METHODS AND APPLICATIONS OF ANALYSIS.
Vol. 24, No. 1, pp. 011–032, March 2017

002

# LOGARITHMIC DIFFERENTIAL FORMS ON COHEN-MACAULAY VARIETIES*

A. G. ALEKSANDROV†

*Dedicated to Henry Laufer on the occasion of his 70th birthday*

**Abstract.** The purpose of the paper is to introduce a notion of logarithmic differential forms on singular varieties. We also compute the Poincaré series and generators of the corresponding modules in a few particular cases, including quasihomogeneous complete intersections, normal varieties, determinantal varieties, and others.

**Key words.** Logarithmic differential forms, de Rham complex, graded singularities, Poincaré series, complete intersections, normal varieties, determinantal varieties, fans.

**AMS subject classifications.** 32S25, 14F10, 14F40, 58K45, 58K70.

**Introduction.** The concept of logarithmic differential forms with poles along a reduced divisor $D$ in a complex manifold $M$ has appeared in contemporary theory in 1960s in relation with studies of Hodge structure and Gauss-Manin connection on algebraic varieties. More precisely, by traditional definition, a meromorphic differential form $\omega$ with poles along such a divisor $D$ is called *logarithmic* if the two forms $\omega$ and $d\omega$ have at worst simple poles on $D$ only. The corresponding sheaves are usually denoted by $\Omega^p_M(\log D)$, $p \geqslant 0$. For instance, P. Deligne, N. Katz, Ph. Griffiths exploited this notion for a union of smooth subvarieties which are *normally crossing*, K.Saito and his successors studied the case of arbitrary *reduced* divisors (see [20]), etc.

In his previous works, the author elaborates another approach to this subject, which is mainly based on an original interpretation of the above definition with the use of a modified version of the classical de Rham lemma adapted to the case of singular hypersurfaces (see [2, 4, 6]). In this article, we shall extend the notion of logarithmic differential forms with poles along Cartier divisors on Cohen-Macaulay varieties in the course of these ideas. Among other things, we compute the Poincaré series and generators of the modules of logarithmic differential forms in the case of divisors given on graded isolated complete intersection singularities (graded ICIS), normal varieties, determinantal varieties, and others.

In the first two sections, we discuss basic notions and definitions, involving some important properties of regular differential forms on complex varieties, and describe a few simple methods for computing the module of Kähler differentials and the Poincaré series of related objects on quasihomogeneous singularities. Then we apply this techniques for computing the Poincaré series of modules of regular differential forms on the quotient surface singularities of embedding codimension 2. The next two sections contain a brief discussion of the main principle of our approach which is based on a variant of the classical de Rham lemma. This leads to an alternative definition of logarithmic differential forms which is better suited for the use in a rather general setting. In the sixth section, we consider some simple properties of logarithmic forms on normal varieties. Then we compute explicitly the Poincaré series and generators

*Received March 30, 2016; accepted for publication October 13, 2016.
†Institute for Control Sciences RAS, 65 Profsoyunaya str., B-342, GSP-7, Moscow, 117997, Russian Federation (ag_aleksandrov@mail.ru).

of the modules of logarithmic differential forms for divisors given on complete intersections with isolated singularities and determinantal varieties. In the final section, we analyze an example of 2-dimensional fan, the simplest example of rigid surface singularity which is neither normal nor Cohen-Macaulay. Almost all results are illustrated by concrete examples, containing explicit computations, nonformal remarks and comments.

I would like to thank the organizing committee of the Conference, especially Professor Stephen Yau, for their complimentary invitation to visit the Tsinghua Sanya International Mathematics Forum and hospitality there. I am also deeply indebted to all participants of the meeting for many fruitful conversations and stimulating scientific discussions. I am especially grateful to the referee for careful reading of the manuscript and useful remarks.

**1. Differential forms and the Poincaré complex.** Let $(X, \mathfrak{o})$ be a germ of complex space. We choose one of its representatives embedded in an open neighborhood $U$ of the point $0 \in \mathbb{C}^m$ with coordinates $z_1, \dots, z_m$, and denote it by $X$. Then $X$ is determined by an ideal $I$ of the structure sheaf $\mathscr{O}_U$, generated by a sequence of holomorphic functions $f_1, \dots, f_k \in \mathscr{O}_U$. In what follows, to simplify notation, we shall refer to any germ $(X, \mathfrak{o})$ of complex space, as well as to its suitable representative as a *singularity* or the germ of a singularity (even if the distinguished point $\mathfrak{o}$ is nonsingular).

By definition, the coherent sheaves of *regular* (holomorphic) $p$-forms $\Omega^p_X$, $p \geq 1$, on $X$ are determined by the restriction to $X$ of the corresponding quotient modules:

$$\Omega^p_X = \Omega^p_U/((f_1, \dots, f_k)\Omega^p_U + df_1 \wedge \Omega^{p-1}_U + \dots + df_k \wedge \Omega^{p-1}_U)\big|_X, \qquad (1)$$

so that $\Omega^p_X = \bigwedge^p \Omega^1_X$, $p \geqslant 1$, where $\Omega^1_X$ is the coherent sheaf of regular *Kähler differentials* on $X$, $\Omega^0_X = \mathscr{O}_X$ and $\Omega^p_X = 0$ for $p < 0$ and $p > m$. Next, if $\dim X = n$ then the *support* of the sheaf $\Omega^p_X$ is contained in the singular locus of $X$, whereas $n < p \leqslant m$. We note also that in the case of smooth manifolds or nonsingular spaces the elements of these sheaves are usually called *holomorphic* functions, holomorphic differential forms, etc. Sometimes we shall use these terms in the singular case as well.

At that, the usual differential $d$, extended to this quotient, is also denoted by the same symbol; it endows all this family of sheaves with the structure of an *increasing* complex, which is called the *Poincaré complex*, or the *de Rham complex*, of $X$; it is denoted by $(\Omega^\bullet_X, d)$. The $\mathscr{O}_X$-module of regular *vector* fields on $X$ is denoted by $\mathrm{Der}(X) \cong \mathrm{Hom}_{\mathscr{O}_X}(\Omega^1_X, \mathscr{O}_X)$.

It should be remarked that for *nonsingular* varieties one of the main results of the theory of differential forms is the classical Poincaré lemma, which asserts the acyclicity of the complex $(\Omega^\bullet_X, d)$.

Let us also give an equivalent definition of regular $p$-forms in slightly different notations. In short, we use the notation $A$ to denote the dual local analytic $\mathbb{C}$-algebra $\mathcal{O}_{X,\mathfrak{o}}$ of the germ $X$, so that $A$ is the localization at the distinguished point of the quotient algebra $H/I$, where $H = \mathbb{C}\langle z_1, \dots, z_m\rangle$ is the ring of convergent power series in $z_1, \dots, z_m$. In what follows, for simplicity, instead of analytic objects we shall often consider their *affine* counterparts; in this case the ring $H$ can be replaced by the ring of polynomials $P = \mathbb{C}[z_1, \dots, z_m]$, the ideal $I$ is generated by a sequence of polynomials $f_1, \dots, f_k$, and so on. We shall often denote the ground field of characteristic 0 by $\mathsf{k}$.

Then the $A$-module $\Omega^1_{A/\mathsf{k}}$ of Kähler differentials of $\mathsf{k}$-algebra $A$ is determined by the exact sequence

$$I/I^2 \xrightarrow{D} \Omega^1_{P/\mathsf{k}} \otimes_P A \longrightarrow \Omega^1_{A/\mathsf{k}} \longrightarrow 0, \tag{2}$$

where the homomorphism $D$ is given by the rule $D(\overline{f}) = d(f) \otimes 1$ for an element $f \in I$ and its class $\overline{f}$ in the *conormal* module $I/I^2$, and where $d\colon P \to \Omega^1_{P/\mathsf{k}}$ is the universal differentiation. Thus, $\Omega^0_{A/\mathsf{k}} = A$, $\Omega^p_{A/\mathsf{k}} = \bigwedge^p \Omega^1_{A/\mathsf{k}}$ for all $p \geq 1$, and $\Omega^p_{A/\mathsf{k}} = 0$ for $p < 0$ and $p > m$. In fact, $D$ can be represented by the jacobian matrix of the map determined by the sequence of functions $(f_1, \dots, f_k)$. Recall that, for any exact sequence of $A$-modules $N \to M \to M/N \to 0$, there exists a representation

$$\wedge^p(M/N) \cong \wedge^p M/\{n \wedge m_1 \wedge \ldots \wedge m_{p-1}\}$$

for all $n \in N, m_i \in M, i = 1, \dots, p-1$. This yields the equivalence of both definitions.

Remark also that, in the standard notations of deformation theory, there is a canonical isomorphism $\Omega^1_A \cong T_0(A)$, and the kernel of the map $D$ from the fundamental sequence (2) is isomorphic to $T_1(A)$, where $T_i(A)$, $i = 0, 1$, are the *lower* cotangent homology modules of the $\mathsf{k}$-algebra $A$. In particular, we can complete the sequence (2) to a left exact sequence in the following way:

$$0 \longrightarrow T_1(A) \longrightarrow I/I^2 \xrightarrow{D} \Omega^1_{P/\mathsf{k}} \otimes_P A \longrightarrow \Omega^1_{A/\mathsf{k}} \longrightarrow 0. \tag{3}$$

Example 1. Suppose that the singularity $X$ is determined in $\mathbb{C}^2_{x,y}$ by the two functions $f_1 = x^2$ and $f_2 = xy$. That is, $X$ is the germ of the union of coordinate axes with an *embedded* point at the origin, since $(x^2, xy) = (x) \cap (x, y)^2 = (x) \cap (x^2, y)$. Hence, $X$ is an isolated nonreduced 1-dimensional singularity, which is *not* Cohen-Macaulay. One can readily verify that $\Omega^1_A \cong \mathbb{C}\langle dx, dy, ydx, ydy, y^2dy, y^3dy, \dots\rangle$, and $T_1(A) \cong \mathbb{C}\langle x^3, x^2y\rangle$.

Example 2. If the ideal $I$ is generated by a *regular* sequence of functions $f_1, \dots, f_k$, then the corresponding germ is called a *complete intersection*. It is well-known that, in this case, $T_i(A) = T^i(A) = 0$ for all $i \geq 2$. If such a germ is, in addition, *reduced*, then $T_1(A) = 0$ and the sequence (2) is left-exact (cf. [21, §4, Example (1)]).

Note also that a local $\mathsf{k}$-algebra $A$ for which $\mathrm{T}_1(A) = 0$, is called an $L$-algebra (see [21, (1.1), (2.2)]).

It should be mentioned that one can find in [7, Sections 11 & 15] a more detail discussion of the basic properties of the first cotangent homology $\mathrm{T}_1(A)$, as well as some useful applications.

**2. Quasihomogeneous singularities.** Suppose now that the germ of an $n$-dimensional singularity $X$ is determined by an ideal $I \subset P$ generated by a sequence $f_1, \dots, f_k$ of functions in variables $z_1, \dots, z_m$, which are quasihomogeneous of degrees $d_1, \dots, d_k$ relatively to weights $w_1, \dots, w_m$. Then this germ is said to be of *type* $\pi(X) = (d_1, \dots, d_k; w_1, \dots, w_m) \in \mathbb{Z}^k \times \mathbb{Z}^m$.

Under the same conditions, the modules $\Omega^p_X$ for all $p \geq 0$, as well as $\mathrm{T}_i(X)$ and $\mathrm{T}^i(X)$ for $i \geq 0$, are equipped with a *natural* $\mathbb{Z}$-grading in which $\deg(df_j) = d_j$, $j = 1, \dots, k$, and $\deg(dz_i) = w_i$, $i = 1, \dots, m$. The elements of the homogeneous component $\mathrm{T}^0(X)_v \cong \mathrm{Der}(X)_v$ are called vector fields of *weight* $v$. In particular, the weight of the partial derivatives $\partial/\partial z_i$ is equal to $-w_i$, $i = 1, \dots, m$; the element $\mathcal{V}_0 = \sum_{i=1}^{m} w_i z_i \partial/\partial z_i \in \mathrm{Der}(X)_0$ of *zero* weight is usually called the Euler vector field. If the grading is *positive*, then $\pi(X) \in \mathbb{Z}_+^k \times \mathbb{Z}_+^m$, and the germ is called a singularity with an *effective* $\mathbb{C}^*$-action.

Suppose that $M = \bigoplus_{\nu\in\mathbb{Z}} M_\nu$ is a $\mathbb{Z}$-graded $A$-module whose homogeneous components are *finite-dimensional*. Then the formal Laurent series

$$\mathscr{P}(M;t) = \sum_{\nu\in\mathbb{Z}} \dim_{\mathsf{k}}(M_\nu)\, t^\nu$$

is called the Poincaré series of $M$. If the sum is finite, then $P(M;t)$ is usually called the Poincaré *polynomial*; in this case $\mathscr{P}(M;1) = \dim_{\mathsf{k}}(M)$.

In the case of Example 1 one can easily verify

$$\begin{array}{rcl} \mathscr{P}(\mathscr{O}_X;t) &=& (1+t-t^2)/(1-t) \;=\; 1+2t+t^2+t^3+t^4+t^5+\dots, \\ \mathscr{P}(\Omega^1_X;t) &=& t\,(2-t^2)/(1-t) \qquad = \; 2t+2t^2+t^3+t^4+t^5+\dots, \\ \mathscr{P}(\mathrm{T}_1(X);t) &=& 2t^3, \ \text{ and so on.} \end{array}$$

Now, we proceed to a description of the modules of differential forms $\Omega^p_X$, $p \ge 0$, for certain types of graded singularities using the technique of Poincaré series calculus.

