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# REMARKS ON LAUFER'S FORMULA FOR THE MILNOR NUMBER, ROCHLIN'S SIGNATURE THEOREM AND THE ANALYTIC EULER CHARACTERISTIC OF COMPACT COMPLEX MANIFOLDS*

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**Introduction.** There are several classical approaches to studying the geometry and topology of isolated singularities $(V, \underline{0})$ defined by a holomorphic map-germ $(\mathbb{C}^{n+1}, \underline{0}) \xrightarrow{f} (\mathbb{C}, 0)$. One of these is by looking at resolutions of the singularity, $\pi : \widetilde{V} \to V$. Another is by considering the non-critical levels of the function $f$ and the way how these degenerate to the special fiber $V$. Laufer's formula for the Milnor number establishes a beautiful bridge between these two points of view. The formula says that if $n = 3$, then one has:

$$\mu + 1 = \chi(\widetilde{V}) + K^2 + 12\rho_g \ ,$$

where $\mu$ is the Milnor number, *i.e.*, the number of vanishing cycles in a nearby local non-critical level; $\chi$ is the usual Euler Characteristic, $K^2$ is the self-intersection number of the canonical class of the resolution, and $\rho_g$ is the geometric genus (see the text).

There is a generalistion of this formula by J. Steenbrink in [44] for normal surface singularities which are Gorenstein and smoothable. There is also a generalization by E. Looijenga in [26] to higher dimensions.

In this article we look at Laufer's formula from various viewpoints. In Section 1 we discuss some basic facts about surface singularities and carefully describe the various invariants appearing in Laufer's formula. Section 2 puts together various known geometric and topological facts about surface singularities. The main point is that if the singularity is Gorenstein and we set $V^* = V \setminus P$, then the structure group of the tangent bundle $TV^*$ has a reduction to SU(2), which is isomorphic to the group of unit quaternions. This has a number of geometric and topological implications.

In Section 3 we discuss relations amongst Laufer's formula and one of the classical theorems in low dimensional topology: Rochlin's signature theorem, saying that if $M$ is a closed oriented 4-manifold and $W$ is a "characteristic submanifold", then one has:

$$\sigma(M) - W^2 \equiv 8 \operatorname{Arf} W \mod (16) \ ,$$

where $\operatorname{Arf} W$ is the Arf invariant of a certain quadratic form over $H_1(W; \mathbb{Z}_2)$ (see the text for explanations). This section is based on work by H. Esnault, E. Viehweg and the author in [14], carrying Rochlin's theorem to the framework of algebraic geometry. Here we refine slightly the arguments in [14] to prove a formula over the integers, whose reduction modulo 2 is, essentially, Rochlin's theorem. No doubt this

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formula itself needs a refinement, aiming toward relations with more recent invariants of low dimensional manifolds.

In section 4 we revisit Laufer's formula in the context of Gorenstein normal surface singularities. We use the material in the previous sections to discuss briefly the non-smoothable case. We give a weak answer using a cobordism invariant; we hope this may serve to point out a path to follow. A basic point is: Who plays the role of the Milnor number when this is not naturally defined?

This article grew up from the talk I gave at the International Conference on Singularity Theory in honor of Henry Laufer's 70th Birthday, held at the Tsinghua Sanya International Mathematics Forum in December 2015. I am indebted to the organizers for inviting me to participate in this meeting in honor of one of the mathematicians I most admire.

**1. Laufer's Formula for the Milnor number.** Consider a holomorphic map-germ

$$f : (\mathbb{C}^{n+1}, \underline{0}) \to (\mathbb{C}, 0)\,,$$

with a critical point at $\underline{0}$. Let $V = f^{-1}(0)$ and let $L_V = V \cap \mathbb{S}_\varepsilon$ be the link. Milnor's classical theorem in [29] says that we have a locally trivial fibration:

$$\phi := \frac{f}{|f|} : \mathbb{S}_\varepsilon \setminus L_V \longrightarrow \mathbb{S}^1.$$

There is an alternative description of this fibration, essentially due to Milnor too. Given $\varepsilon > 0$ as above, choose $0 < \delta << \varepsilon$ and set $N(\varepsilon, \delta) = f^{-1}(\partial \mathbb{D}_\delta) \cap \mathbb{B}_\varepsilon$. Then:

$$f : N(\varepsilon, \delta) \longrightarrow \partial \mathbb{D}_\delta \cong \mathbb{S}^1$$

is a locally trivial fibration, equivalent to the previous one.

When $f$ has an isolated critical point, Milnor proved that the fiber $F_t := f^{-1}(t) \cap \mathbb{B}_\varepsilon$ has the homotopy type of a bouquet of spheres of middle dimension: $F_t \simeq \bigvee \mathbb{S}^n$. The number of spheres in this wedge is, by definition, the Milnor number of $f$; usually denoted $\mu(f)$ or simply $\mu$. By definition one has: $\mu = \operatorname{Rank} H_n(F_t)$.

Milnor also proved that $\mu$ equals the Poincaré-Hopf local index of the gradient vector field $\nabla f$. Hence it equals the intersection number: $\mu = \dim_{\mathbb{C}} \frac{\mathcal{O}_{n+1,\underline{0}}}{\operatorname{Jac}(f)}$, where $\operatorname{Jac}(f)$ is the Jacobian ideal of $f$ (generated by its partial derivatives).

This number is also known as the Milnor number of the hypersurface germ $(V, \underline{0})$ where $V = f^{-1}(0)$. It is an important invariant that has played a key-role in singularity theory.

These results were soon generalized by H. Hamm to complex isolated complete intersection singularities (ICIS), see [19, 27]:

$$f := (f_1, \cdots, f_k) : (\mathbb{C}^{n+k}, \underline{0}) \to (\mathbb{C}^k, 0)\,.$$

Such a germ also has an associated Milnor fibration and the fibre has the homotopy type of a bouquet of spheres of middle dimension. So this has too a well-defined Milnor number, which is the rank of the middle-homology of the Milnor fibre.

Extending the notion of Milnor number to more general singularities is a topic on which there is a vast literature and we will come back to this point in Section 4.

Now consider a normal surface singularity germ $(V, \underline{0})$. A natural and classical way to study the germ $(V, \underline{0})$ is by considering resolutions of it. This leads, for instance, to several important invariants. We now introduce some of these, and we refer to the literature for details (see for instance [5] or [41, Chapter IV]).

Let $\widetilde{V}$ be a resolution of $(V, \underline{0})$. This means that $\widetilde{V}$ is non-singular and we have a proper analytic map:

$$\pi : \widetilde{V} \longrightarrow V\,,$$

which is a biholomorphism away from the exceptional divisor $E := \pi^{-1}(\underline{0})$. The resolution is good if it further satisfies that the irreducible components $E_i$ of $E$ are non-singular and they meet normally, *i.e.*, they meet transversally and no three of them meet at a point. Given any resolution of $(V, \underline{0})$, we can make it good by a further sequence of blow ups. And there is always a minimal resolution, characterized by Castelnuovo's criterium: that no irreducible component of $E := \pi^{-1}(\underline{0})$ is rational with self-intersection -1.

