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CERTAIN NORMAL SURFACE SINGULARITIES OF GENERAL TYPE*

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Dedicated to Professor Henry B. Laufer on his seventieth birthday

Abstract. Koyama's inequality for normal surface singularities gives the upper bound on the self-intersection number of the canonical cycle in terms of the arithmetic genus. For those singularities of fundamental genus two attaining the bound, a formula for computing the geometric genus is shown and the resolution dual graphs are roughly classified. In Gorenstein case, the multiplicity and the embedding dimension are also computed.

Key words. Even singularity, canonical cycle, Yau cycle, maximal ideal cycle.

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Introduction. Various studies have done on normal singular points of a complex surface. Among others, most fruitful achievements are beautiful theorems on classes of small genera such as rational and elliptic singular points.

In this article, we study a class of normal (numerically) Gorenstein surface singularities of fundamental genus ≥ 2, which may be regarded as "singularities on the Noether line" on the analogy of the geography of global surfaces of general type. To be more precise, let (V, o) be a normal surface singularity and $\pi : X \to V$ a resolution. The arithmetic genus of the fundamental cycle is called the fundamental genus of (V, o) and we denote it by $p_f(V, o)$. The arithmetic genus $p_a(V, o)$ of (V, o) is the maximum of the arithmetic genus of effective divisors with support in $\pi^{-1}(o)$. Let Z_K be the canonical cycle on $\pi^{-1}(o)$, that is, the $\mathbb{Q}$-divisor numerically equivalent to $-K_X$. Then Yoichi Koyama obtained the inequality $-Z_K^2 \geq 8p_a(V, o) - 8$ around 1984 (unpublished). If the equality sign holds here, then one has $Z_K = 2D$ with the unique effective $\mathbb{Z}$-divisor D computing $p_a(V, o)$ and, in particular, (V, o) is numerically Gorenstein. We call such a singular point *even* and study those of fundamental genus 2 in this paper.

In §1, we summarize mostly known results about normal surface singularities for the later use. Some facts are reproved from the point of view of [5]. We discuss in §2 Koyama's inequality and introduce the notion of even singularities. Then we restrict ourselves to those of fundamental genus 2. In this case, a half of the canonical cycle is nothing more than the Yau cycle introduced in [6]. Using such information, we can roughly classify the resolution dual graphs. A general picture of the exceptional set is the core part attached two ADE-branches of (-2)-curves. Here, the core part is the *minimally even cycle*, i.e., the minimal model of the fundamental cycle Z, which itself is the fundamental cycle of an even singular point with $p_f = p_a = 2$. Similarly as in the case of elliptic singularities, minimally even cycles of genus 2 is closely related to the singular fibers in pencils of curves of genus 2 (cf. [9]). Our classification is much coarser than [9] and the cycles fall into five classes (0), (i.a), (i.b), (ii.a) and (ii.b) according to the numerical connectivity and the base locus of the canonical linear

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system. In §3, we establish a formula computing the geometric genus that is modeled on Okuma's formula for elliptic singularities [11]. In §4 and §5, we compute the numerical invariants of (V, o), such as the multiplicity and the embedding dimension, when (V, o) is Gorenstein.

In the course of the study, basic results for 1-connected curves or chain-connected curves are used freely. For these, we refer the readers to [1, Appendix] and [5].

1. Preliminaries. In this paper, a *curve* means a non-zero effective divisor (with support compact) on a smooth surface. For a curve D, the arithmetic genus $p_a(D)$ is defined by $p_a(D) = 1 - \chi(D, \mathcal{O}_D)$, where $\chi(D, \mathcal{O}_D) = h^0(D, \mathcal{O}_D) - h^1(D, \mathcal{O}_D)$. Then $2p_a(D) - 2 = K_X D + D^2$ by the adjunction formula.

1.1. Chain-connected curves. A curve D is said to be *chain-connected* if $\mathcal{O}_{D-\Gamma}(-\Gamma)$ is not nef for any proper subcurve $\Gamma \prec D$. If D is chain-connected and $p_a(D) > 0$, then there exists the unique chain-connected subcurve $D_{\min}$ of D such that $p_a(D_{\min}) = p_a(D)$ and $K_{D_{\min}}$ is nef. We call it the *minimal model* of D.

A maximal chain-connected subcurve of a curve is called a chain-connected component. Every curve D decomposes into a sum of chain-connected curves as $D = \Gamma_1 + \cdots + \Gamma_n$, where Γ_i is a chain-connected component of $D - \sum_{j<i} \Gamma_j$. Then, $\mathcal{O}_{\Gamma_j}(-\Gamma_i)$ is nef when $i < j$, which implies that either $\mathrm{Supp}(\Gamma_i) \cap \mathrm{Supp}(\Gamma_j) = \emptyset$ or $\Gamma_j \preceq \Gamma_i$. Such an ordered decomposition is essentially unique and is called a *chain-connected component decomposition* (a CCCD for short) of D. For the properties, see [5].

Lemma 1.1. *Let D be a chain-connected curve on a smooth surface X.*

(1) *If $\mathcal{O}_D(-D)$ is nef and $p_a(D) \leq 1$, then $p_a(\Gamma) \leq p_a(D)$ for any curve Γ with support in* $\mathrm{Supp}(D)$.

(2) *Let $p \in D$ and put $\nu = \min\{\mathrm{mult}_p(A) \mid A \preceq D$ is a component, $p \in A\}$ If $\rho : \tilde{X} \to X$ is the blowing-up at p and $E = \rho^{-1}(p)$, then $\rho^* D - kE$ is chain-connected for $0 \leq k \leq \nu$.*

Proof. (1) Assume first that Γ is chain-connected. Since $\mathcal{O}_\Gamma(-D)$ is nef, $\Gamma \preceq D$. Then $p_a(\Gamma) \leq h^1(\mathcal{O}_\Gamma) \leq h^1(\mathcal{O}_D) = p_a(D)$. Consider the general case and let $\Gamma = \sum_{i=1}^n \Gamma_i$ be a CCCD. Then $p_a(\Gamma) - 1 = \sum_{i=1}^n (p_a(\Gamma_i) - 1) + \sum_{i<j} \Gamma_i \Gamma_j$. Since $\Gamma_i \Gamma_j \leq 0$ when $i < j$, and $p_a(\Gamma_i) \leq p_a(D)$ by the first step, we get $p_a(\Gamma) \leq p_a(D)$ when $p_a(D) = 0, 1$.

(2) We fix a component A satisfying $\mathrm{mult}_p(A) = \nu$ and take a connecting chain (cf. [5]) $D_0 = A, \ldots, D_N = D$ starting from A, where $A_i = D_i - D_{i-1}$ is an irreducible curve with $A_i D_{i-1} > 0$. Let $\overline{A}$ be the proper transform of A. Then $\rho^* A = \overline{A} + \nu E$. Put $\overline{D}_i = \rho^* D_i - \nu E$ for any i. Since $\overline{D}_i - \overline{D}_{i-1} = \rho^* A_i$, $\overline{D}_i$'s are effective and non-zero for any i. We construct a connecting chain for $\rho^* D - \nu E$ by refining $\{\overline{D}_i\}$. For this purpose, it suffices to consider i such that $\rho^* A_i$ is reducible, and to construct a connecting chain from $\overline{D}_{i-1}$ to $\overline{D}_i$. Since $\rho^* A_i$ is reducible, $p \in A_i$. If $\overline{A}_i$ denotes the proper transform of A_i by ρ, then $\rho^* A_i = \overline{A}_i + \nu_i E$, where $\nu_i = \mathrm{mult}_p(A_i)$. We have $\nu_i \geq \nu$ by the definition of ν. Put

$$\Gamma_j = \begin{cases} \overline{D}_{i-1} + jE, & \text{if } 0 \leq j \leq \nu, \\ \rho^* D_{i-1} + \overline{A}_i, & \text{if } j = \nu + 1, \\ \Gamma_{\nu+1} + (j - \nu - 1)E, & \text{if } \nu + 2 \leq j \leq \nu_i + 1 \text{ and } \nu_i > \nu. \end{cases}$$

Then $\Gamma_0 = \overline{D}_{i-1}, \Gamma_1, \ldots, \Gamma_{\nu_i+1} = \overline{D}_i$ is the desired connecting chain, since $\Gamma_j - \Gamma_{j-1}$, which is either E or $\overline{A}_i$, is irreducible and $\Gamma_{i-1}(\Gamma_i - \Gamma_{i-1}) > 0$. Therefore, $\rho^* D - \nu E$

is chain-connected. Since $(\rho^*D - (\nu - i)E)E > 0$ when $0 \le i < \nu$, we see inductively that $\rho^*D - (\nu - i - 1)E$ is chain-connected by [5, Proposition 1.5]. □

PROPOSITION 1.2. *Let D be a chain-connected curve with $p_a(D) > 0$ and $D_{\min}$ its minimal model. Assume that* $\mathrm{Supp}(D - D_{\min})$ *contracts to several rational double points and that $-D$ is nef on* $\mathrm{Supp}(D - D_{\min})$. *Put $(D - D_{\min})^2 = -2n$. Then the following hold.*

(1) *$-(D - D_{\min})$ is nef on its support.*

(2) *$D - D_{\min}$ decomposes as $\Gamma_1 + \cdots + \Gamma_n$, where each Γ_i is the fundamental cycle on its support, $-\Gamma_i^2 = 2$, $D_{\min}\Gamma_i = 1$ for any i and $\mathcal{O}_{\Gamma_j}(-\Gamma_i) \simeq \mathcal{O}_{\Gamma_j}$ for any pair (i, j) with $i < j$. In particular, either $\Gamma_j \prec \Gamma_i$ or* $\mathrm{Supp}(\Gamma_i) \cap \mathrm{Supp}(\Gamma_j) = \emptyset$ *for $i < j$.*

(3) *If $n \ge 2$ and $\Gamma_2 \prec \Gamma_1$ in (2), then* $(\mathrm{type}(\Gamma_1), \mathrm{type}(\Gamma_2)) = (\mathsf{A}_{l+2}, \mathsf{A}_l)$ $(l \ge 1)$, $(\mathsf{D}_{l+2}, \mathsf{D}_l)$ $(l \ge 3)$, $(\mathsf{D}_l, \mathsf{A}_1)$ $(l \ge 4)$, $(\mathsf{E}_6, \mathsf{A}_5)$, $(\mathsf{E}_7, \mathsf{D}_6)$ *or* $(\mathsf{E}_8, \mathsf{E}_7)$, *with an obvious convention that* $\mathsf{D}_3 = \mathsf{A}_3$.

Proof. (1) Assume that there exists an irreducible component $C \preceq D - D_{\min}$ such that $C(D - D_{\min}) > 0$. By the assumption, we have $CD \le 0$ and, hence, $CD_{\min} < 0$. In particular, C is a component of $D_{\min}$. Then, since $K_{D_{\min}}$ is nef, we have $0 \le \deg K_{D_{\min}}|_C = \deg K_C + C(D_{\min} - C)$, from which we get the contradiction that $CD_{\min} \ge 0$ by $\deg K_C = C^2 = -2$. Therefore, $-(D - D_{\min})$ is nef on $\mathrm{Supp}(D - D_{\min})$.

(2) By [5, Lemma 3.6], $D - D_{\min}$ decomposes as $D - D_{\min} = \Gamma_1 + \cdots + \Gamma_n$, where Γ_i is a chain-connected curve, $\mathcal{O}_{\Gamma_j}(-\Gamma_i)$ is numerically trivial for $i < j$, and $D_{\min} + \Gamma_i$ is a chain-connected curve satisfying $D_{\min}\Gamma_i = 1 - p_a(\Gamma_i)$ for $i \in \{1, 2, \ldots, n\}$. In our case, $p_a(\Gamma_i) = 0$ and, hence, $D_{\min}\Gamma_i = 1$, $\Gamma_i^2 = -2$ and $\mathcal{O}_{\Gamma_j}(-\Gamma_i) \simeq \mathcal{O}_{\Gamma_j}$ for $i < j$. Furthermore, $(D - D_{\min})^2 = (\sum_{i=1}^n \Gamma_i)^2 = \sum_{i=1}^n \Gamma_i^2 = -2n$. So, it suffices to see that every Γ_i is the fundamental cycle on its support.

Since $-(D - D_{\min})$ is nef by (1), every chain-connected component of $D - D_{\min}$ is the fundamental cycle on a connected component of $\mathrm{Supp}(D - D_{\min})$. Hence so is Γ_1. Assume that Γ_i is the fundamental cycle on its support for $i \le k$. Now, Γ_{k+1} is a chain-connected component of $D - D_{\min} - \sum_{i=1}^k \Gamma_k$. Since $-(D - D_{\min} - \sum_{i=1}^k \Gamma_i)$ is nef on its support by the property $\mathcal{O}_{\Gamma_j}(-\Gamma_i) \simeq \mathcal{O}_{\Gamma_j}$ for $i < j$, we see that Γ_{k+1} is the fundamental cycle on its support. Therefore, we are done by induction.

(3) We have $\Gamma_2 \prec \Gamma_1$ and $(\Gamma_1 - \Gamma_2)\Gamma_2 = 2$. Let C be a component of Γ_1 with $C\Gamma_2 = 1$. Since Γ_2 is the fundamental cycle and, either $\Gamma_i \prec \Gamma_2$ or $\mathrm{Supp}(\Gamma_2) \cap \mathrm{Supp}(\Gamma_i) = \emptyset$ for $i \ge 3$, we see that $C \not\preceq \Gamma_i$ for $i \ge 2$. Then we get $C\Gamma_1 < 0$ from $0 \ge CD = C\Gamma_1 + 1 + C\sum_{i \ge 3} \Gamma_i \ge C\Gamma_1 + 1$. If Γ_1 is of type A_l for some $l \ge 2$, then C is one of the end-components of Γ_1 and it follows that Γ_2 is of type A_{l-2}, because, by $(\Gamma_1 - \Gamma_2)\Gamma_2 = 2$, we have to remove both end-components from Γ_1 in order to obtain Γ_2. As to the other types of Γ_1, note that C as above is the unique component of multiplicity 2 of Γ_1 which is located at an end of the Dynkin diagram except in the case D_l where C is the "second" component. Furthermore, $\Gamma_1 - 2C$ has connected support unless Γ_1 is of type D_l. Then one can determine the possible type of Γ_2 by examining the weighted dual graph of Γ_1. We leave the detail to the reader. □

1.2. Cycles over a normal surface singularity. Let (V, o) be a normal surface singular point and $\pi : X \to V$ a resolution. Let Z be the *fundamental cycle* on $\pi^{-1}(o)$, that is, the smallest curve with support in $\pi^{-1}(o)$ such that $-Z$ is nef on $\pi^{-1}(o)$. It is chain-connected. Its arithmetic genus is called the *fundamental genus* and denoted by $p_f(V, o)$. The *arithmetic genus* of (V, o) which we denote by $p_a(V, o)$ is the maximum of $p_a(D) := D(K_X + D)/2 + 1$ when D runs over all curves with support in $\pi^{-1}(o)$. Clearly, we have $p_f(V,o) \le p_a(V,o)$. The *geometric genus* is defined by $p_g(V,o) = \dim_{\mathbb{C}}(R^1\pi_*\mathcal{O}_X)_o$. Then $p_a(V,o) \le p_g(V,o)$. When $p_g(V,o) = 0$ (resp. $p_a(V,o) = 1$), (V,o) is called rational (resp. elliptic). It is known (and follows from Lemma 1.1) that (V,o) is rational (resp. elliptic) if and only if $p_f(V,o) = 0$ (resp. 1).

According to [12], we put

$$Z_1 = \min\{D \mid h^1(D,\mathcal{O}_D) = p_g(V,o),\ D \text{ is a curve}\}$$

and call it the *cohomological cycle* on $\pi^{-1}(o)$. If $p_g(V,o) > 0$, then we have $h^1(D,\mathcal{O}_D) = p_g(V,o)$ when $Z_1 \preceq D$, and $h^1(D,\mathcal{O}_D) < p_g(V,o)$ when $Z_1 \not\preceq D$.

Since the intersection form is negative definite on $\pi^{-1}(o)$, there is a $\mathbb{Q}$-divisor Z_K with support in $\pi^{-1}(o)$ such that $-Z_K$ is numerically equivalent to a canonical divisor K_X. We call Z_K the *canonical cycle*. If it is integral, then (V,o) is said to be *numerically Gorenstein*. In this case, it is shown in [12] that $Z_1 \preceq Z_K$ on the minimal resolution (see also [3]). In particular, we have $p_a(Z_K) = 1$ and $h^1(Z_K,\mathcal{O}_{Z_K}) = h^0(Z_K,\mathcal{O}_{Z_K}) = p_g(V,o)$. The singular point is Gorenstein if and only if $-Z_K$ is linearly equivalent to K_X. When $p_f(V,o) > 0$, $Z_1 = Z_K$ holds if and only if (V,o) is Gorenstein (see, e.g., [3]).