Let $X$ be the germ of an $n$-dimensional *isolated* complete intersection singularity given by a *regular* sequence of quasihomogeneous functions $f_1, \dots, f_k$ of weighted degrees $d_1, \dots, d_k$ relative to variables with weights $w_1, \dots, w_m$, so that $m - k = n$.

If $n \ge 1$, that is, the singularity $X$ has *positive* dimension, then the type of quasihomogeneity is determined *uniquely*, except in the case of a hypersurface with multiplicity 2, and $\pi(X) \in \mathbb{Z}_+^k \times \mathbb{Z}_+^m$. In other words, the singularity $X$ is $\mathbb{Z}_+$-graded, or positively graded, and the Euler vector field *generates* the $\mathscr{O}_X$-module $\mathrm{Der}(X)$ modulo *Hamiltonian* vector fields (see [1, Theorem 6.1]). In the exceptional case, a grading is defined not uniquely, but there always exists the normalized *canonical* $\mathbb{Z}_+$-grading introduced by K.Saito (see further details in [1, (6.3), (6.4)]).

Lemma 1 ([1, Lemma 3.2]). *Let $X$ be the germ of an $n$-dimensional isolated complete intersection singularity. Then the Poincaré series of modules of regular (holomorphic) forms of order $p$, $0 \le p \le n$, on $X$ can be represented as follows:*

$$\mathscr{P}(\Omega^p_X;t) = \mathscr{P}(\mathscr{O}_X;t) \operatorname*{res}_{\xi=0} \xi^{-p-1} \prod (1+\xi t^{w_i}) \Big/ \prod (1+\xi t^{d_j}),$$

*where* $\mathscr{P}(\mathscr{O}_X;t) = \prod(1-t^{d_j})/\prod(1-t^{w_i})$.

Remark 1. In the author's paper cited above, the Poincaré series of the modules $\mathrm{Der}(X)$ and $\mathrm{T}^1(X)$ were also computed.

A detail proof of this statement with the use of a generalized de Rham lemma can also be found in [5]. Yet another approach, based on the use of an explicit form of the *Lebelt resolvents* for the modules of holomorphic forms on complete intersections with isolated singularities, is discussed in [7]. Modifying these methods in an appropriate direction, one can adapt them for computing the Poincaré series of modules of regular holomorphic forms on varieties of divers kinds.

For example, in describing the modules $\Omega^p_X$ on hypersurfaces with *nonisolated* singularities, it is convenient to use the *four-term* exact sequences (13) (see Section 5 below), which are closely related to the modules of logarithmic differential forms.

**3. Graded L-algebras.** Developing the methods mentioned above, one can also learn how to compute the Poincaré series of modules of differential forms given on *noncomplete* intersections: on Cohen-Macaulay curves, normal, determinantal and toric varieties, and others.

Hereinafter, for the sake of simplicity, we shall examine a few examples of non-complete intersection singularities whose first cotangent homology is *trivial.* In this case, the exact sequence (2) can be presented in the same form as in the case of *reduced* complete intersections, namely (in view of Example 2) we have

$$0 \longrightarrow I/I^2 \xrightarrow{D} \Omega^1_{P/\mathsf{k}} \otimes_P A \longrightarrow \Omega^1_{A/\mathsf{k}} \longrightarrow 0, \tag{4}$$

with the only difference that the *conormal* $A$-module $I/I^2$ *is not free.*

The class of such singularities contains the *normal* Cohen-Macaulay germs of embedding codimension 2 (see [22, (4.6)]), some types of *determinantal* singularities (see [14]), singular loci of germs with nonisolated singularities of linear type, etc.

As an important example we shall compute the Poincaré series of modules of differential forms for a series of surface singularities with embedding dimension 4, which are quotients of smooth manifolds by cyclic groups. Deformation theory of such singularities were investigated in detail in [19].

More precisely, let $(x : y : z : u)$ denote homogeneous coordinates in projective space $\mathbb{P}^3$. Then $A \cong k[x, y, z, u]/I$, and the prime ideal $I = (f_{12}, f_{13}, f_{23})$ is generated by the maximal minors of the matrix

$$M_{a,b} = \begin{bmatrix} x & y & z^{b-1} \\ y^{a-1} & z & u \end{bmatrix}, \tag{5}$$

where $a, b \geqslant 2$, so that

$$f_{12} = xz - y^a, \quad f_{13} = y^{a-1}z^{b-1} - xu, \quad f_{23} = yu - z^b.$$

It should be remarked that the first element of this series for $a = b = 2$ is a remarkable determinantal surface of embedding codimension 2, which occurs in many works and is called the Hilbert cubic. In fact, it is the affine cone over the rational curve of degree 3 in projective space $\mathbb{P}^3$, given by the embedding $\mathbb{P}^1 \to \mathbb{P}^3$ by the sheaf $\mathscr{O}_{\mathbb{P}^1}(3)$ (see [17, §8]).

It is readily seen that a *rational* parameterization $\rho\colon (\mathbb{C}^2, 0) \to X$ of this *normal* cubic surface singularity is determined by the formulas

$$x = \xi^{ab-1}, \quad y = \xi^b\zeta, \quad z = \xi\zeta^a, \quad u = \zeta^{ab-1}, \tag{6}$$

so that $\rho^*(A) \cong \mathsf{k}\langle \xi^{ab-1}, \xi^b\zeta, \xi\zeta^a, \zeta^{ab-1}\rangle \subset \mathsf{k}\langle \xi, \zeta\rangle$. Next, the germ $X$ is a *determinantal* isolated singularity of embedding codimension 2. In particular, $X$ is a normal Cohen-Macaulay germ, and hence $\mathrm{T}_1(A) = 0$. We also note that the germ $X$ is an *almost* complete intersection and, consequently, it is not a Gorenstein germ.

First show how to compute the Poincaré series $\mathscr{P}(A; t)$ of the structure analytic algebra $A$. In what follows, to simplify notation, for a graded $A$-module $M$ we shall denote by $M(\lambda)$ the graded $A$-module, which is obtained from $M$ by shifting the grading to the left by $\lambda \in \mathbb{Z}$, i.e. $[M(\lambda)]_\nu = M_{\nu+\lambda}$. It is well-known (see, e.g., [19, Section 4, Remark]), that there exists a $P$-free Hilbert resolvent of the ideal $I$:

$$0 \longrightarrow P^2(-(a+b-1)) \xrightarrow{\varphi_1} P(-a) \oplus P(-(a+b)-2) \oplus P(-b) \xrightarrow{\varphi_0} I \longrightarrow 0,$$

where $\varphi_0 = [f_{12}, f_{13}, f_{23}]^T$, and the rows of the matrix

$$\varphi_1 = \begin{bmatrix} z^{b-1} & y & x \\ u & z & y^{a-1} \end{bmatrix}, \tag{7}$$

are given by the two evident syzygies of the first order between the generators $f_{12}$, $f_{13}$, and $f_{23}$. Here we have used the usual grading:

$$\deg x = a - 1,\ \deg y = \deg z = 1,\ \deg u = b - 1,$$

so that the weighted degrees of the generators $f_{12}, f_{13}, f_{23}$ of the ideal $I$ are equal to $a$, $a + b - 2$, and $b$, respectively. Taking into account the shifting of grading, we see that $\deg(\varphi_0) = \deg(\varphi_1) = 0$. As a result,

$$\mathscr{P}(I;t) = (t^a + t^{a+b-2} + t^b - 2t^{a+b-1})/\vartheta(t),$$

where $\vartheta(t) = (1-t^{a-1})(1-t^{b-1})(1-t)^2$. Next, since $A = P/I$ and $\mathscr{P}(P;t) = 1/\vartheta(t)$, we obtain

$$\mathscr{P}(A;t) = 1/\vartheta(t) - \mathscr{P}(I;t) = (1 - t^a - t^{a+b-2} - t^b + 2t^{a+b-1})/\vartheta(t).$$

The Poincaré series $\mathscr{P}(I^2;t)$ can be computed in a similar way. Indeed, the ideal $I^2$ is generated by the six elements:

$$f_{12}^2,\ f_{12}f_{13},\ f_{12}f_{23},\ f_{13}^2,\ f_{13}f_{23},\ f_{23}^2.$$

The first row of the matrix $\varphi_1$ determines a syzygy between the first three generators, since they have a common factor $f_{12}$. The same procedure yields syzygies between the second, fourth, and fifth generators, as well as between the third, fifth, and sixth generators, because the former triple has a common factor $f_{13}$, while the latter has a common factor $f_{23}$. Similar relations are derived from the second row of the matrix $\varphi_1$ by using cyclic permutations of the variables in the first row. As a result, we obtain a complete system of six syzygies of the first order, which can be represented by the matrix

$$S_1 = \begin{bmatrix} z^{b-1} & 0 & 0 & u & 0 & 0 \\ y & z^{b-1} & 0 & z & u & 0 \\ x & 0 & z^{b-1} & y^{a-1} & 0 & u \\ 0 & y & 0 & 0 & z & 0 \\ 0 & x & y & 0 & y^{a-1} & z \\ 0 & 0 & x & 0 & 0 & y^{a-1} \end{bmatrix}. \tag{8}$$

Finally, there exists the only syzygy of the second order between the six columns of the matrix: $S_2 = [u, z, y^{a-1}, -z^{b-1}, -y, -x]$. Let us now denote by $\nu_i$ the weighted degrees of the generators of the ideal $I^2$, that is,

$$\nu_1 = 2a,\ \nu_2 = 2a + b - 2,\ \nu_3 = a + b,\ \nu_4 = 2a + 2b - 4,\ \nu_5 = a + 2b - 2,\ \nu_6 = 2b.$$

Similarly to the above we obtain a $P$-free resolvent

$$\begin{aligned} 0 \to P(-2(a+b-1)) &\xrightarrow{\beta_2} \\ P^2(-(2a+b-1)) \oplus P^2(-(2a+2b-3)) \oplus P^2(-(a+2b-1)) &\xrightarrow{\beta_1} \\ \oplus_{i=1}^6 P(-\nu_i) &\xrightarrow{\beta_0} I^2 \to 0, \end{aligned}$$

where $\beta_0$ is a vector-column determined by the six generators of the ideal $I^2$, while $\beta_1$ and $\beta_2$ are given by the matrices $S_1^T$ and $S_2$, respectively. It is clear that in our grading $\deg\beta_0 = \deg\beta_1 = \deg\beta_2 = 0$. Hence,

$$\mathscr{P}(I^2;t) = (\textstyle\sum_{i=1}^{6} t^{\nu_i} - 2t^{2a+b-1} - 2t^{2a+2b-3} - 2t^{a+2b-1} + t^{2a+2b-2})/\vartheta(t).$$

Taking into account the exact sequence (4), where $\deg(D) = 0$, we get

$$\mathscr{P}(\Omega_A^1;t) = (t^{a-1} + 2t + t^{b-1})\mathscr{P}(A;t) - \mathscr{P}(I;t) + \mathscr{P}(I^2;t).$$

To obtain $\mathscr{P}(\Omega_A^2;t)$ without tedious computations, we apply a simple trick (see [7, Example 15.9]). Namely, it is well known (see [16, Lemma 2.1.1]) that, for isolated singularities with an *effective* $\mathbb{C}^*$-action, the *generalized* Koszul complex

$$0 \longrightarrow \Omega_A^4 \xrightarrow{\iota_E} \Omega_A^3 \xrightarrow{\iota_E} \Omega_A^2 \xrightarrow{\iota_E} \Omega_A^1 \xrightarrow{\iota_E} A \xrightarrow{\varepsilon} \mathsf{k} \longrightarrow 0 \tag{9}$$

is *acyclic* in all dimensions. Here $\iota_E$ denotes the contraction along the Euler vector field $E = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + u\frac{\partial}{\partial u}$ and $\varepsilon$ is the augmentation epimorphism. In particular, $\chi(\Omega_A^\bullet, \iota_E) = 0$. This immediately implies the following identity for the Poincaré series:

$$\textstyle\sum_{p\geq 0}(-1)^p\mathscr{P}(\Omega_A^p;t) = 1. \tag{10}$$

Moreover, the modules $\Omega_A^3$ and $\Omega_A^4$, as well as the corresponding Poincaré series, are very easy to compute. Indeed, both modules are finite-dimensional vector spaces over the ground field $\mathsf{k}$. Next, the space $\Omega_A^4$, as a $\mathsf{k}$-module, is generated by the differential form $dx\wedge dy\wedge dz\wedge du$, so that $\mathscr{P}(\Omega_A^4;t) = t^{a+b}$.

Further calculations show that the module $\Omega_A^3$ is generated over $\mathsf{k}$ by the $a-1$ forms $\langle 1, y, y^2, \dots, y^{a-2}\rangle dx\wedge dy\wedge dz$, the $b$ forms $\langle 1, x, z, z^2, \dots, z^{b-2}\rangle dy\wedge dz\wedge du$ and the two forms $dx\wedge dy\wedge du$ and $dx\wedge dz\wedge du$. That is, $\dim_{\mathsf{k}}\Omega_A^3 = a+b+1$ and

$$\mathscr{P}(\Omega_A^3;t) = t^{a+1}(1+t+\ldots+t^{a-2}) + t^{a+b} + t^{b+1}(1+t+\ldots+t^{b-2}) + 2t^{a+b-1}.$$

It is useful to note that the 3-form of weight $a+b$ is equal to $x dy\wedge dz\wedge du \equiv \iota_E(dx\wedge dy\wedge dz\wedge du)$. Finally, making use of the identity

$$\mathscr{P}(\Omega_A^2;t) = 1 - \mathscr{P}(A;t) + \mathscr{P}(\Omega_A^1;t) + \mathscr{P}(\Omega_A^3;t) - \mathscr{P}(\Omega_A^4;t),$$

one can derive the desired explicit formula.

