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THE STEADY NAVIER-STOKES AND STOKES SYSTEMS WITH MIXED BOUNDARY CONDITIONS INCLUDING ONE-SIDED LEAKS AND PRESSURE*

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Abstract. In this paper we are concerned with the steady Navier-Stokes and Stokes problems with mixed boundary conditions involving Tresca slip, leak condition, one-sided leak conditions, velocity, pressure, rotation, stress and normal derivative of velocity together. Relying on the relations among strain, rotation, normal derivative of velocity and shape of boundary surface, we have variational formulations for the problems, which consist of five formulae with five unknowns. We get the variational inequalities equivalent to the formulated variational problems, which have one unknown. Then, we study the corresponding variational inequalities and relying the results for variational inequalities, we get existence, uniqueness and estimates of solutions to the Navier-Stokes and Stokes problems with the boundary conditions. Our estimates for solutions do not depend on the thresholds for slip and leaks.

Key words. Navier-Stokes equations, variational inequality, mixed boundary condition, Tresca slip, leak boundary conditions, one-sided leak, pressure boundary condition, existence, uniqueness.

AMS subject classifications. 35Q30, 35J87, 76D03, 76D05, 49J40.

1. Introduction.

As mathematical models of steady flows of incompressible viscous Newtonian fluids the Stokes equations

$$-\nu\Delta v + \nabla p = f, \;\; \nabla \cdot v = 0 \quad \text{in } \Omega, \tag{1.1}$$

and Navier-Stokes equations

$$-\nu\Delta v + (v\cdot\nabla)v + \nabla p = f, \;\; \nabla \cdot v = 0 \quad \text{in } \Omega, \tag{1.2}$$

are used. For these systems, different natural and artificial boundary conditions are considered(cf. Introduction of [32] and references therein).

Recently several papers are devoted to problems with Tresca slip boundary condition or leak boundary condition. All these boundary conditions are called the boundary conditions of friction type, which are nonlinear.

Tresca slip boundary condition (threshold slip condition) means that if absolute value of tangent stress on a boundary is less than a given threshold, then there is not any slip on the boundary surface, but the absolute value is same with the threshold, then slip on the boundary surface may occur. Physical and experimental backgrounds of such boundary conditions are mentioned in several papers(cf. [19], [7], [5], and especially [27]). When v is a solution to (1.1) or (1.2), the strain tensor is one with the components $\varepsilon_{ij}(v) = \frac{1}{2}(\partial_{x_i}v_j + \partial_{x_j}v_i)$ and stress tensor $S(v,p)$ is one with components $S_{ij} = -p\delta_{ij} + 2\nu\varepsilon_{ij}(v)$. Let n be the outward normal unit vector on a boundary surface and τ tangent vectors. Then, stress vector on the surface is $\sigma(v,p) = Sn$ and normal stress $\sigma_n(v,p) = \sigma\cdot n$. Under such notations Tresca slip boundary condition

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is expressed by

$$|\sigma_\tau(v)| \leq g_\tau, \quad \sigma_\tau(v) \cdot v_\tau + g_\tau |v_\tau| = 0, \tag{1.3}$$

where and in what follows $\sigma_\tau = \sigma - \sigma_n n$ and $v_\tau = v - (v \cdot n)n$.

Leak boundary condition means that if absolute value of normal stress on a boundary is less than a given threshold, then there is not any leak through the boundary surface, but the absolute value is same with the threshold, then leak through the boundary surface may occur. For physical backgrounds of this boundary condition refer to [19], [22], [4]. Under notations above leak boundary condition is expressed by

$$|\sigma_n(v)| \leq g_n, \quad \sigma_n(v) v_n + g_n |v_n| = 0, \tag{1.4}$$

where and in what follows $v_n = v \cdot n$.

Till now, for the Stokes and Navier-Stokes problems with friction type boundary conditions rather simple cases are studied. More clearly, one deal with problems with the Dirichlet boundary condition on a portion of boundary and one of friction type conditions on other portion.

In [19] existence of solutions to the steady Stokes and Navier-stokes equations with the homogeneous Dirichlet boundary condition on a portion of boundary and leak or threshold slip boundary condition on other portion is studied. Also, [20]-[22] concerned with the steady or non-steady Stokes equations with the homogeneous Dirichlet boundary condition and leak boundary condition.

When a portion of boundary with Dirichlet boundary condition and other moving portion where nonlinear slip occurs are separated, existence, uniqueness and continuous dependence on the data are studied for the steady Stokes equations in [43] and for the steady Navier-Stokes equations in [45]. In [47] when a portion of boundary with Dirichlet boundary condition and another portion with slip condition are separated, existence of strong solution to the steady Stokes equations is studied. In [48] when a portion with homogeneous Dirichlet boundary condition and other portion with nonlinear boundary condition are separated, for the steady Stokes equations a relation between a regularized problem and the original problem, regularity of solution are studied.

In [45] for the steady Navier-Stokes equations, existence, uniqueness and continuous dependence on the data are studied when a portion of boundary with Dirichlet boundary condition and another moving portion where nonlinear slip occurs are separated. In [46] local unique existence of solution to the steady Navier-Stokes problem with homogeneous Dirichlet boundary condition and one of friction boundary conditions is studied. In [3] existence and uniqueness of solution to the steady rotating Navier-Stokes equations are studied when boundary consists of a portion with homogeneous Dirichlet boundary condition and other portions where there is flow and threshold slip. In [40] under similar boundary condition the steady Navier-Stokes problem is studied.

In [4] existence of weak solution and local existence of a strong solution to the non-steady Navier-Stokes problem are studied when boundary consists of a portion with homogeneous Dirichlet boundary condition and another portion with leak condition. In [30] existence of a strong solution to the non-steady Navier-Stokes equation is studied when boundary consists of a portion with homogeneous Dirichlet boundary condition and another portion with nonlinear slip or leak condition.

For other kinds of non-steady fluid equations with friction slip boundary conditions and Dirichlet condition, refer to [9], [10], [11] and [15]. Numerical solution methods are studied for the Stokes and Navier-Stokes problems with friction boundary conditions. For the 2-D steady Stokes problems refer to [5], [28], [38], [39], [41] and for the 3-D steady Stokes problems [29]. For the 2-D steady Navier-Stokes problem refer to [2], [35], [36] and [37]. For the 2-D non-steady Navier-Stokes problem refer to [34].

In practice we deal with mixture of some kinds of boundary conditions. Especially, when there is flux through a portion of boundary, we can deal with pressure boundary conditions. There are many papers dealing with the total pressure (Bernoulli's pressure) $\frac{1}{2}|v|^2 + p$ (cf. [13], [14]) or static pressure p (cf. [1], [44]). It is also known that the total stress $\sigma^t(v,p)$ on the boundary is a natural boundary condition, where $\sigma^t(v,p) = S^t n$, and total stress tensor S^t is one with components $S^t_{ij} = -(p + \frac{1}{2}|v|^2)\delta_{ij} + 2\nu\varepsilon_{ij}(v)$. (see [17], [18]).

Also, in practice we deal with one-sided leak of fluid. The condition (1.4) means that according to direction of normal stress, fluid penetrates out or into through boundary. If the fluid can only leak out through boundary when $-\sigma_n(v)$ is same with a threshold $g_{+n}(> 0)$, then we can describe that by

$$v_n \geq 0, \quad \sigma_n(v) + g_{+n} \geq 0, \quad (\sigma_n(v) + g_{+n})v_n = 0. \tag{1.5}$$

In contrast, if the fluid can only leak into through boundary when $-\sigma_n(v)$ is same with a threshold $-g_{-n}(g_{-n} > 0)$, then we can describe that by

$$v_n \leq 0, \quad \sigma_n(v) - g_{-n} \leq 0, \quad (\sigma_n(v) - g_{-n})v_n = 0. \tag{1.6}$$

For one-sided flow condition depending on a threshold of total pressure refer to [12]. For similar one-sided boundary conditions of elasticity refer to [31], Section 5.4.1, ch. 3 in [16].

In the present paper, we are concerned with the the systems (1.1) and (1.2) with mixed boundary conditions involving Tresca slip condition (1.3), leak boundary condition (1.4), one-sided leak boundary conditions (1.5) and (1.6), velocity, static pressure, rotation, stress and normal derivative of velocity together. And also without discussing whether static pressure or total pressure (correspondingly stress or total stress) is suitable for real phenomena which is over our knowledge, we consider the problems with total pressure and total stress instead of static pressure and stress. Relying on the result in [32], we reflect all these boundary conditions into variational formulations of problems. Overcoming difficulty from one-sided leak boundary conditions, we get variational inequalities equivalent to the variational formulation for the problems. We study some variational inequalities concerned with the Navier-Stokes problems. Using the results for the variational inequalities, we prove existence, uniqueness and estimates of weak solutions to the Navier-Stokes problems with such boundary conditions. Also using the previous results for elliptic variational inequality, we get some results for the Stokes problem with such boundary conditions.