We remark the following:

Lemma 1.3. *Let $\pi : X \to V$ be a resolution of a normal surface singular point (V,o) with $p_g(V,o) > 0$ and Z the fundamental cycle on $\pi^{-1}(o)$. Let L be a line bundle on X that is numerically trivial on $\pi^{-1}(o)$. Then L is trivial if and only if the restriction map $H^0(X,L) \to H^0(Z,L)$ is not the zero map. In particular, when (V,o) is numerically Gorenstein and π is minimal, (V,o) is Gorenstein if and only if $\omega_{Z_K} \simeq \mathcal{O}_{Z_K}$.*

Proof. One direction is trivial. So, we assume that L is non-trivial and show that $H^0(X,L) \to H^0(Z,L)$ is the zero map. We may assume that $H^0(X,L) \neq 0$. Take any non-zero $s \in H^0(X,L)$ and write its zero divisor as $(s) = A + B$, where $\mathrm{Supp}(A) \subseteq \pi^{-1}(o)$ and $\mathrm{Supp}(B) \cap \pi^{-1}(o)$ is at most a finite set. Since L is numerically trivial and B is nef on $\pi^{-1}(o)$, we see that $-A$ is nef on $\pi^{-1}(o)$. Note that A is a non-zero effective divisor, because $A = 0$ implies $\mathrm{Supp}(B) \cap \pi^{-1}(o) = \emptyset$ and, thus, L is trivial. Then, by the nefness of $-A$, we have $Z \preceq A$. Hence s vanishes identically on Z.

In order to see the last assertion, we apply the above argument to $L = K_X + Z_K$. Note that $\mathcal{O}_{Z_K}(L) \simeq \omega_{Z_K}$ and $H^0(X,L) \to H^0(Z_K,L)$ is surjective by $H^1(X,K_X) = 0$. Since Z is the chain-connected component of Z_K, the restriction map $H^0(Z_K,L) \to H^0(Z,L)$ is not the zero map if and only if $\mathcal{O}_{Z_K}(L) \simeq \mathcal{O}_{Z_K}$. □

The following is a special case of [13, (2.11)].

Lemma 1.4. *If (V,o) is Gorenstein and $p_g(V,o) \ge 2$, then $p_a(V,o) < p_g(V,o)$.*

Proof. We work on the minimal resolution. We assume that there exists a curve D with support in $\pi^{-1}(o)$ satisfying $p_a(D) = p_g(V,o)$, and show that this eventually leads us to a contradiction.

Recall the general inequality $p_a(D) \le h^1(D,\mathcal{O}_D) \le p_g(V,o)$. Hence, in the present case, $p_a(D) = h^1(D,\mathcal{O}_D) = p_g(V,o)$ and it follows $h^0(D,\mathcal{O}_D) = 1$. Furthermore, since $h^1(D,\mathcal{O}_D) = p_g(V,o)$, D contains the cohomological cycle Z_1 as a subcurve. Since (V,o) is Gorenstein, we have $Z_1 = Z_K$. So, $Z_K \preceq D$. Note that we cannot have $D = Z_K$, because $h^0(D, \mathcal{O}_D) = 1$ but $h^0(Z_K, \mathcal{O}_{Z_K}) = p_g(V, o) > 1$. We consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_{Z_K}(K_{Z_K}) \to \mathcal{O}_D(K_D) \to \mathcal{O}_{D-Z_K}(K_D) \to 0.$$

We have $\mathcal{O}_{Z_K}(K_{Z_K}) \simeq \mathcal{O}_{Z_K}$ and $H^0(Z_K, K_{Z_K}) \to H^0(D, K_D)$ is an isomorphism. Furthermore, $H^1(Z_K, K_{Z_K}) \to H^1(D, K_D)$ is non-trivial, being the dual of the restriction map $H^0(D, \mathcal{O}_D) \to H^0(Z_K, \mathcal{O}_{Z_K})$. Hence $h^0(D - Z_K, K_D) = p_g(V, o) - 1 > 0$. On the other hand, one has $\mathcal{O}_{D-Z_K}(K_D) \simeq \mathcal{O}_{D-Z_K}(D - Z_K)$ by the adjunction formula. Since the intersection form is negative definite on $\pi^{-1}(o)$, we get $H^0(D - Z_K, D - Z_K) = 0$. This contradiction implies that we cannot have $p_a(D) = p_g(V, o)$. Therefore, $p_a(V, o) < p_g(V, o)$. □

The *Yau sequence* for Z is the longest sequence of curves

$$D_m \prec D_{m-1} \prec \cdots \prec D_2 \prec D_1 = Z, \tag{1.1}$$

where D_{i+1} is the biggest (non-trivial) subcurve of D_i such that $p_a(D_{i+1}) = p_a(D_i)$ and $\mathcal{O}_{D_{i+1}}(-D_i)$ is numerically trivial ($1 \le i \le m-1$). If $Z_{\min}$ denotes the minimal model of Z, then $Z_{\min} \preceq D_m$ and $Z_{\min} D_m < 0$. It is shown in [6] that each D_i is the fundamental cycle on its support and

$$\mathrm{Supp}(D_k) \cap \mathrm{Supp}(D_i - D_j) = \emptyset \text{ for } i < j < k. \tag{1.2}$$

We put $Y = \sum_{i=1}^m D_i$ and call it the *Yau cycle*. When $p_f(V, o) = 2$, we have $p_a(V, o) = p_a(Y) = m + 1$ by [6, Corollary 2.5].

Lemma 1.5. *Put $Y_\nu = \sum_{i=\nu}^m D_i$ and let $L \in \mathrm{Pic}^0(Y_\nu)$. If $H^0(Y_\nu, L) \ne 0$, then $\mathcal{O}_{Y_i}(L) \simeq \mathcal{O}_{Y_i}(Y_\nu - Y_i)$ for some i, $\nu \le i \le m$.*

Proof. Take a non-zero section $s \in H^0(Y_\nu, L)$. If s does not vanish identically on any components, then L is trivial and the assertion holds with $i = \nu$. Assume not and let C_s be the biggest subcurve of Y_ν on which s vanishes identically. We may assume that there are no sections t such that $C_s \prec C_t$. Then $\mathcal{O}_{Y_\nu - C_s}(L - C_s)$ is nef, because s induces a section of it that does not vanish identically on any components. Applying [6, Lemma 1.3] to $\Delta = Y_\nu - C_s$, we see $\Delta = Y_i$ for some $\nu < i \le m$. Since $\mathcal{O}_{Y_i}(L - (Y_\nu - Y_i))$ is numerically trivial, the section induced by s is nowhere vanishing and we get $\mathcal{O}_{Y_i}(L) \simeq \mathcal{O}_{Y_i}(Y_\nu - Y_i)$. □

1.3. Koyama's inequality. The author was informed by Masataka Tomari that Yoichi Koyama obtained the following inequality around 1984:

Proposition 1.6 (Koyama's inequality). *Let (V, o) be a normal surface singular point with $p_f(V, o) > 1$ and $\pi : X \to V$ a resolution. Then*

$$-Z_K^2 \ge 8p_a(V, o) - 8$$

with equality holding if and only if there exists a curve D such that $Z_K = 2D$ and $p_a(D) = p_a(V, o)$. In particular, (V, o) is a numerically Gorenstein singularity if $-Z_K^2 = 8p_a(V, o) - 8$.

Proof. Let D be any curve with support in $\pi^{-1}(o)$. We have $(D - Z_K/2)^2 \le 0$, because the intersection form is negative definite. Hence,

$$2p_a(D) - 2 = K_X D + D^2 = -Z_K D + D^2 = (D - Z_K/2)^2 - Z_K^2/4 \le -Z_K^2/4$$

and we get the inequality $-Z_K^2 \geq 8p_a(V,o) - 8$ as wished. We have the equality sign if and only if $D - Z_K/2 = 0$ and $p_a(D) = p_a(V,o)$. Then $Z_K = 2D$ is a $\mathbb{Z}$-divisor and, hence (V,o) is numerically Gorenstein. □

DEFINITION 1.7. A normal surface singularity (V,o) with $p_f(V,o) > 1$ is called an *even singularity* if there exist a resolution $\pi : X \to V$ and a curve D on X such that $Z_K = 2D$. For an even singularity, the minimal model (cf. [5]) of the fundamental cycle (on X) is called the *minimally even cycle*.

We collect some fundamental properties of even singularities.

LEMMA 1.8. *Let (V,o) be an even singular point and $\pi : X \to V$ a resolution on which $Z_K = 2D$ holds for a curve D. Then the following hold.*

(1) *D is the unique curve satisfying $p_a(D) = p_a(V,o)$.*

(2) *The self-intersection number E^2 is even for any divisor E with support in $\pi^{-1}(o)$. In particular, π is the minimal resolution.*

(3) *If L is a line bundle on X such that $L - K_X$ is nef, then L is generated by its global sections.*

Proof. (1) is clear from the proof of Proposition 1.6. (2) Since $K_X E + E^2 = -2DE + E^2$ is even, so is E^2. In particular, we have no (-1)-curves on X. Hence π is the minimal resolution. (3) follows from (2), because, if the linear system $|L|$ has a base point p, then there exists a curve Δ passing through p and $\Delta^2 = -1$ by [4, Proposition 5.1]. □

LEMMA 1.9. *Let (V,o) be an even singular point with $p_a(V,o) = p_f(V,o) > 1$. Then $Z_K = 2Z$ holds on the minimal resolution space, where Z_K and Z are the canonical cycle and the fundamental cycle, respectively. Furthermore, Z is a minimally even cycle and, either $p_g(V,o) = p_f(V,o) + 1$ or $p_g(V,o) = p_f(V,o)$ holds according to whether (V,o) is Gorenstein or not.*

Proof. The first assertion is clear from the previous lemma, since $p_a(V,o) = p_a(Z)$. As to the second, we consider the exact sequence

$$0 \to \omega_Z \to \omega_{Z_K} \to \mathcal{O}_Z(K_{Z_K}) \to 0$$

noting that $Z_K = 2Z$. By Lemma 1.3, $H^0(Z_K, K_{Z_K}) \to H^0(Z, K_{Z_K})$ is not the zero map if and only if (V,o) is Gorenstein. Hence, $h^0(\omega_{Z_K})$ equals $p_a(Z)+1$ when (V,o) is Gorenstein and, otherwise, it equals $p_a(Z)$. Furthermore, Z is the minimally even cycle, because K_Z is nef being numerically equivalent to $-Z$. □

EXAMPLE 1.10. Let C be a non-singular projective curve of genus $g \geq 2$. Take an invertible sheaf $\mathcal{L}$ on C with $\deg \mathcal{L} = 2g-2$. We consider the $\mathbb{P}^1$-bundle $X := \mathbb{P}(\mathcal{O}_C \oplus \mathcal{L})$ over C. Let $\varphi : X \to C$ be the projection and H a tautological divisor on X such that $\varphi_* \mathcal{O}_X(H) \simeq \mathcal{O}_C \oplus \mathcal{L}$. We choose the unique element $Z \in |H - \varphi^* \mathcal{L}|$. Then Z is isomorphic to C and $Z^2 = -\deg \mathcal{L} = 2 - 2g < 0$. We contract Z to obtain a normal surface singularity (V,o). Since the canonical bundle of X is induced by $-2H + \varphi^*(K_C + \mathcal{L})$ and it is numerically equivalent to $-2Z$ on Z, we see that (V,o) is an even singularity. More precisely, we have $\mathcal{O}_Z(K_X + 2Z) = \mathcal{O}_Z(\varphi^*(K_C - \mathcal{L}))$. Hence, (V,o) is Gorenstein if and only if $\mathcal{L} \simeq \mathcal{O}_C(K_C)$. On the other extreme, it tells us that $\mathcal{O}_Z(K_X + Z_K)$ is not necessarily of finite order.

LEMMA 1.11. *Let (V,o) be an even singularity with $p_f(V,o)=2$. Then $Z_K=2Y$ on the minimal resolution, where $Y=\sum_{i=1}^m D_i$ denotes the Yau cycle; $D_1=Z$, $m=p_a(V,o)-1$. Furthermore, the following hold.*

(1) *D_i is a 1-connected curve with $D_i^2=-2$ for $1\le i\le m$.*

(2) *$D_m=Z_{\min}$. Any component $A\preceq D_m$ satisfying $AD_m<0$ is not a (-2)-curve.*

(3) *Put $Y_\nu:=\sum_{i=\nu}^m D_i$ for $\nu\in\{1,2,\dots,m\}$. Then Y_ν and $2Y_\nu$ are the Yau cycle and the canonical cycle on $\mathrm{Supp}(Y_\nu)$, respectively. In particular, the singularity (V_ν,o_ν) obtained by contracting Y_ν is an even singularity with $p_f(V_\nu,o_\nu)=2$ and $p_a(V_\nu,o_\nu)=m-\nu+2$.*

Proof. Since Y computes the arithmetic genus of (V,o) as $p_a(V,o)=p_a(Y)=m+1$ by [6], we have $Z_K=2Y$ by the uniqueness of such a curve.

(1) For any i, we have $p_a(D_i)=2$ by the definition of the Yau sequence. Hence $2=K_XD_i+D_i^2=-2YD_i+D_i^2=-D_i^2$, because we have $D_iD_j=0$ when $j\ne i$. Let Γ be any non-trivial subcurve of D_i. Then $-2=D_i^2=\Gamma^2+(D_i-\Gamma)^2+2(D_i-\Gamma)\Gamma$. Since Γ^2 and $(D_i-\Gamma)^2$ are negative even integers, we get $(D_i-\Gamma)\Gamma>0$. Hence D_i is numerically 1-connected.

(2) For $i<m$, $\mathcal{O}_{D_m}(-D_i)$ is numerically trivial. So, K_X+D_m, which is numerically equivalent to $-2Y+D_m$ on $\pi^{-1}(o)$, is numerically equivalent to $-D_m$ on D_m. Since D_m is the fundamental cycle on its support, this shows that K_{D_m} is nef. Therefore, D_m is the minimal model of $Z=D_1$. Assume that A is a (-2)-curve and $AD_m<0$. It follows that D_m is reducible and we have $0<A(D_m-A)=AD_m+2$ by the 1-connectivity of D_m. Then $A(D_m-A)=1$, which implies that A is a $(-1)_{D_m}$-curve, contradicting that D_m is the minimal model.

(3) Since $\mathcal{O}_{D_j}(-D_i)$ is numerically trivial for $i<j$, K_X is numerically equivalent to $-2Y_\nu$ on $\mathrm{Supp}(Y_\nu)$. This is sufficient to imply the assertion. □

2. Resolution dual graphs. In the sequel, unless otherwise stated explicitly, we let (V,o) be an even singular point of fundamental genus 2. We denote by Z, Y and Z_K the fundamental cycle, the Yau cycle and the canonical cycle, respectively, on the minimal resolution $\pi:X\to V$ of (V,o).

The purpose of this section is to classify the weighted dual graph of the fundamental cycle Z (modulo some 2-connected curves).

Recall that Z is numerically 1-connected by Lemma 1.11 (1). Numerical decompositions and the canonical algebra of such a curve were extensively studied in [8]. If Z is 2-connected, then the canonical linear system $|K_Z|$ is free from base points by [1, Proposition (A.7)]. If Z is not 2-connected, then, letting Δ be a minimal subcurve of Z with respect to the condition that $\Delta(Z-\Delta)=1$, we see that Δ is 2-connected and $Z-\Delta$ is 1-connected ([1, Lemma (A.4)]). We have $\Delta^2=(Z-\Delta)^2=-2$ by $Z^2=-2$, and $p_a(\Delta)=p_a(Z-\Delta)=1$. In particular, $\omega_\Delta\simeq\mathcal{O}_\Delta$ and it follows that $\mathcal{O}_\Delta(-\Delta)$ is nef. Then, Δ is the fundamental cycle of a minimally elliptic singular point (cf. [7]).

2.1. Minimally even cycles. We first consider the case $p_a(V,o)=2$. Then, Z is a minimally even cycle (of fundamental genus 2) and vice versa, by Lemmas 1.9 and 1.11.

