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SINGULARITIES OF LOW DEGREE COMPLETE INTERSECTIONS*

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Dedicated to Henry Laufer on the occasion of his 70th birthday

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1. Introduction. When discussing the singularities of the minimal model program (see [Kol13, 2.8]) arguably one of the most frequently used examples is a cone over a smooth projective cubic plane curve. This is the simplest example of a log canonical singularity that is not rational (see [Kol13, 2.76]) and hence in particular not log terminal. It is also an example for an extreme case of these singularities in the sense that a cone over a smooth projective plane curve of degree d is log canonical if and only if $d \leq 3$.

In fact this last statement generalizes to arbitrary dimension:

LEMMA 1.1. *Let $H \subseteq \mathbb{P}^n$ be a smooth hypersurface of degree d and $X \subseteq \mathbb{P}^{n+1}$ the projective cone over H and assume that $n \geq 2$. Then X has log canonical singularities if and only if $d \leq n+1$.*

Proof. By the adjunction formula $\omega_H \simeq \mathscr{O}_{\mathbb{P}^n}(d-n-1)\big|_H$. Blowing up the vertex of X is a resolution, $\sigma : \widetilde{X} \to X$ with exceptional set $E \simeq H$. Then, using this isomorphism, and the properties of blowing up implies that $\mathscr{O}_{\widetilde{X}}(E)\big|_E \simeq \mathscr{O}_{\mathbb{P}^n}(-1)\big|_H$. Writing $\omega_{\widetilde{X}} \simeq \sigma^*\omega_X(aE)$ and using adjunction for $E \subseteq \widetilde{X}$ we obtain that

$$\mathscr{O}_{\mathbb{P}^n}(d-n-1)\big|_H \simeq \omega_E \simeq \omega_{\widetilde{X}}(E)\big|_E \simeq \mathscr{O}_{\widetilde{X}}((a+1)E)\big|_E \simeq \mathscr{O}_{\mathbb{P}^n}(-a-1))\big|_H,$$

and hence that $n-d = a$. Now X has log canonical singularities if and only if $n-d = a \geq -1$ and therefore if and only if $d \leq n+1$. □

The purpose of this note is to prove that a similar condition exists under much more general conditions. In particular we will allow X to be a complete intersection and instead of requiring that X is a cone we only restrict the size of its singular set. We will also show that the condition is sharp.

The main tool used in the proof is the equivalence of log canonical and Du Bois singularities for complete intersections (see [Kov99, KSS10, KK10]) and a characterization of Du Bois singularities on projective varieties (see [Kov12]). For some of the background on these see Section 2.

DEFINITION 1.2. A *complete intersection of multidegree* $(d_1,\dots,d_r)$ is a complete intersection $X = H_1 \cap \dots \cap H_r \subseteq \mathbb{P}^N$ where $H_i \subseteq \mathbb{P}^N$ is a degree $\deg H_i = d_i$ hypersurface for each $i = 1,\dots,r$. Note that then

$$\omega_X \simeq \mathscr{O}_X\left(-(N+1)+\sum d_i\right) \simeq \mathscr{O}_{\mathbb{P}^N}\left(-(N+1)+\sum d_i\right)\Big|_X. \tag{1}$$

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Definition 1.3. Let $X \subseteq \mathbb{P}^N$ be a projective variety of $\dim X = n$ and define the *canonical sheaf* as $\omega_X := \mathcal{E}xt^{N-n}_{\mathbb{P}^N}(\mathscr{O}_X, \omega_{\mathbb{P}^N})$ cf. [Har77, III.7.5].

Theorem 1.4. *Let $X \subseteq \mathbb{P}^N$ be an irreducible complete intersection of multidegree $(d_1, \dots, d_r)$. Then*

(i) if $\dim \operatorname{Sing} X \leq N - \sum d_i$, then X is log canonical, and

(ii) for any triple $n, s, d \in \mathbb{N}$, such that $n > s > n+1-d$ there exists an irreducible hypersurface $X \subseteq \mathbb{P}^{n+1}$ of degree d such that

(a) $\dim \operatorname{Sing} X = s$, and

(b) X is not log canonical.

Note that if in addition $n > s+1$, then X is automatically normal.