This paper consists of 5 sections.

In Section 2, some previous results for variational formulation of our problems are stated. Also, three problems to study are described. For the Navier-Stokes equations, according to the pressure or the total pressure (correspondingly stress or the total stress) two problems are distinguished.

In Section 3, for the stationary Navier-Stokes and Stokes problems with mixture of 11 kinds of boundary conditions we have the variational formulations which consist of five formulae with five unknown functions, that is, using velocity, tangent stress on slip surface, normal stress on leak surface, normal stresses on one-sided leak surfaces together as unknown functions. Except friction type conditions, other boundary conditions are reflected in a variational equation as usual(Problems I-VE, II-VE, III-VE). When the solution smooth enough, these variational formulations are equivalent to the original PDE problems(Theorems 3.1, 3.4). Then, we get variational inequalities equivalent to the variational formulations above, which have one unknown function-velocity(Theorems 3.3, 3.5). In proof of equivalence, to overcome difficulties from the one-sided leak conditions Lemma 3.2 is used.

In Section 4 we study 3 kinds of variational inequality which are for the problems in Section 3. With an exception [46] studying local unique existence, in all previous papers dealing with friction boundary conditions one approximate the functionals in the considering variational inequalities with smooth one resulting to study of operator equation and it's convergence. Owing to the one-sided leak conditions such approximation for our problem may be complicated. Without such approximation we first get existence, uniqueness and estimates of solutions to the variational inequalities(Theorems 4.1, 4.2). In addition, for a special case excluding flux through boundary we also show approximation way of the functional(Theorem 4.3).

In Section 5, relying the results in Section 4, we study existence, uniqueness and estimates of solutions to the Navie-Stokes problems with 11 kinds of boundary condition. For the Navier-Stokes problem with boundary condition (2.7), which is including static pressure and stress, local unique existence is proved(Theorem 5.1). For the Navier-Stokes problem with boundary condition (2.8), which is including total static pressure and total stress, existence and estimate of solutions are proved(Theorem 5.2). For a special case of the Navier-Stokes problem with boundary condition (2.7) in which there is no any flux across boundary except Γ_1, Γ_8, existence and estimate of solutions are proved(Theorem 5.3, 5.5). Also, relying the previous results in elliptic variational inequality, we study unique existence, an estimate and continuous dependence on data of solutions to the Stokes problem with the boundary condition (2.7)(Theorem 5.6).

Throughout this paper we will use the following notation.

Let Ω be a connected bounded open subset of R^l, $l = 2, 3$. $\partial\Omega \in C^{0,1}$, $\partial\Omega = \cup_{i=1}^{11}\overline{\Gamma}_i, \Gamma_i \cap \Gamma_j = \varnothing$ for $i \neq j$, $\Gamma_i = \bigcup_j \Gamma_{ij}$, where Γ_{ij} are connected open subsets of $\partial\Omega$ and $\Gamma_{ij} \in C^2$ for $i = 2, 3, 7$ and $\Gamma_{ij} \in C^1$ for others. When X is a Banach space, $\mathbf{X} = X^l$. Let $W_\alpha^k(\Omega)$ be Sobolev spaces, $H^1(\Omega) = W_2^1(\Omega)$, and so $\mathbf{H}^1(\Omega) = \{H^1(\Omega)\}^l$. Let 0_X be the zero element of space X and $\mathscr{O}_M(0_X)$ be M-neighborhood of 0_X in space X. Compact continuous imbedding of a space X into a space Y is denoted by $X \hookrightarrow\hookrightarrow Y$.

An inner product and norm in the space $\mathbf{L}_2(\Omega)$ are, respectively, denoted by $(\cdot\,,\cdot)$ and $\|\cdot\|$; and $\langle\cdot\,,\cdot\rangle$ means the duality pairing between a Sobolev space X and its dual one. Also, $(\cdot\,,\cdot)_{\Gamma_i}$ is an inner product in the $\mathbf{L}_2(\Gamma_i)$ or $L_2(\Gamma_i)$; and $\langle\cdot\,,\cdot\rangle_{\Gamma_i}$ means the duality pairing between $\mathbf{H}^{\frac{1}{2}}(\Gamma_i)$ and $\mathbf{H}^{-\frac{1}{2}}(\Gamma_i)$ or between $H^{\frac{1}{2}}(\Gamma_i)$ and $H^{-\frac{1}{2}}(\Gamma_i)$. The inner product and norms in R^l, respectively, are denoted by $(\cdot\,,\cdot)_{R^l}$ and $|\cdot|$. Sometimes the inner product between a and b in R^l is denoted by $a \cdot b$. For convenience, in the case that $l = 2$, $y = (y_1(x_1, x_2),\, y_2(x_1, x_2))$ is identified with $\bar{y} = (y_1,\, y_2,\, 0)$, and so $\operatorname{rot} y = \operatorname{rot} \bar{y}$. Thus, for $y = (y_1,\, y_2)$ and $v = (v_1,\, v_2)$, $\operatorname{rot} y \times v$ is the 2-D vector consisted of the first two components of $\operatorname{rot} \bar{y} \times \bar{v}$.

Let $n(x)$ and $\tau(x)$ be, respectively, outward normal and tangent unit vectors at x in $\partial\Omega$. When for $u \in H^1(\Omega)$ such that $u_\tau = 0$ on Γ_i, sometimes for convenience we use notation $u|_{\Gamma_i}$ instead $u_n|_{\Gamma_i}$. If when $f \in H^{-1/2}(\Gamma_i)$, $\langle f, w\rangle_{\Gamma_i} \geq 0$ (≤ 0) $\forall w \in C_0^\infty(\Gamma_i)$ with $w \geq 0$, then we denoted by $f \geq 0$ (≤ 0).

2. Preliminary and problems. Let Γ be a surface (curve for $l=2$) of C^2 and v be a vector field of C^2 on a domain of R^l near Γ. In this paper the surfaces concerned by us are pieces of boundary of 3-D or 2-D bounded connected domains, and so we can assume the surfaces are oriented.

THEOREM 2.1. *(Theorem 2.1 in [32]) Suppose that $v \cdot n|_\Gamma = 0$. Then, on the surface Γ the following holds.*

$$(\varepsilon(v)n, \tau)_{R^l} = \frac{1}{2}(rot\, v \times n, \tau)_{R^l} - (S\tilde{v}, \tilde{\tau})_{R^{l-1}}, \tag{2.1}$$

$$(rot\, v \times n, \tau)_{R^l} = \left(\frac{\partial v}{\partial n}, \tau\right)_{R^l} + (S\tilde{v}, \tilde{\tau})_{R^{l-1}}, \tag{2.2}$$

$$(\varepsilon(v)n, \tau)_{R^l} = \frac{1}{2}\left(\frac{\partial v}{\partial n}, \tau\right)_{R^l} - \frac{1}{2}(S\tilde{v}, \tilde{\tau})_{R^{l-1}}, \tag{2.3}$$

where $\varepsilon(v)$ denotes the matrix with the components $\varepsilon_{ij}(v)$, S is the shape operator of the surface Γ (the matrix (A.1) in [32]) for $l=3$ and the curvature of Γ for $l=2$, and $\tilde{v}, \tilde{\tau}$ are expressions of the vectors v, τ in a local curvilinear coordinates on Γ.

REMARK 2.1. *Assuming Γ be a surface of C^2, let us introduce a local curvilinear coordinate system on Γ which is orthogonal at all points each other. Then, the shape operator S is expressed by the following matrix*

$$S = \begin{pmatrix} L & K \\ M & N \end{pmatrix},$$

where

$$L = \left(e_1, \frac{\partial n}{\partial e_1}\right)_{R^l}, \quad K = \left(e_2, \frac{\partial n}{\partial e_1}\right)_{R^l}, \quad M = \left(e_1, \frac{\partial n}{\partial e_2}\right)_{R^l}, \quad N = \left(e_2, \frac{\partial n}{\partial e_2}\right)_{R^l}$$

and $e_i, i = 1, 2$, are unit vector of the local coordinate system. The bilinear form $(S\tilde{v}, \tilde{u})_{R^{l-1}}$ for vector u, v tangent to the surface is independent from curvilinear coordinate system which is orthogonal at all points each other(cf. Appendix in [32]).