EXAMPLE 3. In the case of Hilbert cubic ($a = b = 2$), one obtains

$$\begin{aligned}
&\mathscr{P}(I;t) = (3t^2 - 2t^3)/(1-t)^4,\ \mathscr{P}(A;t) = (1 - 3t^2 + 2t^3)/(1-t)^4 = (1+2t)/(1-t)^2,\\
&\mathscr{P}(I^2;t) = (6t^4 - 6t^5 + t^6)/(1-t)^4,\\
&\mathscr{P}(\Omega_A^1;t) = 4t\mathscr{P}(A;t) - \mathscr{P}(I;t) + \mathscr{P}(I^2;t) = t(4 + 5t - 4t^2 + t^3)/(1-t)^2,\\
&\mathscr{P}(\Omega_A^3;t) = 4t^3 + t^4,\ \mathscr{P}(\Omega_A^4;t) = t^4,\ \mathscr{P}(\Omega_A^2;t) = t^2(6 - 7t^2 + 4t^3)/(1-t)^2.
\end{aligned}$$

As a consequence, in the notation of [11] or [5], we get the following value of the *homological index* for a suitable Euler vector field $\mathscr{V}_0$ of weight 0 on the Hilbert cubic:

$$\begin{aligned}
\mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathscr{V}_0) &= (\mathscr{P}(A;t) - \mathscr{P}(\Omega_A^1;t) + \mathscr{P}(\Omega_A^2;t))\big|_{t=1}\\
&= (1 + \mathscr{P}(\Omega_A^3;t) - \mathscr{P}(\Omega_A^4;t))\big|_{t=1} = (1+4t^3)\big|_{t=1} = 5.
\end{aligned}$$

It is possible to continue these calculations and analyze a lot of others, more complicated examples just as easily.

Example 4. Set $a = k+1$, $b = 2$, so that the quotient surface singularity $X_{k+1,2}$ is determined by the maximal minors of the matrix

$$M_{k+1,2} = \begin{bmatrix} x & y & z \\ y^k & z & u \end{bmatrix}.$$

In this case, we obtain

$$\begin{aligned} \mathscr{P}(\Omega^3_A;t) &= t^3 + t^{k+3} + t^{k+2}(3 + t + \ldots + t^{k-1}),\ \mathscr{P}(\Omega^4_A;t) = t^{k+3}, \\ \mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathscr{V}_0) &= (1 + \mathscr{P}(\Omega^3_A;t) - \mathscr{P}(\Omega^4_A;t))\big|_{t=1} \\ &= (1 + t^3 + t^{k+2}(3 + t + \ldots + t^{k-1}))\big|_{t=1} = k+4. \end{aligned}$$

Remark 2. In general, for arbitrary $n$-dimensional isolated singularity $X$ one may consider the Euler characteristic of the torsion subcomplex $(\mathrm{Tors}\,\widehat{\Omega}^\bullet_X, d)$ of the *truncated* de Rham complex $(\widehat{\Omega}^\bullet_X, d)$ (see [5, 7]) as a basic analytic invariant of the set of *vanishing* cocycles of different dimensions. Let denote it by $\tau'(X)$. Then

$$\tau'(X) = (-1)^n \sum_{i=1}^{n} (-1)^i \tau'_i(X),$$

where $\tau'_i(X) = \dim_{\mathsf{k}} \mathrm{Tors}\,\Omega^i_X$, $i = 1, \ldots, n$.

Assume that the singularity $X$ is endowed with an effective $\mathbb{C}^*$-action. Then, in virtue of relation (10),

$$\tau'(X) = (-1)^{n+1} \sum_{i=n+1}^{m} (-1)^i \dim_{\mathsf{k}} \mathrm{Tors}\,\Omega^i_X,$$

where $m$ is the *embedding* dimension of the singularity $X$.

As a result,

$$\mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathscr{V}_0) = 1 + (-1)^n \tau'(X).$$

In particular, this implies $\tau'(X_{k+1,2}) = k+3$. Moreover, it is well known also (see, e.g., [13]) that, for graded isolated *complete* intersection singularities, $\tau'_i(X) = 0$ for all $0 \le i < n$, so that

$$\mu(X) = \tau(X) = \tau'(X) = \tau'_n(X),$$

where $\mu(X)$ and $\tau(X)$ are the Milnor and Tjurina numbers of $X$, respectively.

It should be also underlined that in the case of an isolated complete intersection singularity the homological index $\mathrm{Ind}_{\mathrm{hom},\mathfrak{o}}(\mathscr{V}_0)$ coincides with the Poincaré index of the vector field $\mathscr{V}_0$ on the singularity (see [11]). Developing this idea, one can compute other important topological and analytic invariants of singularities of divers kinds (see, e.g., [7]).

In the same vein, we can compute the Poincaré series of modules of differential forms for many types of affine *toric* varieties and, more generally, of varieties associated with *semigroups*.

**4. The Poincaré index and the Cauchy integral.** First recall that, one of the best familiar problems considered by Poincaré in his pioneer research, is the analysis of the system of differential equations

$$\frac{dx}{dt} = P(x,y), \quad \frac{dy}{dt} = Q(x,y),$$

where $P(x,y)$ and $Q(x,y)$ are continuous functions, given on an open domain $G$ of the Euclidean plane $\mathbb{R}^2$ with coordinates $x, y$, and having continuous partial derivatives in this domain. It should be remarked that such systems, whose right parts do not depend on the parameter $t$ in an explicit way, are often called *autonomous* or *dynamical* systems on the plane.

Thus, at each point of $G$ we can define a vector whose components are values of the functions $P(x,y)$ and $Q(x,y)$, so that the dynamical system determines the *vector field* in the domain $G$

$$\mathscr{V} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}.$$

If at least one coefficient of the field does not vanish at some point of the domain then the length of the vector $r = \sqrt{P^2 + Q^2}$ at this point differs from zero and the angle $\vartheta$ between the axis $x$ and the vector at this point (in the standard orientation) is given by the following expressions:

$$\sin\vartheta = \frac{Q}{\sqrt{P^2+Q^2}}, \quad \cos\vartheta = \frac{P}{\sqrt{P^2+Q^2}}.$$

Otherwise, if $P(x,y) = Q(x,y) = 0$, then the direction of the vector is undefined and the corresponding point is called the *singular* point of the vector field or the *equilibrium* point of the system.

Let us now assume that the boundary of the domain $G$ is a *smooth* closed curve $\gamma = \partial G$, that is, $\gamma = \gamma(t)$, $t \in [0, 2\pi]$, $\gamma(0) = \gamma(2\pi)$, and the vector field has no singular points on the curve, that is, $\mathscr{V}(g) \neq 0$ for all $g \in \gamma$. One can regard the vector filed $\mathscr{V}$ on the boundary as a *periodical* vector-function $\mathscr{V}(t) = \mathscr{V}(\gamma(t))$, depending on the parameter $t$ on the curve $\gamma$. By definition, the *Poincaré index* of the vector field $\mathscr{V}$ on the boundary $\partial G$ is the number of rotations of the vector $\mathscr{V}(t)$, bypassing along the boundary when the parameter $t$ varies from 0 till $2\pi$. In fact, it is the index of the curve $\gamma$, which can be presented in terms of the famous Poincaré formula (1886):

$$\operatorname{Ind}\gamma = \frac{1}{2\pi}\oint_\gamma d\varphi, \quad \textit{where } \varphi = \arctan\frac{Q}{P}, \quad d\varphi = \frac{QdP - PdQ}{P^2+Q^2}. \tag{11}$$

In addition, if the vector field $\mathscr{V}$ has no singular points on $G \setminus \mathfrak{o}$, then $\operatorname{Ind}\gamma$ is called the Poincaré index of $\mathscr{V}$ at the distinguished point.

The celebrated Poincaré theorem proclaims that, in the case when the index does not vanish, there is an *equilibrium* point of the dynamical system in the interior part of $G$. If the index vanishes then the vector field can be prolong from the boundary into the domain, so that the interior part of $G$ does not contain equilibrium points at all. That is, analyzing the behavior of the vector field on the boundary, one can understand whether equilibrium points exist *inside* the domain.

Example 5. Set $P = x$ and $Q = y$. In complex analysis, the differential form $d\varphi = \frac{ydx - xdy}{x^2+y^2}$ is usually referred (e.g., in Cauchy's works) as the *imaginary* part of the complex differential form $\frac{dz}{z}$, where $z = x + iy$, or, in other terms, $\oint_\gamma d\varphi$ is "the variation of the argument of the complex number $z$ along the path $\gamma$"(see [9, Pt.II, §1, n. 5]).

Among other things we shall see below (cf. Example 7) that the differential form $d\varphi$ can be regarded as the imaginary part of a rational differential form in the plane with *logarithmic* poles along the divisor $D$, determined by the equation $P^2 + Q^2 = 0$, in the usual complexification of the real plane $\mathbb{R}^2$. Thus, the form $\frac{dz}{z}$ or its imaginary part, as well as the form $d\varphi$ from (11) can be regarded as the first examples (or even prototypes) of complex and real logarithmic differential forms.

**5. The de Rham lemma towards logarithmic forms.** First recall an observation due to de Rham, which plays an important role in analysis and geometry.

Theorem 1 (see [10]). *Assume that $X$ is the germ of a smooth manifold $M$ of dimension $m \geqslant 1$ and all the coefficients of a differential 1-form $\omega \in \Omega^1_{M,\mathfrak{o}}$ generate a regular sequence in the local ring $\mathcal{O}_{M,\mathfrak{o}}$. Then the increasing complex $(\Omega^\bullet_{M,\mathfrak{o}}, \wedge\omega)$ is acyclic in all dimensions $0 \leqslant p < m$. In particular, there are exact sequences of $\mathcal{O}_{M,\mathfrak{o}}$-modules*

$$0 \longrightarrow \omega\wedge\Omega^{p-1}_{M,\mathfrak{o}} \longrightarrow \Omega^p_{M,\mathfrak{o}} \xrightarrow{\wedge\omega} \Omega^{p+1}_{M,\mathfrak{o}} \longrightarrow \Omega^{p+1}_{M,\mathfrak{o}}/\omega\wedge\Omega^p_{M,\mathfrak{o}} \longrightarrow 0\ , \quad 0 \leqslant p < m. \qquad (12)$$

Under the same assumptions, the classical de Rham lemma is usually formulated as follows: if a differential form $\eta \in \Omega^p_{M,\mathfrak{o}}$ satisfies the condition $\omega \wedge \eta = 0$, then $\eta = \omega \wedge \xi$ for a suitable holomorphic form $\xi \in \Omega^{p-1}_{M,\mathfrak{o}}$.

The following assertion from [2, 4] can be regarded as a modified version of the above statement for singular hypersurfaces in a smooth manifold.

Lemma 2. *Suppose that a reduced divisor $D$ in a complex manifold $M$ is determined in a suitable neighborhood $U$ of the distinguished point $\mathfrak{o} \in M$ by arbitrary function $h \in \mathcal{O}_{M,\mathfrak{o}}$ without multiple factors. Then for $p = 0, 1, \dots, m-1$ there are exact sequences of $\mathcal{O}_{M,\mathfrak{o}}$-modules*

$$0 \longrightarrow \Omega^p_{M,\mathfrak{o}}(\log D) \xrightarrow{\cdot h} \Omega^p_{M,\mathfrak{o}} \xrightarrow{\wedge dh} \Omega^{p+1}_{M,\mathfrak{o}}/h{\cdot}\Omega^{p+1}_{M,\mathfrak{o}} \longrightarrow \Omega^{p+1}_{D,\mathfrak{o}} \longrightarrow 0. \qquad (13)$$

*Proof.* Taking $\vartheta \in \mathrm{Ker}(\wedge dh) \subseteq \Omega^p_{M,\mathfrak{o}}$, we see that $dh\wedge\vartheta = h\xi$, where $\xi$ is contained in $\Omega^{p+1}_{M,\mathfrak{o}}$. Then, in view of [20, (1.1)], the meromorphic form $\omega = \vartheta/h$ is contained in $\Omega^p_{M,\mathfrak{o}}(\log D)$, because $dh \wedge \omega = \xi$. Therefore $hd\omega = d(h\omega) - dh \wedge \omega = d\vartheta - \xi \in \Omega^{p+1}_{M,\mathfrak{o}}$ is holomorphic on $M$. Thus, $\mathrm{Ker}(\wedge dh) \subseteq h{\cdot}\,\Omega^p_{M,\mathfrak{o}}(\log D)$ and vice versa. □

Remark 3. It is not difficult to see that the preceding lemma, as well as the traditional definition of logarithmic differential forms, has a significant meaning over the ground field of real numbers $\mathbb{R}$. In fact, making use of the approach developed by the author in [2, 4], one can prove that many results from the theory of complex logarithmic differential forms are valid in the real case with minor changes (see, for example, [12]).

Example 6. Suppose that $M = \mathbb{C}^2_{x,y}$. Let us consider the hypersurface $D$ given by the equation $xy = 0$, that is, $h = xy$, and $D$ is a plane curve with a *node*. In other words, it is an $A_1$-singularity, a very particular case of *strongly* normal crossings.