It is well known, by [30], that the intersection pairing in $H_2(\widetilde{V})$ is negative definite. This has many important consequences, one of these being that if the resolution is good, then its canonical class $K$ is characterized uniquely by the adjunction formula

$$2g_{E_i} - 2 = E_i \cdot (K + E_i)\,.$$

Let us be more precise. The irreducible components $E_i$, $i = 1, ...r$, of $E$ generate the homology group $H_2(\widetilde{V})$. One has the canonical bundle $\mathcal{K}_{\widetilde{V}}$ of $\widetilde{V}$, which is the bundle of holomorphic 2-forms, and $K$ is a divisor of this bundle. Thence $K$ represents a homology class in $H_2(\widetilde{V}; \mathbb{R})$ which "morally" is the dual of minus the Chern class of $\widetilde{V}$. This statement would be correct if $\widetilde{V}$ were a compact surface; in our setting we need something else. A way to make this precise is to consider $\widetilde{V}$ as a compact smooth 4-manifold with boundary the link $L_V$ and a complex structure in its interior. By Poincaré-Lefschetz duality we have an isomorphism $H_2(\widetilde{V}) \cong H^2(\widetilde{V}, L_V)$. Hence the class represented by the divisor $K$ is dual to a class $-\hat{c}_i(\widetilde{V})$ in the relative cohomology $H^2(\widetilde{V}, L_V)$; the image of $-\hat{c}_i(\widetilde{V})$ in $H^2(\widetilde{V})$ is the usual Chern class $c_1(T\widetilde{V})$.

Since each irreducible component $E_i$ of the divisor $E$ is a smooth Riemann surface embedded in $\widetilde{V}$, we have a natural $C^\infty$ splitting:

$$T\widetilde{V}|_{E_i} \cong TE_i \oplus \nu E_i\,,$$

where the latter is the normal bundle. Thus one has:

$$c_1(T\widetilde{V}|_{E_i}) = c_1(TE_i) + c_1(\nu E_i)\,,$$

or equivalently:

$$-K \cdot E_i = (2 - 2g_{E_i}) + E_i \cdot E_i\,,$$

which is the adjunction formula. The remarkable fact is that satisfying this formula for all the $E_i$ characterizes uniquely the canonical class $K$; this happens because the $E_i$ generate the homology of $\widetilde{V}$ and the intersection pairing in $\widetilde{V}$ is non-degenerate. Yet, it can happen that $K$ is not an integral class, since the coefficients one gets as solutions of the adjunction formulae can be rational numbers. For instance the surface singularity one gets by taking a holomorphic line bundle $\mathcal{L}$ with Chern class $-c < 0$ over the Riemann sphere and blowing down to a point the zero section $S$, has canonical class $K = (-1 + \frac{2}{c})\,S$, which is never integral if $c > 2$.

For normal surface singularities, we can define Gorenstein singularities as follows. The concept of numerically Gorenstein is due to Durfee in [11].

DEFINITION 1.1. The germ of $V$ at $p$ is Gorenstein if its canonical bundle $\mathcal{K} = \wedge^2(T^*(V \setminus \{\underline{0}\})$ is holomorphically trivial. The germ $(V, \underline{0})$ is numerically Gorenstein if the complex bundle $\mathcal{K}$ is topologically trivial away from $\underline{0}$.

It is well known that every hypersurface (or complete intersection) germ is Gorenstein. For instance, if $V$ is defined by a holomorphic map germ $(\mathbb{C}^3, \underline{0}) \xrightarrow{f} (\mathbb{C}, 0)$, then contracting the canonical 3-form

$$\Omega = dz_1 \wedge dz_2 \wedge dz_3$$

by the gradient vector field $(\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \frac{\partial f}{\partial z_3})$ yields a holomorphic 2-form which is never-vanishing on $V \setminus \{\underline{0}\}$. In local coordinates this 2-form can be written as:

$$\omega(z_1, z_2, z_3) = \frac{dz_1 \wedge dz_2}{\partial f / \partial z_3} = \frac{dz_2 \wedge dz_3}{\partial f / \partial z_1} = \frac{dz_3 \wedge dz_1}{\partial f / \partial z_2} \,.$$

A. Durfee proved in [11]:

PROPOSITION 1.2. *The following are equivalent:*
(1) *The germ $(V, \underline{0})$ is numerically Gorenstein;*
(2) *The complex bundle $TV^*$ is topologically trivial, where $V^* := V \setminus \{\underline{0}\}$.*
(3) *The canonical class $K$ is integral.*

The equivalence between the first two statements follows because $c_1(TV^*) = -c_1(\mathcal{K}_{V^*})$ and by elementary homotopy reasons, a 2-dimensional complex vector bundle over $V^*$ is trivial if and only if its first Chern class vanishes. The next statement is more subtle and we refer to [23] for a further discussion on numerically Gorenstein singularities. The point is that the integrality of $K$ corresponds to being able to find a representative of the 1st Chern class of the resolution $\widetilde{V}$ whose support is contained in the exceptional divisor $\pi^{-1}(\underline{0})$; and this is independent of the choice of resolution.

REMARK 1.3. [Zariski-Lipman conjecture] *For every normal surface singularity $(V, \underline{0})$ one has that the tangent bundle $T(V \setminus \{\underline{0}\})$ is trivial as a real vector bundle. If the singularity is Gorenstein, then the canonical bundle $\mathcal{K}$ of $V \setminus \{\underline{0}\}$, which is the second exterior product of $T(V \setminus \{\underline{0}\})$, actually is holomorphically trivial. Hence it is natural to ask whether there exist normal surface singularities such that $T(V \setminus \{\underline{0}\})$ is holomorphically trivial. The Zariski-Lipman conjecture claims that if this is so, then the germ $(V, \underline{0})$ actually is regular.*

Notice that the Euler characteristic $\chi(\widetilde{V})$ is easily computable from the genera of the $E_i$ and the way they meet: Each $E_i$ contributes to $\chi(\widetilde{V})$ with its Euler characteristic $\chi(E_i)$, and then each meeting point of an $E_i$ with an $E_j$ contributes with $-1$. Of course $\chi(\widetilde{V})$ depends on the choice of resolution. Similarly, the self intersection number

$$K^2 = K \cdot K = a_i^2 E_i^2 + 2 a_i a_j (E_i \cdot E_j) \,,$$

where $K = a_1 E_1 + \cdots a_r E_r$, also depends on the choice of resolution. Yet, it is an exercise to show that the sum:

$$\chi(\widetilde{V}) + K^2$$

is independent of the choice of resolution, and so is an invariant of the germ $(V, \underline{0})$. Moreover, this invariant depends only on the topology of the germ $(V, \underline{0})$ and not on its complex structure, by [32].

There is another important invariant of the germ $(V, \underline{0})$ which plays a major role in the sequel, the geometric genus:

$$\rho_g := \dim H^1(\widetilde{V}, \mathcal{O}) .$$

This too is independent of the choice of resolution, but this invariant does depend on the choice of the complex structure on $(V, \underline{0})$. For instance the Pham-Brieskorn singularities:

$$z_1^2 + z_2^7 + z_3^{14} \quad \text{and} \quad z_1^3 + z_2^4 + z_3^{12} ,$$

have the same topology but different geometric genus. Yet, there is remarkable work by A. Nemethi and others showing that under certain conditions the geometric genus is topological (see for instance [31]).

Laufer in [22] proved:

THEOREM 1.4. *Assume the germ $(V, \underline{0})$ is a hypersurface germ. Then*

$$\mu(V) + 1 \;=\; \chi(\widetilde{V}) + K^2 + 12\rho_g(V) .$$

In fact the same statement, with essentially the same proof, holds for all Gorenstein surface singularities with a smoothing that can be put in a projective family with no other singularities. This includes all ICIS.