THEOREM 2.2. *(Theorem 2.2 in [32]) On the surface Γ the following holds.*

$$(\varepsilon(v)n, n)_{R^l} = \left(\frac{\partial v}{\partial n}, n\right)_{R^l}. \tag{2.4}$$

If $v \cdot \tau|_\Gamma = 0$, then

$$(\varepsilon(v)n, n)_{R^l} = \left(\frac{\partial v}{\partial n}, n\right)_{R^l} = -(k(x)v, n)_{R^l} - div_\Gamma\, v_\tau + div\, v, \tag{2.5}$$

where $k(x) = div\, n(x)$, v_τ is the tangential component of v and div_Γ is the divergence of a tangential vector field in the tangential coordinate system on Γ.

DEFINITION 2.1. *(Definition A.2 in [32]) If a piece of boundary on a neighborhood of $x \in \partial\Omega$ is on the opposite (same) side of the outward normal vector with respect* *to tangent plane (line for l=2) at x or coincides with the tangent plane, then piece of the boundary called convex (concave) at x. If at all $x \in \Gamma \subset \partial\Omega$ the boundary convex (concave), then Γ called convex (concave).*

LEMMA 2.3. *(Lemma A.3 in [32]) If Γ_{ij} are convex (concave), then quadratic forms $(S\tilde{v}, \tilde{v})|_{\Gamma_i}$ and $(k(x)v, v)_{\Gamma_i}$ are positive (negative).*

DEFINITION 2.2. *A functional $f : X \to \overline{R} \equiv R \cup +\infty$ is said to be proper if it is not identically equal to ∞. If $f(x) \in (-\infty, +\infty)$ $\forall x \in X$, then it is said to be finite.*

We are concerned the problems I and II for the Navier-Stokes equations

$$-\nu\Delta v + (v \cdot \nabla)v + \nabla p = f, \;\; \nabla \cdot v = 0 \quad \text{in } \Omega, \tag{2.6}$$

which are distinguished according to boundary conditions. Problem I is one with the boundary conditions

$$\begin{aligned}
&(1) \quad v|_{\Gamma_1} = h_1,\\
&(2) \quad v_\tau|_{\Gamma_2} = 0, \; -p|_{\Gamma_2} = \phi_2,\\
&(3) \quad v_n|_{\Gamma_3} = 0, \; \operatorname{rot} v \times n|_{\Gamma_3} = \phi_3/\nu,\\
&(4) \quad v_\tau|_{\Gamma_4} = h_4, \; (-p + 2\nu\varepsilon_{nn}(v))|_{\Gamma_4} = \phi_4,\\
&(5) \quad v_n|_{\Gamma_5} = h_5, \; 2(\nu\varepsilon_{n\tau}(v) + \alpha v_\tau)|_{\Gamma_5} = \phi_5, \;\; \alpha : \text{a matrix},\\
&(6) \quad (-pn + 2\nu\varepsilon_n(v))|_{\Gamma_6} = \phi_6,\\
&(7) \quad v_\tau|_{\Gamma_7} = 0, (-p + \nu\frac{\partial v}{\partial n} \cdot n)|_{\Gamma_7} = \phi_7,\\
&(8) \quad v_n|_{\Gamma_8} = h_8, |\sigma_\tau(v)| \le g_\tau, \sigma_\tau(v) \cdot v_\tau + g_\tau|v_\tau| = 0 \text{ on } \Gamma_8,\\
&(9) \quad v_\tau|_{\Gamma_9} = h_9, |\sigma_n(v)| \le g_n, \sigma_n(v)v_n + g_n|v_n| = 0 \text{ on } \Gamma_9,\\
&(10) \quad v_\tau = 0, \; v_n \ge 0, \quad \sigma_n(v) + g_{+n} \ge 0, \quad (\sigma_n(v) + g_{+n})v_n = 0 \text{ on } \Gamma_{10},\\
&(11) \quad v_\tau = 0, v_n \le 0, \quad \sigma_n(v) - g_{-n} \le 0, \quad (\sigma_n(v) - g_{-n})v_n = 0 \text{ on } \Gamma_{11},
\end{aligned} \tag{2.7}$$

and Problem II is one with the conditions

$$\begin{aligned}
&(1) \quad v|_{\Gamma_1} = h_1,\\
&(2) \quad v_\tau|_{\Gamma_2} = 0, \; -(p + \frac{1}{2}|v|^2)|_{\Gamma_2} = \phi_2,\\
&(3) \quad v_n|_{\Gamma_3} = 0, \; \operatorname{rot} v \times n|_{\Gamma_3} = \phi_3/\nu,\\
&(4) \quad v_\tau|_{\Gamma_4} = h_4, \; (-p - \frac{1}{2}|v|^2 + 2\nu\varepsilon_{nn}(v))|_{\Gamma_4} = \phi_4,\\
&(5) \quad v_n|_{\Gamma_5} = h_5, \; 2(\nu\varepsilon_{n\tau}(v) + \alpha v_\tau)|_{\Gamma_5} = \phi_5, \;\; \alpha : \text{a matrix},\\
&(6) \quad (-pn - \frac{1}{2}|v|^2 n + 2\nu\varepsilon_n(v))|_{\Gamma_6} = \phi_6,\\
&(7) \quad v_\tau|_{\Gamma_7} = 0, (-p - \frac{1}{2}|v|^2 + \nu\frac{\partial v}{\partial n} \cdot n)|_{\Gamma_7} = \phi_7,\\
&(8) \quad v_n|_{\Gamma_8} = h_8, |\sigma^t_\tau(v)| \le g_\tau, \sigma^t_\tau(v) \cdot v_\tau + g_\tau|v_\tau| = 0 \text{ on } \Gamma_8,\\
&(9) \quad v_\tau|_{\Gamma_9} = h_9, |\sigma^t_n(v)| \le g_n, \sigma^t_n(v)v_n + g_n|v_n| = 0 \text{ on } \Gamma_9,\\
&(10) \quad v_\tau = 0, \; v_n \ge 0, \quad \sigma^t_n(v) + g_{+n} \ge 0, \quad (\sigma^t_n(v) + g_{+n})v_n = 0 \text{ on } \Gamma_{10},\\
&(11) \quad v_\tau = 0, v_n \le 0, \quad \sigma^t_n(v) - g_{-n} \le 0, \quad (\sigma^t_n(v) - g_{-n})v_n = 0 \text{ on } \Gamma_{11},
\end{aligned} \tag{2.8}$$

where $\varepsilon_n(v) = \varepsilon(v)n$, $\varepsilon_{nn}(v) = (\varepsilon(v)n, n)_{R^l}$, $\varepsilon_{n\tau}(v) = \varepsilon(v)n - \varepsilon_{nn}(v)n$ and h_i, ϕ_i, α_{ij}(components of matrix α) are given functions or vectors of functions. And σ_n^t is the normal component of total stress on surface, that is, $\sigma_n^t = \sigma^t \cdot n$. Also, $\sigma_\tau^t(v,p) = \sigma^t(v,p) - \sigma_n^t(v,p)n$ and $g_\tau \in L^2(\Gamma_8)$, $g_n \in L^2(\Gamma_9)$, $g_{+n} \in L^2(\Gamma_{10})$, $g_{-n} \in L^2(\Gamma_{11})$, $g_\tau > 0$, $g_n > 0$, $g_{+n} > 0$, $g_{-n} > 0$, at a.e.

For Problem II the static pressure p and stress in the boundary conditions for Problem I are changed with the total pressure and the total stress. Note

$$\sigma_\tau(v,p) = \sigma_\tau^t(v,p) = 2\nu\varepsilon_{n\tau}(v).$$

We also consider the Stokes equations

$$-\nu\Delta v + \nabla p = f,\ \nabla \cdot v = 0 \quad \text{in } \Omega \tag{2.9}$$

with the boundary conditions (2.7), which is Problem III.

3. Variational formulations and equivalent variational inequalities. In this section we give variational formulations for Problems I, II, III above and get variational inequalities equivalent to the formulations.