Then the exact sequence (13) for $p = 0$ implies $\mathrm{Ker}(\wedge dh) = h\mathcal{O}_M$, because

$$\varphi \wedge dh = h{\cdot}\xi,\ \xi \in \Omega^1_M \iff \varphi = \psi h,\ \psi \in \mathcal{O}_M.$$

Next, for $p = 1$ one gets $\mathrm{Ker}(\wedge dh) \cong \mathcal{O}_M\langle dh, ydx - xdy\rangle \cong \mathcal{O}_M\langle dx, dy\rangle$, while $\mathrm{Ker}(\wedge dh) \cong \Omega^2_M \cong \mathcal{O}_M\langle dx \wedge dy\rangle$ for $p = 2$. As a result, we derive

$$\Omega^0_{M,\mathfrak{o}}(\log D) \cong \mathcal{O}_{M,\mathfrak{o}}, \quad \Omega^1_{M,\mathfrak{o}}(\log D) \cong \mathcal{O}_{M,\mathfrak{o}}\langle \tfrac{dh}{h}, \tfrac{ydx-xdy}{h}\rangle \cong \mathcal{O}_{M,\mathfrak{o}}\langle \tfrac{dx}{x}, \tfrac{dy}{y}\rangle,$$

$$\Omega^2_{M,\mathfrak{o}}(\log D) \cong \mathcal{O}_{M,\mathfrak{o}}\langle \tfrac{dx\wedge dy}{h}\rangle.$$

All these $\mathcal{O}_{M,\mathfrak{o}}$-modules are *free* of rank 1, 2 and 1, respectively. It is not difficult to verify that $\vartheta = ydx - xdy \in \mathrm{Tors}\,\Omega^1_{D,\mathfrak{o}}$. Indeed, taking a non-zero divisor $x+y \in \mathcal{O}_{D,\mathfrak{o}}$, one obtains the following identities in $\Omega^1_{D,\mathfrak{o}}$ :

$$(x+y){\cdot}\vartheta = xydx - x^2dy + y^2dx - xydy = -(x-y)dh + 2h(dx - dy) \equiv 0.$$

Moreover, in this case, $\mathrm{Tors}\,\Omega^1_{D,\mathfrak{o}} \cong \mathcal{O}_{D,\mathfrak{o}}\langle\vartheta\rangle \cong \mathbb{C}\langle\vartheta\rangle$, and the Milnor number $\mu(D) = 1$.

Example 7 (cf. [18, Pt. III, Ch. 2, §3]). In the notation and under assumptions of Section 4, we set $p = P + iQ$, $q = P - iQ$, so that the function $h = p\,q = P^2 + Q^2$ determines the divisor $D$ in the usual complexification of $\mathbb{R}^2$. Then

$$d\varphi = \frac{QdP - PdQ}{P^2 + Q^2} = \tfrac{i}{2}{\cdot}\frac{qdp - pdq}{p\,q}\ ,$$

so that $hd\varphi = QdP - PdQ$ and $dh \wedge d\varphi = -2dP \wedge dQ = -idp \wedge dq$ are regular (continuous) everywhere. Thus, one can regard the real 1-form $\vartheta = QdP - PdQ$ as the imaginary part of the complex (torsion) 1-form $\frac{1}{2}(pdq - qdp) \in \Omega^1_D$. Similarly, $d\varphi$ is a real rational 1-form and one can regard this form as the imaginary part of a complex differential form with poles along the divisor $D \subset \mathbb{C}^2$.

Example 8. Let $D \subset M$ be a complex plane curve with a *cusp* given by the *quasihomogeneous* polynomial $h = x^2 - y^3$. In other words, $D$ is an $A_2$-singularity. Routine calculations show that

$$\Omega^1_{M,\mathfrak{o}}(\log D) \cong \mathcal{O}_{M,\mathfrak{o}}\langle \tfrac{dh}{h}, \tfrac{2ydx-3xdy}{h}\rangle, \quad \Omega^2_{M,\mathfrak{o}}(\log D) \cong \mathcal{O}_{M,\mathfrak{o}}\langle \tfrac{dx\wedge dy}{h}\rangle$$

are free $\mathcal{O}_{M,\mathfrak{o}}$-modules of rank 2 and 1, respectively. Notice that the numerator of the second generator of $\Omega^1_{M,\mathfrak{o}}(\log D)$, the differential 1-form $\vartheta = 2ydx - 3xdy$, represents an element of the *torsion* submodule $\mathrm{Tors}\,\Omega^1_{D,\mathfrak{o}} \subset \Omega^1_{D,\mathfrak{o}}$. Indeed, in our case $\mathcal{O}_{D,\mathfrak{o}} \cong \mathbb{C}\langle t^2, t^3\rangle$, the normalization $\varrho\colon \widetilde{D} \to D$ is given by formulas $x = t^3$, $y = t^2$. Thus,

$$\varrho^*(\vartheta) = \varrho^*(2ydx - 3xdy) = 2t^2dt^3 - 3t^3dt^2 = 0,$$

that is, $\vartheta \in \mathrm{Ker}(\varrho^*) \cong \mathrm{Tors}\,\Omega^1_{D,\mathfrak{o}}$. Equivalently, taking a non-zero divisor $x \in \mathcal{O}_{D,\mathfrak{o}}$, one then obtains $x \cdot \vartheta = 2xydx - 3x^2dy = ydh - 3hdy \equiv 0$ in the quotient module $\Omega^1_{D,\mathfrak{o}} = \Omega^1_{M,\mathfrak{o}}/(h \cdot \Omega^1_{M,\mathfrak{o}} + dh \wedge \mathcal{O}_{M,\mathfrak{o}})$. Further calculations show that $\mathrm{Tors}\,\Omega^1_{D,\mathfrak{o}} \cong \mathcal{O}_{D,\mathfrak{o}}\langle\vartheta\rangle \cong \mathbb{C}\langle\vartheta,\ y{\cdot}\vartheta\rangle$, that is, $\mu(D) = 2$.

EXAMPLE 9 (cf. [3, §1]). For completeness, let now consider the unimodal *semi*-quasihomogeneous singularity $E_{12}$ given by the polynomial $h = x^3 + y^7 + xy^5$. Then more complicated calculations show that the *free* $\mathcal{O}_{M,\mathfrak{o}}$-module $\Omega^1_{M,\mathfrak{o}}(\log D)$ is generated by the two differential forms $\omega_i = \frac{1}{h}\vartheta_i$, $i = 1, 2$, where

$$\begin{aligned} \vartheta_1 &= \quad (3y^2 - 21xg - 5y^3g)\,dx \;-\; (7xy + 35y^4g)\,dy, \\ \vartheta_2 &= (3xy - 5xy^2g + 7y^4g)\,dx \;-\; (7x^2 - 49y^5g)\,dy, \end{aligned}$$

and $g = 1/(147 + 25y)$. In particular, it is not difficult to verify the following useful relations: $gdh = -\frac{1}{7}(x\vartheta_1 - y\vartheta_2)$, $\vartheta_1 \wedge \vartheta_2 = 147gh\,dx \wedge dy$, and so on.

**6. Normal varieties.** Now let us suppose that $(X, \mathfrak{o})$ is the germ of complex space of dimension $n \geqslant 1$ determined locally by a sequence of functions $f_1, \ldots, f_k$. Let $X$ be its suitable representative embedded in an open neighborhood $U$ of the point $0 \in \mathbb{C}^m$. Then the local equation of any *effective* Cartier divisor $D \subset X$ is given, in the neighborhood $U$ of the point 0, by a non-zero divisor $h \in \mathcal{O}_{X,\mathfrak{o}}$ (see [15, Lecture 9]).

Thus, if $z = (z_1, \ldots, z_m)$ is a system of local coordinates in a neighborhood $U$ of $0 \in \mathbb{C}^m$, then $h(z)\big|_X = 0$ is a local equation of the divisor $D$ at $\mathfrak{o} \in X$, so that $\mathcal{O}_{D,\mathfrak{o}} \cong \mathcal{O}_{X,\mathfrak{o}}/(h) \cong \mathcal{O}_{\mathbb{C}^m,0}/(f_1, \ldots, f_k, h)$.

In what follows, unless otherwise specified, we shall suppose that the *depth* of a Cartier divisor along its singular locus is *positive*, that is, in the standard notation, $\operatorname{depth}(\operatorname{Sing} D, D) \geqslant 1$. In particular, this means that the image of the Jacobi ideal $\operatorname{Jac}(h)$ under the canonical epimorphism $\mathcal{O}_{X,\mathfrak{o}} \to \mathcal{O}_{D,\mathfrak{o}}$ contains at least one *regular* element $g$, which certainly is not a zero divisor. Such divisors do not have multiple components of the *maximal* dimension. However, they are *nonreduced* in the commonly accepted sense, since they may have embedded components of lower dimensions.

It should be noted that if $D$ is a Cartier divisor in any complex space $X$ such that $D$ is a *Cohen-Macaulay* space or variety of *positive* dimension, then there is at least *one* non-zero divisor in the local ring $\mathcal{O}_{D,\mathfrak{o}}$. In particular, the inequality $\operatorname{codim}(\operatorname{Sing} D, D) \geqslant 1$ yields that $D$ is *reduced* and vice versa (see [13, Lemma 1.1]); in this case the inequality $\operatorname{depth}(\operatorname{Sing} D, D) \geqslant 1$ is also fulfilled.

One can regard the next assertion as an analog of the Lemma 2, as well as a variant of the fundamental sequence (2) or sequence (13), in the case of Cartier divisors.

LEMMA 3. *In the same notation and under the same assumptions for every $p \geqslant 0$ there exists the following exact sequence*

$$0 \longrightarrow \operatorname{Ker}^p_{X,\mathfrak{o}}(\wedge dh) \longrightarrow \Omega^p_{X,\mathfrak{o}} \xrightarrow{\wedge dh} \Omega^{p+1}_{X,\mathfrak{o}}/h{\cdot}\Omega^{p+1}_{X,\mathfrak{o}} \longrightarrow \Omega^{p+1}_{D,\mathfrak{o}} \longrightarrow 0. \tag{14}$$

REMARK 4. Recall, in addition, the following useful observation: if $X$ is a complete intersection, then the generalized de Rham lemma from [13, Lemma 1.6] implies that there are the following exact sequences of $\mathcal{O}_{X,\mathfrak{o}}$-modules

$$0 \longrightarrow \Omega^p_{D,\mathfrak{o}} \xrightarrow{\wedge dh} \Omega^{p+1}_{X,\mathfrak{o}}/h{\cdot}\Omega^{p+1}_{X,\mathfrak{o}} \longrightarrow \Omega^{p+1}_{D,\mathfrak{o}} \longrightarrow 0 \tag{15}$$

for all $0 \leqslant p < c - 1$, where $c = \operatorname{codim}(\operatorname{Sing} D, D)$. In particular, in these cases the kernel of exterior multiplication by the total differential $dh$ in the sequence (14) is isomorphic to the sum $(h)\Omega^p_{X,\mathfrak{o}} + dh \wedge \Omega^{p-1}_{X,\mathfrak{o}}$.

Definition 1. Given a Cartier divisor $D$ on a complex space $X$, the analytic sheaves of *logarithmic differential forms* $\Omega^p_X(\log D)$, $p \geqslant 0$, are locally defined via the kernel of the operator of exterior multiplication as follows:

$$\Omega^p_{X,\mathfrak{o}}(\log D) = \frac{1}{h}\operatorname{Ker}^p_{X,\mathfrak{o}}(\wedge dh)\colon = \frac{1}{h}\operatorname{Ker}(\wedge dh\colon \Omega^p_{X,\mathfrak{o}} \longrightarrow \Omega^{p+1}_{X,\mathfrak{o}}/h\cdot\Omega^{p+1}_{X,\mathfrak{o}}),$$

where by $\wedge dh$ we denote the homomorphism of exterior multiplication by the total differential of the function $h$. In particular, the analytic sheaves $\Omega^p_X(\log D)$ are *coherent* for all $p \geqslant 0$.

Tautologically, the set of regular $p$-forms, annihilated by the exterior multiplication by $dh$, coincides with $h\cdot\Omega^p_{X,\mathfrak{o}}(\log D)$ (cf. Lemma 2).

Assertion 1. *In the notation of the above definition, the* $\mathcal{O}_{X,\mathfrak{o}}$*-module* $\Omega^p_{X,\mathfrak{o}}(\log D)$ *consists of germs of meromorphic* $p$*-forms* $\omega$ *on* $X$ *such that* $\omega$ *and* $d\omega$ *have at worst simple poles along* $D$*. In other words,* $h\omega$ *and* $hd\omega$ *are regular at* $\mathfrak{o}$*, that is,*

$$h\cdot\Omega^p_{X,\mathfrak{o}}(\log D) \subseteq \Omega^p_{X,\mathfrak{o}}, \quad h\cdot d(\Omega^p_{X,\mathfrak{o}}(\log D)) \subseteq \Omega^{p+1}_{X,\mathfrak{o}}.$$

*In addition,* $dh\wedge\Omega^p_{X,\mathfrak{o}}(\log D) \subseteq h\Omega^{p+1}_{X,\mathfrak{o}}(\log D)$*, that is,* $\frac{dh}{h}\wedge\Omega^p_{X,\mathfrak{o}}(\log D) \subseteq \Omega^{p+1}_{X,\mathfrak{o}}(\log D)$*.*

*Proof.* Taking $\vartheta \in \operatorname{Ker}^p_{X,\mathfrak{o}}(\wedge dh) \subseteq \Omega^p_{X,\mathfrak{o}}$, we see that $dh \wedge \vartheta = h\cdot\xi$ for a suitable regular form $\xi \in \Omega^{p+1}_{X,\mathfrak{o}}$. Since $h$ is not a zero divisor, then one can examine the form $\omega = \vartheta/h$, which, evidently, is contained in $\Omega^p_{X,\mathfrak{o}}(\log D)$. Indeed, the condition $dh \wedge \vartheta = 0$ implies $dh \wedge \omega = 0$. Hence, $hd\omega = d(h\omega) = d\vartheta \in \Omega^{p+1}_{X,\mathfrak{o}}$ is regular on $X$ at the distinguished point. Next, since the equality $dh\wedge\vartheta = h\vartheta'$ for a suitable $\vartheta' \in \Omega^{p+1}_{X,\mathfrak{o}}$ follows from the definition immediately, the last inclusion is evident. □

Remark 5. In fact, Definition 1 is closely related with another application of Theorem 1 in the context of the theory of *generalized functions*: any distribution $T$ of degree zero, satisfying the condition $T \wedge \omega = 0$, is equal to a multiple of the *Dirac* distribution (see [10], eq.(5)).

