = \Sigma_{k=0}^{n}(T^{\psi}_{\varepsilon})^k (h_{n-k}) .$$

Theorem (cf. section 3, Theorem 2). *Suppose there exists a positive function $c(\rho)$ such that*

$$\sup_{X_{\rho,c}} |df_{\varepsilon,n}| \leq c(\rho) ,$$

independently of $0 < \varepsilon < \rho < c$ and $n \geq 0$.

In addition, for all $0 < \varepsilon < \rho < c$, and all $n \geq 0$, let there be a strictly positive function $C(\rho) \in L^2(0,c)$ satisfying

$$\sup_{X_{\rho,c}} |\psi(f_{\varepsilon,n})| \leq C(\rho) .$$

Then the power series corresponding to (*) admits a uniform limit, as ε approaches zero, which is also uniformly convergent on compact subsets of $X_{0,c}$ for $|t| < 1$. The corresponding function on $X^{t\psi}_{0,c}$ moreover satisfies the deformed Cauchy-Riemann equation, i.e.,

$$\bar{\partial}^{t\psi}\left(\frac{1}{1+T^{t\psi}_{0}}(h)\right) = 0 .$$

We consequently obtain a holomorphic extension of the function f to the family of spaces determined by the involutive deformation $t\psi$. As we will see in section 5, the criteria of Arzela's Theorem for uniform convergence on compact subsets can be met without explicitly assuming the uniform estimate for $df_{\varepsilon,n}$ above when the Kähler metric form $\omega = \mathbf{i}\bar{\partial}\partial\rho$ is assumed to be flat. On the other hand, when F denotes the ensemble of holomorphic functions which determine an embedding of $X_{0,c}$ in $\mathbf{C}^N$, we show in section 4 that the same hypotheses provide a sufficient condition for stable embedding and Stein completion of $X^{t\psi}_{0,c}$. Here $H = \Sigma_{k=0}^{\infty}(-1)^k H_k \cdot t^k$

denotes the corresponding uniformly convergent power series of a holomorphic map $H : X \times \Delta \to \mathbf{C}^N$, for which $H_0 = F$. As above, we will also write

$$\frac{1}{1+T_\varepsilon^{t\psi}}(H) = \Sigma_{n=0}^{\infty}(-1)^n F_{\varepsilon,n} \cdot t^n \ .$$

Theorem (cf. section 4, Theorem 3). *Consider a normal Stein surface* $F : X \hookrightarrow \mathbf{C}^N$, *with a strongly plurisubharmonic function* ρ *and associated Kähler form* $\omega = \mathbf{i}\bar{\partial}\partial\rho$, *defined in a neighbourhood of the singularity at* x_0 *and having curvature in* $L^2(X_{0,c})$ *etc. Let* ψ *be an involutive deformation of the regular neighbourhood* $X_{0,c}$, *and* $H : X \times \Delta \to \mathbf{C}^N$ *a holomorphic map with* $H_0 = F$, *such that there exists a positive function* $c(\rho)$ *satisfying*

$$\sup_{X_{\rho,c}} |dF_{\varepsilon,n}| \leq c(\rho) \ ,$$

independently of $0 < \varepsilon < \rho < c$ and $n \geq 0$. *In addition, suppose there is a positive function* $C(\rho) \in L^2(0,c)$ *such that*

$$\sup_{X_{\rho,c}} |\psi(F_{\varepsilon,n})| \leq C(\rho) \ ,$$

independently of both n *and* $0 < \varepsilon < \rho < c$. *Then each fibre in the* t*-parameter family* $\mathcal{X}^\psi$ *of deformations is stably embedded by the map* $\frac{1}{1+T_0^{t\psi}}(H)$ *and admits a unique Stein completion in a neighbourhood of the origin in* $\mathbf{C}^N$ *for* $|t|$ *sufficiently small.*

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This article is dedicated to my doctoral supervisor and teacher, Henry Laufer, in honour of his seventieth birthday.

2. Involutive deformations of surfaces. At first, let M be an arbitrary complex manifold, with $\varphi \in C^\infty(M, T_M^{1,0} \otimes \bigwedge^q (T_M^{0,1})^*)$ written locally in the form

$$\varphi = \Sigma_{(\lambda)}\Sigma_{1\leq\alpha\leq n+1}\varphi_{(\lambda)}^{\alpha}\partial_\alpha \otimes d\bar{w}_{(\lambda)} \ .$$

Here $T_M^{1,0}$, as usual, denotes the holomorphic tangent bundle, with local basis $\{\partial_\alpha\}_{\alpha=1}^n$, (λ) is a multi-index $(\lambda_{i_1}, ..., \lambda_{i_q})$ and $\varphi_{(\lambda)}^\alpha$ is smooth in any local chart. The following calculation, reproduced for the reader's convenience from [5], is carried out for $q = 1$, though the general case is essentially well-known. A vector field $\xi^{0,1}$ will be referred to as "anti-holomorphic" if the Lie bracket $[\xi^{0,1}, \vartheta] = 0$ for any holomorphic vector field ϑ, and will be written locally in the form

$$\xi^{0,1} = \Sigma_{\nu=1}^n \xi^\nu \bar{\partial}_\nu \ .$$

Now consider the contraction

$$\iota_{\xi^{0,1}}\varphi^\alpha = \Sigma_{\lambda=1}^n \varphi_\lambda^\alpha \xi^\lambda \qquad \text{such that}$$

$$\left(\bar{\partial}\iota_{\xi^{0,1}}\varphi^\alpha\right)_\nu = \Sigma_\lambda \left(\frac{\partial\varphi_\lambda^\alpha}{\partial\bar{w}_\nu}\xi^\lambda + \varphi_\lambda^\alpha \frac{\partial\xi^\lambda}{\partial\bar{w}_\nu}\right) \ .$$