Let

$$\begin{aligned}\mathbf{V}(\Omega) &= \{u \in \mathbf{H}^1(\Omega) : \operatorname{div} u = 0,\ u|_{\Gamma_1} = 0,\ u_\tau|_{(\Gamma_2\cup\Gamma_4\cup\Gamma_7\cup\Gamma_9\cup\Gamma_{10}\cup\Gamma_{11})} = 0,\\ &\qquad u_n|_{(\Gamma_3\cup\Gamma_5\cup\Gamma_8)} = 0\},\\ \mathbf{V}_{\Gamma 237}(\Omega) &= \{u \in \mathbf{H}^1(\Omega) : \operatorname{div} u = 0,\ u_\tau|_{(\Gamma_2\cup\Gamma_7)} = 0,\ u_n|_{\Gamma_3} = 0\},\end{aligned}$$

and

$$K(\Omega) = \{u \in \mathbf{V}(\Omega) : u_n|_{\Gamma_{10}} \ge 0,\ u_n|_{\Gamma_{11}} \le 0\}.$$

By Theorem 2.1 and 2.2 for $v \in \mathbf{H}^2(\Omega) \cap \mathbf{V}_{\Gamma 237}(\Omega)$ and $u \in \mathbf{V}(\Omega)$

$$\begin{aligned}-(\Delta v, u) &= 2(\varepsilon(v), \varepsilon(u)) - 2(\varepsilon(v)n, u)_{\cup_{i=2}^{11}\Gamma_i}\\ &= 2(\varepsilon(v), \varepsilon(u)) + 2(k(x)v, u)_{\Gamma_2} - (\operatorname{rot} v \times n, u)_{\Gamma_3} + 2(S\tilde{v}, \tilde{u})_{\Gamma_3}\\ &\quad - 2(\varepsilon_{nn}(v), u_n)_{\Gamma_4} - 2(\varepsilon_{n\tau}(v), u)_{\Gamma_5} - 2(\varepsilon_n(v), u)_{\Gamma_6} - \left(\frac{\partial v}{\partial n}, u\right)_{\Gamma_7}\\ &\quad + (k(x)v, u)_{\Gamma_7} - 2(\varepsilon_{n\tau}(v), u)_{\Gamma_8} - 2(\varepsilon_{nn}(v), u_n)_{\Gamma_9}\\ &\quad - 2(\varepsilon_{nn}(v), u_n)_{\Gamma_{10}} - 2(\varepsilon_{nn}(v), u_n)_{\Gamma_{11}}.\end{aligned} \tag{3.1}$$

Also, for $p \in H^1(\Omega)$ and $u \in \mathbf{V}(\Omega)$ we have

$$(\nabla p, u) = (p, u_n)_{\cup_{i=2}^{11}\Gamma_i} = (p, u_n)_{\Gamma_2} + (p, u_n)_{\Gamma_4\cup\Gamma_7\cup\Gamma_9\cup\Gamma_{10}\cup\Gamma_{11}} + (pn, u)_{\Gamma_6}, \tag{3.2}$$

where $u_n\ |_{\Gamma_3\cup\Gamma_5\cup\Gamma_8} = 0$ was used.

We assume that the following holds.

Assumption 3.1. *1) There exists a function $U \in \mathbf{H}^1(\Omega)$ such that*

$$\begin{aligned}&div\, U = 0, U|_{\Gamma_1} = h_1, U_\tau|_{(\Gamma_2\cup\Gamma_7)} = 0, U_n|_{\Gamma_3} = 0, U_\tau|_{\Gamma_4} = h_4,\\ &U_n|_{\Gamma_5} = h_5, U|_{\Gamma_8} = h_8 n,\ U|_{\Gamma_9} = h_9,\ U|_{\Gamma_{10}} = 0,\ U|_{\Gamma_{11}} = 0.\end{aligned}$$

2) $f \in \mathbf{V}(\Omega)^*$, $\phi_i \in H^{-\frac{1}{2}}(\Gamma_i), i = 2,4,7$, $\phi_i \in \mathbf{H}^{-\frac{1}{2}}(\Gamma_i), i = 3,5,6$, $\alpha_{ij} \in L_\infty(\Gamma_5)$, *and* $\Gamma_1 \neq \varnothing$.

3) If Γ_i, *where* i *is 10 or 11, is nonempty, then at least one of* $\{\Gamma_j : j \in \{2,4,7,9-11\}\backslash i\}$ *is nonempty and there exist a diffeomorphisms in* C^1 *between* Γ_i *and* Γ_j.

Having in mind Assumption 3.1 and putting $v = w + U$, by (3.1), (3.2) we can see that smooth solutions v of problem (2.6), (2.7) satisfy the following.

$$\begin{cases} v - U = w \in K(\Omega), \\ 2\nu(\varepsilon(w), \varepsilon(u)) + \langle (w\cdot\nabla)w, u\rangle + \langle (U\cdot\nabla)w, u\rangle + \langle (w\cdot\nabla)U, u\rangle \\ \qquad + 2\nu(k(x)w,u)_{\Gamma_2} + 2\nu(S\tilde{w},\tilde{u})_{\Gamma_3} + 2(\alpha(x)w,u)_{\Gamma_5} + \nu(k(x)w,u)_{\Gamma_7} \\ \qquad - 2(\nu\varepsilon_{n\tau}(w+U),u)_{\Gamma_8} + (p - 2\nu\varepsilon_{nn}(w+U), u_n)_{\Gamma_9\cup\Gamma_{10}\cup\Gamma_{11}} \\ \quad = -2\nu(\varepsilon(U),\varepsilon(u)) - \langle (U\cdot\nabla)U,u\rangle - 2\nu(k(x)U,u)_{\Gamma_2} - 2\nu(S\tilde{U},\tilde{u})_{\Gamma_3} \\ \qquad - 2(\alpha(x)U,u)_{\Gamma_5} - \nu(k(x)U,u)_{\Gamma_7} + \langle f,u\rangle + \sum_{i=2,4,7}\langle \phi_i, u_n\rangle_{\Gamma_i} \\ \qquad + \sum_{i=3,5,6}\langle \phi_i, u\rangle_{\Gamma_i} \quad \forall u \in \mathbf{V}(\Omega), \\ |\sigma_\tau(v)| \le g_\tau,\ \sigma_\tau(v)\cdot v_\tau + g_\tau|v_\tau| = 0 \quad \text{on } \Gamma_8, \\ |\sigma_n(v)| \le g_n,\ \sigma_n(v)v_n + g_n|v_n| = 0 \quad \text{on } \Gamma_9, \\ \sigma_n(v) + g_{+n} \ge 0, \quad (\sigma_n(v) + g_{+n})v_n = 0 \quad \text{on } \Gamma_{10}, \\ \sigma_n(v) - g_{-n} \le 0, \quad (\sigma_n(v) - g_{-n})v_n = 0 \quad \text{on } \Gamma_{11}. \end{cases} \tag{3.3}$$

Define $a_{01}(\cdot,\cdot), a_{11}(\cdot,\cdot,\cdot)$ and $F_1 \in V^*$ by

$$\begin{aligned} a_{01}(w,u) &= 2\nu(\varepsilon(w),\varepsilon(u)) + \langle (U\cdot\nabla)w,u\rangle + \langle (w\cdot\nabla)U,u\rangle + 2\nu(k(x)w,u)_{\Gamma_2} \\ &\qquad + 2\nu(S\tilde{w},\tilde{u})_{\Gamma_3} + 2(\alpha(x)w,u)_{\Gamma_5} + \nu(k(x)w,u)_{\Gamma_7} \quad \forall w,u \in \mathbf{V}(\Omega), \\ a_{11}(w,u,v) &= \langle (w\cdot\nabla)u, v\rangle \quad \forall w,u,v \in \mathbf{V}(\Omega), \\ \langle F_1, u\rangle &= -2\nu(\varepsilon(U),\varepsilon(u)) - \langle (U\cdot\nabla)U,u\rangle - 2\nu(k(x)U,u)_{\Gamma_2} - 2\nu(S\tilde{U},\tilde{u})_{\Gamma_3} \\ &\qquad - 2(\alpha(x)U,u)_{\Gamma_5} - \nu(k(x)U,u)_{\Gamma_7} + \langle f,u\rangle + \sum_{i=2,4,7}\langle \phi_i,u_n\rangle_{\Gamma_i} \\ &\qquad + \sum_{i=3,5,6}\langle \phi_i,u\rangle_{\Gamma_i} \quad \forall u \in \mathbf{V}(\Omega). \end{aligned} \tag{3.4}$$

Then, taking into account

$$\sigma_\tau(v) = 2\nu\varepsilon_{n\tau}(v), \quad \sigma_n(v) = -p + 2\nu\varepsilon_{nn}(v)$$

and (3.3), we introduce the following variational formulation for problem (2.6), (2.7).