Observe that the left hand side in Laufer's formula is the Euler characteristic of the Milnor fibre and therefore has no *a priori* meaning if the singularity is not an ICIS. Yet, the right hand side is an integer defined always for all normal numerically Gorenstein surface singularities, and it is an invariant of $(V, \underline{0})$.

DEFINITION 1.5. Let $(V, \underline{0})$ be a numerically Gorenstein normal surface singularity germ. We call:

$$\mathrm{La}(V, \underline{0}) := \chi(\widetilde{V}) + K^2 + 12\rho_g(V)$$

the Laufer invariant of $(V, \underline{0})$.

So a natural question is:

QUESTION 1.6. *What is the Laufer invariant when the germ is not an ICIS? In other words, what is there on the left hand side of the equation in Theorem 1.4 when the singularity is not an ICIS?*

I will consider two cases:

a) The singularity germ is Gorenstein and smoothable;

b) The germ is non-smoothable.

The answer in the first case is due to Steenbrink (Theorem 1.8 below). The non-smoothable case is open and this is the subject we discuss in Section 4.

We recall that the surface singularity germ $(V, \underline{0})$ is smoothable if there exists a 3-dimensional complex analytic space $\mathcal{W}$ and a flat map $\mathcal{F} : \mathcal{W} \to \mathbb{C}$ such that

$\mathcal{F}^{-1}(0)$ is isomorphic to $(V, \underline{0})$ and $\mathcal{F}^{-1}(t)$ is smooth for $t \neq 0$. It is well known that there exist normal surface Gorenstein singularities which are non-smoothable. And there exist also normal surface singularities which have non-equivalent smoothings. We refer, for instance, to [47] for more on smoothings of surface singularities.

The following two theorems are important in the sequel (see [18, 44]):

THEOREM 1.7 (Greuel-Steenbrink). *Every smoothable Gorenstein normal surface singularity $(V, \underline{0})$ has a well-defined Milnor number $\mu_{\mathrm{GS}}$: The 2nd Betti-number of a smoothing. Furthermore, the first Betti number $b_1$ vanishes, hence the Euler characteristic of every smoothing equals $\mu_{\mathrm{GS}} + 1$.*

THEOREM 1.8 (Steenbrink). *This invariant satisfies Laufer's formula:*

$$\mu_{\mathrm{GS}} + 1 \;=\; \chi(\widetilde{V}) + K^2 + 12\rho_g(V)\;.$$

In other words, for normal surface singularities which are Gorenstein and smoothable, the Laufer invariant is: $\mathrm{La}(V, \underline{0}) = \mu_{\mathrm{GS}} + 1$.

**2. Geometry and topology on Gorenstein singularities.** Let $(V, \underline{0})$ be a normal surface singularity germ. Let $L_V$ be its link and set $V^* = V \setminus \{\underline{0}\}$. Then $V^*$ is a complex manifold of dimension 2, so its tangent bundle $TV^*$ has $\mathrm{GL}(2, \mathbb{C})$ as structure group. This can always be reduced to $\mathrm{U}(2)$ by endowing $V^*$ with a Riemannian metric. If we want to reduce the structure group further to $\mathrm{SU}(2)$ then we meet an obstruction: The action of $\mathrm{U}(2)$ must be trivial on the canonical bundle $\mathcal{K}_{V^*}$ of holomorphic 2-forms on $V^*$, since the action of the structure group $\mathrm{U}(2)$ on $TV^*$ is by the determinant. Hence the structure group can be reduced to $\mathrm{SU}(2)$ if and only if the canonical bundle of $V^*$ is topologically trivial. So we arrive to the first statement in the following theorem from [38, 39], since for Gorenstein singularities the canonical bundle of $V^*$ is trivial. Recall that $\mathrm{SU}(2)$ consists of the $2 \times 2$ matrices of the form:

$$\begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \quad , \quad z_1, z_2 \in \mathbb{C} \text{ and } |z_1|^2 + |z_2|^2 = 1\;,$$

so it is isomorphic to the 3-sphere $\mathbb{S}^3$, which is the group $\mathrm{Sp}(1)$ of unit quaternions.

THEOREM 2.1. *Let $(V, \underline{0})$ be a normal Gorenstein surface singularity germ. Let $L_V$ be its link and set $V^* = V \setminus \{\underline{0}\}$. Then:*

(1) *A choice of a never-vanishing holomorphic 2-form $\omega$ on $V^*$ determines:*
- *A reduction to $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$ of the structure group of the tangent bundle $TV^*$*
- *A canonical trivialization $\mathcal{P}$ of the tangent bundle $TL_V$ which is compatible with the complex structure on $V$. We call $\mathcal{P}$ the canonical framing of $L_V$.*

(2) *The element in the framed cobordism group $\Omega_3^{fr} \cong \mathbb{Z}_{24}$ represented by the pair $(L_V, \mathcal{P})$ depends only on the analytic structure of the germ $(V, \underline{0})$ and not on the other choices.*

We thus have, from the first statement, that for Gorenstein singularities, the tangent bundle $TV^*$ actually is a bundle over the quaternions and we have at each point in $V^*$ multiplication by the quaternions $i, j, k$. This implies the second statement in the theorem above: At each point $x \in L_V$ we can multiply the unit outward normal vector $\nu(x)$ by $i, j, k$, thus getting three linearly independent vector fields on $L_V$ that provide a trivialization of its tangent bundle. By construction this parallelism on the link is compatible with the complex structure on $V^*$.

The proof of the last statement in Theorem 2.1 follows easily from Theorem 2.2 below.

We recall that if $X$ is a closed oriented 4-manifold, its Pontryagin class $p_1(X)$ is the $2^{nd}$ Chern class of the complexification of its tangent bundle: $p_1(X) = -c_2(TX \otimes \mathbb{C})$.

For every 2-dimensional complex bundle one has $p_1 = c_1^2 - 2c_2$. If $X$ is a complex manifold then: $c_2(X)[X] = \chi(X)$. Hence in this case we have:

$$p_1(X)[X] = c_1^2(X)[X] - 2c_2(X)[X] = K_X^2 - 2\chi(X)\,,$$

where $K_X$ is the canonical class, dual of $c_1$ of the canonical bundle $\mathcal{K}_X := \wedge^2(T^*(X)$.

We now let $X$ be a compact oriented 4-manifold with non-empty boundary $M$. Suppose further that $M$ is equipped with a trivialization $\tau$ of its tangent bundle $TM$. Since $X$ is oriented, the normal bundle $\nu(M)$ of $M$ in $X$ is trivial and, endowing $X$ with a Riemannian metric, we have a canonical trivialization $\nu$ of $\nu(M)$. Then $\tau \oplus \nu$ determines a trivialization of $TX|_M$.

Using the isomorphism $TX|_M \cong M \times \mathbb{R}^4$ determined by $\tau \oplus \nu$, we get a vector bundle over the quotient $X/M$. The relative Pontryagin class $p_1(X;\tau) \in H^4(X, M; \mathbb{Z})$ is defined as that of the bundle we get over $X/M$ using that $H^4(X/M) \cong H^4(X, M)$. This depends on the choice of $\tau$. We refer to [45] and to [7, Chapter 1] for a thorough discussion of relative characteristic classes.