On the other hand,

$$(\bar{\partial}\varphi)_{\lambda,\nu} = \Sigma_{\lambda<\nu}\left(\frac{\partial\varphi_\nu^\alpha}{\partial\bar{w}_\lambda} - \frac{\partial\varphi_\lambda^\alpha}{\partial\bar{w}_\nu}\right)$$

so that

$$\begin{aligned}\left(\iota_{\xi^{0,1}}\bar{\partial}\varphi\right)_\nu &= \Sigma_{\lambda<\nu}\ \xi^\lambda\left(\frac{\partial\varphi_\nu^\alpha}{\partial\bar{w}_\lambda} - \frac{\partial\varphi_\lambda^\alpha}{\partial\bar{w}_\nu}\right) - \Sigma_{\lambda>\nu}\ \xi^\lambda\left(\frac{\partial\varphi_\lambda^\alpha}{\partial\bar{w}_\nu} - \frac{\partial\varphi_\nu^\alpha}{\partial\bar{w}_\lambda}\right)\\ &= \Sigma_{\lambda\neq\nu}\ \xi^\lambda\frac{\partial\varphi_\nu^\alpha}{\partial\bar{w}_\lambda} - \Sigma_{\lambda\neq\nu}\ \xi^\lambda\frac{\partial\varphi_\lambda^\alpha}{\partial\bar{w}_\nu}\ ,\end{aligned}$$

and therefore

$$\begin{aligned}\left(\bar{\partial}\iota_{\xi^{0,1}}\varphi^\alpha\right)_\nu + \left(\iota_{\xi^{0,1}}\bar{\partial}\varphi^\alpha\right)_\nu &= \frac{\partial\varphi_\nu^\alpha}{\partial\bar{w}_\nu}\xi^\nu + \Sigma_\lambda\ \varphi_\lambda^\alpha\frac{\partial\xi^\lambda}{\partial\bar{w}_\nu} + \Sigma_{\lambda\neq\nu}\ \xi^\nu\frac{\partial\varphi_\nu^\alpha}{\partial\bar{w}_\lambda}\\ &= \xi^{0,1}(\varphi_\nu^\alpha) + \Sigma_\lambda\ \varphi_\lambda^\alpha\frac{\partial\xi^\lambda}{\partial\bar{w}_\nu}\ .\end{aligned}$$

Writing $\varphi_\nu = \Sigma_\alpha\varphi_\nu^\alpha\partial_\alpha$, we note that the assumption $\xi^{0,1}$ is anti-holomorphic implies specifically that

$$[\xi^{0,1}, \varphi_\nu]^\alpha = \xi^{0,1}(\varphi_\nu^\alpha)\ .$$

In addition,

$$[\xi^{0,1}, \bar{\partial}_\nu] = -\Sigma_\lambda\ \frac{\partial\xi^\lambda}{\partial\bar{w}_\nu}\bar{\partial}_\lambda\ ,$$

so that

$$\Sigma_\lambda\ \varphi_\lambda^\alpha\frac{\partial\xi^\lambda}{\partial\bar{w}_\nu} = -\varphi^\alpha([\xi^{0,1}, \bar{\partial}_\nu])$$

implies

$$\left(\bar{\partial}\iota_{\xi^{0,1}}\varphi^\alpha\right)_\nu + \left(\iota_{\xi^{0,1}}\bar{\partial}\varphi^\alpha\right)_\nu = \xi^{0,1}(\varphi_\nu^\alpha) - \varphi^\alpha([\xi^{0,1}, \bar{\partial}_\nu]) = \left(L_{\xi^{0,1}}\varphi^\alpha\right)_\nu\ ,$$

which recalls the well-known formula for the Lie derivative due to E. Cartan.

In particular, let $Z \in H^0(M, T_M^{1,0})$ be a holomorphic vector field on M, and consider

$$\psi \in C^\infty(M, T_M^{1,0} \otimes (T_M^{0,1})^*)$$

such that $\bar{\partial}\psi = 0$ and $\psi = L_{\bar{Z}}\varphi$, for some smooth $\varphi \in C^\infty(M, T_M^{1,0} \otimes (T_M^{0,1})^*)$. It follows from the formula above that ψ is cohomologous to $\iota_{\bar{Z}}\bar{\partial}\varphi$.

Writing ψ in local form as $\Sigma_{\alpha,\lambda}\psi_\lambda^\alpha\partial_\alpha \otimes d\bar{w}_\lambda$ we recall the specific formula for the Frölicher-Nijenhuis bracket

$$[\psi, \psi]_{FN} = \Sigma_{\alpha,\beta=1}^n 2\left(\psi^\alpha \wedge \partial_\alpha\psi^\beta\right)\partial_\beta = \Sigma_{1\leq\lambda<\nu\leq n}[\psi_\lambda, \psi_\nu]d\bar{w}_\lambda \wedge d\bar{w}_\nu\ ,$$

where $[\psi_\lambda, \psi_\nu]$ denotes the standard Lie bracket of vector fields. Let $\eta \in C^\infty(M, T_M^{1,0} \otimes \bigwedge^2(T_M^{0,1})^*)$ be represented locally as $\eta_{\alpha,\beta}^\gamma\partial_\gamma \otimes d\bar{w}_\alpha \wedge d\bar{w}_\beta$. Now

$$(\iota_{\bar{Z}}\eta)_\lambda^\gamma = \Sigma_{\alpha<\lambda}\bar{Z}^\alpha\eta_{\alpha,\lambda}^\gamma - \Sigma_{\beta>\lambda}\bar{Z}^\beta\eta_{\lambda,\beta}^\gamma\ ,$$

and in particular, $n = 2$ implies

$$(\iota_{\bar{Z}}\eta)_1^\gamma = -\bar{Z}^2\eta_{12}^\gamma\ , \quad (\iota_{\bar{Z}}\eta)_2^\gamma = \bar{Z}^1\eta_{12}^\gamma\ ,$$

so that

$$[\iota_{\bar{Z}}\eta, \iota_{\bar{Z}}\eta]_{FN} = (-\bar{Z}^2\eta_{12}(\bar{Z}^1) + \bar{Z}^1\eta_{12}(\bar{Z}^2))\eta_{12} = 0\ ,$$

given $\bar{Z}$ is anti-holomorphic.