2. Review of log canonical and Du Bois singularities. If X is a smooth proper variety, then Hodge theory tells us that there is a strong link between topological (say singular) and analytic (say Dolbeault) cohomology. In particular, there is a surjective map

$$H^i(X, \mathbb{C}) \twoheadrightarrow H^i(X, \mathscr{O}_X). \tag{2}$$

This seemingly innocent fact has far reaching consequences: it plays an important role in the proof of the Kodaira vanishing theorem [Kol87] and has some nice consequences for deformations of smooth proper varieties.

Because of the usefulness of this map we are interested in finding out how this could be extended to (some) singular varieties. Let us first recall where this map comes from.

For a smooth proper variety, the Hodge-to-de-Rham (a.k.a. Frölicher) spectral sequence degenerates at E_1 hence the singular cohomology group $H^i(X, \mathbb{C})$ admits a Hodge filtration

$$H^i(X, \mathbb{C}) = F^0 H^i(X, \mathbb{C}) \supseteq F^1 H^i(X, \mathbb{C}) \supseteq \dots \tag{3}$$

and in particular there exists a natural surjective map

$$H^i(X, \mathbb{C}) \twoheadrightarrow Gr^0_F H^i(X, \mathbb{C}) \tag{4}$$

where

$$Gr^0_F H^i(X, \mathbb{C}) \simeq H^i(X, \mathscr{O}_X). \tag{5}$$

Deligne's theory of (mixed) Hodge structures implies that even if X is singular (but still proper) there still exists a Hodge filtration and (4) remains true, but in general (5) fails.

However, there is something one can still say in general: Even if X is singular (but still proper) there exist natural maps between these groups; namely the map from (4) factors through $H^i(X, \mathscr{O}_X)$ (see [Kov12, 2.3] for a more precise statement):

$$H^i(X, \mathbb{C}) \xrightarrow[\beta]{} H^i(X, \mathscr{O}_X) \xrightarrow[\gamma]{} Gr^0_F H^i(X, \mathbb{C}), \qquad \alpha = \gamma \circ \beta. \tag{6}$$

Du Bois singularities were introduced by Steenbrink to identify the class of singularities for which γ in the above diagram is an isomorphism, that is, those for which (5) remains true as well. However, naturally, one does not define a class of singularities by properties of proper varieties. Singularities should be defined by local properties and Du Bois singularities are indeed defined locally. For the precise definition see [Kol13, §6.1].

It is known that rational singularities are Du Bois (conjectured by Steenbrink and proved in [Kov99]) and so are log canonical singularities (conjectured by Kollár and proved in [Kov99], [KSS10] in special cases and in [KK10] in full generality). These properties make Du Bois singularities very important in higher dimensional geometry, especially in moduli theory (see [Kol13] for more details on applications). For the definition of rational and log canonical singularities see [Kol13, 2.76] and [Kol13, 2.8] respectively.

Unfortunately, the definition of Du Bois singularities is rather technical. The most important and useful fact about them is the consequence of (4) and (5) that if X is a proper variety over $\mathbb{C}$ with Du Bois singularities, then the natural map

$$H^i(X,\mathbb{C}) \twoheadrightarrow H^i(X,\mathscr{O}_X) \tag{7}$$

is surjective.

One could try to take this as a definition, but it would not lead to a good result for two reasons. As mentioned earlier, singularities should be defined locally and it is not at all likely that a global cohomological assumption would turn out to be a local property. Second, this particular condition could obviously hold "accidentally" and lead to the inclusion of singular spaces that should not be included, thereby further lowering the chances of having a local description of this class of singularities.

Therefore the reasonable approach is to keep Steenbrink's original definition [Ste83, (3.5)] (for a more general definition see [Kol13, §6.1]), after all it has been proven to define a useful class. It does satisfy the first requirement above: it is defined locally. Once that is accepted, one might still wonder if proper varieties with Du Bois singularities could be characterized with a property that is close to requiring that (7) holds.

It turns out that there exists a characterization like that.

As we have already observed, simply requiring that (7) holds is likely to lead to a class of singularities that is too large. A more natural requirement is to ask that (5) holds, or in other words that γ is an isomorphism. Clearly, (5) implies (7) by (6), so our goal requirement is indeed satisfied.