The following statement from basic commutative algebra (see, e.g., [8, Ch. IV, §1, Corollaire 2]) is very useful for explicit computations of the quotient modules $\Omega^{p+1}_{X,\mathfrak{o}}/h\cdot\Omega^{p+1}_{X,\mathfrak{o}}$ in the exact sequence (14).

Assertion 2. *Given a Noetherian ring* $A$*, the homothety of an* $A$*-module* $M$*, corresponding to any element* $h \in A$*, is injective if and only if the element* $h$ *does not belong to any of the prime ideals associated with the module* $M$*.*

The next proposition is concerned an extension of the basic properties of logarithmic differential forms for divisors given on smooth manifolds to the case of normal varieties. Although we do not use this statement later on, however, it is essential to the understanding of the subject. For convenience, we reproduce the corresponding results from [20, (1.1)] with relevant changes.

Proposition 1. *Let* $V$ *be a domain of* $\mathbb{C}^m$*, let* $X$ *be a normal subspace of dimension* $n \geqslant 2$*,* $U = X \cap V$ *and* $D \subset X$ *a Cartier divisor determined by an equation* $h(z) = 0$*, where* $h \in \mathcal{O}_{X,\mathfrak{o}}$ *is regular on* $U$*, such that* $\operatorname{depth}(\operatorname{Sing} D, D) \geqslant 1$*. Then for a meromorphic* $q$*-form* $\omega$ *on* $X$ *with poles along* $D$ *the following four conditions are equivalent:*

(*i*) *differential forms* $h\omega$ *and* $hd\omega$ *are regular on* $X$*,* *(ii) differential forms $h\omega$ and $dh \wedge \omega$ are regular on $X$,*

*(iii) there exists a regular function $g \in \mathcal{O}_{X,\mathfrak{o}}$, a regular $(q-1)$-form $\xi \in \Omega^{q-1}_{X,\mathfrak{o}}$ and a regular q-form $\eta \in \Omega^q_{X,\mathfrak{o}}$ on $X$, such that:*

*a) $\dim_{\mathbb{C}} D \cap \{z \in V\colon g(z) = 0\} \leqslant n-2$, that is, $g$ is not a zero divisor in the local ring $\mathcal{O}_{D,\mathfrak{o}}$,*

*b) $g\omega = \frac{dh}{h} \wedge \xi + \eta$,*

*(iv) there exists an $(n-2)$-dimensional analytic subset $W \subset D$ such that the germ of $\omega$ at any point $p \in D \setminus W$ belongs to $\frac{dh}{h} \wedge \Omega^{q-1}_{U,p} + \Omega^q_{U,p}$, where $\Omega^q_{U,p} = \Omega^p_{X,\mathfrak{o}}|_U$ denotes the module of regular q-forms on $U$ at p.*

*Proof.* First note, that the local ring $\mathcal{O}_{X,\mathfrak{o}}$ of any normal variety is a *domain*. In particular, this ring does not contain zero divisors at all. The equivalence $(i) \Leftrightarrow (ii)$ is evident, the proof of implications $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (iv)$ goes similarly to the proof in [20], taking into account that, by our assumptions, the image of the ideal $\mathrm{Jac}(h)$ under the canonical epimorphism $\mathcal{O}_{X,\mathfrak{o}} \to \mathcal{O}_{D,\mathfrak{o}}$ contains a *non-zero* divisor $g$. Finally, let $\omega$ be a differential form satisfying the condition $(iv)$. Then $h\omega$ and $dh \wedge \omega$ are regular on $D \setminus W$. Since $\mathrm{codim}(W, U) \geqslant 2$, we can apply Hartogs' theorem: any regular function outside of an analytic subset of codimension at least two in a normal variety can be extended to the whole variety in a regular way. As a result, both differential forms $h\omega$ and $dh \wedge \omega$ can be extended on $U$ in such a way. □

It should be also noted that normal varieties are characterized by the latter property completely. Moreover, in complex analysis and geometry it is often taken as a definition of normal varieties.

Corollary 1. *Assume that $X$ is a normal variety. Then the direct sum $\Omega^{\bullet}_{X,\mathfrak{o}}(\log D) = \oplus_{p=0}^m \Omega^p_{X,\mathfrak{o}}(\log D)$ is an $\mathcal{O}_{X,\mathfrak{o}}$-exterior algebra.*

Remark 6. It should be underlined that this property is not true in general (see Remark 8 in §9). Nevertheless, the direct sum $\mathrm{Ker}^{\bullet}_{X,\mathfrak{o}}(\wedge dh) = \oplus_{p=0}^m \mathrm{Ker}^p_{X,\mathfrak{o}}(\wedge dh)$ is an $\mathcal{O}_{X,\mathfrak{o}}$-exterior algebra for *any* variety $X$. In view of Assertion 1, the algebra $\Omega^{\bullet}_{X,\mathfrak{o}}(\log D)$ is always closed under the exterior differentiation $d$ and the exterior multiplication by $dh/h$ (cf. [20, (1.3)]).

**7. Graded ICIS.** The case of graded complete intersections with isolated singularities one can analyze in the manner of Examples 6 and 8. Let us describe the corresponding procedure in detail.

Thus, let us take the germ $M = (\mathbb{C}^3, \mathfrak{o})$ with coordinates $x, y, z$, and let $X$ be a hypersurface determined by the equation $f = x^2 + y^2 + z^2 = 0$. Indeed, it is a *normal* cone over the rational quadric in $\mathbb{P}^2$ with an $A_1$-singularity at the vertex. In particular, $\mathcal{O}_{X,\mathfrak{o}}$ is a domain.

We shall consider the divisor $D \subset X$ given by the function $h = yz$. In fact, the divisor $D$ is a *hypersurface section* of $X$ in terms of [16, 1.1.3, p.19]. On the other side, the germ $D \subset M$ is an ICIS. In view of Lemma 1 we get

$$\mathscr{P}(\Omega^0_X;t) = \mathscr{P}(\mathcal{O}_X;t) = (1-t^2)/(1-t)^3 = (1+t)/(1-t)^2,$$
$$\mathscr{P}(\Omega^1_X;t) = \mathscr{P}(\mathcal{O}_X;t)\mathrm{res}_{\xi=0}\xi^{-2}(1+\xi t)^3/(1+\xi t^2) = t(3-t)(1+t)/(1-t)^2,$$
$$\mathscr{P}(\Omega^2_X;t) = \mathscr{P}(\mathcal{O}_X;t)\mathrm{res}_{\xi=0}\xi^{-3}(1+\xi t)^3/(1+\xi t^2) = t^2(3-3t+t^2)(1+t)/(1-t)^2,$$

since $(1+\xi t)^3/(1+\xi t^2) = 1 + (3t - t^2)\xi + (3t^2 - 3t^3 + t^4)\xi^2 + \cdots$. Next, it is clear that $\Omega^3_X \cong \mathcal{O}_X/\mathrm{Jac}(f)$. Hence $\Omega^3_X = \mathbb{C}\langle dx\wedge dy\wedge dz\rangle$, and

$$\mathscr{P}(\Omega^3_X;t) = \mathscr{P}(\mathrm{Tors}\,\Omega^3_X;t) = \mathscr{P}(\mathrm{Tors}\,\Omega^2_X;t) = t^3,$$

so that $\tau(X) = \mu(X) = 1$ as required.

Since $(1+\xi t)^3/(1+\xi t^2)^2 = 1 + (3t - 2t^2)\xi + (3t^2 - 6t^3 + 3t^4)\xi^2 + \cdots$, similarly to the above we get

$$\begin{aligned}&\mathscr{P}(\Omega^0_D;t) = \mathscr{P}(\mathcal{O}_D;t) = (1-t^2)^2/(1-t)^3 = (1+t)^2/(1-t),\\ &\mathscr{P}(\Omega^1_D;t) = \mathscr{P}(\mathcal{O}_D;t)\mathrm{res}_{\xi=0}\xi^{-2}(1+\xi t)^3/(1+\xi t^2)^2 = t(3-2t)(1+t)^2/(1-t).\end{aligned}$$

Moreover, $\mathscr{P}(\mathrm{Tors}\,\Omega^1_D;t) = 3t^2 + 2t^3$ by [1, Theorem (3.4)], so that $\tau(D) = \mu(D) = 5$ as required. Since $\Omega^3_D \cong \mathcal{O}_D/(\mathrm{Jac}(f),\mathrm{Jac}(h))$, simple observations show

$$\begin{aligned}&\mathscr{P}(\Omega^3_D;t) = \mathscr{P}(\mathrm{Tors}\,\Omega^3_D;t) = t^3,\\ &\mathscr{P}(\Omega^2_D;t) = \mathscr{P}(\mathrm{Tors}\,\Omega^2_D;t) = 3t^2 + 3t^3 = 3t^2(1+t).\end{aligned}$$

For every $p \geqslant 0$ the exact sequence (14)

$$0 \longrightarrow \mathrm{Ker}^p_{X,\mathfrak{o}}(\wedge dh) \longrightarrow \Omega^p_{X,\mathfrak{o}} \xrightarrow{\wedge dh} \Omega^{p+1}_{X,\mathfrak{o}}/h\cdot\Omega^{p+1}_{X,\mathfrak{o}} \longrightarrow \Omega^{p+1}_{D,\mathfrak{o}} \longrightarrow 0,$$

implies the following identity for the Poincaré series:

$$\mathscr{P}(\Omega^{p+1}_{D,\mathfrak{o}};t) = \mathscr{P}(\Omega^{p+1}_{X,\mathfrak{o}}/h\cdot\Omega^{p+1}_{X,\mathfrak{o}};t) - t^2\mathscr{P}(\Omega^p_{X,\mathfrak{o}};t) + t^2\mathscr{P}(\mathrm{Ker}^p_{X,\mathfrak{o}}(\wedge dh);t)$$

since $\deg(\wedge dh) = 2$. Hence,

$$\mathscr{P}(\mathrm{Ker}^p_{X,\mathfrak{o}}(\wedge dh);t) = \mathscr{P}(\Omega^p_{X,\mathfrak{o}};t) - t^{-2}\mathscr{P}(\Omega^{p+1}_{X,\mathfrak{o}}/h\cdot\Omega^{p+1}_{X,\mathfrak{o}};t) + t^{-2}\mathscr{P}(\Omega^{p+1}_{D,\mathfrak{o}};t).$$

Recall (see [13, Proposition (1.11)]) that if $X$ is an ICIS of dimension $n \geqslant 1$, then the torsion modules $\mathrm{Tors}\,(\Omega^p_X)$ are trivial for any $0 \leqslant p < n$. In our case $n = 2$, so that the relation $h\omega = 0$ in the module $\Omega^1_{X,\mathfrak{o}}$ implies $\omega = 0$, since $h$ is a *non-zero divisor* in $\mathcal{O}_{X,\mathfrak{o}}$ and $\mathrm{Tors}\,(\Omega^1_X) = 0$. Hence,

$$\begin{aligned}\mathscr{P}(\Omega^1_{X,\mathfrak{o}}/h\cdot\Omega^1_{X,\mathfrak{o}};t) = (1-t^2)P(\Omega^1_{X,\mathfrak{o}};t) &= (1-t^2)t(3-t)(1+t)/(1-t)^2\\ &= t(3-t)(1+t)^2/(1-t).\end{aligned}$$

As a result, $\mathscr{P}(\mathrm{Ker}^0_{X,\mathfrak{o}}(\wedge dh);t) =$

$$\begin{aligned}&= (1+t)/(1-t)^2 - (t^{-2}-1)t(3-t)(1+t)/(1-t)^2 + t^{-2}t(3-2t)(1+t)^2/(1-t)\\ &= \{(1+t) - (1-t^{-2})t(3-t)(1+t) + t^{-1}(3-2t)(1+t)^2(1-t)\}/(1-t)^2\\ &= t^2(1+t)/(1-t)^2 = t^2\mathscr{P}(\mathcal{O}_X;t).\end{aligned}$$

On the other hand, by [13, Lemma 1.6] there is an exact sequence

$$0 \longrightarrow \Omega^0_{D,\mathfrak{o}} \xrightarrow{\wedge dh} \Omega^1_{X,\mathfrak{o}}/h\cdot\Omega^1_{X,\mathfrak{o}} \longrightarrow \Omega^1_{D,\mathfrak{o}} \longrightarrow 0.$$

Since $\Omega^0_{D,\mathfrak{o}} \cong \Omega^0_{X,\mathfrak{o}}/h\Omega^0_{X,\mathfrak{o}}$, i.e., $\mathcal{O}_{D,\mathfrak{o}} \cong \mathcal{O}_{X,\mathfrak{o}}/h\mathcal{O}_{X,\mathfrak{o}}$, then $\mathrm{Ker}^0_{X,\mathfrak{o}}(\wedge dh) \cong h\Omega^0_{X,\mathfrak{o}}$. This implies

$$\mathscr{P}(\mathrm{Ker}^0_{X,\mathfrak{o}}(\wedge dh);t) = t^2(1+t)/(1-t)^2 = t^2\mathscr{P}(\mathcal{O}_{X,\mathfrak{o}},t);$$

that confirms an accuracy of our computations. As a result, we obtain the isomorphism $\Omega^0_{X,\mathfrak{o}}(\log D) \cong \mathcal{O}_{X,\mathfrak{o}}$ which agrees with the usual case of divisors in a manifold.