We summarize with the following

THEOREM 1. *Let M be a smooth complex surface, and $Z \in H^0(M, T_M)$ a holomorphic vector field. Let*

$$L_{\bar{Z}} : C^\infty(M, T_M^{1,0} \otimes (T_M^{0,1})^*) \to C^\infty(M, T_M^{1,0} \otimes (T_M^{0,1})^*)$$

represent the natural first-order operator defined by Lie differentiation along the anti-holomorphic vector field $\bar{Z}$. Then a sufficient condition for any class $[\psi] \in H^1(M, T_M)$ to be an involutive deformation is that it belong to the quotient

$$\Gamma_{\bar{Z}}^1 := \frac{im(L_{\bar{Z}}) \cap ker(\bar{\partial})}{im(\bar{\partial})}\ .$$

In this case we note that for any $f \in C^\infty(M, \mathbf{C})$ we have the additional properties

$$\bar{\partial}(\psi(f)) = -\psi(\bar{\partial}f) \text{ and}$$

$$\psi(\psi(f)) \equiv 0\ .$$

Proof. If $\psi = L_{\bar{Z}}\varphi$ then $[\psi] \in H^1(M, T_M)$ is also represented by $\iota_{\bar{Z}}\bar{\partial}\varphi$, which is clearly involutive when $n = 2$. The natural action of ψ as a derivation on functions and forms of higher degree yields the first of the properties above as an immediate consequence of the Leibniz rule. For the second, we may write

$$\psi(\psi(f)) = \Sigma_{\alpha,\beta,\lambda,\nu}\left(\psi_\lambda^\alpha\frac{\partial\psi_\nu^\beta}{\partial z_\alpha} - \psi_\nu^\alpha\frac{\partial\psi_\lambda^\beta}{\partial z_\alpha}\right)\frac{\partial f}{\partial z_\beta}\ d\bar{z}_\lambda \wedge d\bar{z}_\nu$$
$$+\Sigma_{\alpha,\beta,\lambda,\nu}\left(\psi_\lambda^\alpha\psi_\nu^\beta - \psi_\nu^\alpha\psi_\lambda^\beta\right)\frac{\partial^2 f}{\partial z_\alpha\partial z_\beta}\ d\bar{z}_\lambda \wedge d\bar{z}_\nu\ .$$

The first of these summations is simply $[\psi, \psi]_{FN}(f)$, which vanishes by our basic assumption. The second also vanishes, more specifically as a consequence of the assumption $\psi = \iota_{\bar{Z}}\bar{\partial}\varphi$. This concludes the proof.

3. Extension of Holomorphic Functions. Let X denote a Stein surface with isolated singularity at x_0, and $\rho : X \to [0, \infty)$ a strongly plurisubharmonic function, satisfying the conditions $(i)-(iii)$ outlined in section 1, such that $\rho(x_0) = 0$. Consider now the surfaces $X_{\varepsilon,c} = \{p \in X \mid 0 < \varepsilon < \rho(p) < c\}$. It is a consequence of [6], Theorem 1 that for any $\eta \in C^\infty \cap L^2(X_{0,c}, (T_{X_0}^{1,0})^{\otimes n} \otimes (T_{X_0}^{0,1})^*)$ $(n \geq 0)$, there exists a unique $(0,1)$-form μ_ε, satisfying the $\bar{\partial}$-Neumann-Dirichlet boundary conditions on $X_{\varepsilon,c}$, such that

$$\Box\mu_\varepsilon = \eta \quad \text{ and } \quad \|\mu_\varepsilon\|_{L^2(X_{\varepsilon,c})} \leq \|\eta\|_{L^2(X_{0,c})}\ .$$

This estimate, together with boundary regularity of solutions derived from the basic estimate of [6], Lemma 1, implies the existence of a uniform limit $\mu_0 \in C^\infty \cap L^2_{\mathbf{C}}(X_{0,c})$. Let G denote the associated Green-operator of the Laplace-Beltrami equation above, and for any smooth, complex-valued function f on $X_{0,c}$, we may then define, for any $0 < \varepsilon < c$, a complex t-parameter family of $\mathbf{C}$-linear operators $T^{t\psi}_\varepsilon$ such that $T^{t\psi}_\varepsilon(f) = t\bar{\partial}^* G\psi(f)$, relative to an involutive deformation ψ. The sequence of successive iterations of this operation on functions is denoted by $(T^{t\psi}_\varepsilon)^k$, in terms of which we define the formal expansion of the *Neumann-Dirichlet operator* $\frac{1}{1+T^{t\psi}_\varepsilon}$, which acts on a power series of the form $h = \Sigma_{k=0}^\infty (-1)^k h_k \cdot t^k$, where h_k is holomorphic for all k, to produce the "deformed series"

$$\frac{1}{1+T^{t\psi}_\varepsilon}(h) = \Sigma_{n=0}^\infty (-1)^n f_{\varepsilon,n} \cdot t^n$$

such that

$$f_{\varepsilon,n} = \Sigma_{k=0}^n (T^{\psi}_\varepsilon)^k (h_{n-k}) \ .$$

Two key hypotheses assumed in all of the following will be, for a given holomorphic h, with $h_0 = f \in \mathcal{O}_X \cap L^2_{\mathbf{C}}(X_{0,c})$, the existence of a strictly positive function $c(\rho)$, such that

$$\sup_{\overline{X_{\rho,c}}} |df_{\varepsilon,n}| \leq c(\rho) \qquad (\dagger)$$

independently of $0 < \varepsilon < \rho < c$ and $n \geq 0$, and a strictly positive continuous function $C(\rho) \in L^2(0,c)$, such that

$$\sup_{\overline{X_{\rho,c}}} |\psi(f_{\varepsilon,n})| \leq C(\rho) \qquad (1)\ ,$$

uniformly in both ε and n.

Our first task is to show that (1) together with ($\dagger$) implies a corresponding inequality for $\varepsilon = 0$, i.e., for all $0 < \rho < c$

$$\sup_{\overline{X_{\rho,c}}} |\psi(f_{0,n})| \leq C(\rho) \qquad (2)$$

uniformly in $n \geq 0$. Note first that for each fixed n equicontinuity of the family $\{f_{\varepsilon,n}\}_{0<\varepsilon<\rho}$ on $\overline{X_{\rho,c}}$ follows immediately, for if γ is a distance-minimizing arc joining two points $p, q \in \overline{X_{\rho,c}}$, then for any $\epsilon > 0$, the arc-length

$$\delta(p,q) := l_\gamma(p,q) < \delta := \frac{\epsilon}{c(\rho)}$$

implies, for any fixed $n \geq 0$, and any $\varepsilon < \rho$, that

$$|f_{\varepsilon,n}(p) - f_{\varepsilon,n}(q)| = |\int_0^1 df_{\varepsilon,n}(\gamma'(s))\ ds\ | \ \leq \delta(p,q)c(\rho) < \epsilon \quad (3).$$