If $X$ has a complex structure then we may ask the trivialization $\tau$ of $TM$ to be compatible with that complex structure, *i.e.*, that $\tau \oplus \nu$ trivializes $TX|_M$ as a complex bundle. In that case, if $\tau$ is compatible with complex structure on $X$, we have:

$$p_1(X;\tau)[X] = c_1^2(X;\tau)[X] - 2c_2(X;\tau)[X] = K_{X,\tau}^2 - 2\chi(X)$$

where $K_{X,\tau}$ is the canonical class relative to $\tau$, dual of $c_1$ of the canonical bundle relative to $\tau$. In other words, $\tau$ defines a trivialization of the canonical bundle restricted to $M$, and $K_{X,\tau}$ is the obstruction to extending that trivialization over the interior of $X$. Alternatively, we may use $\tau$ to identify the fibers of the canonical bundle over points in $M$, and get a line bundle over the quotient space $X/M$. Then $K_{X,\tau}$ is the dual in $H_2(X)$ of the Chern class of the line bundle over $X/M$ via the isomorphism $H^2(X/M) \cong H^2(X, M)$.

In order to determine the element in $\mathbb{Z}_{24}$ that $(L_V, \mathcal{P})$ represents we may use a classical invariant coming from algebraic topology: the Adams e-invariant. In this dimension it is an integer modulo 24 and it provides a group isomorphism:

$$e_{\mathbb{R}} : \Omega_3^{fr} \longrightarrow \mathbb{Z}_{24}\,,$$

where $\Omega_3^{fr}$ is the cobordism group of stably framed 3-manifolds

The original definition of this invariant by J. F. Adams is via homotopy theory. Conner and Floyd gave an interpretation using spin cobordism and the $\widehat{A}$-genus, which is an integer for closed spin manifolds. There is also a slightly weaker complex Adams e-invariant $e_{\mathbb{C}}$, which in this dimension is the reduction modulo 2 of $e_{\mathbb{R}}$ and can be defined in terms of complex cobordism, using the Todd genus. Later Seade [40] improved the interpretation of the real Adams e-invariant in terms of cobordism by using the Todd genus together with a correction term in $\mathbb{Z}_2$ for the lack of a spin structure. In the setting of singularities, the complex Adams e-invariant was first used by A. Durfee in [11]. Then Seade used it in various papers, as briefy explained below.

Let $(V,\underline{0})$ be again a normal Gorenstein surface singularity, and let $(L_V,\mathcal{P})$ be its link equipped with its canonical framing. The following theorem spring easily from [38, 39, 40]:

Theorem 2.2. *If $X$ is a compact 4-manifold with boundary $L_V$, whose interior has a complex structure compatible with $\mathcal{P}$. Then:*

$$e_{\mathbb{R}}([L_V,\mathcal{P}]) \;=\; K^2_{X,\mathcal{P}} + \chi(X) + 12\,\mathrm{Arf}(K_X) \qquad \mathrm{mod}\ (24)\,,$$

*where $K_{X,\mathcal{P}} \in H_2(X)$ is the dual of the Chern class of the canonical bundle of $X$ relative to $\mathcal{P}$ and $\mathrm{Arf}(K_X) \in \{0,1\}$ is the Arf invariant of a certain quadratic form associated to $K_X$. Furthermore:*

- *If $X = \widetilde{V}$ is a good resolution of $(V,\underline{0})$, then $K_X$ is the canonical class, independently of the choice of $\mathcal{P}$.*

- *If $(V,\underline{0})$ is smoothable and $X = F_V$ is a smoothing, then $K_X = 0$ and $\mathrm{Arf}(K_X) = 0$.*

We say more about the invariant $\mathrm{Arf}(K_X)$ in the following section. Recall that just as the signature classifies the non-degenerate quadratic forms on finite-dimensional vector spaces over $\mathbb{R}$, so too the Arf invariant classifies the non-degenerate quadratic forms on finite dimensional vector spaces over $\mathbb{Z}_2 := \{0,1\}$. The Arf invariant of such a form is 0 if and only if it carries more elements to 0 than to 1.

Corollary 2.3. *Let $(V,\underline{0})$ be Gorenstein and smoothable, and $\widetilde{V}$ a good resolution. Then:*

$$\mu_{GS} + 1 \equiv K^2 + \chi(\widetilde{V}) + 12\mathrm{Arf}(K) \quad \mathrm{mod}\ (24)$$

*where $\mu_{GS}$ is the Milnor number and $K := K_{\widetilde{V}}$ is the canonical class. Thence by the Laufer-Steenbrink formula, $\mathrm{Arf}(K)$ equals the parity of the geometric genus:*

$$\mathrm{Arf}(K) \;=\; \dim H_1(\widetilde{V},\mathcal{O}_{\widetilde{V}}) \quad \mathrm{mod}\ (2)\,.$$

We remark that $\mathrm{Arf}(K)$ and the geometric genus are defined even if the germ $(V,\underline{0})$ is non-smoothable. It was thus asked in [40] whether or not the last congruence adobe holds for non-smoothable singularities. The answer is positive, as proved in [14] and discussed in the following section.

**3. Rochlin's signature theorem revisited for complex manifolds.** The classical Rochlin's signature theorem in [35] states that the signature of a closed spin 4-manifold is divisible by 16. In the context of complex manifolds this is equivalent to saying that for compact complex surfaces with even canonical class, the Todd genus is an even integer. In this section we briefly discuss improvements of this theorem.

Recall that if $X$ is a closed oriented 4-manifold, the cup product determines a non-degenerate bilinear form:

$$H^2(X;\mathbb{R}) \cup H^2(X;\mathbb{R}) \longrightarrow H^4(X;\mathbb{R}) \cong \mathbb{R}\,.$$

The signature of $X$, $\sigma(X) \in \mathbb{Z}$, is by definition the signature of this quadratic form. By Thom's theorem in [46] we have:

$$\sigma(X) = \frac{1}{3}\,p_1(X)[X]\,,$$

where $p_1$ is the Pontryagin class and $[X]$ is the orientation cycle. Notice that this is a special case of Hirzebruch theorem [20], saying that the signature of closed oriented manifolds of dimension $4k$ is given by the corresponding $L$-genus.

Rochlin signature theorem was later refined and improved by a number of people, culminating in the theorem below (see for instance [36, 15, 34]). We recall that a characteristic submanifold is an oriented submanifold of codimension 2 representing a homology class in $H_2(X;\mathbb{Z})$ whose reduction modulo 2 is dual to the 2nd Stiefel-Whitney class.

THEOREM 3.1 (Rochlin). *Let $W$ be a characteristic submanifold of $X$, then*

$$\sigma(X) - W^2 \equiv 8\,\mathrm{Arf}\,W \mod (16)$$

*where* $\mathrm{Arf}\,W \in \{0,1\}$ *is the Arf invariant of a certain quadratic form on* $H_1(W;\mathbb{Z}_2)$.

In fact it is well-known that every closed oriented 4-manifold is spin$^c$ and its spin$^c$ structures are classified by the homology classes represented by its characteristic submanifolds. Moreover, if $W$ is characteristic in $X$, then $W$ actually determines a spin$^c$ structure on $X$, which in turn determines a spin structure on $W$ (see [34]). One has that $\mathrm{Arf}\,W = 0$ if and only if $W$, equipped with its induced spin structure, is a spin boundary.

If we now consider a compact complex manifold $M$ of complex dimension 2, then its Pontryagin class can be expressed in terms of the Chern classes:

$$p_1(M)[M] = c_1^2(M)[M] - 2c_2(M)[M]\,.$$

By definition one has $c_2(M)[M] = \chi(M)$ where $\chi(M)$ is the usual Euler characteristic. Hence, if $K_M$ is the canonical class, Poincaré dual of $c_1$ of the canonical bundle $\mathcal{K}_M := \wedge^2(T^*(M)$, then we get:

$$p_1(M)[M] = K_M^2 - 2\chi(M)\,.$$

We we may actually take $K_M$ as being a divisor, and $-K_M$ is the anti-canonical divisor, dual of $c_1(TM)$.