Now one can compute the modules $\Omega^1_{X,\mathfrak{o}}(\log D)$ and $\Omega^2_{X,\mathfrak{o}}(\log D)$ as follows. The main difference with the above calculations lies in the fact that, *in general*, $\operatorname{Tors}\Omega^p_X \neq 0$ and the homothety of the module $\Omega^2_{X,\mathfrak{o}}$ (as well as of the module $\Omega^3_{X,\mathfrak{o}}$), determined by the multiplication by $h$, has a *nontrivial* kernel (cf. also Assertion 2).

Let us consider the exact sequence (14) in the case $p = 1$:

$$0 \longrightarrow \operatorname{Ker}^1_{X,\mathfrak{o}}(\wedge dh) \longrightarrow \Omega^1_{X,\mathfrak{o}} \xrightarrow{\wedge dh} \Omega^2_{X,\mathfrak{o}}/h\Omega^2_{X,\mathfrak{o}} \longrightarrow \Omega^2_{D,\mathfrak{o}} \longrightarrow 0 \tag{16}$$

and compute the Poincaré series $\mathscr{P}(\Omega^2_{X,\mathfrak{o}}/h\Omega^2_{X,\mathfrak{o}}; t)$. If $\omega \in \Omega^2_{X,\mathfrak{o}}$, then the relation $h\omega = 0$ implies $\omega \in \operatorname{Tors}\Omega^2_{X,\mathfrak{o}}$, since $h$ is a *non-zero divisor* of $\mathcal{O}_{X,\mathfrak{o}}$.

On the other hand, the differential form

$$\vartheta_0 = \iota_E(dx \wedge dy \wedge dz) = xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$$

is contained in $\operatorname{Tors}\Omega^2_{X,\mathfrak{o}}$, where $E$ is the Euler vector field on $X$; this expression corresponds to the *unique* possible relation between the coordinate 2-forms, since $\dim_{\mathsf{k}} \operatorname{Tors}\Omega^2_{X,\mathfrak{o}} = \mu(X) = 1$. That is, $\operatorname{Tors}\Omega^2_{X,\mathfrak{o}} = \mathbb{C}\langle\vartheta_0\rangle$. It is not difficult to verify that, in fact, the non-zero divisor $h$ annihilates $\vartheta_0$, so that $h\vartheta_0 = 0$ in $\Omega^2_{X,\mathfrak{o}}$, $\deg(\vartheta_0) = 3$ and $\mathscr{P}(\operatorname{Tors}\Omega^2_{X,\mathfrak{o}}; t) = t^3$. As a result, we obtain

$$\mathscr{P}(\Omega^2_{X,\mathfrak{o}}/h\Omega^2_{X,\mathfrak{o}}; t) = (1-t^2)\mathscr{P}(\Omega^2_{X,\mathfrak{o}}; t) + t^5 = t^2(3+3t-2t^2)/(1-t).$$

From the exact sequence (16) we deduce the following identities:

$$\begin{aligned}
&\mathscr{P}(\operatorname{Ker}^1_{X,\mathfrak{o}}(\wedge dh); t) = \mathscr{P}(\Omega^1_{X,\mathfrak{o}}; t) - t^{-2}\{(1-t^2)\mathscr{P}(\Omega^2_{X,\mathfrak{o}}; t) + t^5\} + t^{-2}\mathscr{P}(\Omega^2_{D,\mathfrak{o}}; t)\\
&= \mathscr{P}(\Omega^1_{X,\mathfrak{o}}; t) - t^{-2}t^2(3+3t-2t^2)/(1-t) + 3t^{-2}t^2(1+t)\\
&= t(3-t)(1+t)/(1-t)^2 - (3+3t-2t^2)/(1-t) + 3(1+t)\\
&= \{t(3-t)(1+t) - (1-t)(3+3t-2t^2) + 3(1-t)^2(1+t)\}/(1-t)^2\\
&= 4t^2/(1-t)^2 \not\approx \mathscr{P}(\mathcal{O}_{X,\mathfrak{o}}; t).
\end{aligned}$$

In particular, this implies that the $\mathcal{O}_{X,\mathfrak{o}}$-modules $\operatorname{Ker}^1_{X,\mathfrak{o}}(\wedge dh)$ and $\Omega^1_{X,\mathfrak{o}}(\log D)$ are not free. It remains to compute explicitly a system of generators of $\operatorname{Ker}^1_{X,\mathfrak{o}}(\wedge dh)$.

Proposition 2. *A system of generators of the $\mathcal{O}_{X,\mathfrak{o}}$-module $\operatorname{Ker}^1_{X,\mathfrak{o}}(\wedge dh)$ consists of the four differential forms of weight two*

$$\vartheta_1 = zdy,\ \vartheta_2 = ydz,\ \vartheta_3 = xdy - ydx,\ \vartheta_4 = xdz - zdx,$$

*so that the module $\Omega^1_{X,\mathfrak{o}}(\log D)$ has four generators $\vartheta_i/h$, $i = 1,2,3,4$, and*

$$\mathscr{P}(\Omega^1_{X,\mathfrak{o}}(\log D); t) = t^{-2}\mathscr{P}(\operatorname{Ker}^1_{X,\mathfrak{o}}(\wedge dh); t) = 4/(1-t)^2 \not\approx \mathscr{P}(\mathcal{O}_{X,\mathfrak{o}}; t).$$

*Moreover, since $\mathcal{O}_{X,\mathfrak{o}}$-module $\Omega^1_{X,\mathfrak{o}}(\log D)$ is not free, then there are nontrivial syzygies between these generators.*

*Proof.* Let us take $\vartheta \in \Omega^1_X$, $\vartheta = adx + bdy + cdz$, where $a, b, c \in \mathcal{O}_{X,\mathfrak{o}}$. Then one can transform the congruence $\vartheta \wedge dh \equiv 0 \,(\mathrm{mod}\ h)$ in the quotient module $\Omega^2_X = \Omega^2_{\mathbb{C}^3}/(f\Omega^2_{\mathbb{C}^3} + df \wedge \Omega^1_{\mathbb{C}^3})$ into the following system of equations:

$$\begin{aligned}
az &= (Bx - Ay) + \phi_1 f + \psi_1 h,\\
ay &= (Cx - Az) + \phi_2 f + \psi_2 h,\\
by - cz &= (Cy - Bz) + \phi_3 f + \psi_3 h,
\end{aligned}$$

where $A, B, C, \phi_i, \psi_i \in \mathcal{O}_{\mathbb{C}^3,\mathfrak{o}}$, $i = 1, 2, 3$. If $a = 0$ one gets the two forms $\vartheta_1$ and $\vartheta_2$, as a result of simple calculations of syzygies between $x, y, f, h$. If $b = 0$, then $c = x$, $C = 0$, $B = x$, and the first equation transforms as follows: $az = x^2 - Ay + \phi_1 f + \psi_1 h$. This yields $a = -z$, $A = -y$, $\phi_1 = -1$. In other words, the differential form $xdz - zdx$ is contained in the kernel of exterior multiplication by the total differential $dh$, and so on. □

For completeness, it should be noted that similarly to Example 6 one can choose another system of generators of the module $\Omega^1_{X,\mathfrak{o}}(\log D)$, using the following 2-forms from $\mathrm{Ker}^1_{X,\mathfrak{o}}(\wedge dh)$:

$$\theta_0 = \vartheta_1 + \vartheta_2 = dh,\ \theta_1 = \vartheta_1 - \vartheta_2 = zdy - ydz,\ \theta_2 = \vartheta_3,\ \theta_3 = \vartheta_4. \tag{17}$$

Computing the Poincaré series $\mathscr{P}(\mathrm{Ker}^2_{X,\mathfrak{o}}(\wedge dh); t)$ and the generators of the module $\mathrm{Ker}^2_{X,\mathfrak{o}}(\wedge dh)$ is a more easy exercise. Thus, it is clear that $\Omega^3_{X,\mathfrak{o}} = \mathrm{Tors}\,\Omega^3_{X,\mathfrak{o}} \cong \mathrm{Tors}\,\Omega^2_{X,\mathfrak{o}} = \mathbb{C}\langle dx \wedge dy \wedge dz\rangle$ and $h\Omega^3_{X,\mathfrak{o}} = 0$, since the polynomial $h$ is contained in the maximal ideal of $\mathcal{O}_{X,\mathfrak{o}}$ and $\mathrm{Jac}(f) \cong (x, y, z)\mathcal{O}_{X,\mathfrak{o}}$. Hence, the exact sequence (14) looks like this:

$$0 \longrightarrow \mathrm{Ker}^2_{X,\mathfrak{o}}(\wedge dh) \longrightarrow \Omega^2_{X,\mathfrak{o}} \xrightarrow{\wedge dh} \Omega^3_{X,\mathfrak{o}} \longrightarrow \Omega^3_{D,\mathfrak{o}} \longrightarrow 0.$$

Moreover, $\mathscr{P}(\Omega^3_{X,\mathfrak{o}}; t) = \mathscr{P}(\Omega^3_{D,\mathfrak{o}}; t)$ in virtue of previous computations and, consequently,

$$\mathscr{P}(\mathrm{Ker}^2_{X,\mathfrak{o}}(\wedge dh); t) = \mathscr{P}(\Omega^2_{X,\mathfrak{o}}; t) - t^{-2}\mathscr{P}(\Omega^3_{X,\mathfrak{o}}; t) + t^{-2}\mathscr{P}(\Omega^3_{D,\mathfrak{o}}; t) = \mathscr{P}(\Omega^2_{X,\mathfrak{o}}; t).$$

On the other hand, $dh\wedge\Omega^2_{X,\mathfrak{o}} = 0$, since $\mathrm{Jac}(h) \subset \mathrm{Jac}(f)$, that is, $\mathrm{Ker}^2_{X,\mathfrak{o}}(\wedge dh) = \Omega^2_{X,\mathfrak{o}}$. Further, the $\mathcal{O}_{X,\mathfrak{o}}$-module $\Omega^2_{X,\mathfrak{o}}$ is generated by the three coordinate forms $dx \wedge dy$, $dy \wedge dz$ and $dx \wedge dz$; there are nontrivial syzygies of the first and the second orders between these forms. As a result, we obtain

$$\begin{aligned}
&\Omega^2_{X,\mathfrak{o}}(\log D) \cong \frac{1}{h}\Omega^2_{X,\mathfrak{o}} \subset \mathcal{O}_{X,\mathfrak{o}}\Big\langle \frac{dx \wedge dy}{h}, \frac{dy \wedge dz}{h}, \frac{dx \wedge dz}{h}\Big\rangle,\\
&\mathscr{P}(\Omega^2_{X,\mathfrak{o}}(\log D); t) = t^{-2}\mathscr{P}(\Omega^2_X; t),
\end{aligned}$$

so that $\Omega^2_{X,\mathfrak{o}}(\log D)$ is *not free* and $\mathrm{Tors}\,\Omega^2_{X,\mathfrak{o}}(\log D) \cong \mathrm{Tors}\,\Omega^2_{X,\mathfrak{o}}$.

Remark 7. It should be underlined that $\vartheta_i \wedge \vartheta_j \equiv 0\,(\mathrm{mod}\ h)$ in $\Omega^2_{X,\mathfrak{o}}$. More precisely, $\vartheta_3 \wedge \vartheta_4 = x \cdot \vartheta_0 = 0$ in $\Omega^2_{X,\mathfrak{o}}$, where $\vartheta_0 = \iota_E(dx \wedge dy \wedge dz) \in \mathrm{Tors}\,\Omega^2_{X,\mathfrak{o}}$. Next, $\vartheta_1 \wedge \vartheta_4 = z \cdot \vartheta_0 + hdx \wedge dy = hdx \wedge dy$, $\vartheta_2 \wedge \vartheta_3 = -y \cdot \vartheta_0 + hdx \wedge dy = hdx \wedge dy$, and so on. As a result, the $\mathcal{O}_{X,\mathfrak{o}}$-module $\Omega^{\bullet}_{X,\mathfrak{o}}(\log D) = \oplus_{p=0}^{n}\Omega^p_{X,\mathfrak{o}}(\log D)$ is, in fact, an *exterior* algebra as Corollary 1 asserts.

Above all, it is not difficult to verify that the differential 1-forms $\theta_1, \theta_2$ and $\theta_3$ in the presentation (17) are contained in the torsion module $\mathrm{Tors}\,\Omega^1_{D,\mathfrak{o}}$ (cf. Example 6, Example 8, [2, Lemma 1] or [4, §1, Proposition]).