Hence the family of functions $f_{\varepsilon,n}$, $0 < \varepsilon < \rho$, is equicontinuous at each $p \in \overline{X_{\rho,c}}$. Moreover

$$\|f_{\varepsilon,n}\|^2_{L^2(X_{\varepsilon,c})} = \|T^\psi_\varepsilon(f_{\varepsilon,n-1}) + h_n\|^2_{L^2(X_{\varepsilon,c})}$$

$$= \|\bar{\partial}^* G\psi(f_{\varepsilon,n-1})\|^2_{L^2(X_{\varepsilon,c})} + 2\mathfrak{Re}(h_n, \bar{\partial}^* G\psi(f_{\varepsilon,n-1})) + \|h_n\|^2_{L^2(X_{\varepsilon,c})} \ ,$$

and since h_n is holomorphic, the vanishing of the middle term implies

$$\begin{aligned} &\|f_{\varepsilon,n}\|^2_{L^2(X_{\varepsilon,c})} \\ &= \int_{X_{\varepsilon,c}} \langle \bar{\partial}\bar{\partial}^* G\psi(f_{\varepsilon,n-1}), G\psi(f_{\varepsilon,n-1})\rangle \ \det(g) \ + \ \|h_n\|^2 \\ &= \int_{X_{\varepsilon,c}} \langle \psi(f_{\varepsilon,n-1}), G\psi(f_{\varepsilon,n-1})\rangle \ \det(g) - \int_{X_{\varepsilon,c}} |\bar{\partial} G\psi(f_{\varepsilon,n-1})|^2 \ \det(g) \ + \ \|h_n\|^2 \\ &\le \|\psi(f_{\varepsilon,n-1})\| \|G\psi(f_{\varepsilon,n-1})\| + \|h_n\|^2 \le \|\psi(f_{\varepsilon,n-1})\|^2_{L^2(X_{\varepsilon,c})} + \|h_n\|^2_{L^2(X_{\varepsilon,c})} \ . \end{aligned}$$

Now,

$$\|\psi(f_{\varepsilon,n-1})\|_{L^2(X_{\varepsilon,c})} \le K \sup_{X_{\varepsilon,c}} |\psi(f_{\varepsilon,n-1})| \le K \cdot C(\varepsilon) \ ,$$

where K can be chosen as a uniform constant, independently of n and ε. Hence

$$\|f_{\varepsilon,n}\|_{L^2(X_{\rho,c})} \le \left(\frac{K^2}{\rho}\int_0^\rho C(s)^2 \ ds + \|h_n\|^2_{L^2(X_{\varepsilon,c})}\right)^{\frac{1}{2}} < \infty \ .$$

Since h_n is holomorphic on X, and moreover uniformly bounded in $X_{0,c}$ independently of n (given that h is holomorphic on $X \times \Delta$), we conclude

$$\|f_{\varepsilon,n}\|_{L^2(X_{\rho,c})} \le \left(\frac{K^2}{\rho}\int_0^\rho C(s)^2 \ ds + B\right)^{\frac{1}{2}} < \infty \quad (4),$$

and hence $\|f_{\varepsilon,n}\|_{L^2(X_{\rho,c})}$ is uniformly bounded for all $0 < \varepsilon < \rho < c$.

Proposition 1. *For each $n \ge 0$ the family $\{|f_{\varepsilon,n}|\}_{\varepsilon<\rho}$ is uniformly bounded.*

Proof. Suppose not, i.e., for all $M > 0$ there exists $\varepsilon < \rho$ such that

$$\sup_{X_{\rho,c}} |f_{\varepsilon,n}| > M \ .$$

Let

$$\bar{\delta} := \sup_{X_{\rho,c}\times X_{\rho,c}} \delta(p,q) \ , \ \text{and } L := \bar{\delta} c(\rho) \ .$$

A slight manipulation of inequality (3) above implies

$$\inf_{X_{\rho,c}} |f_{\varepsilon,n}| > M - L \ \text{ and hence}$$

$$\int_{X_{\rho,c}} |f_{\varepsilon,n}|^2 \det(g) > K(M-L)^2 \ ,$$

where K denotes the same uniform constant as in (4). But notice now that (4) is contradicted when

$$M > L + \sqrt{\frac{K}{\rho}\int_0^\rho C(s)^2 \ ds + B} \ .$$

This completes the proof.

Uniform boundedness, together with the equicontinuity above, implies that the family $\{f_{\varepsilon,n}\}_{\varepsilon\leq\rho}$ contains a subsequence which is uniformly convergent on $\overline{X_{\rho,c}}$ to $f_{0,n}$, for any $n \geq 0$. Moreover

$$\lim_{k\to\infty} df_{\varepsilon_k,n} = df_{0,n} \text{ on } \overline{X_{\rho,c}} \ ,$$

and hence, for any $\epsilon > 0$ we can find k sufficiently large that

$$\sup_{\overline{X_{\rho,c}}} |\psi(f_{0,n})| \leq \sup_{\overline{X_{\rho,c}}} |\psi(f_{\varepsilon_k,n})| + \epsilon \ \leq C(\rho) + \epsilon \ .$$

This gives the required inequality (2), with which we can now prove the following

Proposition 2. *If the smooth functions $f_{0,n}$ all satisfy inequality (2) on $X_{0,c}$, then for all $n \geq 0$, $\bar\partial^*\bar\partial G\psi(f_{0,n}) = 0$.*

Proof. We proceed by induction. When $n = 0$, note that $\psi(f) \in L^2(X_{0,c})$ and $\bar\partial\psi(f) = \psi(\bar\partial f) = 0$, hence the argument of [6], Theorem 1, allows us to conclude that $\bar\partial^*\bar\partial G\psi(f) = 0$.

Suppose now that $\bar\partial^*\bar\partial G\psi(f_{0,n}) = 0$, and note

$$\bar\partial\psi(f_{0,n+1}) = \psi(\bar\partial f_{0,n+1})$$

$$= \psi(\bar\partial\bar\partial^* G\psi(f_{0,n}))$$

$$= \psi(\psi(f_{0,n})) - \psi(\bar\partial^*\bar\partial G\psi(f_{0,n})) = 0 \ ,$$

due to the involutivity of ψ and the induction hypothesis. This, together with inequality (2) applied to $\psi(f_{0,n+1})$, ensures once more that $\bar\partial^*\bar\partial G\psi(f_{0,n+1}) = 0$. This completes the proof of the proposition.