We remark that for complex manifolds, the 2nd Stiefel-Whitney class is the reduction modulo 2 of its 1st Chern class. Hence in this case we can say that a characteristic submanifold of $M$ is a smooth submanifold $W$ of real dimension 2 that represents the same class as $K_M$ in $H_2(M;\mathbb{Z}_2)$.

Since $K_M$ is the zero-set of a section of the canonical bundle $\mathcal{K}_M$, if we forget about holomorphicity, we can make that section transverse to the zero section of $\mathcal{K}_M$ and get that $K_M$ can always be smoothed $C^\infty$; then the smoothing is a characteristic submanifold. If $K_M$ is even, the empty set $\emptyset$ is characteristic and $M$ is a spin manifold.

Recall that the Thom-Hirzebruch signature theorem in [20] says:

$$\sigma(M) = \frac{1}{3}\,p_1(M)[M]$$

where $p_1$ = Pontryagin class. Thus, if $W = \widetilde{K}$ is a $C^\infty$ smoothing of the canonical divisor $K_M$, then Rohlin's theorem can be restated as:

$$(c_1^2(M) + c_2(M))[M] \equiv 12\,\mathrm{Arf}\,\widetilde{K} \quad (24).$$

Recall the 2nd Todd polynomial is $\frac{1}{12}(c_1^2 + c_2)$. That is:

$$Td(M) = \frac{1}{12}(c_1^2(M) + c_2(M))[M]$$

so Rohlin's theorem can be restated as:

$$Td(M) \equiv \operatorname{Arf} \widetilde{K} \mod (2)\ .$$

On the other hand the Riemann-Roch-Hirzebruch theorem [20] says the Todd genus equals the analytic Euler characteristic: $Td(M) = \chi(M, \mathcal{O}_M)$. Thus we arrive to the following formulation of Rochlin's theorem:

THEOREM 3.2. *For compact complex surfaces, the parity of the analytic Euler characteristic is determined by the invariant* Arf $\widetilde{K}$*:*

$$\chi(M, \mathcal{O}_M) \equiv \operatorname{Arf} \widetilde{K} \mod (2)\ .$$

In this theorem we can essentially replace $K$ by any other divisor whose reduction modulo 2 coincides with that of $K$. In that case we must consider the analytic Euler characteristic with coefficients in some appropriate holomorphic bundle (see below for details).

We want a similar expression in algebraic geometry, not with a topological smoothing of $K$ but in terms of actual divisors. We follow [14]. We have:

DEFINITION 3.3. [Esnault-Seade-Viehweg] A characteristic divisor $W$ of $M$ is a divisor of a bundle $\mathcal{L}$ of the form $\mathcal{L} = \mathcal{D}^2 \otimes \mathcal{K}_M^{-1}$.

Observe that in this case $W$ is a divisor of the form $W = 2D - K_M$ where $D$ is a divisor of some holomorphic bundle $\mathcal{D}$. In this case, if $W \neq 0$, the restriction of $\mathcal{D}$ to $W$ is a theta-characteristic on $W$. That is (see [14, Lemma 1.3]):

$$\mathcal{D}^2|_W \cong \mathcal{K}_W\ .$$

Notice too that the reduction of $W$ modulo 2 coincides with that of $K_M$. If $W = 0$, then we have that the bundle $\mathcal{D}$ is a square root of the canonical bundle $\mathcal{K}_M$.

DEFINITION 3.4. Let $W$ be a characteristic divisor of $M$. Define its mod (2)-index by:

$$\mathfrak{h}(W) = \dim H^0(W, \mathcal{D}|_W) \mod (2)\,,$$

if $W \neq 0$. If $W = 0$, then define $\mathfrak{h}(W) = 0$.

We remark that by [4, 25], if $W$ is non-singular, this is the invariant Arf($W$) in Rochlin's theorem. In the particular case of the canonical class $K := K_M$ one has $\mathcal{D} = \mathcal{K}_M$, so that its mod (2)-index is:

$$\mathfrak{h}(K) = \dim H^0(K, \mathcal{K}_M|_K) \mod (2)\ .$$

We have the following theorem from [14], which is a version of Rochlin's signature theorem for complex manifolds, with the additional feature that it uses divisors instead of smooth submanifolds.

THEOREM 3.5. [Esnault-Seade-Viehweg] *Let $M$ be a compact complex manifold of complex dimension 2, and let $W$ be a characteristic divisor in $M$, say $W = K_M - 2D$ where $D$ is a divisor of some holomorphic bundle $\mathcal{D}$. Then:*

$$\mathfrak{h}(W) \equiv \chi(M, \mathcal{D}) \mod (2)\,,$$

*where* $\chi(M,\mathcal{D}) = \sum_{i=0}^{2}(-1)^i h^i(M,\mathcal{D})$ *is the analytic Euler characteristic of* $M$ *with coefficients in* $\mathcal{D}$*. In particular, the parity of the analytic Euler characteristic of* $M$ *coincides with the mod (2) index* $\mathfrak{h}(K)$*:*

$$\mathfrak{h}(K) \equiv \chi(M,\mathcal{O}_M) \mod (2)\ .$$

This theorem is proved in [14] in a more general setting, for complex manifolds of dimension of the form $4k+2$. Here we restrict to the case $k=0$.

If we can take $W=0$, this means that $M$ admits a spin structure, and in this case the theorem is very easy to prove: If $W=0$, then $\mathcal{K}_M = \mathcal{D}^2$ and therefore Serre duality yields:

$$H^0(M;\mathcal{D}) \cong H^2(M;\mathcal{K}_M \otimes \mathcal{D}^{-1}) \cong H^2(M;\mathcal{D})\,.$$

Thence

$$\chi(M,\mathcal{D}) \equiv h^1(M,\mathcal{D}) \mod (2)\ ,$$

and the result follows because the cup product on $H^1(M,\mathcal{D})$ is skew, so this space must be even dimensional.

The proof of (3.7) in [14] can be refined slightly to prove the theorem below, which is stronger.

Theorem 3.6. *Let* $M$ *and* $W = 2D - K_M$ *be as in Theorem 3.7. If* $W \neq 0$*, then:*

$$h^0(W;\mathcal{D}|_W)\ =\ \chi(M,\mathcal{D}) + R\,,$$

*with* $R$ *an even integer:* $R = h^1(M;\mathcal{D}) - 2h^2(M;\mathcal{D}) + \dim \operatorname{Ker}(\hat\beta)$, *where* $\hat\beta$ *is a skew symmetric bilinear form on* $H^1(M;D)$*.*

*Proof.* Consider the exact sequence of sheaves over $M$,

$$0 \to \mathcal{K}_M \otimes \mathcal{D}^{-1} \xrightarrow{s^*} \mathcal{D} \xrightarrow{r} \mathcal{D}|_W \to 0\,,$$

where $s^*$ is multiplication by the section $s$ of $\mathcal{L}$ that defines $W$ and $r$ is the restriction to $W$. We have the corresponding long exact cohomology sequence:

$$0 \to H^0(M;\mathcal{K}_M \otimes \mathcal{D}^{-1}) \longrightarrow H^0(M;\mathcal{D}) \longrightarrow H^0(W;\mathcal{D}|_W) \xrightarrow{\alpha}$$

$$\xrightarrow{\alpha} H^1(M;\mathcal{K}_M \otimes \mathcal{D}^{-1}) \xrightarrow{\beta} H^1(M;\mathcal{D}) \to \cdots$$