**8. Determinantal varieties.** Now we are able to describe the logarithmic forms for divisors given on the Hilbert cubic from Example 3. In the same notation, let $(x : y : z : u)$ be homogeneous coordinates in $\mathbb{P}^3$. Then $\Omega^p_{X,\mathfrak{o}} \cong k\langle x, y, z, u\rangle/I$, where the prime ideal $I = (f_1, f_2, f_3)$ is generated by the maximal minors of the Hankel matrix $M = \left[\begin{smallmatrix} x & y & z \\ y & z & u\end{smallmatrix}\right]$, so that

$$f_1 = xz - y^2,\ f_2 = yz - xu,\ f_3 = yu - z^2.$$

The germ $X$ is a determinantal Cohen-Macaulay surface singularity of codimension 2 and the local ring $\mathcal{O}_{X,\mathfrak{o}}$ is a domain. In Example 3 we computed the following Poincaré series:

$$\begin{aligned}
&\mathscr{P}(\mathcal{O}_{X,\mathfrak{o}};t) = (1-3t^2+2t^3)/(1-t)^4 = (1+2t)/(1-t)^2,\\
&\mathscr{P}(\Omega^1_{X,\mathfrak{o}};t) = t(4+5t-4t^2+t^3)/(1-t)^2,\\
&\mathscr{P}(\Omega^2_{X,\mathfrak{o}};t) = t^2(6-7t^2+4t^3)/(1-t)^2,\\
&\mathscr{P}(\Omega^3_{X,\mathfrak{o}};t) = 4t^3+t^4 = t^3(4+t),\\
&\mathscr{P}(\Omega^4_{X,\mathfrak{o}};t) = t^4,
\end{aligned}$$

so that $\mathscr{P}(\Omega^3_{X,\mathfrak{o}};t) = \mathscr{P}(\operatorname{Tors}\Omega^3_{X,\mathfrak{o}};t)$ and $\mathscr{P}(\Omega^4_{X,\mathfrak{o}};t) = \mathscr{P}(\operatorname{Tors}\Omega^4_{X,\mathfrak{o}};t)$.

Taking the divisor $D \subset X$, which is determined by a quasihomogeneous polynomial $h \in \mathcal{O}_{X,\mathfrak{o}}$, one can compute the modules of logarithmic differential forms $\Omega^p_{X,\mathfrak{o}}(\log D)$, $p \geqslant 0$, and the corresponding Poincaré series similarly to the method described above. More precisely, let us take $h = x^2 - u^2$. The corresponding divisor $D$ consists of two pairs of straight lines and of two pairs of a line and a quadric. Further computations show

$$\begin{aligned}
&\mathscr{P}(\mathcal{O}_{D,\mathfrak{o}};t) = (1+t)(1+2t)/(1-t),\\
&\mathscr{P}(\Omega^1_{D,\mathfrak{o}};t) = 2t(2-t^2)(1+2t)/(1-t),\\
&\mathscr{P}(\Omega^2_{D,\mathfrak{o}};t) = 2t^2(3+4t),\\
&\mathscr{P}(\Omega^3_{D,\mathfrak{o}};t) = t^3(4+t),\\
&\mathscr{P}(\Omega^4_{D,\mathfrak{o}};t) = t^4.
\end{aligned}$$

so that $\mathscr{P}(\Omega^2_{D,\mathfrak{o}};t) = \mathscr{P}(\operatorname{Tors}\Omega^2_{D,\mathfrak{o}};t)$, and so on. Now we shall compute $\operatorname{Ker}^0_{X,\mathfrak{o}}(\wedge dh)$. First note, that

$$\mathscr{P}(\Omega^1_{X,\mathfrak{o}}/h{\cdot}\Omega^1_{X,\mathfrak{o}};t) = t(1+2t)(4+t-t^2)/(1-t) \neq (1-t^2)\mathscr{P}(\Omega^1_{X,\mathfrak{o}};t).$$

In particular, $\operatorname{Tors}\Omega^1_{X,\mathfrak{o}} \neq 0$. Hence,

$$\begin{aligned}
&\mathscr{P}(\operatorname{Ker}^0_{X,\mathfrak{o}}(\wedge dh);t) = \mathscr{P}(\Omega^0_{X,\mathfrak{o}};t) - t^{-2}\mathscr{P}(\Omega^1_{X,\mathfrak{o}}/h{\cdot}\Omega^1_{X,\mathfrak{o}};t) + t^{-2}\mathscr{P}(\Omega^1_{D,\mathfrak{o}};t)\\
&= (1+2t)/(1-t)^2 - t^{-2}t(1+2t)(4+t-t^2)/(1-t) + t^{-2}2t(2-t^2)(1+2t)/(1-t)\\
&= t^{-1}\{t(1+2t) - (1+2t)(4+t-t^2)(1-t) + 2(2-t^2)(1+2t)(1-t)\}/(1-t)^2\\
&= t^{-1}(t^3+2t^4)/(1-t)^2 = t^2(1+2t)/(1-t)^2 = t^2\mathscr{P}(\mathcal{O}_{X,\mathfrak{o}};t).
\end{aligned}$$

that is, $\operatorname{Ker}^0_{X,\mathfrak{o}}(\wedge dh)$ is a *free* $\mathcal{O}_X$-module, similarly to the case of divisors in smooth manifolds. Next,

$$\mathscr{P}(\Omega^2_{X,\mathfrak{o}}/h{\cdot}\Omega^2_{X,\mathfrak{o}};t) = 6t^2(1+t-t^2)/(1-t) \neq (1-t^2)\mathscr{P}(\Omega^2_{X,\mathfrak{o}};t),$$

that is, $\operatorname{Tors}\Omega^2_{X,\mathfrak{o}} \neq 0$. Hence,

$$\begin{aligned}
&\mathscr{P}(\operatorname{Ker}^1_{X,\mathfrak{o}}(\wedge dh);t) = \mathscr{P}(\Omega^1_{X,\mathfrak{o}};t) - t^{-2}\mathscr{P}(\Omega^2_{X,\mathfrak{o}}/h{\cdot}\Omega^2_{X,\mathfrak{o}};t) + t^{-2}\mathscr{P}(\Omega^2_{D,\mathfrak{o}};t)\\
&= t(4+5t-4t^2+t^3)/(1-t)^2 - t^{-2}6t^2(1+t-t^2)/(1-t) + t^{-2}2t^2(3+4t)\\
&= \{t(4+5t-4t^2+t^3) - 6(1+t-t^2)(1-t) + 2(3+4t)(1-t)^2\}/(1-t)^2\\
&= (7t^2-2t^3+t^4)/(1-t)^2 = t^2(7-2t+t^2)/(1-t)^2 \not\approx \mathscr{P}(\mathcal{O}_{X,\mathfrak{o}};t).
\end{aligned}$$

As a result, the module $\mathrm{Ker}^1_{X,\mathfrak{o}}(\wedge dh)$ is not free. Similarly to the proof of Proposition 2 one can verify that a system of generators of the $\mathcal{O}_{X,\mathfrak{o}}$-module $\mathrm{Ker}^1_{X,\mathfrak{o}}(\wedge dh)$ consists of the following differential 2-forms of weight two

$$\vartheta_0 = xdx - udu = \tfrac{1}{2}dh,\ \vartheta_1 = xdy - ydx,\ \vartheta_2 = xdz - zdx,\ \vartheta_3 = xdu - udx,$$
$$\vartheta_4 = ydz - zdy,\ \vartheta_5 = ydu - udy,\ \vartheta_6 = udz - zdu.$$

Making use of rational parametrization (6), we see that $\vartheta_i \in \mathrm{Tors}\,\Omega^1_{D,\mathfrak{o}}$ for all $i = 1, \dots, 6$. Moreover, $\vartheta_i \wedge \vartheta_j \equiv 0 \,(\mathrm{mod}\ h)$ in $\Omega^2_{X,\mathfrak{o}}$ for all $0 \leqslant i, j \leqslant 6$. For example, $\vartheta_1 \wedge \vartheta_2 \equiv x \cdot \iota_E(dx \wedge dy \wedge dz) = 0$ in $\Omega^2_{X,\mathfrak{o}}$, and so on (cf. Remark 7).

The computation of the Poincaré series of $\mathrm{Ker}^2_{X,\mathfrak{o}}(\wedge dh)$ goes similarly to Section 7. Again, it is clear that $\Omega^3_{X,\mathfrak{o}} = \mathrm{Tors}\,\Omega^3_{X,\mathfrak{o}}$ and one can easily verify that $h\Omega^3_{X,\mathfrak{o}} = 0$. Since $\mathscr{P}(\Omega^3_{X,\mathfrak{o}}; t) = \mathscr{P}(\Omega^3_{D,\mathfrak{o}}; t)$, one obtains

$$\mathscr{P}(\mathrm{Ker}^2_{X,\mathfrak{o}}(\wedge dh); t) = \mathscr{P}(\Omega^2_{X,\mathfrak{o}}; t) - t^{-2}\mathscr{P}(\Omega^3_{X,\mathfrak{o}}; t) + t^{-2}\mathscr{P}(\Omega^3_{D,\mathfrak{o}}; t) = \mathscr{P}(\Omega^2_{X,\mathfrak{o}}; t).$$

On the other hand, it is not difficult to verify that $dh \wedge \Omega^2_{X,\mathfrak{o}} = 0$. Hence, one gets the identification $\mathrm{Ker}^2_{X,\mathfrak{o}}(\wedge dh) = \Omega^2_{X,\mathfrak{o}}$, so that the module $\Omega^2_{X,\mathfrak{o}}(\log D)$ is *not free* and $\mathrm{Tors}\,\Omega^2_{X,\mathfrak{o}}(\log D) \cong \mathrm{Tors}\,\Omega^2_{X,\mathfrak{o}}$. The same reasonings show that $\mathrm{Ker}^3_{X,\mathfrak{o}}(\wedge dh) = \Omega^3_{X,\mathfrak{o}}$, $\mathrm{Tors}\,\Omega^3_{X,\mathfrak{o}}(\log D) \cong \mathrm{Tors}\,\Omega^3_{X,\mathfrak{o}}$, and so on.

**9. Fans.** In fact, the above methods are suitable and highly effective for computing the modules $\mathrm{Ker}^p_{X,\mathfrak{o}}(\wedge dh)$ in very different situations, involving the case of divisors on discriminants or, more generally, on (non-normal) Saito free divisors, as well as on non-Cohen-Macaulay varieties of divers types whose structure algebra contains at least *one* non-zero divisor. In the former situation the Poincaré series for $\Omega^p_{X,\mathfrak{o}}$ were described in [4], while in the latter case the corresponding Poincaré series can be easily computed in the case of *fans* (which are rigid singularities); one of the simplest examples has been studied in detail in [7, Example 15.9]).

Now, let $X \subset (\mathbb{C}^4, 0)$ be the coproduct of two copies of complex plane over the origin. Then $A = \mathsf{k}\langle x, y\rangle \times_{\mathsf{k}} \mathsf{k}\langle z, u\rangle$ is the structure local algebra of the corresponding germ $X$, so that $A \cong \mathsf{k}\langle x, y, z, u\rangle/I$, where $I = (xz, xu, yz, yu)$. It is clear, that $X$ is an *isolated* surface singularity. Moreover, $X$ is not normal because it is not Cohen-Macaulay. In particular, the singularity $X$ is not determinantal and has no nontrivial infinitesimal deformations of the first order, so that $X$ is a *rigid* singularity.

In this case the local ring $\mathcal{O}_{X,\mathfrak{o}}$ is not a domain. However, it contains enough non-zero divisors such that $x + z$, $x + y + z + u$, $x^2 + y^2 + z^2 + u^2$, and so on. It is not difficult to check that the corresponding 1-dimensional germs $D \subset X$ are neither *reduced*, nor even equal-dimensional. Moreover, every such germ contains at least one *embedded* component at the origin. Nevertheless, one can compute the modules $\mathrm{Ker}^p_{X,\mathfrak{o}}(\wedge dh)$ with the help of our methods, making use of exact sequence (14). For explicit computations we need the Poincaré series

$$\begin{aligned}
&\mathscr{P}(\mathcal{O}_{X,\mathfrak{o}}; t) = (1 + 2t - t^2)/(1 - t)^2,\\
&\mathscr{P}(\Omega^1_{X,\mathfrak{o}}; t) = 4t(1 + t - 2t^2 + t^3)/(1 - t)^2,\\
&\mathscr{P}(\Omega^2_{X,\mathfrak{o}}; t) = 2t^2(3 - 2t - 2t^2 + 2t^3)/(1 - t)^2,\\
&\mathscr{P}(\Omega^3_{X,\mathfrak{o}}; t) = t^3(4 + t),\\
&\mathscr{P}(\Omega^4_{X,\mathfrak{o}}; t) = t^4,
\end{aligned}$$

which have been obtained in [7, 15.9].