Proposition 3. *The family $\{|f_{0,n}|\}_{n\geq 0}$ is uniformly bounded on $X_{\rho,c}$.*

Proof. We essentially mimic the argument of proposition 1, noting that the estimate (4) also applies to the L^2-norm of $f_{0,n}$, for any n. For any $M > 0$, suppose there exists n such that

$$\sup_{\overline{X_{\rho,c}}} |f_{0,n}| > M \ .$$

With the same manipulation of (3) we see that

$$\inf_{\overline{X_{\rho,c}}} |f_{0,n}| > M - L \ , \text{ and so}$$

$$\int_{X_{\rho,c}} |f_{0,n}|^2 \det(g) > K(M-L)^2 \ .$$

With

$$M > L + \sqrt{\frac{K}{\rho}\int_0^\rho C(s)^2 \ ds + B} \ ,$$

we obtain the same contradiction of (4). This completes the proof.

Note that uniform boundedness of $f_{0,n}$ on $X_{\rho,c}$ guarantees uniform convergence of the power series corresponding to $\frac{1}{1+T_0^{t\psi}}(h)\mid_{\overline{X}_{\rho,c}}$, for $|t|<1$. Our next step is to show that for $|t|<1$, $\frac{1}{1+T_0^{t\psi}}(h)$ is a holomorphic function with respect to the family of complex structures determined by $t\psi$, i.e., it is in the kernel of the deformed Cauchy-Riemann operator

$$\bar{\partial}^{t\psi} = \bar{\partial} + t\psi \ .$$

Proposition 4. $\bar{\partial}^{t\psi}\left(\frac{1}{1+T_0^{t\psi}}(h)\right) = 0$

Proof.

$$\begin{aligned}\bar{\partial}^{t\psi}\left(\frac{1}{1+T_0^{t\psi}}(h)\right) &= (\bar{\partial}+t\psi)\left(\Sigma_{n=0}^{\infty}(-1)^n f_{0,n}\cdot t^n\right)\\ &= \Sigma_{n=1}^{\infty}(-1)^{n-1}\left\{-\bar{\partial} f_{0,n} + \psi(f_{0,n-1})\right\} t^n \ .\end{aligned}$$

From proposition 2 we see that

$$\begin{aligned}\bar{\partial} f_{0,n} &= \bar{\partial}\bar{\partial}^* G\psi(f_{0,n-1})\\ &= \psi(f_{0,n-1}) \ ,\end{aligned}$$

Hence $-\bar{\partial} f_{0,n} + \psi(f_{0,n-1}) = 0$ for all n. This completes the proof.

We may summarize the discussion of this section with the following

Theorem 2. *Let f be a holomorphic function on $X_{0,c}$, such that there exist, for all $0<\varepsilon\le\rho<c$, and all $n\ge 0$, positive functions $c(\rho)$ and $C(\rho)$ $(C(\rho)\in L^2(0,c))$ such that*

$$\sup_{X_{\rho,c}} |df_{\varepsilon,n}| \le c(\rho) \ ,$$

and

$$\sup_{X_{\rho,c}} |\psi(f_{\varepsilon,n})| \le C(\rho) \ .$$

Then there exists a holomorphic function $\hat{f} = \frac{1}{1+T_0^{t\psi}}(h)$, defined on the family of complex manifolds $X_{0,c}^{t\psi}$, for $|t|<1$, such that $\hat{f}\mid_{X_{0,c}} = f$.

4. Stable embedding of deformations. Now let f^i , $1\le i\le N$ denote an ensemble of holomorphic functions on the normal Stein surface X, which together define an embedding $F: X_c \hookrightarrow \mathbf{C}^N$, such that $F(x_0)=\mathbf{0}$. Up to this point we have assumed the existence of a strongly plurisubharmonic function $\rho: X\to[0,\infty)$ satisfying the hypotheses required to prove [6], Theorem 1. Though in many instances ρ may be identified with the pullback of the Euclidean norm-squared function, it is not generally the same. Hence we define

$$\sigma: X\to[0,\infty)\ ,\ \text{such that}\ \ \sigma(p) := \Sigma_{i=1}^N |f^i(p)|^2 \ .$$

Just as $X^{t\psi}_{a,b}$, $0 < a < b < c$ has been used to denote the smooth domain $\rho^{-1}(a,b)$ with the additional complex structure induced by $t\psi$, we will now formally distinguish $\tilde{X}^{t\psi}_{a,b}$ as the smooth domain corresponding to $\sigma^{-1}(a,b)$ with the same induced complex structure.

Proposition 5. *Suppose that there exist positive functions $c(\rho)$, and $C(\rho) \in L^2(0,c)$, independently of $0 < \varepsilon \leq \rho < c$, and $n \geq 0$, such that*

$$\sup_{X_{\rho,c}} |dF_{\varepsilon,n}| \leq c(\rho) \ ,$$

and

$$\sup_{X_{\rho,c}} |\psi(F_{\varepsilon,n})| \leq C(\rho) \ .$$

Then there is a holomorphic map $\hat{F}$, defined on the family of complex manifolds $X^{t\psi}_{0,c}$ such that

- $\hat{F}\mid_{X_{0,c}} = F$, *and*
- *for $0 < a < b < c$, and $|t|$ sufficiently small, the relatively compact coronas $X^{t\psi}_{a,b}$ and $\tilde{X}^{t\psi}_{a,b}$ are embedded in $\mathbf{C}^N$.*

Proof. Let $\pi : \mathcal{X}^{\psi} \to \Delta$ denote the complex one-parameter family of surfaces such that $t \in \Delta$ implies $\pi^{-1}(t) = X^{t\psi}_{0,c}$. $\hat{F}$ will then denote the holomorphic map corresponding to the ensemble of extensions $\hat{f}^i$, $1 \leq i \leq N$ guaranteed by the previous theorem. Note that for all $p \in X_{a,b}$ (resp. $\tilde{X}_{a,b}$), $\mathrm{rank}_{\mathbf{C}}(dF)_p = 2$ implies the function

$$\Sigma_{1\leq i<j\leq N} |\partial \hat{f}^i \wedge \partial \hat{f}^j(p)|^2 > 0 \ \text{ for } t = 0 \ .$$

Continuity in t and p, together with relative compactness of the domain, then ensures the existence of an $\epsilon > 0$ such that this function remains positive for all $|t| < \epsilon$, hence the restriction of $\hat{F}$ to fibres $X^{t\psi}_{a.b}$ (resp. $\tilde{X}^{t\psi}_{a,b}$) within this local family is immersive. It remains to show that these restrictions are also injective for sufficiently small $|t|$.