LEMMA 2.1. *Let (V,o) be an even singularity with $p_f(V,o)=p_a(V,o)=2$. Then, $\mathcal{O}_X(-Z)$ is π-free if and only if (V,o) is Gorenstein and Z is 2-connected.*

Proof. Since $Z_K = 2Z$ and $H^1(X, -Z_K) = 0$ by the vanishing theorem, we see from the cohomology long exact sequence for

$$0 \to \mathcal{O}_X(-Z_K) \to \mathcal{O}_X(-Z) \to \mathcal{O}_Z(-Z) \to 0$$

that the restriction map $H^0(X, -Z) \to H^0(Z, -Z)$ is surjective. Since $\mathcal{O}_Z(-Z)$ is numerically equivalent to K_Z and Z is a 1-connected curve with $p_a(Z) = 2$, one easily see that the following three conditions are equivalent: (i) $h^0(Z, -Z) > 1$, (ii) $\mathcal{O}_Z(-Z) = \omega_Z$, (iii) (V, o) is Gorenstein. [To show (ii) $\Rightarrow$ (iii), one may need the exact sequence

$$0 \to \omega_Z \to \omega_{Z_K} \to \mathcal{O}_Z(K_X + 2Z) \to 0$$

and Lemma 1.3.] So, we may assume that (V, o) is Gorenstein.

If Z is 2-connected, then $|\omega_Z|$ is free from base points by [1, Proposition (A.7)]. It follows that $\mathcal{O}_X(-Z)$ is π-free.

Assume that Z is not 2-connected and take a minimal subcurve Δ of Z with respect to the condition that $(Z - \Delta)\Delta = 1$. Then Δ is 2-connected and $Z - \Delta$ is 1-connected by [1, Lemma (A.4)]. Since $\mathcal{O}_\Delta(-Z)$ is nef of degree 1 and Δ is 2-connected with $p_a(\Delta) = 1$, we see that any non-zero element in $H^0(\Delta, -Z) \simeq \mathbb{C}$ vanishes at one non-singular point $p \in \Delta$ by [1, Proposition (A.5)]. This implies that p is a base point of $|\mathcal{O}_Z(-Z)|$. It also means that p is a base point of $|\mathcal{O}_X(-Z)|$. □

Suppose that Z is not 2-connected and take a subcurve Δ as above. Since $\omega_\Delta \simeq \mathcal{O}_\Delta$, we have $\mathcal{O}_\Delta(K_X) \simeq \mathcal{O}_\Delta(-\Delta)$. From

$$0 \to \omega_{Z-\Delta} \to \omega_Z \to \mathcal{O}_\Delta(K_X + Z) \to 0,$$

$h^0(\omega_Z) = 2$ and $h^0(\omega_{Z-\Delta}) = 1$, the restriction map $H^0(Z, \omega_Z) \to H^0(\Delta, K_X + Z)$ is non-trivial. Since $\mathcal{O}_\Delta(K_X + Z) \simeq \mathcal{O}_\Delta(Z - \Delta)$ is nef of degree 1 and Δ is 2-connected, we see that $\mathcal{O}_\Delta(Z - \Delta) \simeq \mathcal{O}_\Delta(x)$ with a non-singular point x of Δ. Then, either $\Delta \cap (Z - \Delta) = \{x\}$ or $\Delta \prec Z - \Delta$. Note that we have $\mathcal{O}_\Delta(Z - \Delta) \equiv \mathcal{O}_\Delta(-Z)$, where $\equiv$ means the numerical equivalence. Let A be the unique irreducible component of Δ through x.

LEMMA 2.2. *Let the notation be as above. Either Δ is irreducible and reduced, or A is a (-4)-curve and $\Delta - A$ is the fundamental cycle of a rational double point. In the latter case, an irreducible component B of $\Delta - A$ satisfies $AB > 0$ if and only if $B(\Delta - A) < 0$.*

Proof. Since $K_X \equiv -2Z$, we have $\mathcal{O}_\Delta(K_X) \equiv \mathcal{O}_\Delta(2x)$. Then, $K_X A = -2ZA = 2$. It follows that either $p_a(A) = 1$ and $A^2 = -2$ (so $\Delta = A$), or A is a (-4)-curve. Assume the latter. Then $A\Delta = -2$, $K_X(\Delta - A) = 0$ and $(\Delta - A)^2 = -2$. Hence $\Delta - A$ is a 1-connected curve consisting of (-2)-curves. Since $\mathcal{O}_{\Delta-A}(-(\Delta - A)) \simeq \mathcal{O}_{\Delta-A}(A)$ is nef, $\Delta - A$ is the fundamental cycle on its support. The last assertion for B is clear from $\mathcal{O}_{\Delta-A}(-\Delta) \simeq \mathcal{O}_{\Delta-A}$. Since $(\Delta - A)A = 2$, there are at most two irreducible components of $\Delta - A$ meeting A. □

REMARK 2.3. The possible dual graphs of Δ's are in one-to-one correspondence with Kodaira's list of non-multiple fibers in elliptic surfaces [2, Theorem 6.2], with an obvious modification in the reducible case: choose a component of multiplicity one and replace it by a (-4)-curve. See also [7].

We turn our attention to the 2-disconnected Z itself.

Case 1. Assume that $\Delta \cap (Z-\Delta) = \{x\}$. We let Δ_1 be a minimal subcurve of $Z-\Delta$ with respect to the condition that $(Z-\Delta_1)\Delta_1 = 1$, and put $\Gamma := Z - \Delta - \Delta_1$. Then Δ_1 is a 2-connected curve with $p_a(\Delta_1) = 1$, $\Delta_1^2 = -2$ and $\mathcal{O}_{\Delta_1}(-Z) \simeq \mathcal{O}_{\Delta_1}(y)$ for a non-singular point $y \in \Delta_1$. We have $\Delta\Delta_1 = 0, 1$. If $\Delta\Delta_1 = 1$, then $\Gamma^2 = 0$ and, thus, $\Gamma = 0$. In this case, we have $Z = \Delta + \Delta_1$ and $x = y$. If $\Delta\Delta_1 = 0$, then $\Delta \cap \Delta_1 = \emptyset$ and $\Gamma^2 = -2$. Since $K_X\Gamma = 0$, Γ is a (-2)-cycle. We have $\Delta \cap \Gamma = \{x\}$ and $\Delta_1 \cap \Gamma = \{y\}$. Since $\mathcal{O}_\Gamma(-\Gamma) \simeq \mathcal{O}_\Gamma(-Z+\Delta+\Delta_1) \simeq \mathcal{O}_\Gamma(x+y)$, Γ is the fundamental cycle of a rational double point of type A_ℓ for some ℓ. The dual graph of Z would be as in Fig. 1 if we consider Δ and Δ_1 as if they were irreducible curves. It is clear that Γ is the fixed part of $|K_Z|$, and that $\mathcal{O}_{Z-\Gamma}(-Z-\Gamma) \simeq \mathcal{O}_{Z-\Gamma}$, $h^0(Z-\Gamma, \mathcal{O}_{Z-\Gamma}) = 2$.

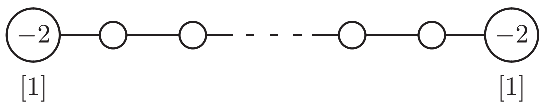

FIG. 1.

Case 2. Assume that $\Delta \prec Z - \Delta$. Then Δ is the minimal model of $Z - \Delta$. We have $(Z-2\Delta)^2 = -6$. It follows from $K_X(Z-2\Delta) = -2Z(Z-2\Delta) = 0$ that $Z - 2\Delta$ consists of (-2)-curves. Note also that $\mathcal{O}_{\Delta-A}(-\Delta) \simeq \mathcal{O}_{\Delta-A}$. In particular, we see that Δ is nef on $Z-2\Delta$, even when $Z-2\Delta$ has a common component with Δ. Then, by Proposition 1.2, $Z - 2\Delta$ decomposes as $\Gamma_1 + \Gamma_2 + \Gamma_3$, where each Γ_i is the fundamental cycle on its support, $\Gamma_i^2 = -2$, $\Delta\Gamma_i = 1$ and $\mathcal{O}_{\Gamma_j}(-\Gamma_i) \simeq \mathcal{O}_{\Gamma_j}$ when $i < j$. We have the following three cases after re-labeling if necessary:

(a) Γ_1, Γ_2 and Γ_3 are mutually disjoint,

(b) $\Gamma_2 \prec \Gamma_1$ and $\Gamma_1 \cap \Gamma_3 = \emptyset$,

(c) $\Gamma_3 \prec \Gamma_2 \prec \Gamma_1$.

LEMMA 2.4. *For each $i = 1, 2, 3$, Γ_i is the fundamental cycle of a rational double point. If Γ_i is a minimal curve* (*with respect to $\preceq$*) *in $\{\Gamma_1, \Gamma_2, \Gamma_3\}$, then it is a (-2)-curve that is not a component of Δ. The possible Dynkin types* $(\mathrm{type}(\Gamma_1), \mathrm{type}(\Gamma_2), \mathrm{type}(\Gamma_3))$ *are as follows:*

(a) $(\mathsf{A}_1, \mathsf{A}_1, \mathsf{A}_1)$.

(b) $(\mathsf{A}_3, \mathsf{A}_1, \mathsf{A}_1)$ *or* $(\mathsf{D}_l, \mathsf{A}_1, \mathsf{A}_1)$, $l \geq 4$.

(c) $(\mathsf{A}_5, \mathsf{A}_3, \mathsf{A}_1)$ *or* $(\mathsf{D}_6, \mathsf{D}_4, \mathsf{A}_1)$.

Proof. For each i, since $\mathcal{O}_{\Gamma_i}(\Delta)$ is nef of degree 1, there exists a unique component C_i of Γ_i, along which Γ_i is of multiplicity one, such that $C_i\Delta = 1$ and $\Delta(\Gamma_i - C_i) = 0$. Recall that $Z - 2\Delta = \Gamma_1 + \Gamma_2 + \Gamma_3$, $\mathcal{O}_{\Gamma_i}(Z) \simeq \mathcal{O}_{\Gamma_i}$ and $\mathcal{O}_{\Gamma_j}(-\Gamma_i) \simeq \mathcal{O}_{\Gamma_j}$ when $i < j$. So, if Γ_i is a minimal curve in $\{\Gamma_1, \Gamma_2, \Gamma_3\}$, then $\mathcal{O}_{\Gamma_i}(2\Delta) \simeq \mathcal{O}_{\Gamma_i}(-\Gamma_i)$. Hence $C_i\Gamma_i = -2$ and it follows $\Gamma_i = C_i$, since $(\Gamma_i - C_i)^2 = 0$. Note then that Γ_i is not a component of Δ, since Δ is the fundamental cycle and $\Delta\Gamma_i = 1$.

We restrict ourselves to the case (c), since the other cases are similar and easier. In this case, we have $C_1 = C_2 = C_3$, and $\Gamma_3 = C_3$ as we saw above. We have $\mathcal{O}_{\Gamma_2}(-\Gamma_2) \simeq \mathcal{O}_{\Gamma_2}(2\Delta+\Gamma_3)$ which is trivial on Γ_3. So, Γ_2 has a non-multiple component Γ_3 and any component B with $B\Gamma_2 < 0$ has to meet Γ_3. From this property and the A-D-E classification, we see that Γ_2 is of type A_3 or D_l. Similarly, we have $\mathcal{O}_{\Gamma_1}(-\Gamma_1) \simeq \mathcal{O}_{\Gamma_1}(2\Delta + \Gamma_2 + \Gamma_3)$ which is trivial on Γ_2. Then Γ_1 has a non-multiple component Γ_3 and any component B with $B\Gamma_1 < 0$ satisfies $B(\Gamma_2 + \Gamma_3) > 0$. From this, we see that Γ_1 is of type A_5 when Γ_2 is of type A_3. When Γ_2 is of type D_l, it is not so hard to see that Γ_2 is of type D_4 and Γ_1 is of type D_6. □

Since $\mathcal{O}_\Delta(-Z) \equiv \mathcal{O}_\Delta(x)$ and $\mathcal{O}_\Delta(-\Delta) \equiv \mathcal{O}_\Delta(2x)$, we get $\mathcal{O}_\Delta(\Gamma_1+\Gamma_2+\Gamma_3) \equiv \mathcal{O}_\Delta(3x)$. So, $A(\Gamma_1+\Gamma_2+\Gamma_3) = 3$ and $\mathcal{O}_{\Delta-A}(\Gamma_1+\Gamma_2+\Gamma_3) \simeq \mathcal{O}_{\Delta-A}$. We have either $(\Delta - A)\cap(\Gamma_1+\Gamma_2+\Gamma_3) = \emptyset$ or $\Delta - A \preceq \Gamma_1+\Gamma_2+\Gamma_3$, because $\Delta - A$ is 1-connected even if $\Delta \neq A$. When $(\Delta - A)\cap(\Gamma_1+\Gamma_2+\Gamma_3) = \emptyset$, the dual graph of Z would be as in Fig. 2 if we treat Δ as if it were a single elliptic curve.

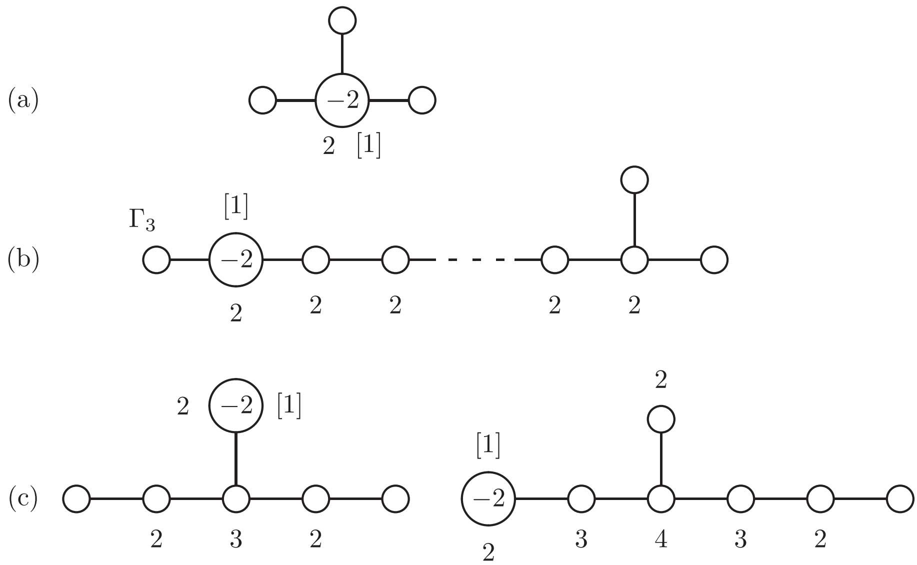

FIG. 2. *Dual graph of Z when* $\gcd(\Delta, Z - 2\Delta) = 0$

Assume that $\Delta - A \preceq \Gamma_1+\Gamma_2+\Gamma_3$ ($\Delta \neq A$). We cannot have the case (a), because, if (a) is the case, each Γ_i is a (-2)-curve which is not a component of Δ by Lemma 2.4.

If (b) is the case, then Lemma 2.4 implies that Γ_2 and Γ_3 are (-2)-curves not contained in Δ, because they are minimal curves in $\{\Gamma_i\}_{i=1}^3$. So, $\Delta - A \preceq \Gamma_1 - \Gamma_2$. We have $A\Gamma_3 = 1$ by $(\Delta - A)\cap\Gamma_3 \subseteq (\Gamma_1 - \Gamma_2)\cap\Gamma_3 = \emptyset$ and $\Delta\Gamma_3 = 1$. Since $2 = (\Delta - A)A \le (\Gamma_1 - \Gamma_2)A$ and $A(\Gamma_1+\Gamma_2) = 2$, we get $A\Gamma_1 = 2$, $A\Gamma_2 = 0$. So, $A\cap\Gamma_2 = \emptyset$ and $(\Delta - A)\Gamma_2 = 1$. There exists a component $B \preceq \Delta - A$ of multiplicity one with $B\Gamma_2 = 1$ and $B\Gamma_1 = -1$. Since $\mathcal{O}_{\Gamma_1-\Gamma_2}(-\Delta)$ is nef of degree 0, any chain-connected subcurve of $\Gamma_1 - \Gamma_2$ is either a subcurve of Δ or disjoint from Δ. This implies that $\Delta - A$ is nothing but a chain-connected component of $\Gamma_1 - \Gamma_2$. By using this fact, one can easily see the following by considering the CCCD of $\Gamma_1 - \Gamma_2$: If Γ_1 is of type A_3, then $\Delta - A = B$ is a (-2)-curve and, either A meets B normally at two distinct points or A contacts B at a point to the second order. If Γ_1 is of type D_l, then $\Delta - A$ is either B or $\Gamma_1 - \Gamma_2 - B$ which is of type D_{l-1}. Here, we cannot have $\Delta - A = B$, because, if so, we have $\mathrm{mult}_A(Z) = 2$, $\mathrm{mult}_B(Z) = 2\mathrm{mult}_B(\Delta) + \mathrm{mult}_B(\Gamma_1) = 4$ and $AB = 2$ which leads us to a contradiction: $AZ = A(2A + 4B + \Gamma_3) = 1$.