PROBLEM I-VE. Find $(v, \sigma_\tau, \sigma_n, \sigma_{+n}, \sigma_{-n}) \in \big(U + K(\Omega)\big) \times \mathbf{L}^2_\tau(\Gamma_8) \times L^2(\Gamma_9) \times$

$H^{-1/2}(\Gamma_{10}) \times H^{-1/2}(\Gamma_{11})$ such that

$$\begin{cases} v - U = w \in K(\Omega), \\ a_{01}(w,u) + a_{11}(w,w,u) - (\sigma_\tau, u_\tau)_{\Gamma_8} - (\sigma_n, u_n)_{\Gamma_9} \\ \qquad - \langle \sigma_{+n}, u_n \rangle_{\Gamma_{10}} - \langle \sigma_{-n}, u_n \rangle_{\Gamma_{11}} = \langle F_1, u \rangle \quad \forall u \in \mathbf{V}(\Omega), \\ |\sigma_\tau| \le g_\tau,\ \sigma_\tau \cdot v_\tau + g_\tau |v_\tau| = 0 \quad \text{on } \Gamma_8, \\ |\sigma_n| \le g_n,\ \sigma_n v_n + g_n |v_n| = 0 \quad \text{on } \Gamma_9, \\ \sigma_{+n} + g_{+n} \ge 0, \quad \langle \sigma_{+n} + g_{+n}, v_n \rangle_{\Gamma_{10}} = 0 \quad \text{on } \Gamma_{10}, \\ \sigma_{-n} - g_{-n} \le 0, \quad \langle \sigma_{-n} - g_{-n}, v_n \rangle_{\Gamma_{11}} = 0 \quad \text{on } \Gamma_{11}, \end{cases} \tag{3.5}$$

where $\mathbf{L}^2_\tau(\Gamma_8)$ is the subspace of $\mathbf{L}^2(\Gamma_8)$ consisted of functions such that $(u,n)_{\mathbf{L}^2(\Gamma_8)} = 0$.

REMARK 3.1. *If $u \in H^1(\Omega)$, then $v|_{\Gamma_i} \in H^{\frac{1}{2}}(\Gamma_i)$, however if $u|_{\partial\Omega} = 0$ on $O(\Gamma_i)\backslash\overline{\Gamma}_i$, where $O(\Gamma_i)$ is an open subset of $\partial\Omega$ such that $\overline{\Gamma}_i \subset O(\Gamma_i)$, then $u|_{\Gamma_i} \in H^{\frac{1}{2}}_{00}(\Gamma_i)$ (cf. (c) of Theorem 1.5.2.3 in [26]). Since $H^{\frac{1}{2}}_{00}(\Gamma_i) \hookrightarrow H^{\frac{1}{2}}_0(\Gamma_i)$ and $H^{\frac{1}{2}}(\Gamma_i) = H^{\frac{1}{2}}_0(\Gamma_i)$, (cf. Theorems 11.7 and 11.1 of ch. 1 in [42])*

$$H^{\frac{1}{2}}_{00}(\Gamma_i) \hookrightarrow H^{\frac{1}{2}}(\Gamma_i) \hookrightarrow H^{-\frac{1}{2}}(\Gamma_i) \hookrightarrow (H^{\frac{1}{2}}_{00}(\Gamma_i))'.$$

Thus, under condition $u|_{\partial\Omega} = 0$ on $O(\Gamma_i)\backslash\overline{\Gamma}_i$, for $\phi_i \in (H^{\frac{1}{2}}_{00}(\Gamma_i))'$ a dual product $\langle \phi_i, u \rangle_{\Gamma_i}$ has meaning. But, without knowing that $u|_{\partial\Omega} = 0$ on $O(\Gamma_i)\backslash\overline{\Gamma}_i$, for $\phi_i \in H^{-\frac{1}{2}}(\Gamma_i)$ the dual product $\langle \phi_i, u \rangle_{\Gamma_i}$ has meaning. Therefore, under 2) of Assumption 3.1 the dual products on Γ_i in (3.3) *have meaning.*

THEOREM 3.1. *Assume 1), 2) of Assumption 3.1. If a solution smooth enough ($v \in \boldsymbol{H}^2(\Omega), f \in \boldsymbol{L}^2(\Omega)$), then Problem I-VE is equivalent to problem* (2.6), (2.7). *In addition, if among $\Gamma_i, i = 2,4,6,7,9-11$, at least one is nonempty, then p of problem* (2.6), (2.7) *is unique.*

Proof. It is enough to prove conversion from Problem I-VE to problem (2.6), (2.7).

Let v is a solution smooth enough to Problem I-VE. From (3.4), (3.5) we have

$$\begin{aligned} &2\nu(\varepsilon(v), \varepsilon(u)) + \langle (v \cdot \nabla)v, u \rangle + 2\nu(k(x)v, u)_{\Gamma_2} + 2\nu(S\tilde{v}, \tilde{u})_{\Gamma_3} \\ &\quad + 2(\alpha(x)v, u)_{\Gamma_5} + \nu(k(x)v, u)_{\Gamma_7} - (\sigma_\tau, u_\tau)_{\Gamma_8} - (\sigma_n, u_n)_{\Gamma_9} \\ &\quad - \langle \sigma_{+n}, u_n \rangle_{\Gamma_{10}} - \langle \sigma_{-n}, u_n \rangle_{\Gamma_{11}} - \sum_{i=2,4,7} \langle \phi_i, u_n \rangle_{\Gamma_i} - \sum_{i=3,5,6} \langle \phi_i, u \rangle_{\Gamma_i} \\ &= \langle f, u \rangle \quad \forall u \in \mathbf{V}(\Omega). \end{aligned} \tag{3.6}$$

From (3.1) we get

$$\begin{aligned} &2\nu(\varepsilon(v), \varepsilon(u)) \\ &= -\nu(\Delta v, u) - 2\nu(k(x)v, u)_{\Gamma_2} + \nu(\operatorname{rot} v \times n, u)_{\Gamma_3} - 2\nu(S\tilde{v}, \tilde{u})_{\Gamma_3} \\ &+ 2\nu(\varepsilon_{nn}(v), u \cdot n)_{\Gamma_4} + 2\nu(\varepsilon_{n\tau}(v), u)_{\Gamma_5} + 2\nu(\varepsilon_n(v), u)_{\Gamma_6} + \nu\left(\frac{\partial v}{\partial n}, u\right)_{\Gamma_7} \\ &- \nu(k(x)v, u)_{\Gamma_7} + 2\nu(\varepsilon_{n\tau}(v), u)_{\Gamma_8} + 2\nu(\varepsilon_{nn}(v), u)_{\Gamma_9} \\ &+ 2\nu(\varepsilon_{nn}(v), u)_{\Gamma_{10}} + 2\nu(\varepsilon_{nn}(v), u)_{\Gamma_{11}}. \end{aligned} \tag{3.7}$$

From (3.6), (3.7) we have

$$\begin{aligned}
&(-\nu\Delta v+(v\cdot\nabla)v-f,u)+\nu(\operatorname{rot} v\times n,u)_{\Gamma_3}+2\nu(\varepsilon_{nn}(v),u\cdot n)_{\Gamma_4}\\
&+2\nu(\varepsilon_{n\tau}(v),u)_{\Gamma_5}+2(\alpha(x)v,u)_{\Gamma_5}+2\nu(\varepsilon_n(v),u)_{\Gamma_6}+\nu\left(\frac{\partial v}{\partial n},u\right)_{\Gamma_7}\\
&+2\nu(\varepsilon_{n\tau}(v),u)_{\Gamma_8}+2\nu(\varepsilon_{nn}(v),u_n)_{\Gamma_9}+2\nu(\varepsilon_{nn}(v),u_n)_{\Gamma_{10}}+2\nu(\varepsilon_{nn}(v),u_n)_{\Gamma_{11}}\\
&-(\sigma_\tau,u_\tau)_{\Gamma_8}-(\sigma_n,u_n)_{\Gamma_9}-\langle\sigma_{+n},u_n\rangle_{\Gamma_{10}}-\langle\sigma_{-n},u_n\rangle_{\Gamma_{11}}\\
&-\sum_{i=2,4,7}\langle\phi_i,u_n\rangle_{\Gamma_i}-\sum_{i=3,5,6}\langle\phi_i,u\rangle_{\Gamma_i}=0
\end{aligned}\tag{3.8}$$

Taking any $u\in C_0^\infty$ with $div\,u=0$, we have

$$(-\nu\Delta v+(v\cdot\nabla)v-f,u)=0,$$

which implies existence of a unique $P\in H^1(\Omega)$ such that $\int_\Omega P\,dx=0$ and

$$-\nu\Delta v+(v\cdot\nabla)v-f=-\nabla P. \tag{3.9}$$

(cf. Proposition 1.1, ch. 1 of [49]).