Choose $p \in \rho^{-1}(a,b)$ (resp. $\sigma^{-1}(a,b)$) and let p_t denote its representative in relation to any of the complex structures determining $X^{t\psi}_{a,b}$ (resp. $\tilde{X}^{t\psi}_{a,b}$). For $|t| < \epsilon$, we note that $\hat{F}^{-1}(\hat{F}(p_t))$ is a finite set, and the union over t forms a covering space

$$\varpi : V_p \to \Delta_\epsilon \ .$$

If $\hat{F}$ does not restrict injectively to each corona for $|t|$ sufficiently small, then V_p has a branch point at p_0. Note, however, that the holomorphic map $p_t \mapsto (\hat{F}(p_t), t)$, from some neighbourhood of $p_0 \in \mathcal{X}^{\psi}$ into $\mathbf{C}^{N+1}$, has a derivative of maximal rank at p_0 and hence is locally biholomorphic to its image. It follows at once that V_p is a simple cover for sufficiently small $|t|$, and $\hat{F}$ is locally injective. Again by virtue of the relative compactness of $\rho^{-1}(a,b)$ and $\sigma^{-1}(a,b)$, and the arbitrariness of p, we conclude there is in fact an $\epsilon > 0$ such that $\hat{F}$ embeds the corona $X^{t\psi}_{a,b}$ (resp. $\tilde{X}^{t\psi}_{a,b}$) for each $|t| < \epsilon$.

This completes the proof of the proposition.

According to Milnor's conical structure theorem there is a natural diffeomorphism $\eta : \Gamma \times (0,b) \to F(\tilde{X}_{0,b})$, assigning to each $p \in \tilde{X}_{0,b}$ a unique $q \in \Gamma := F(\tilde{X}_{0,c}) \cap \mathbf{S}^{2N-1}_{\sqrt{b}}$ and $\sigma \in (0,b)$. While there is clearly a Stein completion of $F(\tilde{X}_{a,b})$ corresponding to

$F(X)\cap B_{\sqrt{b}}(\mathbf{0})\subset \mathbf{C}^N$, the same is not automatically true of $\hat{F}(\tilde{X}^{t\psi}_{a,b})$ for $t\neq 0$. It will hold as a consequence of Hartogs extension, however, if each embedded corona forms an open neighbourhood of its intersection with $\mathbf{S}^{2N-1}_{\sqrt{d}}$, for a fixed $d\in(a,b)$ (cf. [4]).

Proposition 6. *For $|t|<\epsilon$ the embedded corona $\hat{F}(\tilde{X}^{t\psi}_{a,b})$ admits a Stein completion inside the ball $B_{\sqrt{b}}(\mathbf{0})\subset\mathbf{C}^N$.*

Proof. Define a smooth function $\phi : X_{0,c}\times\Delta\to\mathbf{R}$ such that

$$\phi(p,t)=|\hat{F}(p_t)|^2\ .$$

With respect to the conical structure diffeomorphism, let $p=\eta(q,\sigma)$ and hence write $\tilde{\phi}(q,\sigma,t)=\phi(\eta(q,\sigma),t)$. For each $t\in\Delta$, fix $d\in(a,b)$ and let

$$\Sigma_t := \{p\in X_{0,c}\mid \phi(p,t)=d\}\cong\{(q,\sigma)\in\Gamma\times(0,b)|\ \tilde{\phi}(q,\sigma,t)=d\}\ .$$

Note that $\tilde{\phi}(q,\sigma,0)=\sigma$, hence

$$\frac{\partial\tilde{\phi}}{\partial\sigma}\ |_{t=0}=\ 1\ .$$

It follows from the implicit function theorem that for each $q\in\Gamma$, there exists an open neighbourhood $\mathcal{U}_q\subseteq\Gamma\times\Delta$ around $(q,0)$, and a smooth function $h:\mathcal{U}_q\to(0,b)$ such that for any $(q',t)\in\mathcal{U}_q$,

$$\tilde{\phi}(q',\sigma,t)=d\leftrightarrow\sigma=h(q',t)\ .$$

In particular, $h(q',0)\equiv d$, hence by continuity we may suppose that for ϵ sufficiently small $a<h(q',t)<b$ for all $|t|<\epsilon$. Once again, due to the compactness of Γ and the arbitrariness of q, we may suppose there is a single ϵ for which this boundedness is uniform in q. Hence, for all $|t|<\epsilon$, $\Sigma_t\subset\tilde{X}^{t\psi}_{a,b}$. So $\hat{F}(\tilde{X}^{t\psi}_{a,b})$ forms an open neighbourhood of $\hat{F}(\Sigma_t)$ inside $\mathbf{S}^{2N-1}_{\sqrt{b}}\setminus\overline{\mathbf{S}^{2N-1}_{\sqrt{a}}}$ and as such is a closed irreducible subvariety of a neighbourhood of $\partial B_{\sqrt{d}}(\mathbf{0})$ in $\mathbf{C}^N$. The existence and uniqueness of a Stein completion (up to biholomorphism) is a consequence of [4], Theorem VII D6, and proceeds in much the same way as [10], Theorem 1.2. This completes the proof.

The uniqueness argument of [10], Theorem 1.2 actually entails that for each $|t|<\epsilon$, the holomorphic image $\hat{F}(\tilde{X}^{t\psi}_{0,b})$ is biholomorphically equivalent to the regular part of the above Stein completion, independently of $0<a<b$, hence we have the following:

Theorem 3. *Consider a normal Stein surface $F:X\hookrightarrow\mathbf{C}^N$, with a strongly plurisubharmonic function ρ and Kähler form $\omega=\mathbf{i}\bar{\partial}\partial\rho$, defined in a neighbourhood of the singularity at x_0 and satisfying the basic properties $(i)-(iii)$ above. Let ψ be any involutive deformation of the regular neighbourhood $X_{0,c}$, such that there exist positive functions $c(\rho)$, and $C(\rho)\in L^2(0,c)$, independently of $0<\varepsilon<\rho<c$ and $n\geq 0$, satisfying*

$$\sup_{X_{\rho,c}}|dF_{\varepsilon,n}|\leq c(\rho)\ ,$$

and

$$\sup_{X_{\rho,c}}|\psi(F_{\varepsilon,n})|\leq C(\rho)\ .$$

Then each fibre in the t-parameter family $\mathcal{X}^{\psi}$ of deformations is stably embeddible and admits a unique Stein completion in a neighbourhood of the origin in $\mathbf{C}^N$ for $|t|$ sufficiently small.