From this we get an exact sequence:

$$0 \to H^0(M;\mathcal{K}_M \otimes \mathcal{D}^{-1}) \to H^0(M;\mathcal{D}) \to H^0(W;\mathcal{D}|_W) \xrightarrow{\alpha} \operatorname{Im}(\alpha) \to 0\,.$$

For each cohomology group, let $h^i$ be the dimension of $H^i$. By exactness we get:

$$h^0(M;\mathcal{K}_M \otimes \mathcal{D}^{-1}) - h^0(M;\mathcal{D}) + h^0(W;\mathcal{D}|_W) = \dim \operatorname{Im}(\alpha)\,.$$

By Serre's duality we have:

$$H^0(M;\mathcal{K}_M \otimes \mathcal{D}^{-1}) \cong H^2(M;\mathcal{D})\,.$$

Substituting in the equation above we get:

$$h^2(M;\mathcal{D}) - h^0(M;\mathcal{D}) + h^0(W;\mathcal{D}|_W) = \dim \operatorname{Im}(\alpha)\,.$$

Hence:

$$h^0(W;\mathcal{D}|_W) = \chi(M,\mathcal{D}) + h^1(M;\mathcal{D}) - 2h^2(M;\mathcal{D}) + \dim \operatorname{Ker}(\beta)\,,$$

since by exactness, $\dim \operatorname{Im}(\alpha) = \dim \operatorname{Ker}(\beta)$. We get

$$h^0(W;\mathcal{D}|_W) = \chi(M,\mathcal{D}) + R$$

with $R$ as claimed in Theorem 3.6. It remains to prove that $R$ is even. That $h^1(M;\mathcal{D})$ is even follows because the cup product in $H^1(M;\mathcal{D})$ is a non-degenerate skew form. It remains to show that $\dim \operatorname{Ker}(\beta)$ is even too. In fact, by Serre duality

$$H^1(M;\mathcal{K}_M \otimes \mathcal{D}^{-1}) \cong H^1(M;\mathcal{D})\,,$$

and $\beta$ is a bilinear form: It is defined by the cup product

$$H^1(M;\mathcal{K}_M \otimes \mathcal{D}^{-1}) \times H^1(M;\mathcal{K}_M \otimes \mathcal{D}^{-1}) \longrightarrow H^2(M;\mathcal{K}_M^2 \otimes \mathcal{D}^{-2})\,,$$

followed by the maps:

$$H^2(M;\mathcal{K}_M^2 \otimes \mathcal{D}^{-2}) \xrightarrow{s^*} H^2(M;\mathcal{K}_M) \xrightarrow{tr} \mathbb{C}\,.$$

Since the cup product above is skew, $\beta$ can be written as a matrix of the form:

$$\begin{pmatrix} 0 & b_1 & & & \\ -b_1 & 0 & & & \\ & & 0 & b_2 & \\ & & -b_2 & 0 & \\ & & & & \ddots \end{pmatrix}$$

Hence the kernel of $\beta$ is even dimensional. The result follows by letting $\hat{\beta}$ be the dual form of $\beta$ in $H^1(M;\mathcal{D})$. □

Now consider a normal Gorenstein surface singularity germ $(V,\underline{0})$ that we can assume algebraic. We may take a compactification of it and resolve all its singularities. We are then in the setting envisaged above. It is shown in [14, Sections 3,4] that the previous discution, with some extra work, yields:

Theorem 3.7 (Esnault-Seade-Viehweg). *Let $\pi : \widetilde{V} \to V$ be a resolution of $(V,\underline{0})$, and let $K := K_{\widetilde{V}}$ be a divisor of the canonical bundle $\mathcal{K} := \mathcal{K}_{\widetilde{V}}$. Assume further, with no loss of generality, that the divisor $K$ is vertical, i.e., the support of the divisor is contained in the exceptional curve. Then the parity of the geometric genus coincides with the mod (2) index $\mathfrak{h}(-K)$. That is,*

$$\dim H^1(\widetilde{V},\mathcal{O}_{\widetilde{V}}) \;\equiv\; \dim H^0(-K,\mathcal{K}|_K) \mod (2)\,.$$

*Furthermore, if the resolution $\widetilde{V}$ is minimal, then for all vertical divisors $D \geq 0$ and $W = 2D - K_{\widetilde{V}}$ we have the actual equalities:*

$$\dim H^1(\widetilde{V},\mathcal{O}_{\widetilde{V}}) \;=\; \dim H^0(-K,\mathcal{K}|_K) \;=\; \dim H^0(W,\mathcal{D}|_W) + \frac{1}{8}(W^2 - K_{\widetilde{V}}^2)\,.$$

The last equality essentially comes from the Rieman-Roch theorem for surfaces, which implies that for a compact complex surface $M$ and a divisor $W = K - 2D$ as above we have:

$$\chi(M, \mathcal{D}) = \chi(M, \mathcal{D}) + \frac{1}{8}(W^2 - K^2)\,.$$

**4. Laufer's formula revisited.** We know from Theorem 1.8 that if $(V, \underline{0})$ is a normal Gorenstein singularity which is smoothable, then:

$$\mu_{GS}(V) + 1 \;=\; \chi(\widetilde{V}) + K^2 + 12\rho_g(V)\,.$$

The right hand side is the Laufer invariant of $(V, \underline{0})$. This is well defined even for non-smoothable singularities. It is thus natural to ask who ought to be in left hand side when the singularity is non-smoothable? A weak answer springs from theorems 2.2 and 3.7 above:

THEOREM 4.1. *Let $(V, \underline{0})$ be a normal Gorenstein singularity, let $L_V$ be its link and $\mathcal{P}$ its canonical framing. Let $\widetilde{V}$ be a resolution of $(V, \underline{0})$. Then the Laufer invariant of $(V, \underline{0})$ reduced modulo 24 equals the real e-invariant of the pair $[L_V, \mathcal{P}]$:*

$$e_{\mathbb{R}}[L_V, \mathcal{P}] \;=\; K^2_{\widetilde{V}} + \chi(\widetilde{V}) + 12\,\rho_g \qquad \bmod (24)\,.$$

In order to get a complete answer following this line of thought, we need to define an invariant in $\mathbb{Z}$ associated to the link, such that its reduction modulo 24 equals $e_{\mathbb{R}}[L_V, \mathcal{P}]$. At this moment, I do not know how to do this.

A much related question is: Who ought to be the Milnor number for normal isolated (Gorenstein surface) singularities in general?

There are several possible definitions of the Milnor number in the literature, including a very interesting one by Buchweitz-Greuel for curves (see [9]). This is determined by the Euler characteristic of a smoothing when that exists. Yet, in higher dimensions there are singularities having various smoothings with different topology.

There are two other viewpoints I wish to mention, which have been explored by various people: One is the via indices of vector fields and 1-forms; another is via Chern classes for singular varieties. Let us sketch these. We refer to [7] for a thorough account on the material in this section and bibliography about it.

**4.1. Indices of vector fields and 1-forms: The homological index.** Let $(V, \underline{0})$ be a normal complex isolated singularity germ of dimension $n \geq 1$, defined in some $\mathbb{C}^N$. Let $v$ be a continuous vector field on $V$. That is, $v$ is the restriction to $V$ of a continuous vector field in a neighborhood of $\underline{0}$ in $\mathbb{C}^N$, which is tangent to $V$ at each $x \in V \setminus \{\underline{0}\}$. We assume $v$ is singular at $\underline{0}$ and it has no more singularities on $V$.