Now let us take the divisor $D \subset X$, determined by the homogeneous polynomial $h = x^2+y^2+z^2+u^2$. Remark, that the primary decomposition of the ideal $I$ consists of the two associated primes $\wp_1 = (x,y)$ and $\wp_2 = (z,u)$. Hence, $h$ is not a zero divisor (see Assertion 2), and the germ $D$ consists of 4 straight lines, passing through the origin: $x = y = z^2+u^2 = 0$, $x^2+y^2 = z = u = 0$. Above all it contains an *embedded* zero-dimensional component at the origin. One can verify that there exist the following Poincaré series

$$\begin{aligned}
&\mathscr{P}(\mathcal{O}_{D,\mathfrak{o}};t) = (1+2t-t^2)(1+t)/(1-t),\\
&\mathscr{P}(\Omega^1_{D,\mathfrak{o}};t) = t(4+7t-7t^2)/(1-t),\\
&\mathscr{P}(\Omega^2_{D,\mathfrak{o}};t) = 2t^2(3+2t),\\
&\mathscr{P}(\Omega^3_{D,\mathfrak{o}};t) = t^3(4+t),\\
&\mathscr{P}(\Omega^4_{D,\mathfrak{o}};t) = t^4.
\end{aligned}$$

Next, making use of similar calculations described in the previous sections, one can write down the Poincaré series of the modules $\mathrm{Ker}^p_{X,\mathfrak{o}}(\wedge dh)$, $p \geqslant 0$, in an explicit form. First, we compute the series

$$\mathscr{P}(\Omega^1_{X,\mathfrak{o}}/h{\cdot}\Omega^1_{X,\mathfrak{o}};t) = 4t(1+2t-t^2)/(1-t).$$

This implies that

$$\begin{aligned}
&\mathscr{P}(\mathrm{Ker}^0_{X,\mathfrak{o}}(\wedge dh);t) = \mathscr{P}(\Omega^0_{X,\mathfrak{o}};t) - t^{-2}\mathscr{P}(\Omega^1_{X,\mathfrak{o}}/h{\cdot}\Omega^1_{X,\mathfrak{o}};t) + t^{-2}\mathscr{P}(\Omega^1_{D,\mathfrak{o}};t)\\
&= (1+2t-t^2)/(1-t)^2 - t^{-2}4t(1+2t-t^2)/(1-t) + t^{-2}t(4+7t-7t^2)/(1-t)\\
&= 2t^2/(1-t)^2 \not\approx t^2\mathscr{P}(\mathcal{O}_{X,\mathfrak{o}};t)
\end{aligned}$$

In particular, the module $\mathrm{Ker}^0_{X,\mathfrak{o}}(\wedge dh)$ is not free. Next, let us consider the two polynomials $h_1 = x^2+y^2$ and $h_2 = z^2+u^2$ from the structure algebra $\mathcal{O}_{X,\mathfrak{o}}$, so that $h = h_1+h_2$. First we check that $h_1 \in \mathrm{Ker}^0_{X,\mathfrak{o}}(\wedge dh)$. Let us denote the four generators of the ideal $I = (xz, xu, yz, yu)$ by $f_1, \ldots, f_4$, respectively. Then, in the module $\Omega^1_{X,\mathfrak{o}}$, there exist the following congruences: $zdf_1 \equiv z^2dx$, $udf_2 \equiv u^2dx$, $zdf_3 \equiv z^2dy$, $udf_4 \equiv u^2dy$. Finally, it remains to use formal identities

$$\frac{1}{2}h_1dh \equiv xh_1dx + yh_1dy \equiv -xzdf_1 - xudf_2 - yzdf_3 - yudf_4,$$

taking into account that

$$xh_1dx + xzdf_1 + xudf_2 = xhdx,\ yh_1dy + yzdf_3 - yudf_4 = yhdx.$$

Similar considerations show that $h_2 \in \mathrm{Ker}^0_{X,\mathfrak{o}}(\wedge dh)$, i.e. the module $\mathrm{Ker}^0_{X,\mathfrak{o}}(\wedge dh)$ is generated by the *two* elements $h_1$ and $h_2$ of weighted degree two.

In the same vein one can describe the module $\mathrm{Ker}^1_{X,\mathfrak{o}}(\wedge dh)$. Namely, let us first compute

$$\mathscr{P}(\Omega^2_{X,\mathfrak{o}}/h{\cdot}\Omega^2_{X,\mathfrak{o}};t) = 2t^2(3-2t)(1+t)/(1-t).$$

Therefore,

$$\begin{aligned}
&\mathscr{P}(\mathrm{Ker}^1_{X,\mathfrak{o}}(\wedge dh);t) = \mathscr{P}(\Omega^1_{X,\mathfrak{o}};t) - t^{-2}\mathscr{P}(\Omega^2_{X,\mathfrak{o}}/h{\cdot}\Omega^2_{X,\mathfrak{o}};t) + t^{-2}\mathscr{P}(\Omega^2_{D,\mathfrak{o}};t)\\
&= 4t(1+t-2t^2+t^3)/(1-t)^2 - 2(3-2t)(1+t)/(1-t) + 2(3+2t)\\
&= 4t^2(2-2t+t^2)/(1-t)^2 = -4t^2\mathscr{P}(\mathcal{O}_{X,\mathfrak{o}};t) + 12t^2/(1-t)^2 \not\approx \mathscr{P}(\mathcal{O}_{X,\mathfrak{o}};t).
\end{aligned}$$

Hence, the module $\operatorname{Ker}^1_{X,\mathfrak{o}}(\wedge dh)$ is not free also, it has *eight* generators of weighted degree two.

One can write down a system of generators in an explicit form. Since $dh = dh_1 + dh_2$, then $dh_1 \wedge dh = dh_1 \wedge dh_2 = f_1 dx \wedge dz + f_2 dx \wedge du + f_3 dy \wedge dz + f_4 dy \wedge du = 0$, in the module $\Omega^2_{X,\mathfrak{o}}$. As a consequence, we obtain two generators of the module $\operatorname{Ker}^1_{X,\mathfrak{o}}(\wedge dh)$, the differential 1-forms $dh_1$ and $dh_2$, which are *total* differentials. Remark also that for irreducible divisors there is *one* form of such kind only; namely, it is $dh$.

Applying the scheme of calculations presented in the proof of Proposition 2, we find that the remaining six generators, denoted by $\vartheta_1,\dots,\vartheta_6$, are the following differential 1-forms: $zdx - xdz$, $udx - xdu$, $zdy - ydz$, $udy - ydu$, $ydx - xdy$ and $udz - zdu$; they can be obtained by the contraction of appropriate coordinate 2-forms along the Euler vector field. For example, $zdx - xdz = -\iota_E(dx \wedge dz)$, $udx - xdu = -\iota_E(dx \wedge du)$, and so on.

Let us verify that these forms are contained in the torsion module $\operatorname{Tors}\Omega^1_{D,\mathfrak{o}}$. Thus, it is clear that $(y+u) \in \mathcal{O}_{D,\mathfrak{o}}$ is not a zero divisor, and $(y+u)\vartheta_1 \equiv zudx - xydz \pmod I$. Next, $zudx - xydz = zdf_2 - f_1 du - xdf_3 + f_3 dy = 0$ in $\Omega^1_{D,\mathfrak{o}}$ (as well as in $\Omega^1_{X,\mathfrak{o}}$). Hence, $\vartheta_1 \in \operatorname{Tors}\Omega^1_{D,\mathfrak{o}}$. Similar arguments are valid for the differential forms $\vartheta_2$, $\vartheta_3$ and $\vartheta_4$.

Next, $(x+z)\vartheta_5 \equiv xydx - x^2dy \equiv \frac{1}{2}ydh - y^2dy - x^2dy \pmod I$ and there is the following chain of identities: $\frac{1}{2}ydh - y^2dy - x^2dy = \frac{1}{2}ydh - hdy + (z^2+u^2)dy = \frac{1}{2}ydh - hdy + zdf_3 - f_3dz + udf_4 - f_4du$. Hence, $(x+z)\vartheta_5 = 0$ in $\Omega^1_{D,\mathfrak{o}}$. This means $\vartheta_5 \in \operatorname{Tors}\Omega^1_{D,\mathfrak{o}}$ and, analogously, we deduce that $\vartheta_6 \in \operatorname{Tors}\Omega^1_{D,\mathfrak{o}}$.

Finally, as in the previous section, $\Omega^3_{X,\mathfrak{o}} = \operatorname{Tors}\Omega^3_{X,\mathfrak{o}}$ and $h\Omega^3_{X,\mathfrak{o}} = 0$. Since $\mathscr{P}(\Omega^3_{X,\mathfrak{o}};t) = \mathscr{P}(\Omega^3_{D,\mathfrak{o}};t)$, we have

$$\mathscr{P}(\operatorname{Ker}^2_{X,\mathfrak{o}}(\wedge dh);t) = \mathscr{P}(\Omega^2_{X,\mathfrak{o}};t) - t^{-2}\mathscr{P}(\Omega^3_{X,\mathfrak{o}};t) + t^{-2}\mathscr{P}(\Omega^3_{D,\mathfrak{o}};t) = \mathscr{P}(\Omega^2_{X,\mathfrak{o}};t).$$

In virtue of the equality $dh \wedge \Omega^2_{X,\mathfrak{o}} = 0$, we get $\operatorname{Ker}^2_{X,\mathfrak{o}}(\wedge dh) = \Omega^2_{X,\mathfrak{o}}$, so that the module $\Omega^2_{X,\mathfrak{o}}(\log D)$ is not *free*, and $\operatorname{Tors}\Omega^2_{X,\mathfrak{o}}(\log D) \cong \operatorname{Tors}\Omega^2_{X,\mathfrak{o}}$. Next, as in the above cases, one obtains $\operatorname{Ker}^3_{X,\mathfrak{o}}(\wedge dh) = \Omega^3_{X,\mathfrak{o}}$, $\operatorname{Tors}\Omega^3_{X,\mathfrak{o}}(\log D) \cong \operatorname{Tors}\Omega^3_{X,\mathfrak{o}}$, and so on.

REMARK 8. It should be noted that in the case under consideration the direct sum $\Omega^\bullet_{X,\mathfrak{o}}(\log D) = \oplus_{p=0}^m \Omega^p_{X,\mathfrak{o}}(\log D)$ is *not* an $\mathcal{O}_{X,\mathfrak{o}}$-exterior algebra (cf. Corollary 1 and Remark 6).

## REFERENCES


[1] A. G. ALEKSANDROV, *Cohomology of a quasihomogeneous complete intersection*, Math. USSR, Izv., 26 (1986), pp. 437–477; translation from Izv. Akad. Nauk SSSR, Ser. Mat., 49:3 (1985), pp. 467–510.

[2] A. G. ALEKSANDROV, *The De Rham complex of nonisolated singularities*, Funct. Anal. Appl., 22:2 (1988), pp. 131–133; translation from Funkts. Anal. Prilozh., 22:2 (1988), pp. 59–60.

[3] A. G. ALEKSANDROV, *Nonisolated Saito singularities*, Math. USSR Sbornik, 65:2 (1990), pp. 561–574; translation from Matem. Sbornik, 137(179) (1988), no. 4(12), pp. 554–567.

[4] A. G. ALEKSANDROV, *Nonisolated hypersurface singularities*, Advances in Soviet Mathematics (V.I.Arnol'd, editor) 1, pp. 211–246. Providence, RI: American Mathematical Society (AMS), 1990.

[5] A. G. ALEKSANDROV, *L'indice topologique des champs de vecteurs sur les intersections complètes quasi-homogènes (à la mémoire de Henri Poincaré)*, C. R. l'Acad. Sci. Paris, Sér. I. Math., 350 (2012), no. 19-20, pp. 911–916.

[6] A. G. Aleksandrov, *Residues of logarithmic differential forms in complex analysis and geometry*, Analysis in Theory and Applications, 30:1 (2014), pp. 34-50.

[7] A. G. Aleksandrov, *The Poincaré index and the $\chi_y$-characteristic of Hirzebruch*, Complex Variables and Elliptic Equations, 61:2 (2016), pp. 144–183.

[8] N. Bourbaki, *Algèbre Commutative*, Hermann, Paris, 1961.

[9] H. Cartan, *Théorie élémentaire des fonctions analytiques d'une ou plusieurs variables complexes*, Hermann, Paris, 1961.

[10] G. de Rham, *Sur la division de formes et de courants par une forme linéaire*, Comment. Math. Helvet., 28 (1954), pp. 346–352.

[11] X. Gómez–Mont, *An algebraic formula for the index of a vector field on a hypersurface with an isolated singularity*, J. Alg. Geom., 7 (1988), pp. 731–752.

[12] V. Grandjean, *Coherent vector fields and logarithmic stratification*, In: Real and complex singularities (Bruce, J. W. (ed.) et al.) Proceedings of the 5th workshop, São Carlos, Brazil, July 27-31, 1998. Boca Raton, FL: Chapman & Hall/CRC. Chapman Hall/CRC Res. Notes Math. 412, pp. 46–60 (2000).

[13] G.–M. Greuel, *Der Gauß–Manin–Zusammenhang isolierter Singularitäten von vollständigen Durchscnitten*, Math. Ann., 214:1 (1975), pp. 235–266.

[14] C. Huneke, *The Koszul homology of an ideal*, Adv. in Math., 56:3 (1985), pp. 295–318.

[15] D. Mumford, *Lectures on curves on an algebraic surface*, Princeton University Press, Princeton, New Jersey, 1966.

[16] I. Naruki, *Some remarks on isolated singularities and their application to algebraic manifolds*, Publ. RIMS, Kyoto Univ., 13 (1977), pp. 17–46.

[17] H. C. Pinkham, *Deformations of algebraic varieties with $\mathbb{G}_m$ action*, Astérisque, 20, Société Mathématique de France, Paris, 1974.

[18] G. Pólya and G. Szegö, *Aufgaben und Lehrsätze aus der Analysis*, 1, Verlag von Springer, Berlin, 1925.

[19] O. Riemenschneider, *Deformationen von Quotientensingularitäten (nach zyklischen Gruppen)*, Math. Ann., 209:2 (1974), pp. 211–248.

[20] K. Saito, *Theory of logarithmic differential forms and logarithmic vector fields*, J. Fac. Sci. Univ. Tokyo, ser. IA, 27:2 (1980), pp. 265–291.

[21] P. Seibt, *Infinitesimal extensions of commutative algebras*, J. Pure Appl. Algebra, 16:2 (1980), pp. 197–206.

[22] A. Simis, B. Ulrich, and W. V. Vasconselos, *Tangent algebras*, Trans. Amer. Math. Soc., 364:2 (2012), pp. 571–594.