The definition [Ste83, (3.5)] of Du Bois singularities easily implies that if X has Du Bois singularities and $H \subset X$ is a general member of a basepoint-free linear system, then H has Du Bois singularities as well. Therefore it is reasonable that in trying to give an intuitive definition of Du Bois singularities, one may assume that the defining condition holds for the intersection of general members of a fixed basepoint-free linear system.

In fact, one can make the condition numerical. This is a trivial translation of the "real" statement, but further emphasizes the simplicity of the criterion.

In order to do this we need to define some notation: Let X be a proper algebraic variety over $\mathbb{C}$ and consider Deligne's Hodge filtration $F^\bullet$ on $H^i(X,\mathbb{C})$ as in (3). Let

$$Gr^p_F H^i(X,\mathbb{C}) = {F^p H^i(X,\mathbb{C})}\Big/{F^{p+1} H^i(X,\mathbb{C})}$$

and

$$\mathsf{f}^{\mathsf{p},\mathsf{i}}(X) = \dim_{\mathbb{C}} Gr^p_F H^i(X,\mathbb{C}).$$

I will also use the usual notation

$$\mathsf{h}^{\mathsf{i}}(X, \mathscr{O}_X) = \dim_{\mathbb{C}} H^i(X, \mathscr{O}_X).$$

Recall (cf. (6)) that the natural surjective map from $H^i(X, \mathbb{C})$ factors through $H^i(X, \mathscr{O}_X)$:

$$H^i(X, \mathbb{C}) \longrightarrow H^i(X, \mathscr{O}_X) \longrightarrow Gr_F^0 H^i(X, \mathbb{C}).$$

In particular, the natural morphism

$$H^i(X, \mathscr{O}_X) \twoheadrightarrow Gr_F^0 H^i(X, \mathbb{C}) \tag{8}$$

is also surjective and hence

$$\mathsf{h}^{\mathsf{i}}(X, \mathscr{O}_X) \geq \mathsf{f}^{\mathsf{0,i}}(X). \tag{9}$$

These inequalities inspire the following definition.

Definition 2.1. Let X be a proper algebraic variety over $\mathbb{C}$. Then X is said to be *numerically Du Bois* if $\mathsf{h}^{\mathsf{i}}(X, \mathscr{O}_X) \leq \mathsf{f}^{\mathsf{0,i}}(X)$ for every $i > 0$.

Remark 2.2. Of course, by 9, X is numerically Du Bois if and only if $\mathsf{h}^{\mathsf{i}}(X, \mathscr{O}_X) = \mathsf{f}^{\mathsf{0,i}}(X)$ for every $i > 0$ and by 5 if X is smooth, then it is numerically Du Bois. (In fact, the definition of Du Bois singularities also imply that they are numerically Du Bois as well).

Now we are almost ready to state the characterization we need. It essentially says that if general complete intersections are all numerically Du Bois, then the ambient variety has Du Bois singularities.

We will use the following theorem, which is a direct consequence of [Kov12, 1.10].

Theorem 2.3. *Let $X \subseteq \mathbb{P}^n$ be a projective variety over $\mathbb{C}$. Then X has Du Bois singularities if and only if X_L is numerically Du Bois for any $X_L \subseteq X$ which is an intersection of X with a set of* $\operatorname{codim}(X_L, X)$ *general hyperplanes in $\mathbb{P}^n$.*

Remark 2.4. Note that X is included among the X_L in the theorem as the intersection of the empty set of general hyperplanes with X.

As mentioned above, Du Bois singularities are closely related to log canonical singularities which gives us the following consequence of Theorem 2.3.

Corollary 2.5. *Let $X \subseteq \mathbb{P}^n$ be a normal projective complete intersection variety over $\mathbb{C}$. Then X has log canonical singularities if and only if X_L is numerically Du Bois for any $X_L \subseteq X$ which is an intersection of X with a set of* $\operatorname{codim}(X_L, X)$ *general hyperplanes in $\mathbb{P}^n$.*

Proof. Since X is normal and Gorenstein, it is log canonical if and only if it is Du Bois by [KK10, 1.4] and [Kov99, 3.6]. □

3. Proof of Theorem 1.4.

LEMMA 3.1. *Let* X *be a complete intersection of* $\dim X = n$ *such that* $N - \sum d_i \geq 0$. *Then* $\mathsf{h}^{\mathsf{i}}(X, \mathscr{O}_X) = 0$ *for* $i > 0$.