Substituting (3.9) into (3.8), integrating by parts and taking into account (3.2), we have

$$\begin{aligned}
&(-P-\phi_2,u_n)_{\Gamma_2}+\nu(\operatorname{rot} v\times n-\phi_3/\nu,u)_{\Gamma_3}+(-P+2\nu\varepsilon_{nn}(v)-\phi_4,u_n)_{\Gamma_4}\\
&+(2\nu\varepsilon_{n\tau}(v)+\alpha(x)v_\tau-\phi_5,u)_{\Gamma_5}+(-Pn+2\nu\varepsilon_n(v)-\phi_6,u)_{\Gamma_6}\\
&+\Big(-P+\nu\frac{\partial v}{\partial n}\cdot n-\phi_7,u_n\Big)_{\Gamma_7}+(2\nu\varepsilon_{n\tau}(v)-\sigma_\tau,u)_{\Gamma_8}\\
&+(-P+2\nu\varepsilon_{nn}(v)-\sigma_n,u_n)_{\Gamma_9}+(-P+2\nu\varepsilon_{nn}(v)-\sigma_{+n},u_n)_{\Gamma_{10}}\\
&+(-P+2\nu\varepsilon_{nn}(v)-\sigma_{-n},u_n)_{\Gamma_{11}}=0,
\end{aligned}\tag{3.10}$$

where $(v,u)_{\Gamma_5}=(v_\tau,u)_{\Gamma_5}$ and $(\nu\frac{\partial v}{\partial n},u)_{\Gamma_7}=(\nu\frac{\partial v}{\partial n}\cdot n,u_n)_{\Gamma_7}$ were used.

Taking any $u\in\mathbf{V}$ such that $u_n|_{\partial\Omega}=0$, $u|_{\partial\Omega}=0$ on $\partial\Omega\setminus\Gamma_i$, respectively, for $i=3,5,8$, from (3.10) we get

$$\begin{aligned}
&\operatorname{rot} v\times n=\phi_3/\nu \quad \text{on } \Gamma_3,\\
&2\nu\varepsilon_{n\tau}(v)+\alpha(x)v_\tau-\phi_5=0 \quad \text{on } \Gamma_5,\\
&2\nu\varepsilon_{n\tau}(v)-\sigma_\tau=0 \quad \text{on } \Gamma_8.
\end{aligned}\tag{3.11}$$

If for all $i=2,4,6,7,9-11$, $\Gamma_i=\emptyset$, then putting $p=P+c$, where c is any constant, we get a solution (v,p) to problem (2.6), (2.7).

Assume that among $\Gamma_i, i=2,4,6,7,9-11$, at least one is nonempty. Taking any $u\in\mathbf{V}$ such that $u_\tau|_{\partial\Omega}=0, u|_{\partial\Omega}=0$ on $\partial\Omega\setminus\Gamma_i$, respectively, for $i=2,4,7,9-11$, from (3.10) we have that for some constants c_i, respectively,

$$\begin{aligned}
&-P-\phi_2=c_2 \quad \text{on } \Gamma_2,\\
&-P+2\nu\varepsilon_{nn}(v)-\phi_4=c_4 \quad \text{on } \Gamma_4,\\
&-P+\nu\frac{\partial v}{\partial n}\cdot n-\phi_7=c_7 \quad \text{on } \Gamma_7,\\
&-P+2\nu\varepsilon_{nn}(v)-\sigma_n=c_9 \quad \text{on } \Gamma_9,\\
&-P+2\nu\varepsilon_{nn}(v)-\sigma_{+n}=c_{10} \quad \text{on } \Gamma_{10},\\
&-P+2\nu\varepsilon_{nn}(v)-\sigma_{-n}=c_{11} \quad \text{on } \Gamma_{11}.
\end{aligned}\tag{3.12}$$

Taking any $u \in \mathbf{V}$ such that $u|_{\partial\Omega} = 0$ on $\partial\Omega \setminus \Gamma_6$, from (3.10) we have that for a constant c_6

$$-Pn + 2\nu\varepsilon_n(v) - \phi_6 = c_6 n \quad \text{on } \Gamma_6.$$

Let us prove that all c_i are equal to one constant c. For example, assume that Γ_2 and Γ_4 are nonempty. Taking any $u \in \mathbf{V}$ such that $u|_{\partial\Omega} = 0$ on $\partial\Omega \setminus (\Gamma_2 \cup \Gamma_4)$, from (3.10) we get

$$c_2 \int_{\Gamma_2} u_n \, dx + c_4 \int_{\Gamma_4} u_n \, dx = 0,$$

which implies $c_2 = c_4 = c$ since $\int_{\Gamma_2} u_n \, dx = -\int_{\Gamma_4} u_n \, dx$. Thus, from (3.9), (3.12), we know that uniquely determined $p = P + c$ satisfies

$$-\nu\Delta v + (v \cdot \nabla) + \nabla p = f, \tag{3.13}$$

and

$$\begin{aligned}
&-p = \phi_2 \quad \text{on } \Gamma_2,\\
&-p + 2\nu\varepsilon_{nn}(v) = \phi_4 \quad \text{on } \Gamma_4,\\
&-pn + 2\nu\varepsilon_n(v) = \phi_6 \quad \text{on } \Gamma_6,\\
&-p + \nu\frac{\partial v}{\partial n} \cdot n = \phi_7 \quad \text{on } \Gamma_7,\\
&-p + 2\nu\varepsilon_{nn}(v) = \sigma_n \quad \text{on } \Gamma_9,\\
&-p + 2\nu\varepsilon_{nn}(v) = \sigma_{+n} \quad \text{on } \Gamma_{10},\\
&-p + 2\nu\varepsilon_{nn}(v) = \sigma_{-n} \quad \text{on } \Gamma_{11}
\end{aligned} \tag{3.14}$$

together. By virtue of (3.5), (3.11), (3.14), all conditions in (2.7) are satisfied. Therefore, (v, p) is a solution to problem (2.6), (2.7). □

We will find a variational inequality equivalent to Problem I-VE.

Let $(v, \sigma_\tau, \sigma_n, \sigma_{+n}, \sigma_{-n})$ be a solution of Problem I-VE. From the second formula of (3.5) subtracting the formula putted $u = w$ in the second formula of (3.5), we get

$$\begin{aligned}
&a_{01}(w, u - w) + a_{11}(w, w, u - w) - (\sigma_\tau, u_\tau - w_\tau)_{\Gamma_8} - (\sigma_n, u_n - w_n)_{\Gamma_9}\\
&- \langle\sigma_{+n}, u_n - w_n\rangle_{\Gamma_{10}} - \langle\sigma_{-n}, u_n - w_n\rangle_{\Gamma_{11}} = \langle F_1, u - w\rangle \quad \forall u \in \mathbf{V}(\Omega).
\end{aligned} \tag{3.15}$$

Define the functionals j_τ, j_n, j_+, j_-, respectively, by

$$\begin{aligned}
j_\tau(\eta) &= \int_{\Gamma_8} g_\tau|\eta| \, dx \quad \forall \eta \in \mathbf{L}^2_\tau(\Gamma_8),\\
j_n(\eta) &= \int_{\Gamma_9} g_n|\eta| \, dx \quad \forall \eta \in L^2(\Gamma_9),\\
j_+(\eta) &= \int_{\Gamma_{10}} g_{+n}\eta \, dx \quad \forall \eta \in L^2(\Gamma_{10}),\\
j_-(\eta) &= -\int_{\Gamma_{11}} g_{-n}\eta \, dx \quad \forall \eta \in L^2(\Gamma_{11}).
\end{aligned} \tag{3.16}$$

Since if $u \in K(\Omega)$, then $u|_{\Gamma_8} \in \mathbf{L}^2_\tau(\Gamma_8)$, $u_n|_{\Gamma_9} \in L^2(\Gamma_9)$, $u_n|_{\Gamma_{10}} \in L^2(\Gamma_{10})$, $u_n|_{\Gamma_{11}} \in L^2(\Gamma_{11})$, in what follows for convenience we use the notation

$$j_\tau(u) = j_\tau(u|_{\Gamma_8}),\ j_n(u) = j_n(u_n|_{\Gamma_9}),\ j_+(u) = j_+(u_n|_{\Gamma_{10}}),\ j_-(u) = j_-(u_n|_{\Gamma_{11}}) \quad \forall u \in K(\Omega).$$

Define a functional $J(v) \in (\mathbf{V}(\Omega) \to \overline{R})$ by

$$J(u) = \begin{cases} j_\tau(u) + j_n(u) + j_+(u) + j_-(u) & \forall u \in K(\Omega), \\ +\infty & \forall u \notin K(\Omega). \end{cases} \tag{3.17}$$

Then, J is proper convex lower semi-continuous.