If (c) is the case, then $\Delta - A \preceq (\Gamma_1 - \Gamma_3) + (\Gamma_2 - \Gamma_3)$ and it follows $A(\Gamma_1 - \Gamma_3) + A(\Gamma_2 - \Gamma_3) \ge A(\Delta - A) = 2$. From this and $A(\Gamma_1+\Gamma_2+\Gamma_3) = 3$, we infer that $A\Gamma_1 = 2$, $A\Gamma_2 = 1$ and $A\Gamma_3 = 0$. Also in this case, any chain-connected component of $\Gamma_1 - \Gamma_3$ is either a subcurve of Δ or disjoint from Δ. So, one can conclude similarly as in the previous case: If Γ_1 is of type A_5, then $\Delta - A$ is of type A_2. If Γ_1 is of type

D_6, then $\Delta - A$ is of type A_5.

In each case, the possible dual graph of Z is as in Fig. 3 in which two circles connected by a line with 2 means that the corresponding two curves contact at a point to the second order.

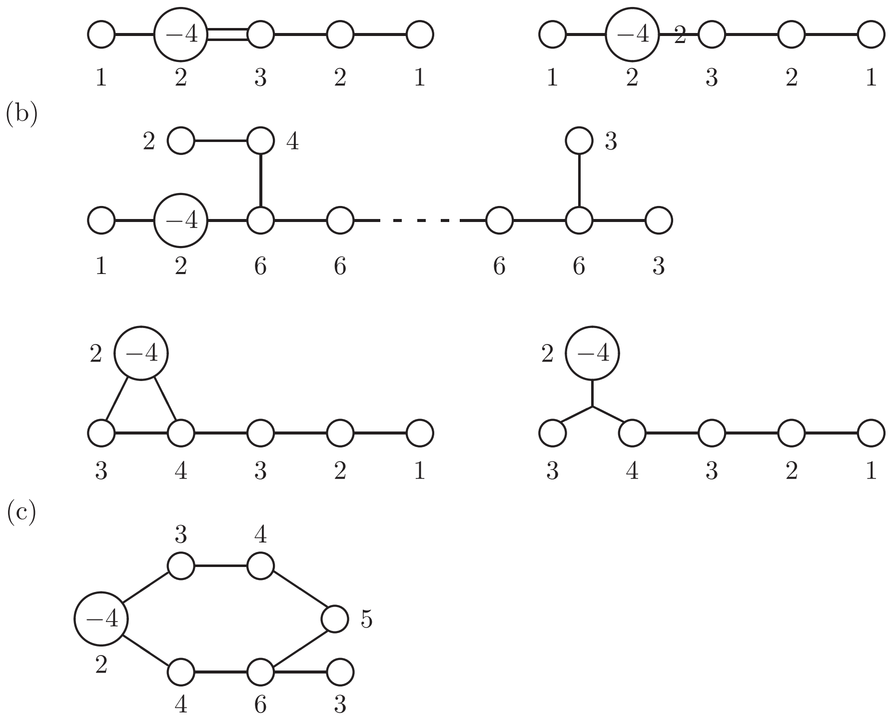

FIG. 3. *Dual graph of* Z *when* $\gcd(\Delta, Z - 2\Delta) \neq 0$

We continue to assume that $\Delta \prec Z - \Delta$ and study the base locus of $|\omega_Z|$ closely. It follows from

$$0 \to \omega_\Delta \to \omega_Z \to \mathcal{O}_{Z-\Delta}(K_X + Z) \to 0$$

that $H^0(Z, \omega_Z) \to H^0(Z - \Delta, K_X + Z)$ is surjective, because $H^1(\omega_\Delta) \to H^1(\omega_Z)$ is injective being the dual map of $H^0(Z, \mathcal{O}_Z) \to H^0(\Delta, \mathcal{O}_\Delta)$. Take a non-zero section $s \in H^0(Z - \Delta, K_X + Z) \simeq \mathbb{C}$. If s does not vanish identically on any components of $Z - \Delta$, then s vanishes only at $x \in \Delta$. Hence x is an isolated base point of $|K_Z|$ in this case. So, we assume that s vanishes identically on some component and let Γ be the biggest curve on which s vanishes identically. Then $\mathcal{O}_{Z-\Delta-\Gamma}(K_X + Z - \Gamma) \equiv \mathcal{O}_{Z-\Delta-\Gamma}(-Z - \Gamma)$ is nef. In particular, $\deg \mathcal{O}_{Z-\Delta-\Gamma}(-Z) \geq (Z - \Delta - \Gamma)\Gamma$. Recall that $\mathcal{O}_{Z-\Delta}(-Z)$ is nef of degree 1 and $Z - \Delta$ is 1-connected. Therefore, we have $\deg \mathcal{O}_{Z-\Delta-\Gamma}(-Z) = (Z - \Delta - \Gamma)\Gamma = 1$. Then Γ is a 1-connected curve on which Z is numerically trivial. This shows that Γ is a (-2)-cycle. We also know that $Z - \Delta - \Gamma$ is 1-connected. We have $\Delta \preceq Z - \Delta - \Gamma$, since Δ must be the minimal model of the 1-connected curve $Z - \Delta - \Gamma$ with $p_a(Z - \Delta - \Gamma) = 1$. Furthermore, since s induces a nowhere vanishing section of $\mathcal{O}_{Z-\Delta-\Gamma}(K_X + Z - \Gamma)$, we see that $\mathcal{O}_{Z-\Delta-\Gamma}(K_X + Z - \Gamma) \simeq \mathcal{O}_{Z-\Delta-\Gamma}$. Note that s is constant on every Γ_i. Therefore, Γ is one of the maximal curves in $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ and $x \in \Gamma \cap \Delta$.

Lemma 2.5. *$\omega_{Z-\Gamma} \simeq \mathcal{O}_{Z-\Gamma}$ and $h^0(Z-\Gamma, \mathcal{O}_{Z-\Gamma}) = 2$.*

Proof. Consider the cohomology long exact sequence for

$$0 \to \omega_{Z-\Gamma} \to \omega_Z \to \mathcal{O}_\Gamma(-Z) \to 0.$$

Since $H^0(Z, \omega_Z) \to H^0(\Gamma, -Z)$ is the zero map, we get $h^0(Z-\Gamma, \omega_{Z-\Gamma}) = h^0(Z, \omega_Z) = 2$. We already know that $\omega_{Z-\Gamma}$ is trivial on $Z - \Delta - \Gamma$. Since $\Delta \preceq Z - \Delta - \Gamma$, we see that $\omega_{Z-\Gamma}$ is numerically trivial. Then, by the Riemann-Roch theorem, we get $h^1(Z-\Gamma, \omega_{Z-\Gamma}) = 2$. By the Serre duality theorem, we get $h^0(Z-\Gamma, \mathcal{O}_{Z-\Gamma}) = 2$.

It remains to show that $\omega_{Z-\Gamma} \simeq \mathcal{O}_{Z-\Gamma}$. For this purpose, we consider the cohomology long exact sequence for

$$0 \to \omega_\Delta \to \omega_{Z-\Gamma} \to \mathcal{O}_{Z-\Delta-\Gamma} \to 0.$$

We have $h^0(\omega_\Delta) = h^0(Z-\Delta-\Gamma, \mathcal{O}) = 1$. It follows that the restriction map $H^0(Z-\Gamma, \omega_{Z-\Gamma}) \to H^0(Z-\Delta-\Gamma, \mathcal{O}_{Z-\Delta-\Gamma})$ is surjective. Since $\Delta \preceq Z-\Delta-\Gamma$, we see that there is a nowhere vanishing section in $H^0(Z-\Gamma, \omega_{Z-\Gamma})$. Therefore, $\omega_{Z-\Gamma} \simeq \mathcal{O}_{Z-\Gamma}$. □

Lemma 2.6. *With the above notation, $Z-\Gamma$ is the canonical cycle of a maximally elliptic singularity of geometric genus two.*

Proof. We shall show that $Z-\Gamma = (Z-\Delta-\Gamma) + \Delta$ gives us the CCCD, i.e., the elliptic sequence. Recall that $Z-\Delta-\Gamma$ and Δ are 1-connected and $\mathcal{O}_\Delta(-(Z-\Delta-\Gamma)) \simeq \mathcal{O}_\Delta(-(K_X+Z-\Gamma)) \simeq \mathcal{O}_\Delta$. So it remains to show that $Z-\Delta-\Gamma$ is the fundamental cycle on its own support. If $C \preceq Z-\Delta-\Gamma$ is a component such that $C \not\preceq \Delta$, then $C\Delta \geq 0$ and, hence, $C(Z-\Delta-\Gamma) = 2ZC - C\Delta \leq 2ZC \leq 0$. This implies that $\mathcal{O}_{Z-\Delta-\Gamma}(-(Z-\Delta-\Gamma))$ is nef and, thus, $Z-\Delta-\Gamma$ is the fundamental cycle on its support. Since the geometric genus $h^0(Z-\Gamma, \mathcal{O}) = 2$ coincides with the length of the CCCD, $Z-\Gamma$ contracts to a maximally elliptic singularity of geometric genus 2. □

Now, as a summary, we state a rough classification of the minimal even cycles according to the base locus of the canonical linear system and the numerical decomposition:

Proposition 2.7. *Minimally even cycles Z of fundamental genus* 2 *fall into the following five classes* (0), (i.a), (i.b), (ii.a) *and* (ii.b).

(0) *$|\omega_Z|$ is free from base points and Z is* 2*-connected.*

(i) *$|\omega_Z|$ has an isolated base point x.*

(i.a) *$Z = \Delta + \Delta_1$, $\Delta \cap \Delta_1 = \{x\}$.*

(i.b) *$Z = 2\Delta + \Gamma_1 + \Gamma_2 + \Gamma_3$, the base point $x \in \Delta$ is a non-singular point of Z_{red}.*

(ii) *The fixed part Γ of $|\omega_Z|$ is the fundamental cycle of a rational double point.*

(ii.a) *$Z = \Delta + \Delta_1 + \Gamma$, where the Γ is a simple chain of (-2)-curves (of type* (A)*) with $\Delta \cap \Gamma = \{x\}$ and $\Delta_1 \cap \Gamma = \{y\}$.*

(ii.b) *$Z = 2\Delta + \Gamma_1 + \Gamma_2 + \Gamma_3$, Γ is a maximal curve in $\{\Gamma_1, \Gamma_2, \Gamma_3\}$.*

We remark that, when (V, o) is Gorenstein, the above substantially gives us the classification of the maximal ideal cycles.

2.2. General case. Let us consider the case $m \geq 2$.

PROPOSITION 2.8. *Let (V, o) be an even singularity with $p_f(V, o) = 2$ and $p_a(V, o) = m + 1 \geq 3$. If $Y = \sum_{i=1}^{m} D_i$ denotes the Yau cycle, then $D_1 - D_m$ decomposes into a numerically disjoint sum of two fundamental cycles Δ_1, Δ_2 of rational double points: $D_1 - D_m = \Delta_1 + \Delta_2$, $D_m \Delta_i = 1$, $\mathcal{O}_{\Delta_2}(-\Delta_1) \simeq \mathcal{O}_{\Delta_2}$. The possible types of Δ_1, Δ_2 are as follows.*

$(2\mathsf{A}_{m-1})$: Δ_1 *and* Δ_2 *are both of type* A_{m-1}, $\mathrm{Supp}(\Delta_1) \cap \mathrm{Supp}(\Delta_2) = \emptyset$.

$(\mathsf{A}_{2m-1}, \mathsf{A}_{2m-3})$: Δ_i *is of type* $\mathsf{A}_{2m-(2i-1)}$ $(i = 1, 2)$ *and* $\Delta_2 \prec \Delta_1$.

$(\mathsf{E}_6, \mathsf{A}_5)$: $m = 3$, Δ_1 *is of type* E_6, Δ_2 *is of type* A_5 *and* $\Delta_2 \prec \Delta_1$.

$(\mathsf{D}_l, \mathsf{A}_1)$: $m = 2$, Δ_1 *is of type* D_l *for some* $l \geq 4$, Δ_2 *is of type* A_1 *and* $\Delta_2 \prec \Delta_1$.

Proof. Suppose that $m \geq 2$. We have $K_X(D_1 - D_m) = 0$, since $p_a(D_1) = p_a(D_m) = -D_1^2 = -D_m^2 = 2$ by Lemma 1.11 (1). Hence $D_1 - D_m$ consists of (-2)-curves. Let A be a component of $D_1 - D_m$ and let $i = i(A)$ be the biggest index such that $A \preceq D_i$. Then $A(D_i + D_{i+1}) = 0$ by (1.2) and $0 = K_X A = -2YA$.

We have $(D_1 - D_m)^2 = -4$ by $D_1 D_m = 0$. It follows from Proposition 1.2 that $D_1 - D_m$ decomposes as $D_1 - D_m = \Delta_1 + \Delta_2$, $D_m \Delta_i = 1$, $\Delta_i^2 = -2$ and $\mathcal{O}_{\Delta_2}(-\Delta_1) \simeq \mathcal{O}_{\Delta_2}$. Since $\mathcal{O}_{\Delta_2}(-D_1)$ is nef of degree one, there exists a non-multiple component C of Δ_2 with $-CD_1 = 1$. Then $CD_i = 0$ for $i \geq 3$ by (1.2) and $CD_2 = 1$. We have $0 > C(D_1 - D_m) = C\Delta_1 + C\Delta_2 = C\Delta_2$ by $CD_m \geq 0$ and $\mathcal{O}_{\Delta_2}(-\Delta_1) \simeq \mathcal{O}_{\Delta_2}$. From the A-D-E classification, we see that Δ_2 is of Dynkin type A_r for some r. Furthermore, we have $r = 1$ (i.e., $C\Delta_2 = -2$) if and only if $CD_m = 1$, i.e., $m = 2$.

Assume that Δ_1 and Δ_2 are disjoint. Then the same argument as above shows that Δ_1 is of Dynkin type A_s for some s. Let C' be a component of Δ_1 with $-C'D_1 = 1$. Then C and C' are $(-1)_{D_1}$-curves. It follows that $D_1 - C - C'$ is a chain-connected curve with $p_a(D_1 - C - C') = 2$. Furthermore, $\mathcal{O}_{D_1 - C - C'}(-D_1)$ is numerically trivial by $D_1^2 = -2$. Hence, $D_2 = D_1 - C - C'$. Now, we can repeat the argument inductively for $D_i - D_m$ $(i = 1, \dots, m-1)$ to find $r = s = m - 1$. This gives $(2\mathsf{A}_{m-1})$.

Assume that $\Delta_2 \prec \Delta_1$. Then it follows from Proposition 1.2 (3) that Δ_1 is of type either A_{r+2} or D_l $(r = 1$ or $l = 5, r = 3)$ or E_6 $(r = 5)$. In this case, C is of multiplicity 2 in $D_1 - D_2$ and $D_2 \preceq D_1 - 2C$. It is easy to see that $-(D_1 - D_2)$ is nef on $D_1 - D_2$. Put $\Gamma = D_1 - C - D_2$. Then $\Gamma^2 = -2$ and $C\Gamma = 0$. Since Γ is the fundamental cycle on its support, as in Proposition 1.2 (3), we see that Γ is of type either A_3 or $\mathsf{D}_{l'}$ for some $l' \geq 4$. From this, we inductively see that $r = 2m - 3$ when Δ_1 is of type A_{r+2}, $m = 3$ when Δ_1 is of type E_6, and that $(\mathsf{D}_5, \mathsf{A}_3)$ is impossible. □

The weighted dual graphs corresponding to the cases in the above Proposition are as in Figures 4–6. Note that, though D_m is treated as if it were a single curve and D_m has no common component with $\Delta_1 + \Delta_2$ in the figures, in reality D_m may well be reducible and, moreover, D_m and $\Delta_1 + \Delta_2$ may have a common component.