5. An example. Let $X = \mathbf{C}^2$ and $X_{\varepsilon,c} = \{(z,w) \mid \varepsilon < \rho(z,w) < c\}$, where $\rho(z,w) = |z|^2 + |w|^2$. The standard Kähler metric associated with $\omega_0 = \mathbf{i}\bar{\partial}\partial\rho$ is of course flat, and the Laplace-Beltrami equation $\Box\mu_\varepsilon = \eta$ for $(0,1)$-forms reduces component-wise to the standard Poisson equation on functions. If we retain the Dirichlet and Neumann conditions for the unique solution to the equation $\Delta u_\varepsilon = \varphi$ we have a formula

$$u_\varepsilon(z,w) = \int_{X_{\varepsilon,c}} \varphi(\xi,\zeta) G_\varepsilon(\xi,\zeta,z,w)\; dVol_{(\xi.\zeta)}\ ,$$

where

$$G_\varepsilon(z,w,\xi,\zeta) = \Gamma(z,w,\xi,\zeta) + H_\varepsilon(z,w,\xi,\zeta)$$

is an ε-family of Green's functions satisfying the same boundary conditions, and comprised of the Fundamental solution Γ in (real) dimension four together with an ε-family of harmonic functions which offset the boundary values of Γ. While it is an elementary consequence of the modern theory of the Laplace operator, we will take a moment to review the reasons for uniform boundedness on compact subsets $\overline{X_{\rho,c}}$ of the family $\{u_\varepsilon\}_{\varepsilon<\rho}$ and its partial derivatives.

Without loss of generality, fix $\varepsilon' \in (\varepsilon,\rho)$ and write, for each $(z,w) \in \overline{X_{\rho,c}}$,

$$u_\varepsilon(z,w) = \int_{X_{\varepsilon,\varepsilon'}} \varphi\cdot\Gamma\; dVol_{(\xi,\zeta)} + \int_{X_{\varepsilon',c}} \varphi\cdot\Gamma\; dVol_{(\xi,\zeta)} + \int_{X_{\varepsilon,c}} \varphi\cdot H_\varepsilon\; dVol_{(\xi,\zeta)}\ .$$

Note that the singularities of Γ lie along the diagonal $(z,w) = (\xi,\zeta)$, so for all $(z,w) \in \overline{X_{\rho,c}}$ the Fundamental solution and its partial (z,w)-derivatives are bounded on the region of the first integral above. If moreover $\varphi \in C^\infty \cap L^2(X_{0,c})$, then this first term is easily seen to be bounded independently of ε. The region of the second integral of course contains the singularities, but it is well-known that Hölder continuity of φ on the bounded region $X_{\varepsilon',c}$ is all that is required to define a function that is at least twice continuously differentiable in z and w, and the supremum on $\overline{X_{\rho,c}}$ in each case provides a bound that is clearly independent of ε. The third integral is itself a harmonic function of z and w, and via the standard estimates for partial derivatives of harmonic functions, we may assume uniform boundedness on $X_{\rho',c}$ for an appropriate $\rho < \rho' < c$. Differentiating all three terms under the integral sign, we may then obtain uniform estimates of the form

$$sup_{\overline{X_{\rho',c}}} \left| \frac{\partial^k u_\varepsilon}{\partial z^l \partial w^{k-l}} \right| \le C(\rho')\ ,$$

and finally apply Arzela's theorem to obtain a uniform limit of the u_ε, and their partial derivatives, on compact sets $\overline{X_{\rho',c}}$, as ε approaches zero.

Now let ψ denote an involutive deformation of the required type on $X_{0,c}$, and let f be any holomorphic function on X, such that $\varphi_j = \psi_j(f) \in L^2(X_{0,c})\ ,\ j = 1,2$. From the above discussion it follows that the family of functions $T^\psi_\varepsilon(f)$ admits a subsequence which is uniformly convergent on compact subsets to a function $T^\psi_0(f)$ such that $\bar{\partial}T^\psi_0(f) = \psi(f)$. This process may clearly be iterated if the family $\{\psi(T^\psi_\varepsilon(f))\}_{0<\varepsilon<\rho}$

is uniformly bounded on relatively compact subsets $X_{\rho,c}$ by a positive function $C(\rho) \in L^2(0,c)$. Note that in the flat context corresponding to $X = \mathbf{C}^2$, there is no need for an explicit assumption of uniform boundedness of $dT^{\psi}_{\varepsilon}(f)$.

Let us instead *suppose there exists a holomorphic function* $h(z,w)$ *such that the family* $\{\psi(T^{\psi}_{\varepsilon}(f)+h)\}_{0<\varepsilon<\rho}$ *is uniformly bounded by* $C(\rho)$. It follows in the same way that we can define uniform limits $\psi(T^{\psi}_0(f)+h)$ and $T^{\psi}_0(T^{\psi}_0(f)+h)$ on any relatively compact subset $X_{\rho,c}$ such that $\bar\partial T^{\psi}_0(T^{\psi}_0(f)+h) = \psi(T^{\psi}_0(f)+h)$. As we saw in the previous sections, if it is possible to iterate this process indefinitely, or until $\psi(f_{\varepsilon,n}) = 0$, then f can be extended holomorphically to all spaces $X^{t\psi}_0$ in the family.