By Assumption 3.1, $w_\tau = v_\tau$ on Γ_8 and $w_n = v_n$ on $\Gamma_9 \sim \Gamma_{11}$. Taking into account the fact that $g_\tau |v_\tau| + \sigma_\tau \cdot v_\tau = 0$, $|\sigma_\tau| \le g_\tau$, we have that

$$\begin{aligned} & j_\tau(u) - j_\tau(w) + (\sigma_\tau, u_\tau)_{\Gamma_8} - (\sigma_\tau, w_\tau)_{\Gamma_8} \\ & \quad = \int_{\Gamma_8} (g_\tau |u_\tau| + \sigma_\tau \cdot u_\tau)\, dx - \int_{\Gamma_8} (g_\tau |w_\tau| + \sigma_\tau \cdot w_\tau)\, dx \\ & \quad = \int_{\Gamma_8} (g_\tau |u_\tau| + \sigma_\tau \cdot u_\tau)\, dx - \int_{\Gamma_8} (g_\tau |v_\tau| + \sigma_\tau \cdot v_\tau)\, dx \ge 0 \quad \forall u \in K(\Omega). \end{aligned} \tag{3.18}$$

Taking into account the fact that $g_n |v_n| + \sigma_n \cdot v_n = 0$ and $|\sigma_n| \le g_n$, in the same way we have

$$j_n(u) - j_n(w) + (\sigma_n, u_n)_{\Gamma_9} - (\sigma_n, w_n)_{\Gamma_9} \ge 0. \tag{3.19}$$

Also,

$$\begin{aligned} & j_+(u) - j_+(w) + \langle \sigma_{+n}, u_n \rangle_{\Gamma_{10}} - \langle \sigma_{+n}, w_n \rangle_{\Gamma_{10}} \\ & = \langle g_{+n} + \sigma_{+n}, u_n \rangle_{\Gamma_{10}} - \langle g_{+n} + \sigma_{+n}, w_n \rangle_{\Gamma_{10}} \ge 0, \end{aligned} \tag{3.20}$$

where the facts that $u_n \ge 0$, $\sigma_{+n} + g_{+n} \ge 0$ and $\langle \sigma_{+n} + g_{+n}, v_n \rangle_{\Gamma_{10}} = 0$, $w_n = v_n$ on Γ_{10} were used. In the same way, we have

$$j_-(u) - j_-(w) + \langle \sigma_{-n}, u_n \rangle_{\Gamma_{11}} - \langle \sigma_{-n}, w_n \rangle_{\Gamma_{11}} \ge 0. \tag{3.21}$$

By virtue of (3.17)-(3.21), we have

$$\begin{aligned} J(u) - J(w) \ge & -(\sigma_\tau, u_\tau - w_\tau)_{\Gamma_8} - (\sigma_n, u_n - w_n)_{\Gamma_9} \\ & - \langle \sigma_{+n}, u_n - w_n \rangle_{\Gamma_{10}} - \langle \sigma_{-n}, u_n - w_n \rangle_{\Gamma_{11}} \quad \forall u \in \mathbf{V}. \end{aligned} \tag{3.22}$$

Therefore, from (3.15) and (3.22) we get

$$\begin{aligned} & a_{01}(w, u - w) + a_{11}(w, w, u - w) + J(u) - J(w) \\ & \ge \langle F_1, u - w \rangle \quad \forall u \in \mathbf{V}(\Omega). \end{aligned} \tag{3.23}$$

Thus, we come to the following formulation associated with Problem I by a variational inequality.

Problem I-VI. Find $v = w + U$ such that

$$\begin{aligned} & a_{01}(w, u - w) + a_{11}(w, w, u - w) + J(u) - J(w) \\ & \ge \langle F_1, u - w \rangle \quad \forall u \in \mathbf{V}(\Omega), \end{aligned} \tag{3.24}$$

where a_{01}, a_{11}, F_1 are in (3.4), U is in Assumption 3.1 and J is in (3.17).

To prove equivalence of Problem I-VI and Problem I-VE we need

LEMMA 3.2. *For $\psi \in C_0^\infty(\Gamma_i), i = 10, 11$, there exists a function $\overline{u} \in \boldsymbol{V}$ such that*

$$\overline{u}_n|_{\Gamma_i} = \psi, \quad \|\overline{u}\|_{\boldsymbol{V}} \leq C_i \|\psi\|_{H^{1/2}(\Gamma_i)},$$

where C_i are independent of ψ.

Proof. By 3) of Assumption 3.1 if $\Gamma_{10} \cup \Gamma_{11} \neq \varnothing$, then, for example, $\Gamma_2 \neq \varnothing$ and there exists a diffeomorphysm $y = f_i(x) \in C^1$ from Γ_i onto Γ_2. Define $\varphi(y)$ at point $y \in \Gamma_2$ corresponding to point $x \in \Gamma_i$ by $\varphi(y) = \frac{1}{Df(x)}\psi(f_i^{-1}(y))$, where $Df_i(x)$ is Jacobian of the transformation f_i. Then,

$$\int_{\Gamma_2} \varphi(y)\, dy = \int_{\Gamma_i} \frac{1}{Df_i(x)} \psi(f_i^{-1}(y)) Df_i(x)\, dx = \int_{\Gamma_i} \psi(x)\, dx, \tag{3.25}$$

and

$$\|\varphi(y)\|_{H^{\frac{1}{2}}(\Gamma_2)} \leq \left\| \frac{1}{Df_i(x)} \right\|_{C(\overline{\Gamma_i})} \|\psi(x)\|_{H^{\frac{1}{2}}(\Gamma_i)} \leq c_i \|\psi(x)\|_{H^{\frac{1}{2}}(\Gamma_i)}. \tag{3.26}$$

When $\psi \in C_0^\infty(\Gamma_i)$, define a function $\overline{\phi} \in \mathbf{H}^{1/2}(\partial\Omega)$ on $\partial\Omega$ as follows.

$$\overline{\phi} \times n|_{\Gamma_2 \cup \Gamma_i} = 0,\ \overline{\phi}_n|_{\Gamma_2} = -\varphi,\ \overline{\phi}_n|_{\Gamma_{10}} = \psi,\ \overline{\phi}|_{(\cup_{i=1,3-9,11}\Gamma_i)} = 0.$$

Thus, by (3.25) $\int_{\partial\Omega} \overline{\phi}_n\, ds = 0$. Then, there exists a solution $\overline{u} \in \mathbf{W}^{1,2}(\Omega)$ to the Stokes problem

$$\begin{cases} -\Delta u + \nabla p = 0, \\ \operatorname{div} u = 0, \\ u|_{\partial\Omega} = \overline{\phi} \end{cases}$$

and

$$\|\overline{u}\|_{\mathbf{V}(\Omega)} \leq c \|\overline{\phi}\|_{\mathbf{H}^{1/2}(\partial\Omega)}.$$

(cf. Theorem IV.1.1 in [24]). Taking into account (3.26), we come to the asserted estimation with $C_i = 1 + c_i$. Thus $\overline{u}$ is the asserted function. □

Problem I-VE and Problem I-VI are equivalent in the following sense.

THEOREM 3.3. *If $(v, \sigma_\tau, \sigma_n, \sigma_{+n}, \sigma_{-n})$ is a solution to Problem I-VE, then v is a solution to Problem I-VI. Inversely, if v is a solution to Problem I-VI, then there exist $\sigma_\tau, \sigma_n, \sigma_{+n}, \sigma_{-n}$ such that $(v, \sigma_\tau, \sigma_n, \sigma_{+n}, \sigma_{-n})$ is a solution to Problem I-VE.*

Proof. We already showed that if $(v, \sigma_\tau, \sigma_n, \sigma_{+n}, \sigma_{-n})$ is a solution to Problem I-VE, then v is a solution to Problem I-VI. Thus, it is enough to prove that if v is a solution to Problem I-VI, then there exist $\sigma_\tau, \sigma_n, \sigma_{+n}, \sigma_{-n}$ such that $(v, \sigma_\tau, \sigma_n, \sigma_{+n}, \sigma_{-n})$ is a solution to Problem I-VE.