EXAMPLE 2.9. Each can be realized by a double point with D_m non-singular. Sample equations are as follows.

$$
\begin{aligned}
(2\mathsf{A}_{m-1}) :&\quad z^2 = x^6 + y^{6m} \\
(\mathsf{A}_{2m-1}, \mathsf{A}_{2m-3}) :&\quad z^2 = x(y + x^{m-1})(y^5 - x^{5(m-1)}),\ m \geq 3 \\
(\mathsf{E}_6, \mathsf{A}_5)\ (m = 3) :&\quad z^2 = (x^2 - y^3)(x^5 - y^{15}) \\
(\mathsf{D}_l, \mathsf{A}_1)\ (m = 2) :&\quad z^2 = (x^{10} - y^5)(x^2 - y^{l-1}),\ l \geq 3
\end{aligned}
$$

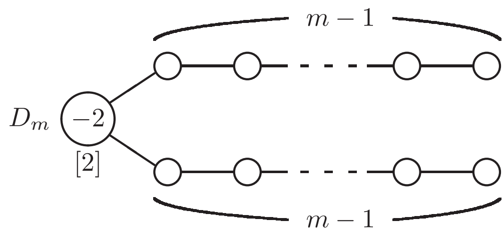

Fig. 4. *Type* $(2\mathsf{A}_{m-1})$

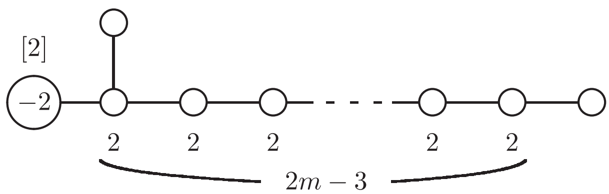

Fig. 5. *Type* $(\mathsf{A}_{2m-1}, \mathsf{A}_{2m-3})$

Not all combinations are possible and we cannot attach an arbitrary type of (Δ_1, Δ_2) to a given minimally even cycle D_m. We call a curve D essentially irreducible, if there is a non-multiple irreducible component C of D such that $D - C$ consists of (-2)-curves at most.

Lemma 2.10. *Assume that* $p_a(V, o) = m + 1 \geq 3$ *and put* $D_1 - D_m = \Delta_1 + \Delta_2$ *as above.*

(1) *If* A *is an irreducible component of* D_m*, then* $AD_m < 0$ *if and only if* $(D_{m-1} - D_m)A > 0$.

(2) *Every irreducible component* B *of* D_{m-1} *satisfying* $BD_{m-1} < 0$ *meets* D_m *at one non-singular point of a non-multiple component* C *of* D_m*. If* $CD_m = 0$*, then* C *is a* (-2)*-curve and* $C \preceq D_{m-1} - D_m$.

(3) *If* (Δ_1, Δ_2) *is of type* $(2\mathsf{A}_{m-1})$*, then* D_m *is of type either* (0), (i.a) *or* (ii.a). *If it is of type* $(\mathsf{E}_6, \mathsf{A}_5)$*, then* D_m *is of type* (0) *and is essentially irreducible. If it is of type either* $(\mathsf{A}_{2m-1}, \mathsf{A}_{2m-3})$ *or* $(\mathsf{D}_l, \mathsf{A}_1)$*, then* D_m *is of type* (0) *and is essentially irreducible, unless* D_m *and* $D_{m-1} - D_m$ *have a common component.*

Proof. (1) is clear. In fact, we have $(D_{m-1} - D_m)A = -D_m A$.

(2), (3): We compare D_{m-1} and D_m. We take an irreducible component B of D_{m-1} satisfying $BD_{m-1} < 0$. Since B is not a component of D_m, it is a (-2)-curve. Then $0 = K_X B = -2YB = 0$, from which we get $(D_{m-1} + D_m)B = 0$. Furthermore, since D_{m-1} is a reducible numerically 1-connected curve, we get $1 \leq (D_{m-1} - B)B = D_m B + 2$ and, thus, $BD_{m-1} = -1$ and $BD_m = 1$. Then, D_m has a unique non-multiple component C meeting B at a point. If $CD_m = 0$, then $(D_{m-1} - D_m)C = 0$ and C is a component of $D_{m-1} - D_m$ (since $BC > 0$ and $B \preceq D_{m-1} - D_m$). In particular, C is a (-2)-curve and $B + C$ is the configuration of type A_2 in this case. We also remark that we have $D_m \cap (D_1 - D_{m-1}) = \emptyset$ for $m \geq 3$ by (1.2) and, thus, C cannot be a component of $D_1 - D_{m-1}$. From these, one easily see that the case $CD_m = 0$ cannot happen when (Δ_1, Δ_2) is of type either $(2\mathsf{A}_{m-1})$ or $(\mathsf{E}_6, \mathsf{A}_5)$.

Assume that $CD_m < 0$. Then, D_m cannot be of types (i.b), (ii.b) because D_m has a non-multiple component C which is not a (-2)-curve. This is the case when

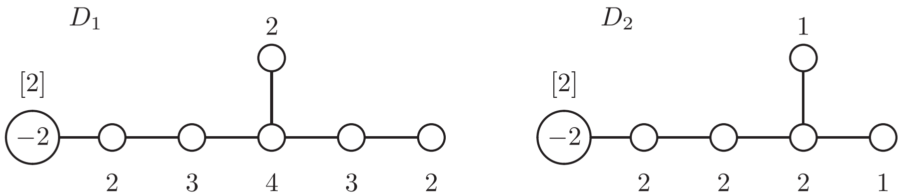

FIG. 6. *Type* $(\mathsf{E}_6, \mathsf{A}_5)$ $(m = 3)$

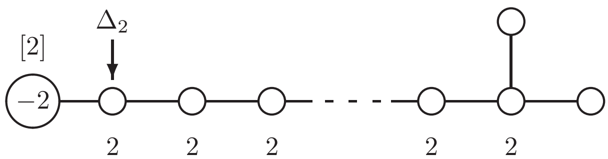

FIG. 7. *Type* $(\mathsf{D}_l, \mathsf{A}_1)$ *(*$m = 2$*)*

$D_1 - D_m$ is of type either $(2\mathsf{A}_{m-1})$ or $(\mathsf{E}_6, \mathsf{A}_5)$ as we saw above. Except when (Δ_1, Δ_2) is of type $(2\mathsf{A}_{m-1})$, B as above is unique and it is a component of multiplicity 2 of D_{m-1}. Then $(D_{m-1} - D_m)C \geq 2BC = 2$, i.e., $-D_mC \geq 2$. This implies that D_m is essentially irreducible, since the other components, if any, are (-2)-curves by $4 = K_XD_m = -2D_mD_m \geq -2D_mC \geq 4$. In particular, when $(\mathsf{E}_6, \mathsf{A}_5)$, the only possible type of D_m is (0) which is essentially irreducible.

We assume that $CD_m = 0$. Assume that $D_{m-1} - D_m$ is of type $(\mathsf{A}_3, \mathsf{A}_1)$. Then C is one of the end components of A_3. A as in (1) does not meet B, but meets C or an another end component C' of A_3. When $m \geq 3$, C has to be the end component of A_{2m-1} adjacent to B, and A meets C. If $D_{m-1} - D_m$ is of type $(\mathsf{D}_l, \mathsf{A}_1)$, then $m = 2$ and C have to be the component adjacent to B in the D_l configuration. □

If $D_1 - D_2$ is of type $(\mathsf{D}_l, \mathsf{A}_1)$ and $CD_2 = 0$, then C is a component of multiplicity 3 in Z. From this, one immediately sees that the only possible case is $l = 4$ and D_2 is of type (ii.a) as in Fig. 8 by the nefness of $-Z$ (examined on the D_l configuration).

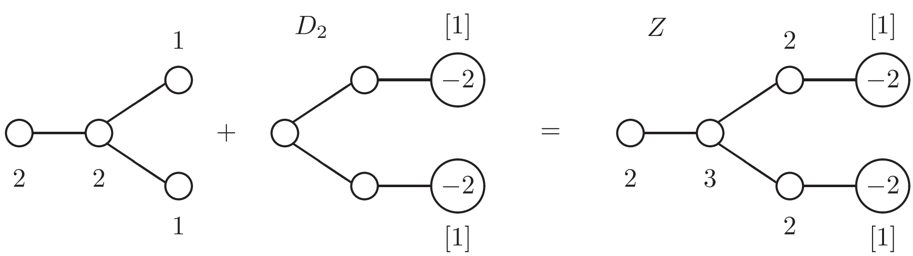

FIG. 8. (ii.a) + $(\mathsf{D}_4, \mathsf{A}_1)$

In Fig. 9, we give an example of the case where D_m and $D_1 - D_m$ have a common component, where D_m is of type (ii.b) (see, Fig. 2 (a)) and $D_1 - D_m$ is of type $(\mathsf{A}_{2m-1}, \mathsf{A}_{2m-3})$.

3. A geometric genus formula. The purpose of this section is to establish a formula computing the geometric genus of the even singular point (V, o) of fundamental genus 2, modeled on Okuma's formula [11] for elliptic singularities.

Recall that $Z_K = 2Y = 2(D_1 + \cdots + D_m)$. By (1.2), $\mathcal{O}_{D_m}(Z) \simeq \mathcal{O}_{D_m}(D_i)$ for

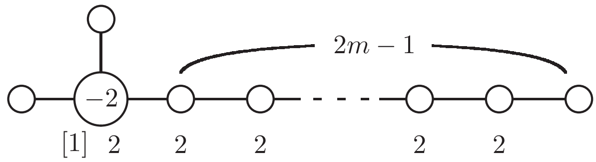

FIG. 9. (ii.b) + $(\mathsf{A}_{2m-1}, \mathsf{A}_{2m-3})$

$1 \leq i \leq m-1$. We put $Y_i = \sum_{j=i}^{m} D_j$ and

$$F_{i-1} = Y - Y_i = \sum_{j=1}^{i-1} D_j \tag{3.1}$$

for $1 \leq i \leq m$. Then Y_i is the Yau cycle on $\mathrm{Supp}(D_i)$ and, as shown in [6, Lemma 1.3], $-F_{i-1}$ is π-nef for $1 \leq i \leq m$. Note that $\mathcal{O}_{Y_i}(F_{i-1})$ is numerically trivial.

We consider two subsets of $\{1, 2, \ldots, m\}$ given by

$$\begin{aligned} \mathcal{A} &= \{i \mid \mathcal{O}_{2Y_i - D_i}(F_{i-1}) \simeq \mathcal{O}_{2Y_i - D_i}, 1 \leq i \leq m\}, \\ \mathcal{B} &= \{i \mid \omega_{2Y_i}(F_{i-1}) \simeq \mathcal{O}_{2Y_i}, 1 \leq i \leq m\}. \end{aligned} \tag{3.2}$$

We always have $1 \in \mathcal{A}$, and we have $1 \in \mathcal{B}$ if and only if (V, o) is Gorenstein.

LEMMA 3.1. *The geometric genus of an even singularity* (V, o) *with* $p_f(V, o) = 2$ *and* $p_a(V, o) = m + 1$ *is given by* $p_g(V, o) = m + {}^{\#}\mathcal{A} + {}^{\#}\mathcal{B}$, *where* ${}^{\#}T$ *denotes the cardinality of the set* T.

Proof. First, we consider the case $m = 1$. Then $Y = D_1 = Z$ and $Z_K = 2Z$. It follows from [14] or [5] that $p_g(V, o) \leq 2p_f(V, o) + 1 = 3$ with equality holding only when (V, o) is Gorenstein. If (V, o) is Gorenstein, then $2 = p_a(V, o) < p_g(V, o)$ by Lemma 1.4. Hence we have $p_g(V, o) = 3$ if and only if (V, o) is Gorenstein and, otherwise, we have $p_g(V, o) = 2$. Hence the formula for $p_g(V, o)$ in the statement holds for $m = 1$.

Next, we consider the case $m \geq 2$. Put $\omega_{Z_K} = \mathcal{O}_{Z_K}(\kappa)$. For $i = 1, 2, \ldots, m$, consider the cohomology long exact sequence for

$$0 \to \omega_{2Y_i - D_i}(F_{i-1}) \to \omega_{2Y_i}(F_{i-1}) \to \mathcal{O}_{D_i}(\kappa - F_{i-1}) \to 0.$$

Note that $\omega_{2Y_i}(F_{i-1})$ is numerically trivial. Since $\mathrm{Supp}(Y_i)$ is connected and D_i is the chain-connected component of $2Y_i$, the restriction map $H^0(2Y_i, \omega_{2Y_i}(F_{i-1})) \to H^0(D_i, \kappa - F_{i-1})$ is not the zero map if and only if $\omega_{2Y_i}(F_{i-1}) \simeq \mathcal{O}_{2Y_i}$. Therefore,

$$h^0(2Y_i, \omega_{2Y_i}(F_{i-1})) = h^0(2Y_i - D_i, \omega_{2Y_i - D_i}(F_{i-1})) + \begin{cases} 1, & \text{if } i \in \mathcal{B}, \\ 0, & \text{if } i \notin \mathcal{B}. \end{cases} \tag{3.3}$$

Next, for $i = 1, 2, \ldots, m-1$, consider the exact sequence

$$0 \to \mathcal{O}_{2Y_{i+1}}(-F_i) \to \mathcal{O}_{2Y_i - D_i}(-F_{i-1}) \to \mathcal{O}_{D_i}(-F_{i-1}) \to 0.$$

By the same reasoning as above, the restriction map $H^0(2Y_i - D_i, -F_{i-1}) \to H^0(D_i, -F_{i-1})$ is not the zero map if and only if $\mathcal{O}_{2Y_i - D_i}(F_{i-1}) \simeq \mathcal{O}_{2Y_i - D_i}$ and we have

$$h^1(2Y_i - D_i, -F_{i-1}) = h^1(2Y_{i+1}, -F_i) + \begin{cases} h^1(D_i, -F_{i-1}), & \text{if } i \in \mathcal{A}, \\ -\chi(D_i, -F_{i-1}), & \text{if } i \notin \mathcal{A}. \end{cases}$$

Since $p_a(D) = 2$, we get $h^1(D_i, -F_{i-1}) = h^0(D_i, -F_{i-1}) + 1$. From this and the duality theorem, we get

$$h^0(2Y_i - D_i, \omega_{2Y_i - D_i}(F_{i-1})) = h^0(2Y_{i+1}, \omega_{2Y_{i+1}}(F_i)) + \begin{cases} 2, & \text{if } i \in \mathcal{A}, \\ 1, & \text{if } i \notin \mathcal{A}. \end{cases} \tag{3.4}$$

Putting $h_i = h^0(2Y_i, \omega_{2Y_i}(F_{i-1}))$, we get

$$h_i = h_{i+1} + 1 + \begin{cases} 2, & \text{if } i \in \mathcal{A} \cap \mathcal{B}, \\ 1, & \text{if } i \in \mathcal{A} \cup \mathcal{B} \setminus (\mathcal{A} \cap \mathcal{B}), \\ 0, & \text{if } i \notin \mathcal{A} \cup \mathcal{B}, \end{cases}$$

by (3.3) and (3.4). This also holds for $i = m$ with $h_{m+1} = 0$. Hence, summing up, we get the desired formula for $p_g(V, o) = h_1$. □

A geometric interpretation of $\mathcal{A}$ is as follows. Note that $\mathcal{O}_{2Y_i - D_i}(F_{i-1}) \simeq \mathcal{O}_{2Y_i - D_i}((i-1)Z)$ by (1.2). If $i > 1$ and $i \in \mathcal{A}$, then this means that $\mathcal{O}_{2Y_i - D_i}(Z)$ is a torsion element and, putting $\alpha = \mathrm{ord}(\mathcal{O}_{2Y_i - D_i}(Z))$, $i - 1$ is a multiple of α. Conversely, if $j = 1 + k\alpha$ with a non-negative integer k and $j \le m$, then we have $j \in \mathcal{A}$. Hence,

$$\mathcal{A} = \left\{ 1 + k\alpha \mid 0 \le k \le \left[\frac{m-1}{\alpha} \right] \right\},$$

where $[x]$ denotes the biggest integer not exceeding $x \in \mathbb{R}$. We remark that, when $m \ge 2$, α does not depend on $i > 1$ in the sense that we have $\mathrm{ord}(\mathcal{O}_{D_m}(Z)) = \alpha$ by [11, Lemma 3.7], since we have $H^1(2Y_i - D_i, \mathbb{Z}) \simeq H^1(D_m, \mathbb{Z})$.