Indices of vector fields on singular varieties appear first in work by M. H. Schwartz [37] and R. MacPherson [28] in relation with Chern classes for singular varieties. They considered vector fields which are radial, which in our setting means that restricted to $V$ they are transversal to every small sphere in $\mathbb{C}^N$ centered at $\underline{0}$.

The first concept of an index of vector fields in general, on singular varieties, was introduced in [7] and is known as the GSV index. This is defined for vector fields as above provided $(V, \underline{0})$ is an ICIS. In that case the GSV index is defined by:

$$\mathrm{Ind}_{\mathrm{GSV}}(v; (V, \underline{0})) = \mathrm{Ind}_{\mathrm{PH}}(v; V_t)\,,$$

where the latter is the total Poincaré-Hopf index of an extension of $v$ to a Milnor fibre of $(V,\underline{0})$. A basic property of this index is that if $v$ is radial at $\underline{0}$, then its index is the Euler characteristic of the Milnor fibre. The GSV index has an interpretation as the "virtual index", defined in terms of Chern-Weil theory (see [7]). And it has also an algebraic interpretation in [17], the homological index, which is particularly interesting for what follows, because is does not need the germ $(V,\underline{0})$ to be an ICIS. We return to this point below.

There is another concept of index which is relevant in the sequel, the radial index, that we denote $\mathrm{Ind}_{\mathrm{rad}}(v;(V,\underline{0}))$. This has several important properties (see for instance [7]). It was first introduced by H. King and D. Trotman in [21] and later (independently) by W. Ebeling and S. Gusein-Zade [12] and by M. Aguilar *et al* in [1].

When the vector field $v$ is radial, then $\mathrm{Ind}_{\mathrm{rad}}(v;(V,\underline{0})) = 1$ by definition. Otherwise we must consider too a contribution for "its lack of radiality": We consider a cylinder $V_{(\epsilon,\epsilon')}$ in $V$ bounded by two sufficiently small spheres $\mathbb{S}_\epsilon$, $\mathbb{S}_{\epsilon'}$, $\epsilon > \epsilon' > 0$. We consider a continuous vector field $\zeta$ on $V_{(\epsilon,\epsilon')}$ with isolated singularities, which restricts to $v$ on $V \cap \mathbb{S}_\epsilon$ and to a radial vector field on $V \cap \mathbb{S}_{\epsilon'}$. Then define

$$\mathrm{Ind}_{\mathrm{rad}}(v;(V,\underline{0})) = 1 + \mathrm{Ind}_{\mathrm{PH}}(\zeta;(V_{(\epsilon,\epsilon')})\,,$$

where the latter is the total Poincaré-Hopf index of $\zeta$ on the manifold $V_{(\epsilon,\epsilon')}$.

An interesting basic property of this index is that it is defined for all continuous vector fields on arbitrary isolated singularities, even real analytic ones, and if the germ $(V,\underline{0})$ is a complex ICIS, then its difference with the GSV index is the Milnor number up to sign, independently of the choice of vector field.

These concepts of indices have been extended to 1-forms by W. Ebeling and S. Gusein-Zade in several articles. The homological index of 1-forms was is introduced in [13] inspired by Gómez-Mont's construction in [17] for vector fields. Let us recall this index for 1-forms:

Consider a normal isolated singularity $(V,\underline{0})$ of dimension $n$ in some $\mathbb{C}^N$, and a germ of a holomorphic 1-form $\omega$ in $\mathbb{C}^n$ with an isolated singularity in $V$ at $p$, *i.e.*, its kernel is transverse to $V$ at every point away from $p$. For each $j \geq 0$, let $\Omega^j(V,\underline{0})$ be the space of $j$-forms on the germ $(V,\underline{0})$. One has a complex $(\Omega^\bullet_{V,p},\wedge\omega)$:

$$0 \longrightarrow \Omega^0(V,\underline{0}) \xrightarrow{\wedge\omega} \Omega^1(V,\underline{0}) \xrightarrow{\wedge\omega} \cdots \xrightarrow{\wedge\omega} \Omega^n(V,\underline{0}) \longrightarrow 0$$

where the arrows are exterior multiplication by $\omega$.

One has the homology of this complex in the usual way, and this is all finite dimensional. We then define:

Definition 4.2. The homological index $\mathrm{Ind}_{\mathrm{hom}}(\omega;(V,\underline{0}))$ is the Euler characteristic of this complex up to sign:

$$\mathrm{Ind}_{\mathrm{hom}}(\omega;(V,\underline{0})) := \sum_{i=0}^{n}(-1)^{n-i}\, h_i(\Omega^\bullet_{V,p},\wedge\omega).$$

This construction is dual to the original one in [17] for vector fields, where the arrows go in reverse sense:

$$\Omega^j(V,\underline{0}) \xrightarrow{\vdash v} \Omega^{j-1}(V,\underline{0})\,,$$

where $\vdash$ is the contraction of forms by the vector field.

If the germ $(V, \underline{0})$ is an ICIS, then the homological index coincides with the GSV-index. The proof of this fact for 1-forms is given in [13] and essentially follows Greuel's computations for the celebrated Lê-Greuel formula for the Milnor number of an ICIS. In the case of vector fields, this was proved in Gómez-Mont's first paper on the topic, [17], when $(V, \underline{0})$ is a hypersurface germ. The proof in the ICIS case is much harder and was proved recently in [10]. It follows that in this case, for a 1-form on $V$ with an isolated singularity at $\underline{0}$, the index $\mathrm{Ind}_{\mathrm{hom}}(\omega; (V, \underline{0}))$ is the number of singularities of the restriction of $\omega$ to a Milnor fibre (counted with their local multiplicities). As noted in [13], we then have:

$$\mu(V, \underline{0}) = \mathrm{Ind}_{\mathrm{hom}}(\omega; (V, \underline{0})) - \mathrm{Ind}_{\mathrm{Rad}}(\omega; (V, \underline{0})) \,. \tag{4.2}$$

That is, the Milnor number is the difference between the homological and the radial indices of the 1-form $\omega$, independently of the choice of $\omega$.

Now, if the germ $(V, \underline{0})$ is not an ICIS, the radial and the homological indices are still defined: What is the left hand side of equation (4.2) in this more general case? Is this the Laufer invariant when $V$ is a two-dimensional Gorenstein singularity?

Remark 4.3. *It is an exercise to show that given an arbitrary isolated singularity germ $(V, \underline{0})$, the difference of the radial and the homological indices of a holomorphic 1-form on $(V, \underline{0})$ is independent of the choice of the 1-form, so it is an invariant $\nu := \nu(V, \underline{0})$ of the germ, introduced in [13]. If this germ is an ICIS, $\nu$ is the Milnor number: What is it in general? If $(V, \underline{0})$ is a Gorenstein surface singularity, is $\nu + 1$ the Laufer invariant?*

Remark 4.4. [The total homological index] *Let $X$ be a compact complex analytic variety of pure dimension $n$ with isolated singularities, and $v$ a continuous vector field on $X$ with isolated singularities. Assume further that $v$ is holomorphic in a neighborhood of each singular point of $X$. Then one has a total homological index* $\mathrm{Ind}_{\mathrm{hom}}(v; X)$ *defined in the obvious way: It is the sum of the homological indices of $v$ at the singularities of $X$, plus the usual Poincaré-Hopf local indices at the smooth points of $X$. It is easy to show that* $\mathrm{Ind}_{\mathrm{hom}}(v; X)$ *is independent of the choice of $v$ and depends only on $X$. What is this invariant, that we may denote* $\mathrm{Ind}_{\mathrm{hom}}(X)$*? If $X$ is a complete intersection, then* $\mathrm{Ind}_{\mathrm{hom}}(X)$ *is the top dimensional Fulton-Johnson class of $X$ evaluated on the orientation cycle. What is it in general?.*

**4.2. The Milnor classes.** Milnor classes were introduced by various authors at about the same time and we refer to [7] for references and a thorough account on the subject. We just sketch here the main ideas in relation with this article.