We turn now to the example of Rossi's unstable family [13] for an illustration (for a different approach to versal deformation of orbifold singularities via CR-structure, see also [12]). This corresponds to a deformation of $\mathbf{C}^2 \setminus \{(0,0)\}$ of the form

$$\psi = \frac{z\bar w}{\rho^2}\partial_z \otimes d\bar z + \frac{|w|^2}{\rho^2}\partial_w \otimes d\bar z - \frac{|z|^2}{\rho^2}\partial_z \otimes d\bar w - \frac{w\bar z}{\rho^2}\partial_w \otimes d\bar w \ .$$

For the vector field $\bar Z = \bar z\partial_{\bar z} + \bar w\partial_{\bar w}$ we can show directly that $L_{\bar Z}\psi = 0$ and hence ψ is of the required form $L_{\bar Z}\phi$, where $\phi = \ln(\rho)\cdot\psi$.

We note that any holomorphic function on $\mathbf{C}^2$ such that $ord_{(0,0)}(f) \geq 2$ satisfies $\psi(f) \in L^2(X_{0,c})$. In particular, if we consider the map $F : \mathbf{C}^2 \to \mathbf{C}^3$ such that $F(z,w) = (z^2, zw, w^2)$, then it is a direct calculation to show that

$$\psi(z^2) = \frac{2}{\rho^2}\left(z^2\bar w d\bar z - z|z|^2 d\bar w\right) = \bar\partial\left(-\frac{2z\bar w}{\rho}\right) \ ;$$

$$\psi(zw) = \frac{2}{\rho^2}\left(z|w|^2 d\bar z - w|z|^2 d\bar w\right) = \bar\partial\left(\frac{|z|^2 - |w|^2}{\rho}\right) \ ;$$

$$\psi(w^2) = \frac{2}{\rho^2}\left(w^2\bar w d\bar z - w^2\bar z d\bar w\right) = \bar\partial\left(\frac{2w\bar z}{\rho}\right) \ .$$

Now let $h_0 := z^2$, and

$$h_1 := -\frac{2z\bar w}{\rho} - T^{\psi}_0(z^2) \ .$$

It follows from a direct calculation that

$$\psi(T^{\psi}_0(z^2) + h_1) = \psi(-\frac{2z\bar w}{\rho}) = 0 \ ,$$

hence the family $\psi(T^{\psi}_{\varepsilon}(z^2) + h_1)$ is uniformly bounded on compact sets. Moreover, by a further direct calculation,

$$\bar\partial^{t\psi}(z^2 - t(T^{\psi}_0(z^2) + h_1)) = 0 \ ,$$

hence if we set $h_k = 0$ for all $k \geq 2$, then $f_{0,0} = z^2$, $f_{0,1} = T^{\psi}_0(z^2) + h_1 = -\frac{2z\bar w}{\rho}$,

$$f_{0,2} = T^{\psi}_0(T^{\psi}_0(z^2) + h_1) = (T^{\psi}_0)^2(z^2) - (T^{\psi}_0)^2(z^2) = 0 \ ,$$

and, by induction, $f_{0,n+1} = T_0^{\psi}(f_{0,n}) + h_{n+1} = 0$ for all $n \geq 1$. The same is true for $T_0^{\psi}(zw)$ and $T_0^{\psi}(w^2)$, hence the extension process for all three functions can be terminated at first-order terms, and we have a map

$$\hat{F} = \left(z^2 + \frac{2z\bar{w}}{\rho}t, zw + \left(\frac{|w|^2 - |z|^2}{\rho}\right)t, w^2 - \frac{2w\bar{z}}{\rho}t\right) = (\zeta_1, \zeta_2, \zeta_3)$$

for which the image satisfies the parametric equation

$$\zeta_2^2 - \zeta_1\zeta_3 = t^2 \text{ in } \mathbf{C}^4 .$$

REFERENCES

[1] A. Andreotti and Y.-T. Siu, *Projective Embeddings of Pseudoconcave spaces*, Ann. Sc. Norm. Sup. Pisa, 24 (1970), pp. 231–278.

[2] S. Bando, *Removable Singularities for Holomorphic Vector Bundles*, Tohoku Math. J., 43 (1991), pp. 61–67.

[3] D. Catlin and L. Lempert, *A Note on the Instability of Embeddings of Cauchy-Riemann Manifolds*, J. Geom. Anal., 2:2 (1992), pp. 99–104.

[4] R. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice Hall, NJ 1965.

[5] A. Harris and M. Kolar, *On Infinitesimal Deformations of the Regular Part of a Complex Cone Singularity*, Kyushu J. Math., 65:1 (2011), pp. 25–38.

[6] A. Harris and Y. Tonegawa, *L^p curvature and the Cauchy-Riemann equation near an isolated complex singularity*, Nagoya Math. J., 164 (2001), pp. 35–51.

[7] J. Kohn and G. Folland *The Neumann Problem for the Cauchy-Riemann Complex*, Ann. Math. Stud. 75, Princeton 1972.

[8] H. B. Laufer, *Deformation of Resolutions of two-dimensional Singularities*, Rice Univ. Stud., 59 (1973), pp. 53–96.

[9] H. B. Laufer, *Normal Two-Dimensional Singularities*, Ann. Math. Stud. 71 , Princeton 1971.

[10] G. Marinescu and T. Dinh, *On the compactification of Hyperconcave Ends and the Theorems of Siu-Yau and Nadel*, Inv. Math., 164 (2006), pp.233–248.

[11] K. Miyajima, *CR Construction of the Flat Deformations of Normal Isolated Singularities*, J. Alg. Geom., 8 (1999), pp. 403–470.

[12] K. Miyajima, *Analytic approach to Deformation of Resolution of Normal Isolated Singularities: Formal Deformations*, J. Korean Math. Soc., 40:4 (2003), pp.709–725.

[13] H. Rossi, *Attaching analytic spaces to an analytic space along a pseudoconvex boundary*, Proc. Conf. on Complex Manifolds (A. Aeppli, E. Calabi, R. Röhrl, eds.), Springer-Verlag, Berlin, Heidelberg, New York 1965.

[14] S. S.-T. Yau, *Kohn-Rossi Cohomology and its application to the Complex Plateau Problem I*, Ann. Math., 113:1 (1981), pp. 67–110.

[15] R. Du and S. S.-T. Yau, *Kohn-Rossi Cohomology and its application to the Complex Plateau Problem, III*, J. Diff. Geom., 90 (2012), pp. 251–266.

[16] H. S. Luk and S. S.-T. Yau, : *Kohn-Rossi Cohomology and its application to the Complex Plateau Problem, II*, J. Diff. Geom., 77 (2007), pp. 135–148.