A similar observation can be done also for $\mathcal{B}$. Assume that $\mathcal{B} \neq \emptyset$ and put $\beta = \min \mathcal{B}$. We have $\omega_{2Y_i}(F_{i-1}) \simeq \mathcal{O}_{2Y_i}(\kappa - (i-1)Z)$. If $i \in \mathcal{B}$ and $i > \beta$, then restricting $\mathcal{O}_{2Y_\beta}(\kappa - (\beta - 1)Z) \simeq \mathcal{O}_{2Y_\beta}$ to $2Y_i$, we get $\mathcal{O}_{2Y_i}(\kappa - (\beta-1)Z) \simeq \mathcal{O}_{2Y_i}$ and, therefore, $\mathcal{O}_{2Y_i}((i - \beta)Z) \simeq \mathcal{O}_{2Y_i}$. Hence $i - \beta$ can be divided by α (which is also the order of $\mathcal{O}_{2Y_i}(Z)$ by the same reasoning as above), and we get

$$\mathcal{B} = \left\{ \beta + k\alpha \mid 0 \le k \le \left[\frac{m - \beta}{\alpha} \right] \right\}$$

as far as $\mathcal{B} \neq \emptyset$. In particular, we have $\mathcal{A} = \mathcal{B}$ if and only if (V, o) is Gorenstein.

We have shown the following:

THEOREM 3.2. *Let (V, o) be an even singularity with $p_f(V, o) = 2$ and $p_a(V, o) = m + 1$. Put*

$$\alpha = \begin{cases} 1, & \text{if } m = 1, \\ \mathrm{ord}(\mathcal{O}_{D_m}(Z)), & \text{if } m \ge 2, \end{cases} \tag{3.5}$$

and $\beta = \min \mathcal{B}$ when $\mathcal{B} \neq \emptyset$. Then,

$$p_g(V, o) = m + 2 + \left[\frac{m-1}{\alpha} \right] + \left[\frac{m - \beta}{\alpha} \right] \qquad \text{if } \mathcal{B} \neq \emptyset$$

and $p_g(V, o) = m + 1 + [(m-1)/\alpha]$ if $\mathcal{B} = \emptyset$.

REMARK 3.3. (1) When $\mathcal{B} = \emptyset$, we have $p_g(V, o) = h^1(Y, \mathcal{O}_Y)$ by [6, Theorem 1.5]. This means that the cohomological cycle is a subcurve of the Yau cycle.

(2) When (V, o) is Gorenstein, α may divide $m-1$.

Recall that (V_ν, o_ν) is the singular point obtained by contracting D_ν $(1 \le \nu \le m)$, $(V_1, o_1) = (V, o)$.

COROLLARY 3.4. *Let (V, o) be an even singularity with $p_f(V, o) = 2$ and $p_a(V, o) = m+1$. Then $p_g(V, o) \le 3m = 3p_a(V, o) - 3$. If the equality sign holds, then (V_ν, o_ν) is Gorenstein and $p_g(V_\nu, o_\nu) = 3(m-\nu+1)$ for all $\nu = 1, \dots, m$.*

DEFINITION 3.5. Let (V, o) be an even singular point with $p_f(V, o) = 2$. It is called a *maximally even singularity* if the equality $p_g(V, o) = 3p_a(V, o) - 3$ holds. Then, it is Gorenstein.

4. Maximal ideal cycles. In this section, we study the maximal ideal cycles of even Gorenstein singularities of fundamental genus 2 and show the following:

THEOREM 4.1. *Let (V, o) be an even Gorenstein singular point with $p_f(V, o) = 2$ and $p_a(V, o) = m+1$, $\mathfrak{m}$ the ideal sheaf of $o \in V$ and $\pi : X \to V$ the minimal resolution. Put $F_\alpha = \sum_{i=1}^{\alpha} D_i$ assuming that $\alpha < m$ when $m \ge 2$, where $Y = \sum_{i=1}^{m} D_i$ is the Yau cycle and α is the number defined in* (3.5)*. Then, one of the following three holds.*

(0) *F_α is the maximal ideal cycle and $\mathfrak{m}\mathcal{O}_X \simeq \mathcal{O}_X(-F_\alpha)$.*

(I) *F_α is the maximal ideal cycle, but there exists a point p such that $\mathfrak{m}\mathcal{O}_X \simeq \mathfrak{m}_p\mathcal{O}_X(-F_\alpha)$.*

(II) *There exists a (-2)-cycle $\Gamma \prec Z$ such that $F_\alpha + \Gamma$ is the maximal ideal cycle and $\mathfrak{m}\mathcal{O}_X \simeq \mathcal{O}_X(-F_\alpha - \Gamma)$. Γ is the fixed part of $|\omega_Z|$ when $m = 1$ and it is a chain-connected component of $Z - D_\alpha$ when $m > \alpha \ge 2$.*

When $m \ge 2$, (I) *occurs only when either $\alpha = 1$ or $\alpha \nmid m-1$;* (II) *occurs only when $\alpha \nmid m-1$.*

Therefore, even Gorenstein singularities of fundamental genus 2 fall into 4 types (0), (I) and (II) in Theorem 4.1 and (III) a class consisting of those with $\alpha \ge m \ge 2$. In the course of the proof, we freely use the notation in the previous sections.

We first consider the case where $m = 1$. Then, $F_\alpha = Z$ and $\omega_Z = \mathcal{O}_Z(-Z)$ since (V, o) is Gorenstein. Therefore, the assertion follows from what we saw in §2.1 (especially, Lemma 2.1) except the statement in (II) for Z of type (ii) in Proposition 2.7.

LEMMA 4.2. *Let (V, o) be an even Gorenstein singularity with $p_f(V, o) = p_a(V, o) = 2$. If it is of type* (ii) *and Γ is the fixed part of $|\omega_Z|$ described in* §2.1*, then $\mathcal{O}_X(-Z-\Gamma)$ is π-free.*

Proof. Since (V, o) is Gorenstein, K_X is linearly equivalent to $-2Z$. So, we have $\omega_{Z-\Gamma} = \mathcal{O}_{Z-\Gamma}(-Z-\Gamma)$. As we have already seen in Case 1 in §2.1 and Lemma 2.5, $-Z-\Gamma$ is trivial on $Z-\Gamma$. We have $\mathcal{O}_\Gamma(-Z-\Gamma) \simeq \mathcal{O}_\Gamma(-\Gamma)$. Hence $-Z-\Gamma$ is nef also on Γ, since Γ is the fundamental cycle of a rational double point. In sum, $\mathcal{O}_Z(-Z-\Gamma)$ is nef.

Using the exact sequences

$$0 \to \mathcal{O}_X(-Z_K - Z - \Gamma) \to \mathcal{O}_X(-Z-\Gamma) \to \mathcal{O}_{Z_K}(-Z-\Gamma) \to 0$$

and

$$0 \to \mathcal{O}_Z(-2Z-\Gamma) \to \mathcal{O}_{Z_K}(-Z-\Gamma) \to \mathcal{O}_Z(-Z-\Gamma) \to 0,$$

one can easily show that the restriction $H^0(X, -Z-\Gamma) \to H^0(Z, -Z-\Gamma)$ is surjective.

We know that $\mathcal{O}_Z(-Z-\Gamma)$ is nef of degree 2. Recall that $\mathcal{O}_{Z-\Gamma}(-Z-\Gamma) \simeq \mathcal{O}_{Z-\Gamma}$ and consider the cohomology long exact sequence for

$$0 \to \mathcal{O}_\Gamma(-2Z) \to \mathcal{O}_Z(-Z-\Gamma) \to \mathcal{O}_{Z-\Gamma} \to 0.$$

Since $\mathcal{O}_\Gamma(-2Z) \simeq \mathcal{O}_\Gamma$, we get $h^0(\Gamma, -2Z) = 1$ and $h^1(\Gamma, -2Z) = 0$. Hence, $H^0(Z, -Z-\Gamma) \to H^0(Z-\Gamma, \mathcal{O}_{Z-\Gamma})$ is surjective, and we get $h^0(Z, -Z-\Gamma) = h^0(\Gamma, -2Z) + h^0(Z-\Gamma, \mathcal{O}_{Z-\Gamma}) = 3$. Since $h^0(Z, -Z-\Gamma) = \deg \mathcal{O}_Z(-Z-\Gamma) + 1$ holds, $|\mathcal{O}_Z(-Z-\Gamma)|$ is free from base point by [5, Theorem 2.1]. This shows that $\mathcal{O}_X(-Z-\Gamma)$ is π-free. ∎

Now, we assume $m \geq 2$ and $\alpha < m$.

For $1 \leq i \leq m$, consider the exact sequence

$$0 \to \mathcal{O}_X(-Z_K) \to \mathcal{O}_X(-F_{i-1}) \to \mathcal{O}_{Y+Y_i}(-F_{i-1}) \to 0.$$

Since $H^1(X, -Z_K) = 0$ by the vanishing theorem, the restriction map $H^0(X, -F_{i-1}) \to H^0(Y+Y_i, -F_{i-1})$ is surjective. Next, consider the exact sequence

$$0 \to \mathcal{O}_{Y+Y_{i+1}}(-F_i) \to \mathcal{O}_{Y+Y_i}(-F_{i-1}) \to \mathcal{O}_{D_i}(-F_{i-1}) \to 0 \tag{4.1}$$

for $1 \leq i \leq m$ with the convention that $Y_{m+1} = 0$, $F_m = Y$. When $i = 1$, $H^0(Z_K, \mathcal{O}_{Z_K}) \to H^0(Z, \mathcal{O}_Z)$ is surjective and we have $h^0(Y+Y_2, -Z) = p_g(V, o) - 1$. When $i \geq 2$ and $i \notin \mathcal{A}$, we have $H^0(D_i, -F_{i-1}) = 0$ and it follows $H^0(Y+Y_{i+1}, -F_i) \simeq H^0(Y+Y_i, -F_{i-1})$. Therefore, when $\alpha \leq m$, we have $H^0(Y+Y_{\alpha+1}, -F_\alpha) \simeq H^0(Y+Y_2, -Z)$. From the commutative diagram

$$\begin{array}{ccccc} H^0(X, -Z_K) & \longrightarrow & H^0(X, -Z) & \longrightarrow & H^0(Y+Y_2, -Z) \\ \| & & \uparrow & & \uparrow \simeq \\ H^0(X, -Z_K) & \longrightarrow & H^0(X, -F_\alpha) & \longrightarrow & H^0(Y+Y_{\alpha+1}, -F_\alpha). \end{array}$$

we see that any holomorphic function on X vanishing identically on Z necessarily vanishes identically on F_α.

We claim that the restriction map $H^0(Y+Y_{\alpha+1}, -F_\alpha) \to H^0(D_{\alpha+1}, \mathcal{O}_{D_{\alpha+1}})$ is surjective. To see this, it suffices to show that the restriction map $H^0(Y+Y_{\alpha+1}, -F_\alpha) \to H^0(Y_{\alpha+1}, -F_\alpha)$ is surjective, since we have $\mathcal{O}_{Y_{\alpha+1}}(-F_\alpha) \simeq \mathcal{O}_{Y_{\alpha+1}}$ and the restriction map $H^0(Y_{\alpha+1}, \mathcal{O}_{Y_{\alpha+1}}) \to H^0(D_{\alpha+1}, \mathcal{O}_{D_{\alpha+1}})$ is surjective. Consider the exact sequence

$$0 \to \mathcal{O}_Y(-Y) \to \mathcal{O}_{Y+Y_{\alpha+1}}(-F_\alpha) \to \mathcal{O}_{Y_{\alpha+1}}(-F_\alpha) \to 0.$$

Since (V, o) is Gorenstein, we have $\mathcal{O}_Y(-Y) \simeq \omega_Y$ and $h^0(Y, -Y) = h^1(Y, \mathcal{O}_Y) = m+1+[(m-1)/\alpha] = p_g(V, o) - 1 - [(m-1)/\alpha]$ by [6, Theorem 1.5]. Similarly, we have $h^0(Y_{\alpha+1}, \mathcal{O}_{Y_{\alpha+1}}) = h^1(Y_{\alpha+1}, \mathcal{O}_{\alpha+1}) - (m-\alpha) = 1 + [(m-\alpha-1)/\alpha] = [(m-1)/\alpha]$ by the Riemann-Roch theorem. Then, $h^0(Y+Y_{\alpha+1}, -F_\alpha) = p_g(V, o) - 1 = h^0(Y, -Y) + h^0(Y_{\alpha+1}, \mathcal{O}_{Y_{\alpha+1}})$. This implies that $H^0(Y+Y_{\alpha+1}, -F_\alpha) \to H^0(Y_{\alpha+1}, \mathcal{O}_{Y_{\alpha+1}})$ is surjective.

We have shown that $|\mathcal{O}_X(-F_\alpha)|$ has no base points on $D_{\alpha+1}$. So our next task is to consider the base points on $Z - D_{\alpha+1}$. Recall that $Z - D_{\alpha+1}$ consists of two fundamental cycles of rational double points. Since the surjective map $H^0(Y+Y_{\alpha+1},-F_\alpha)\to H^0(D_{\alpha+1},\mathcal{O}_{D_{\alpha+1}})$ factors through $H^0(Z,-F_\alpha)$, the restriction map $H^0(Y+Y_{\alpha+1},-F_\alpha)\to H^0(Z,-F_\alpha)$ is non-zero and $H^0(Z,-F_\alpha)\to H^0(D_{\alpha+1},\mathcal{O}_{D_{\alpha+1}})$ is surjective. Note also that we have $\mathcal{O}_{Z-D_{\alpha+1}}(-F_{\alpha+1})\simeq\mathcal{O}_{Z-D_{\alpha+1}}$ being a numerically trivial invertible sheaf on the rational curve $Z-D_{\alpha+1}$. Then, from the exact sequence

$$0\to\mathcal{O}_{Z-D_{\alpha+1}}(-F_{\alpha+1})\to\mathcal{O}_Z(-F_\alpha)\to\mathcal{O}_{D_{\alpha+1}}\to 0, \tag{4.2}$$

we get $h^0(Z,-F_\alpha)=1+h^0(Z-D_{\alpha+1},\mathcal{O})=3$. Since $\mathcal{O}_Z(-F_\alpha)$ is nef of degree 2, it follows from [5, Theorem 2.1] that $|\mathcal{O}_Z(-F_\alpha)|$ is free from base points. We next consider

$$0\to H^0(Y_2+Y_{\alpha+1},-Z-F_\alpha)\to H^0(Y+Y_{\alpha+1},-F_\alpha)\to H^0(Z,-F_\alpha).$$

If $\alpha=1$, then $H^0(Y_2+Y_{\alpha+1},-Z-F_\alpha)=H^0(2Y_2,-2Z)\simeq H^0(2Y_2,\omega_{2Y_2})$ which is of dimension $p_g(V_2,o_2)=3m-3$. It follows that $H^0(Y+Y_2,-Z)\to H^0(Z,-Z)$ is of rank 2. In order to estimate $h^0(Y_2+Y_{\alpha+1},-Z-F_\alpha)$ for $\alpha\ge 2$, we consider

$$0\to\mathcal{O}_{Y_2+Y_{i+1}}(-Z-F_i)\to\mathcal{O}_{Y_2+Y_i}(-Z-F_{i-1})\to\mathcal{O}_{D_i}(-Z-F_{i-1})\to 0$$

for $i\ge 2$ and proceed similarly as what we did with (4.1). Then, one sees $h^0(Y_2+Y_{i+1},-Z-F_i)=h^0(Y_2+Y_i,-Z-F_{i-1})$ for $i<\alpha$, and $h^0(Y_2+Y_{\alpha+1},-Z-F_\alpha)=h^0(Y_2+Y_\alpha,-Z-F_{\alpha-1})-1$ and it follows $h^0(Y_2+Y_{\alpha+1},-Z-F_\alpha)=h^0(2Y_2,-2Z)-1$. On the other hand, since $\mathcal{O}_{2Y_2}(-2Z)\simeq\omega_{2Y_2}$ and the invariant β for (V_2,o_2) is given by $\alpha-1$, we get $h^0(2Y_2,-2Z)=(m-1)+2+[(m-2)/\alpha]+[(m-1-\alpha)/\alpha]=m+[(m-2)/\alpha]+[(m-1)/\alpha]$. Hence,

$$h^0(Y+Y_{\alpha+1},-F_\alpha)-h^0(Y_2+Y_{\alpha+1},-Z-F_\alpha)=2+\left[\frac{m-1}{\alpha}\right]-\left[\frac{m-2}{\alpha}\right]$$

and we get the following:

Lemma 4.3. *Assume that $m\ge 2$ and $\alpha<m$. Then the restriction map $H^0(Y+Y_{\alpha+1},-F_\alpha)\to H^0(Z,-F_\alpha)$ is surjective if $\alpha\ge 2$ and α divides $m-1$; otherwise, its image is 2-dimensional.*

If $\alpha\ge 2$ and $\alpha\mid(m-1)$, then $\mathcal{O}_X(-F_\alpha)$ is π-free, by what we have seen above. Hence $\mathfrak{m}\mathcal{O}_X=\mathcal{O}_X(-F_\alpha)$ in this case.