Recall that a complex manifold $M$ of dimension $n$ has Chern classes $c_1(M), \cdots, c_n(M)$. These can be defined in various equivalent ways, as for instance:

i) As the primary obstructions for constructing appropriate frames of vector fields;

ii) In an axiomatic way;

ii) Using connections, via Chern-Weil theory.

Chern classes are important invariants that play a fundamental role in many areas of mathematics. It is thus natural to search for extensions of these to singular varieties. Nowadays there are several non-equivalent such extensions, each having its own properties and interest. In a way each of these is related to some kind of extension of the concept of tangent bundle to the case of singular varieties.

The first extension of Chern classes to singular varieties was by M. H. Schwartz [37] in the early 1960s. She considered a compact singular variety $X$ of dimension $n$ in some complex manifold $M$, equipped with a Whitney stratification. Let $\mathcal{U}$ be a regular neighborhood of $X$ in $M$; this is itself a complex manifold, union of strata. She then considered stratified vector fields and used obstruction theory to obtain special representatives of the Chern classes of $\mathcal{U}$ adapted to $X$ in some appropriate sense. These classes are elements in $H^*(\mathcal{U},\mathcal{U} \setminus X)$ that depend only on $X$ and not on the manifold $M$. When $X$ is non-singular we have a natural isomorphism $H^*(\mathcal{U},\mathcal{U}\setminus X) \cong H^*(X)$ (with appropriate shifts in dimensions) and the Schwartz classes become the usual Chern classes.

Later, in 1974, R. MacPherson gave in [28] a different construction of Chern classes for singular varieties and showed that these satisfy certain axioms, thus answering a conjecture by Deligne and Grothendieck. These classes live in the homology, and when the space is a compact manifold, they are dual to the usual Chern classes.

MacPherson's construction uses the Nash transform $\widetilde{X} \stackrel{\nu}{\to} X$; the projection $\nu$ is an isomorphism over the regular part $X_{\rm reg}$ of $X$. One has the Nash bundle $\widetilde{T} \to \widetilde{X}$ which essentially coincides with the tangent bundle on $X_{\rm reg}$. The Chern classes of $\widetilde{T}$ live in $H^*(\widetilde{X})$; they can be mapped into $H_*(\widetilde{X})$ by the Alexander homomorphism and then pushed down into the homology of $X$. The classes one gets in $H_*(X)$ are by definition the Mather classes of $X$. The MacPherson classes are obtained from these by putting appropriate weights on each stratum of some Whitney stratification; the weights in question are given by an invariant introduced by MacPherson which is called the local Euler obstruction.

Brasselet and Schwartz in [8] proved that the Alexander duality isomorphism

$$H^*(\mathcal{U},\mathcal{U} \setminus X) \cong H_*(X)$$

carries the Schwatz classes into those defined by MacPherson, thence these became known as the Chern-Schwartz-MacPherson classes, that we may denote by $\hat{c}_i^{SM} \in H_{2i}(X;\mathbb{Z})$, $i = 1, ..., n$. Essentially by definition one has that the 0-degree term is the Euler characteristic.

On the other hand, in [16] the authors gave another construction of Chern classes for singular varieties using Segre's class. This is easy to explain when $X$ is a global complete intersection in some complex manifold $M$, defined by a regular sequence $(s_1, \cdots, s_k)$ of sections of some holomorphic bundle $E$ of rank $k$ over $M$. Restricted to the regular part of $X$ the bundle $E$ is isomorphic to the normal bundle of $X$ (for some Riemannian metric) and therefore one may call:

$$\tau(X) := TM|_X - E|_X \in KU(X)\,,$$

the virtual tangent bundle of $X$. The Chern classes of $\tau(X)$ live in $H^*(X)$ and are known as the Fulton-Johnson classes of $X$.

As mentioned above, when $X$ is not a complete intersection, the definition of its Fulton-Johnson classes uses the Segre class. This is as follows: Consider an arbitrary singular variety $X$ in a compact m-manifold $M$. Its Segre class $s(X, M)$ is defined by considering the blow up of $M$ along $X$:

$$b : B \longrightarrow M\,;$$

let $L$ be the tautological line bundle of $B$ and $D$ the corresponding divisor, which maps onto $X$. Then the Segre class is

$$s(X,M) = b_*\Big(\sum_{i\geq 0}^{m-1} c_1(L)^i \cap D\Big) \in H_*(X)\,.$$

Now consider the total Chern class $c_*(TM)$. Then the total Fulton-Johnson class of $X$ is defined as:

$$c_*^{FJ}(X) := c_*(TM) \cap s(X,M) \in H_*(X)\,.$$

For complete intersections this coincides with the previous definition via the Alexander homomorphism $H^*(X) \to H_*(X)$.

Let us denote the homology Fulton-Johnson classes by $\hat{c}_i^{FJ}(X)$, $i = 1, \cdots, n$. These provide another extension of Chern classes to the case of singular varieties, and it is natural to ask how these relate to the Chern-Schwartz-MacPherson classes. This problem was first studied by P. Aluffi in [2].

One has the following result from [42]:

THEOREM 4.5 (Seade-Suwa). *Let $X$ be an $n$-dimensional compact local complete intersection in a complex manifold, with only isolated singularities. Then $\hat{c}_0^{FJ}(X)$ is the total GSV-index of a vector field on $X$ and one has:*

$$\hat{c}_0^{FJ}(X) = \chi(X) + (-1)^n \sum \mu_j\,,$$

*where the $\mu_j$ are the local Milnor numbers at the singular points of $X$.*

Thence one has that if $X$ is a complete intersections with isolated singularities, then:

$$\hat{c}_0^{FJ}(X) - \hat{c}_0^{SM}(X) = (-1)^{n+1} \sum \mu_j\,.$$

It is thus natural to define:

DEFINITION 4.6. Let $X$ be a compact complex analytic variety of pure dimension $n$. Its $i^{th}$ Milnor class is:

$$\mathcal{M}_i(X) := (-1)^n \Big(\hat{c}_i^{FJ}(X) - \hat{c}_i^{SM}(X)\Big)\,.$$

There is a large literature on Milnor classes and we refer to [7] for more on the topic.

We now focus our attention on one very particular case. Let $(V, \underline{0})$ be a complex isolated singularity germ of dimension $n > 1$ and consider a compactification $\overline{V}$ of this germ, with a unique singular point at $\underline{0}$. Consider the Milnor classes $\mathcal{M}_i(\overline{V})$. Since these have support at the singular set, all Milnor classes vanish, except $\mathcal{M}_0(\overline{V})$ which depends only on the germ $(V, \underline{0})$ and not on the choice of the compactification. If this germ is an ICIS, then we know that $\mathcal{M}_0(\overline{V})$ is the Milnor number. What is the 0-degree Milnor class $\mathcal{M}_0(\overline{V})$ when $(V, \underline{0})$ is not a complete intersection germ?

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