Proof. Since X is a complete intersection, the statement is trivial for $0 < i < n$. To prove that $\mathsf{h}^{\mathsf{n}}(X, \mathscr{O}_X) = 0$, observe that by the definition of ω_X it follows from [Har77, III.7.5] that $\mathsf{h}^{\mathsf{n}}(X, \mathscr{O}_X) = \mathsf{h}^0(\mathsf{X}, \omega_{\mathsf{X}})$ and the latter is zero since ω_X^{-1} is ample. ◻

REMARK 3.2. Note that X is not assumed to be smooth, so we cannot use Kodaira vanishing. At the same time, even though by assumption X is Cohen-Macaulay, we do not need to use Serre duality, only the duality that appears in the definition of a dualizing sheaf.

Proof of Theorem 1.4. Clearly we may assume that $N - \sum d_i \geq 0$, since otherwise the assumptions imply that X is smooth and statement follows trivially. We may also assume that X is not contained in a hyperplane and hence we may assume that $d_i > 1$ for every $i = 1, \dots, r$. Then $\sum d_i \geq r+1$ with equality if and only if $r = 1$ and $d_1 = 2$, i.e., if X is a quadric hypersurface. In that case X is again smooth, so we may actually assume that $\sum d_i \geq r + 2$ and hence

$$\dim \operatorname{Sing} X \leq N - \sum d_i \leq N - r - 2 = \dim X - 2. \tag{10}$$

Since X is a complete intersection, it is Gorenstein, in particular S_2, and so X is normal by (10).

Next, let $L_1, \dots, L_q \subseteq \mathbb{P}^N$ be general hyperplanes for some $q \in \mathbb{N}$, $L = L_1 \cap \cdots \cap L_q$ and $X_L = X \cap L$. By the adjunction formula for complete intersections we obtain that

$$\omega_{X_L} \simeq \mathscr{O}_{X_L}\left(-(N+1) + \sum d_i + q\right) \simeq \mathscr{O}_{\mathbb{P}^N}\left(-(N+1) + \sum d_i + q\right)\big|_{X_L}. \tag{11}$$

By Bertini's theorem and the fact that the L_i are general hyperplanes we obtain that $\operatorname{Sing} X_L = (\operatorname{Sing} X) \cap L$ and hence $\dim \operatorname{Sing} X_L = \dim \operatorname{Sing} X - q$.

If $q > \dim \operatorname{Sing} X$, then X_L is smooth and hence numerically Du Bois by 5 (cf. Remark 2.2). If $q \leq \dim \operatorname{Sing} X \leq N - \sum d_i$, then $\dim \operatorname{Sing} X_L \leq N - (\sum d_i + q)$ and so $\mathsf{h}^{\mathsf{i}}(X_L, \mathscr{O}_{X_L}) = 0$ for all $i > 0$ by Lemma 3.1 and hence X_L is numerically Du Bois trivially. Therefore, X_L is numerically Du Bois for all L and hence Theorem 1.4(i) follows from Corollary 2.5.

To prove Theorem 1.4(ii) first observe that a hypersurface, is S_2, so if $n > s+1$, then it is normal. This proves the last sentence. Let $m := n + 1 - s$ and $Z \subseteq \mathbb{P}^m$ a cone over a degree d smooth hypersurface in $\mathbb{P}^{m-1}$. By Lemma 1.1 Z is log canonical if and only if $d \leq m$, so by the assumption that $s > n + 1 - d$, Z is not log canonical.

Next let $\mathbb{P}^{s-1} \subseteq \mathbb{P}^{n+1}$ be a linear subspace and consider the projection $\pi : \mathbb{P}^{n+1} \setminus \mathbb{P}^{s-1} \to \mathbb{P}^m$. Let $X = \overline{\pi^{-1}(Z)} \subseteq \mathbb{P}^{n+1}$ the closure of the pre-image of Z. Then a general complete intersection of X of codimension s is isomorphic to Z and hence X cannot be log canonical either. By construction X satisfies the requirements of Theorem 1.4(ii). ◻

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