We consider the case that $H^0(Y+Y_{\alpha+1},-F_\alpha)\to H^0(Z,-F_\alpha)$ is of rank 2. Take $s_0,s_1\in H^0(Y+Y_{\alpha+1},-F_\alpha)$ which span the image. We can assume that s_0 is a non-zero constant on $D_{\alpha+1}$, whereas s_1 vanishes identically on $D_{\alpha+1}$. Then s_0 cannot vanish identically on any components of $D_\alpha-D_{\alpha+1}$ and s_1 induces a non-zero element $t_1\in H^0(Z-D_{\alpha+1},-F_{\alpha+1})$ from the exact sequence (4.2) and the fact that $\mathcal{O}_{Z-D_{\alpha+1}}(-F_{\alpha+1})\simeq\mathcal{O}_{Z-D_{\alpha+1}}$.

Assume that s_0 has isolated zeros only. If t_1 does not vanish on any components, then s_0 and s_1 have no common zeros. In this case, $\mathfrak{m}\mathcal{O}_X\simeq\mathcal{O}_X(-F_\alpha)$. If t_1 vanishes identically on a component, then it vanishes identically on a chain-connected component $\Gamma_{\alpha+1}$ of $Z-D_{\alpha+1}$ and is a non-zero constant on $Z-D_{\alpha+1}-\Gamma_{\alpha+1}$; it follows that s_0 and s_1 have common zeros only at a point $p\in\Gamma_{\alpha+1}$, $p\notin D_{\alpha+1}$. That is, $\mathrm{Bs}|\mathcal{O}_X(-F_\alpha)|=\{p\}$.

Assume that s_0 vanishes identically on a component of $Z-D_\alpha$. Then $\alpha\ge 2$ and s_0 vanishes identically on at least one maximal chain-connected component of $Z-D_\alpha$.

If t_1 does not vanish on any components of $Z - D_{\alpha+1}$, then s_0 and s_1 have no common zeros and we have $\mathfrak{m}\mathcal{O}_X \simeq \mathcal{O}_X(-F_\alpha)$. If t_1 vanishes on a component, then it vanishes on $\Gamma_{\alpha+1}$ as above. Therefore, there are two possibilities: either $Z - D_{\alpha+1}$ is of type $(2\mathsf{A}_\alpha)$ and s_0, s_1 has no common zeros, or s_0, s_1 vanishes identically on a maximal chain-connected component Γ of $Z - D_\alpha$ satisfying $\Gamma \prec \Gamma_{\alpha+1}$. In the former case, we have $\mathfrak{m}\mathcal{O}_X \simeq \mathcal{O}_X(-F_\alpha)$. As to the latter, we have the following.

Lemma 4.4. *If s_0 and s_1 vanish identically on Γ, then $\mathfrak{m}\mathcal{O}_X \simeq \mathcal{O}_X(-F)$, where $F = F_\alpha + \Gamma$.*

Proof. Recall that we have $\alpha \geq 2$ and $\alpha \nmid m - 1$.

We first confirm that $\mathcal{O}_X(-F_\alpha - \Gamma)$ is π-nef. Put $\Gamma' = Z - D_\alpha - \Gamma$. Then Γ' is the fundamental cycle of a rational double point and $F_\alpha + \Gamma = F_{\alpha+1} + Z - \Gamma'$. Let C be an irreducible component of Z and assume that $C(F_\alpha + \Gamma) = C(F_{\alpha+1} + Z - \Gamma') > 0$. We know that $-F_\alpha$, $-F_{\alpha+1}$ and $-Z$ are nef on $\pi^{-1}(o)$ by [6, Lemma 1.3]. From $C(F_\alpha + \Gamma) > 0$, we get $C\Gamma > 0$. On the other hand, from $C(F_{\alpha+1} + Z - \Gamma') > 0$, we get $C\Gamma' < 0$ and, thus, C is a component of Γ'. This contradicts $C\Gamma > 0$, because $\mathcal{O}_{\Gamma'}(-\Gamma)$ is trivial. Therefore, $-F$ is nef on $\pi^{-1}(o)$, where $F = F_\alpha + \Gamma$.

It remains to show that $\mathcal{O}_X(-F)$ is π-free. For this purpose, we consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_X(-Z_K - \Gamma) & \longrightarrow & \mathcal{O}_X(-F) & \longrightarrow & \mathcal{O}_{Y+Y_{\alpha+1}}(-F) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\| & & \big\downarrow & & \\
0 & \longrightarrow & \mathcal{O}_X(-Z_K) & \longrightarrow & \mathcal{O}_X(-F) & \longrightarrow & \mathcal{O}_{Y+Y_{\alpha+1}-\Gamma}(-F) & \longrightarrow & 0
\end{array}$$

of sheaves with exact rows. Since there are no curves with support in $\pi^{-1}(o)$ whose self-intersection numbers are odd, we know that ω_X is π-free by a result in [4]. Hence $H^0(X, -Z_K) \to H^0(\Gamma, \mathcal{O}_\Gamma)$ is surjective, and we get $H^1(X, -Z_K - \Gamma) = 0$ by $H^1(X, -Z_K) = 0$. It follows that $H^0(X, -F) \to H^0(Y + Y_{\alpha+1}, -F)$ and $H^0(X, -F) \to H^0(Y + Y_{\alpha+1} - \Gamma, -F)$ are both surjective.

Since Γ is the fixed part of $|\mathcal{O}_X(-F_\alpha)|$, the natural inclusion $H^0(Y + Y_{\alpha+1} - \Gamma, -F) \to H^0(Y + Y_{\alpha+1}, -F_\alpha)$ is an isomorphism. Then, the exact sequence

$$0 \to \mathcal{O}_{Y_2+Y_{\alpha+1}}(-F_\alpha - Z) \to \mathcal{O}_{Y+Y_{\alpha+1}-\Gamma}(-F) \to \mathcal{O}_{Z-\Gamma}(-F) \to 0$$

and the computation done just before Lemma 4.3 show that the image of $H^0(Y + Y_{\alpha+1} - \Gamma, -F) \to H^0(Z - \Gamma, -F)$ is 2-dimensional. We remark that $Z - \Gamma$ is numerically 1-connected, since so is Z and $(Z - \Gamma)\Gamma = 1$. Since $\mathcal{O}_{Z-\Gamma}(-F)$ is nef of degree one, we have $h^0(Z - \Gamma, -F) \leq 2$. Therefore, $H^0(Y + Y_{\alpha+1} - \Gamma, -F) \to H^0(Z - \Gamma, -F)$ is surjective and $\mathcal{O}_{Z-\Gamma}(-F)$ is generated by its global sections by [5, Theorem 2.1]. In particular, we know that $|\mathcal{O}_X(-F)|$ has no base points on $Z - \Gamma$.

Now, we consider the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{O}_\Gamma(-Z_K) & \longrightarrow & \mathcal{O}_{Y+Y_{\alpha+1}}(-F) & \longrightarrow & \mathcal{O}_{Y+Y_{\alpha+1}-\Gamma}(-F) & \longrightarrow & 0 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
0 & \longrightarrow & \mathcal{O}_\Gamma(-F_\alpha - Z) & \longrightarrow & \mathcal{O}_Z(-F) & \longrightarrow & \mathcal{O}_{Z-\Gamma}(-F) & \longrightarrow & 0
\end{array}$$

with exact rows, where the vertical maps are restriction maps. A non-zero element of $H^0(\Gamma, -Z_K) \simeq H^0(\Gamma, \mathcal{O}_\Gamma)$ gives us an element of $H^0(Y + Y_{\alpha*1}, -F)$ that does not vanish on $\Gamma \setminus (D_\alpha \cap \Gamma)$. In sum, $\mathcal{O}_X(-F)$ is π-free. $\square$

LEMMA 4.5. *$h^1(Y+Y_{\alpha+1},\mathcal{O}_{Y+Y_{\alpha+1}}) = p_g(V,o)-1$, $h^1(Y+Y_{\alpha+1},-F_\alpha) = p_g(V,o)-1-\alpha$ and $h^1(Y+Y_{\alpha+1},-nF_\alpha) = p_g(V,o)-2-\alpha$ for any $n\geq 2$.*

Proof. Firstly, we consider the case where $m=1$.

$$0\to\mathcal{O}_Z(-(n+1)Z)\to\mathcal{O}_{Z_K}(-nZ)\to\mathcal{O}_Z(-nZ)\to 0.$$

Since $\mathcal{O}_Z(-Z)=\omega_Z$, by duality, we get $h^1(Z,-(n+1)Z)=h^0(Z,nZ)=0$ for $n\geq 1$. Then, $H^1(Z_K,-nZ)\simeq H^1(Z,\omega_Z(-(n-1)Z)$. We get $h^1(Z_K,-Z)=1$ and $h^1(Z_K,-nZ)=0$ for $n\geq 2$. Since $p_g(V,o)=3$, we are done.

Next, we consider the case $m\geq 2$. The restriction map $H^0(X,-F_\alpha)\to H^0(Y+Y_{\alpha+1},-F_\alpha)$ is surjective and $H^1(X,-F_\alpha)\simeq H^1(Y+Y_{\alpha+1},-F_\alpha)$. Since $h^0(Y+Y_{\alpha+1},-F_\alpha)=p_g(V,o)-1$ as we already saw, we get $h^1(Y+Y_{\alpha+1},-F_\alpha)=p_g(V,o)-1-\alpha$ by the Riemann-Roch theorem. It follows $h^1(Z_K,-F_\alpha)=p_g(V,o)-1-\alpha$. For $n\geq 2$, we consider

$$0\to\mathcal{O}_X(-Z_K-(n-2)F_\alpha)\to\mathcal{O}_X(-nF_\alpha)\to\mathcal{O}_{2Y_{\alpha+1}}(-nF_\alpha)\to 0.$$

We have $H^1(X,-Z_K-(n-2)F_\alpha)=0$ when $n\geq 2$ and $H^1(X,-nF_\alpha)\simeq H^1(2Y_{\alpha+1},-nF_\alpha)\simeq H^1(2Y_{\alpha+1},\mathcal{O})$. Therefore, $h^1(Z_K,-nF_\alpha)=p_g(V_{\alpha+1},o_{\alpha+1})=p_g(V,o)-2-\alpha$ for $n\geq 2$. We note that $\mathcal{O}_{Y+Y_{\alpha+1}}(-F_\alpha)\simeq\omega_{Y+Y_{\alpha+1}}$. Hence $h^1(Y+Y_{\alpha+1},\mathcal{O})=h^0(Y+Y_{\alpha+1},-F_\alpha)=h^1(Y+Y_{\alpha+1},-F_\alpha)+\alpha$ by the Riemann-Roch theorem. □

The following completes the proof of Theorem 4.1.

LEMMA 4.6. *Assume that $\mathrm{Bs}|\mathcal{O}_X(-F_\alpha)|=\{p\}$ and let $\rho:\widetilde{X}\to X$ be the blowing-up at p. Put $E=\rho^{-1}(p)$ and $F:=\rho^*F_\alpha+E$. Then $\mathfrak{m}\mathcal{O}_{\widetilde{X}}\simeq\mathcal{O}_{\widetilde{X}}(-F)$. In particular, $\mathrm{mult}(V,o)=-F^2=2\alpha+1$.*

Proof. Recall that $p\in D_\alpha-D_{\alpha+1}$ when $m\geq 2$. Since every irreducible component through p is non-singular at p and has negative intersection number with F_α, one easily sees that $-F$ is nef on $(\pi\circ\rho)^{-1}(o)$. It remains to show that $\mathcal{O}_{\widetilde{X}}(-F)$ has no base points on E. This will be done along an analogous line to the proof of Lemma 4.4.

Since p is the base point of $|\mathcal{O}_{Y+Y_{\alpha+1}}(-F_\alpha)|$, the restriction map $H^0(\rho^*(Y+Y_{\alpha+1}),-\rho^*F_\alpha)\to H^0(E,\mathcal{O}_E)$ is the zero map and we get $h^0(\rho^*(Y+Y_{\alpha+1})-E,-F)=h^0(\rho^*(Y+Y_{\alpha+1}),-\rho^*F_\alpha)=p_g(V,o)-1$ and $h^1(\rho^*(Y+Y_{\alpha+1})-E,-F)=h^1(\rho^*(Y+Y_{\alpha+1}),-\rho^*F_\alpha)+h^0(E,\mathcal{O}_E)=p_g(V,o)-\alpha$. When $m=1$, this gives us $h^0(\rho^*Z-E,-F)=2$. When $m\geq 2$, we recall that $h^0(Y_2+Y_{\alpha+1},-Z-F_\alpha)=m-1+[(m-2)/\alpha]+[(m-1)/\alpha]=p_g(V,o)-3$ and consider

$$0\to H^0(\rho^*(Y_2+Y_{\alpha+1}),-\rho^*(Z+F_\alpha))\to H^0(\rho^*(Y+Y_{\alpha+1})-E,-F)\to H^0(\rho^*Z-E,-F)$$

to find $H^0(\rho^*(Y+Y_{\alpha+1})-E,-F)\to H^0(\rho^*Z-E,-F)$ is of rank 2 and, hence, $h^0(\rho^*Z-E,-F)\geq 2$. Since $\mathcal{O}_{\rho^*Z-E}(-F)$ is nef of degree 1 and ρ^*Z-E is chain-connected by Lemma 1.1, it follows from [5, Theorem 2.1] that $h^0(\rho^*Z-E,-F)=2$ and $\mathcal{O}_{\rho^*Z-E}(-F)$ is generated by its global sections. From

$$0\to\mathcal{O}_{\widetilde{X}}(-\rho^*Z_K)\to\mathcal{O}_{\widetilde{X}}(-F)\to\mathcal{O}_{\rho^*(Y+Y_{\alpha+1})-E}(-F)\to 0,$$

we see that $H^0(\widetilde{X},-F)\to H^0(\rho^*(Y+Y_{\alpha+1})-E,-F)$ is surjective. As we already remarked in the proof of Lemma 4.4, ω_X is π-free. Thus the restriction map $H^0(\widetilde{X}, -\rho^* Z_K) \to H^0(E, \mathcal{O}_E)$ is surjective and we have $H^1(\widetilde{X}, -\rho^* Z_K - E) \simeq H^1(\widetilde{X}, -\rho^* Z_K) = 0$. Then, it follows from the exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}}(-\rho^* Z_K - E) \to \mathcal{O}_{\widetilde{X}}(-F) \to \mathcal{O}_{\rho^*(Y+Y_{\alpha+1})}(-F) \to 0$$

that $H^0(\widetilde{X}, -F) \to H^0(\rho^*(Y+Y_{\alpha+1}), -F)$ is surjective and $H^1(\widetilde{X}, -F) \simeq H^1(\rho^*(Y+Y_{\alpha+1}), -F)$. We now consider the commutative diagram

$$\begin{array}{ccccc} H^0(E, -\rho^* Z_K) & \hookrightarrow & H^0(\rho^*(Y + Y_{\alpha+1}), -F) & \twoheadrightarrow & H^0(\rho^*(Y + Y_{\alpha+1}) - E, -F) \\ \downarrow \simeq & & \downarrow & & \downarrow \\ H^0(E, -\rho^*(F_\alpha + Z)) & \hookrightarrow & H^0(\rho^* Z, -F) & \twoheadrightarrow & H^0(\rho^* Z - E, -F), \end{array}$$

where the vertical arrows are restriction maps, to conclude that $H^0(\rho^*(Y + Y_{\alpha+1}), -F) \to H^0(\rho^* Z, -F)$ is surjective. Since $\mathcal{O}_{\rho^* Z}(-F)$ is nef of degree 2 and $h^0(\rho^* Z, -F) = 3$, we see that $\mathcal{O}_{\rho^* Z}(-F)$ is generated by its global sections. Since $H^0(\widetilde{X}, -F) \to H^0(\rho^* Z, -F)$ is surjective, we see that $\mathcal{O}_{\widetilde{X}}(-F)$ is $(\pi \circ \rho)$-free and we have $\mathfrak{m}\mathcal{O}_{\widetilde{X}} \simeq \mathcal{O}_{\widetilde{X}}(-F)$. □

By Theorem 4.1 and its proof, we get the following:

COROLLARY 4.7. *Let the situation be as in Theorem* 4.1. *Then, the multiplicity of* (V, o) *is given by*

$$\operatorname{mult}(V, o) = \begin{cases} 2\alpha, & \text{if } \mathfrak{m}\mathcal{O}_X \simeq \mathcal{O}_X(-F_\alpha), \\ 2\alpha + 1, & \text{if } \mathfrak{m}\mathcal{O}_X \simeq \mathfrak{m}_p\mathcal{O}_X(-F_\alpha), \\ 2\alpha + 2 & \text{if } \mathfrak{m}\mathcal{O}_X \simeq \mathcal{O}_X(-F_\alpha - \Gamma). \end{cases}$$

Proof. See, [12, 4.6]. □

REMARK 4.8. It will be interesting to compare the fundamental genus and the arithmetic genus of the maximal ideal cycle described in Theorem 4.1. We have $p_a(F_\alpha) = \alpha + 1$ and $p_a(F_\alpha + \Gamma) = \alpha$. Hence it is smaller than the fundamental genus only when it is of type (II) with $\alpha = 1$.

5. Embedding dimensions. In this section, we compute the embedding dimension of a maximally even singularity of fundamental genus 2. For this purpose, we sometimes need the following:

PROPOSITION 5.1. *Let L be a line bundle of degree $d > 0$ on a curve D such that $|L|$ is free from base points. Then $h^0(D, L) \le d + h^0(D, \mathcal{O}_D)$. Furthermore, the graded ring $R(D, L) = \bigoplus_{n=0}^{\infty} H^0(D, nL)$ is generated in degree one in the following cases.*

(1) $h^0(D, L) = d + h^0(D, \mathcal{O}_D)$.
(2) $h^0(D, L) = d + h^0(D, \mathcal{O}_D) - 1$, $d \ge 3$ *and* $\omega_D \simeq \mathcal{O}_D$.

Proof. Take a general section $s \in H^0(D, L)$. Since $|L|$ is free from base points, $\zeta = (s)$ is an effective Cartier divisor of degree d. By using the identifications $\mathcal{O}_D(L) \simeq \mathcal{O}_D(\zeta)$ and $\mathcal{O}_\zeta(\zeta) \simeq \mathcal{O}_\zeta$ by another general section of L, we get an exact sequence

$$0 \to \mathcal{O}_D \to \mathcal{O}_D(L) \to \mathcal{O}_\zeta \to 0.$$

From this, we get $h^0(D,L) \le h^0(\mathcal{O}_\zeta)+h^0(D,\mathcal{O}_D)$. If (1) is the case, then $H^0(D,L)\to H^0(\mathcal{O}_\zeta)$ is surjective. It follows that $H^0(D,nL)\to H^0(\mathcal{O}_\zeta)$ is surjective for any $n\ge 1$, because so is the composite $\mathrm{Sym}^n H^0(D,L)\to H^0(D,nL)\to H^0(\mathcal{O}_\zeta)$. Then we have $h^0(D,nL)=h^0(D,\mathcal{O}_D)+nd$ for $n\ge 0$ in this case. If (2) is the case, then $H^0(D,L)\to H^0(\mathcal{O}_\zeta)$ is not surjective, but one can prove that $H^0(D,L)\to H^0(\mathcal{O}_{\zeta'})$ is surjective for any effective subscheme $\zeta'\subset\zeta$ of length $d-1$ as in [10, §6]. This shows that ζ is "in general position" with respect to the quadrics (cf. [12]) and it follows that $\mathrm{Sym}^2 H^0(D,L)\to H^0(D,2L)$ and the composite $\mathrm{Sym}^2 H^0(D,L)\to H^0(D,2L)\to H^0(\mathcal{O}_\zeta)$ are surjective. Then $h^0(D,nL)=h^0(D,\mathcal{O}_D)+nd-1$ for $n>0$.

In both of (1) and (2), the rest follows from Castelnuovo's free pencil trick. Choose a general subspace $W\subset H^0(D,L)$ of dimension 2 and consider the Koszul exact sequence

$$0\to \bigwedge^2 W\otimes\mathcal{O}_D((n-1)L)\to W\otimes\mathcal{O}_D(nL)\to\mathcal{O}_D((n+1)L)\to 0$$

for $n>0$. Since $2h^0(D,nL)=h^0(D,(n+1)L)+h^0(D,(n-1)L)$, $W\otimes H^0(D,nL)\to H^0(D,(n+1)L)$ is surjective for $n\ge 1$ when (1), and for $n\ge 2$ when (2). □

We shall show the following with several lemmas:

Theorem 5.2. *Let (V,o) be a maximally even singular point with $p_f(V,o)=2$. Then the embedding dimension is given by*

$$\mathrm{embdim}(V,o)=\begin{cases}3, & \textit{if } \mathrm{mult}(V,o)=2,3,\\ 4, & \textit{if } \mathrm{mult}(V,o)=4.\end{cases}$$

Lemma 5.3. *Let (V,o) be a maximally even singular point with $p_f(V,o)=2$. If* $\mathrm{Bs}|\mathcal{O}_X(-Z)|=\emptyset$, *then $R(Y+Y_2,-Z):=\bigoplus_{n\ge 0}H^0(Y+Y_2,-nZ)$ is generated by $s_0,s_1\in H^0(Y+Y_2,-Z)$ and $s_2\in H^0(Y+Y_2,-3Z)$. In particular,* $\mathrm{embdim}(V,o)=3$.

Proof. We first claim that the rank of the restriction map $H^0(Y+Y_2,-nZ)\to H^0(Z,-nZ)$ is $2n-1$ when $n\ge 2$. This follows from the exact sequence

$$0\to\mathcal{O}_{2Y_2}(-(n+1)Z)\to\mathcal{O}_{Y+Y_2}(-nZ)\to\mathcal{O}_Z(-nZ)\to 0,$$

since we have $h^0(Y+Y_2,-nZ)-h^0(2Y_2,-(n+1)Z)=h^1(Y+Y_2,-nZ)+2n-1-p_g(V_2,o_2)=2n-1$ by Lemma 4.5. Recall that s_0,s_1 generate a $(n+1)$ dimensional subspace of $H^0(Z,-nZ)$ spanned by $s_0^i s_1^{n-i}$ $(0\le i\le n)$. Hence, we find a new element $s_2\in H^0(Y+Y_2,-3Z)$ linearly independent from $s_0^i s_1^{3-i}$ $(0\le i\le 3)$.

Recall that the image of the restriction map $H^0(Y+Y_2,-Z)\to H^0(Z,-Z)$ is of dimension 2. We denote by $W=\langle s_0,s_1\rangle\subset H^0(Y+Y_2,-Z)$ the subspace of dimension 2 which restricts to span the image of $H^0(Y+Y_2,-Z)\to H^0(Z,-Z)$. Then W induces a pencil that is free from base points and we have a Koszul exact sequence

$$0\to\bigwedge^2 W\otimes\mathcal{O}_{Y+Y_2}(-(n-1)Z)\to W\otimes\mathcal{O}_{Y+Y_2}(-nZ)\to\mathcal{O}_{Y+Y_2}(-(n+1)Z)\to 0$$

for any non-negative integer n. We have $2h^1(Y+Y_2,-nZ)=h^1(Y+Y_2,-(n-1)Z)+h^1(Y+Y_2,-(n+1)Z)$ when $n\ge 3$ by Lemma 4.5. It follows that the multiplication map $W\otimes H^0(Y+Y_2,-nZ)\to H^0(Y+Y_2,-(n+1)Z)$ is surjective for $n\ge 3$. Hence, the graded ring $R(Y+Y_2,-Z)$ is generated by elements of degrees ≤ 3. Similarly, we can show that $W \otimes H^0(Y+Y_2,-Z) \to H^0(Y+Y_2,-2Z)$ is surjective and $W \otimes H^0(Y+Y_2,-2Z) \to H^0(Y+Y_2,-3Z)$ has 1-dimensional cokernel. Indeed, as we saw above, we have $s_2 \in H^0(Y+Y_2,-3Z)$. Clearly, there is a relation of the form $s_2^2 = \phi_6(s_0,s_1)$.

Let $x_0, x_1 \in H^0(X,-Z)$ and $x_2 \in H^0(X,-3Z)$ be preimages of s_0, s_1 and s_2. We show that $\mathfrak{m}/\mathfrak{m}^2$ is generated by x_0, x_1, x_2. Indeed, we have $H^0(X,\mathfrak{m}\mathcal{O}_X) \simeq H^0(X,-Z)$ and $H^0(X,-2Z)/H^0(X,\mathfrak{m}^2\mathcal{O}_X)$ is generated by x_2. Then,

$$\begin{aligned}\operatorname{embdim}(V,o) &= \dim H^0(X,-Z)/H^0(X,-2Z) + \dim H^0(X,-2Z)/H^0(X,\mathfrak{m}^2\mathcal{O}_X)\\ &= \dim H^0(X,-Z)/H^0(X,-2Z) + 1.\end{aligned}$$

As is already seen, the restriction map $H^0(X,-nZ) \to H^0(Y+Y_2,-nZ)$ is surjective for any positive integer n. Hence, $\dim H^0(X,-Z)/H^0(X,-2Z) = \dim \operatorname{Im}\{H^0(X,-Z) \to H^0(Z,-Z)\} = \dim \operatorname{Im}\{H^0(Y+Y_2,-Z) \to H^0(Z,-Z)\}$. We have $h^0(Y+Y_2,-Z) = p_g(V,o)-1$ by Lemma 4.5 and the Riemann-Roch theorem. We consider the exact sequence

$$0 \to \mathcal{O}_{2Y_2}(-2Z) \to \mathcal{O}_{Y+Y_2}(-Z) \to \mathcal{O}_Z(-Z) \to 0.$$

Since $\mathcal{O}_{2Y_2}(-2Z) \simeq \omega_{2Y_2}$, we get $h^0(2Y_2,-2Z) = p_g(V_2,o_2) = p_g(V,o)-3$. Therefore, $\operatorname{embdim}(V,o) = (p_g-1)-(p_g-3)+1 = 3$. □

See, Example 2.9 for maximally even double points.

LEMMA 5.4. *Let (V,o) be a maximally even singular point with $p_f(V,o)=2$. If* $\operatorname{Bs}|\mathcal{O}_X(-Z)| = \{p\}$, *then* $\operatorname{embdim}(V,o)=3$.

Proof. Put $F = \rho^*Z - E$. We let $\widetilde{Z}_K = \rho^*Z_K - E$ be the canonical cycle on $\widetilde{X}$. We shall show that $h^0(\widetilde{Z}_K,-F) = \deg(\mathcal{O}_{\widetilde{Z}_K}(-F)) + p_g(V,o) - 1$. We have $H^1(\widetilde{X},-F) \simeq H^1(\widetilde{Z}_K,-F)$. Since $H^1(\widetilde{X},-F) \simeq H^1(\rho^*(Y+Y_2)-E,-F)$, we get $h^1(\widetilde{Z}_K,-F) = p_g(V,o)-1$. Hence, by the Riemann-Roch theorem, we get $h^0(\widetilde{Z}_K,-F) = \deg(\mathcal{O}_{\widetilde{Z}_K}(-F)) + p_g(V,o) - 1$ as wished. Since $\deg(\mathcal{O}_{\widetilde{Z}_K}(-F)) = 3$, it follows from Proposition 5.1 (2) that $R(\widetilde{Z}_K,-F)$ is generated in degree 1. From this, one has $\mathfrak{m}^n \simeq (\pi\circ\rho)_*\mathcal{O}_{\widetilde{X}}(-nF)$ for any positive integer n as in [12, §4]. Then we have $\operatorname{embdim}(V,o) = \dim \operatorname{Im}\{H^0(\widetilde{X},-F) \to H^0(F,-F)\}$.

Since the intersection number of any curve with support in $\pi^{-1}(o)$ is even, we see that $\mathcal{O}_X(K_X - Z)$ is π-free (cf. [4]). This implies that $H^0(\widetilde{X},-\rho^*(Z_K+Z) \to H^0(E,\mathcal{O}_E)$ is surjective and we get $H^1(\widetilde{X},-\rho^*Z_K - F) = 0$ from $H^1(X,-Z_K-Z) = 0$. It follows that $H^0(\widetilde{X},-F) \to H^0(\rho^*Z_K,-F)$ is surjective and $H^1(\widetilde{X},-F) \simeq H^1(\rho^*Z_K,-F)$. In particular, $h^1(\rho^*Z_K,-F) = p_g(V,o)-1$. By the Riemann-Roch theorem, we get $h^0(\rho^*Z_K,-F) = p_g(V,o)+3$. We consider

$$0 \to H^0(\rho^*(Y+Y_2)-E,-2F) \to H^0(\rho^*Z_K,-F) \to H^0(F,-F).$$

Then $\operatorname{embdim}(V,o) = h^0(\rho^*Z_K,-F) - h^0(\rho^*(Y+Y_2)-E,-2F)$. We showed in the proof of Lemma 4.6 that $H^0(\rho^*(Y+Y_2)-E,-F) \to H^0(\rho^*Z-E,-F)$ is surjective. Since $R(\rho^*Z-E,-F)$ is generated in degree 1 by Proposition 5.1 (1), $H^0(\rho^*(Y+Y_2)-E,-2F) \to H^0(\rho^*Z-E,-2F)$ is also surjective. From the exact sequence

$$0 \to H^0(2\rho^*Y_2,-3\rho^*Z-E) \to H^0(\rho^*(Y+Y_2)-E,-2F) \to H^0(\rho^*Z-E,-2F) \to 0,$$

we get $h^0(\rho^*(Y+Y_2)-E,-2F)=p_g(V_2,o_2)+3=p_g(V,o)$. In sum, $\mathrm{embdim}(V,o)=3$. ∎

EXAMPLE 5.5. The minimal resolution of the hypersurface triple point $\{z^3=(x^2-y^4)(x^4-y^2)\}$ has two (-2)-elliptic curves meeting normally at a point as the exceptional set. Hence it is of type (i.a). The exceptional set for $\{z^3=x^4+y^6\}$ consists of one (-2)-elliptic curve E_0 and three (-2)-curves E_1,E_2,E_3 such that $E_0E_i=1$ $(i=1,2,3)$ and $E_iE_j=0$ $(i,j\in\{1,2,3\}, i\neq j)$. Hence it is of type (i.b) (see, Fig. 2 (a)). $\{z^3=x(x^3+y^5)\}$ is another example of type (i.b) (see, Fig. 3 (c)).

If $\alpha=1$, then Type (II) occurs only when $m=1$.

LEMMA 5.6. *Let (V,o) be of type* (II)*, $m=1$. Then the ring $R(Z_K,-F)$ is generated in degree one. Furthermore, $\mathfrak{m}^n\mathcal{O}_X\simeq H^0(X,-nF)$ for any positive integer n and* $\mathrm{embdim}(V,o)=4$.

Proof. We only have to show that $h^0(Z_K,-F)=6=\deg(\mathcal{O}_{Z_K}(-F))-1+h^0(Z_K,\mathcal{O}_{Z_K})$.

$$0\to\mathcal{O}_Z(-2Z-\Gamma)\to\mathcal{O}_{Z_K}(-F)\to\mathcal{O}_Z(-F)\to 0.$$

We have $h^1(Z,-2Z-\Gamma)=0$ and $h^0(Z,-2Z-\Gamma)=3$, since $\omega_Z=\mathcal{O}_Z(-Z)$. As we already saw in the proof of Lemma 4.2, we have $h^0(Z,-F)=3$. Hence $h^0(Z_K,-F)=3+3=6$. By Proposition 5.1, $R(Z_K,-F)$ is generated in degree 1. This shows that $\mathfrak{m}^n\simeq\pi_*\mathcal{O}_X(-nF)$ as in [12]. In particular, $\mathrm{embdim}(V,o)=\varepsilon(V,o)$. Since $H^0(X,-F)\to H^0(Z_K,-F)$ is surjective, by using

$$0\to\mathcal{O}_{Z-\Gamma}(-2F)\to\mathcal{O}_{Z_K}(-F)\to\mathcal{O}_F(-F)\to 0$$

and $h^0(Z-\Gamma,\mathcal{O}_{Z-\Gamma})=2$, we get $\mathrm{embdim}(V,o)=6-2=4$. ∎

REMARK 5.7. For a 1-connected curve D with $p_a(D)=2$ and K_D nef, the structure of the canonical algebra $R(D,K_D)$ was extensively studied in [8]. When D is not 2-connected, her deep analysis shows:

$$R(D,K_D)\simeq\mathbb{C}[x_0,x_1,y,z]/(\varphi_2,\varphi_6),$$

where $\deg x_0=\deg x_1=1$, $\deg y=2$ and $\deg z=3$, and the φ_i's are homogeneous forms of degree i. A normal form is like:

$$\varphi_2=x_0(x_0-\lambda x_1),\quad \varphi_6=z^2-y^3+\cdots\quad(\lambda\in\mathbb{C}).$$

In other words, the canonical model of D is a weighted complete intersection of type $(2,6)$ in $\mathbb{P}(1,1,2,3)$.

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