Since the functional J is proper, from (3.24) we have

$$v - U = w \in K(\Omega) \tag{3.27}$$

because if $w \notin K(\Omega)$, then the left hand side of (3.24) is $-\infty$ which is a contradiction to the fact that the right hand side is finite.

Let $\psi \in \mathbf{V}_{8-11}(\Omega) \equiv \{u \in \mathbf{V}(\Omega) : u|_{\Gamma_8\cup\Gamma_9\cup\Gamma_{10}\cup\Gamma_{11}} = 0\} (\subset K(\Omega))$. Putting $u = w + \psi, u = w - \psi$ and taking into account

$$j_\tau(w) = j_\tau(w+\psi),\ j_n(w) = j_n(w+\psi),\ j_+(w) = j_+(w+\psi),\ j_-(w) = j_-(w+\psi),$$

from (3.17), (3.24) we get

$$\begin{aligned}&a_{01}(w,\psi) + a_{11}(w,w,\psi) \geq \langle F_1, \psi\rangle,\\ &a_{01}(w,-\psi) + a_{11}(w,w,-\psi) \geq \langle F_1, -\psi\rangle \quad \forall \psi \in \mathbf{V}_{8-11}(\Omega),\end{aligned}$$

which imply

$$a_{01}(w,\psi) + a_{11}(w,w,\psi) = \langle F_1, \psi\rangle \quad \forall \psi \in \mathbf{V}_{8-11}(\Omega). \tag{3.28}$$

When $u \in \mathbf{V}_{10-11}(\Omega) \equiv \{u \in \mathbf{V}(\Omega) : u|_{\Gamma_{10}\cup\Gamma_{11}} = 0\} (\subset K(\Omega))$, the set $\{(u|_{\Gamma_8}, u_n|_{\Gamma_9})\}$ is a subspace of $\mathbf{L}^2_\tau(\Gamma_8) \times L^2(\Gamma_9)$, where $u_n|_{\Gamma_9}$ is $u|_{\Gamma_9} \cdot n$.

Define a functional σ^* on the set by

$$\langle \sigma^*, (u|_{\Gamma_8}, u_n|_{\Gamma_9})\rangle = a_{01}(w,u) + a_{11}(w,w,u) - \langle F_1, u\rangle \quad \forall u \in \mathbf{V}_{10-11}(\Omega). \tag{3.29}$$

This functional is well defined. Because if $u, u_1 \in \mathbf{V}_{10-11}(\Omega)$ are such that $(u|_{\Gamma_8}, u|_{\Gamma_9}) = (u_1|_{\Gamma_8}, u_1|_{\Gamma_9})$, then since $u - u_1 \in \mathbf{V}_{8-11}(\Omega)$, by (3.28)

$$a_{01}(w, u-u_1) + a_{11}(w,w,u-u_1) - \langle F_1, u-u_1\rangle = 0,$$

that is,

$$a_{01}(w,u) + a_{11}(w,w,u) - \langle F_1, u\rangle = a_{01}(w,u_1) + a_{11}(w,w,u_1) - \langle F_1, u_1\rangle,$$

and so by (3.29)

$$\langle \sigma^*, (u|_{\Gamma_8}, u_n|_{\Gamma_9})\rangle = \langle \sigma^*, (u_1|_{\Gamma_8}, u_{1n}|_{\Gamma_9})\rangle .$$

This functional is linear.

Putting $u = w + \psi$, where $\psi \in \mathbf{V}_{10-11}(\Omega)$, and taking into account

$$j_+(w+\psi) = j_+(w), \quad j_-(w+\psi) = j_-(w),$$

from (3.29), (3.24) we have

$$\begin{aligned}-\langle \sigma^*, (\psi|_{\Gamma_8}, \psi_n|_{\Gamma_9})\rangle &= -\left[a_{01}(w,\psi) + a_{11}(w,w,\psi) - \langle F_1,\psi\rangle\right]\\ &\leq J(w+\psi) - J(w)\\ &= j_\tau(w+\psi) - j_\tau(w) + j_n(w+\psi) - j_n(w)\\ &\leq \int_{\Gamma_8} g_\tau|\psi|_{\Gamma_8}\,dx + \int_{\Gamma_9} g_n|\psi|_{\Gamma_9}\,dx \quad \forall \psi \in \mathbf{V}_{10-11}(\Omega).\end{aligned} \tag{3.30}$$

Putting $u = w - \psi$, in the same way we have

$$\begin{aligned}\langle \sigma^*, (\psi|_{\Gamma_8}, \psi_n|_{\Gamma_9})\rangle &= \left[a_{01}(w,\psi) + a_{11}(w,w,\psi) - \langle F_1,\psi\rangle\right]\\ &\leq j_\tau(w-\psi) - j_\tau(w) + j_n(w-\psi) - j_n(w)\\ &\leq \int_{\Gamma_8} g_\tau|\psi|_{\Gamma_8}\,dx + \int_{\Gamma_9} g_n|\psi|_{\Gamma_9}\,dx \quad \forall \psi \in \mathbf{V}_{10-11}(\Omega).\end{aligned} \tag{3.31}$$

By (3.30), (3.31), we can know that σ^* is a bounded linear functional with a norm not greater than 1 on a subspace of $\mathbf{L}^1_{g_\tau}(\Gamma_8) \times L^1_{g_n}(\Gamma_9)$, where $\mathbf{L}^1_{g_\tau}(\Gamma_8)$, $L^1_{g_n}(\Gamma_9)$ are, respectively, the spaces of functions integrable with weights g_τ, g_n on Γ_8 and Γ_9. By the Hahn-Banach theorem the functional is extended as a functional on $\mathbf{L}^1_{g_\tau}(\Gamma_8) \times L^1_{g_n}(\Gamma_9)$ norms of which is not greater than 1. Therefore, there exist the elements $\sigma_\tau \in \mathbf{L}^\infty_{\frac{1}{g_\tau}}(\Gamma_8)$, $\|\sigma_\tau\|_{\mathbf{L}^\infty_{\frac{1}{g_\tau}}(\Gamma_8)} \le 1$ and $\sigma_n \in L^\infty_{\frac{1}{g_n}}(\Gamma_9)$, $\|\sigma_n\|_{L^\infty_{\frac{1}{g_n}}(\Gamma_9)} \le 1$, which imply

$$|\sigma_\tau| \le g_\tau, \quad |\sigma_n| \le g_n; \tag{3.32}$$

and

$$\langle \sigma^*, (u|_{\Gamma_8}, u_n|_{\Gamma_9})\rangle = (\sigma_\tau, u|_{\Gamma_8})_{\Gamma_8} + (\sigma_n, u_n|_{\Gamma_9})_{\Gamma_9} \quad \forall u \in \mathbf{V}_{10-11}(\Omega). \tag{3.33}$$

When $u \in \mathbf{V}(\Omega)$, the set $\{(u_n|_{\Gamma_{10}}, u_n|_{\Gamma_{11}})\}$ is a subspace of $H^{\frac{1}{2}}(\Gamma_{10}) \times H^{\frac{1}{2}}(\Gamma_{11})$. Define a functional σ_1^* on the set $\mathbf{V}(\Omega)$ by

$$\begin{aligned}\langle \sigma_1^*, (u_n|_{\Gamma_{10}}, u_n|_{\Gamma_{11}})\rangle &= a_{01}(w,u) + a_{11}(w,w,u) - (\sigma_\tau, u|_{\Gamma_8})_{\Gamma_8} \\ &\quad - (\sigma_n, u|_{\Gamma_9})_{\Gamma_9} - \langle F_1, u\rangle \quad \forall u \in \mathbf{V}(\Omega).\end{aligned} \tag{3.34}$$

This functional is also well defined. Because if $u, u^1 \in \mathbf{V}(\Omega)$ are such that $(u_{\Gamma_{10}}, u|_{\Gamma_{11}}) = (u^1|_{\Gamma_{10}}, u^1|_{\Gamma_{11}})$, then since $u - u^1 \in \mathbf{V}_{10-11}(\Omega)$, by (3.29), (3.33)

$$\begin{aligned}&a_{01}(w, u-u^1) + a_{11}(w,w,u-u^1) - \big(\sigma_\tau, (u-u^1)|_{\Gamma_8}\big)_{\Gamma_8} \\ &\qquad - \big(\sigma_n, (u-u^1)|_{\Gamma_9}\big)_{\Gamma_9} - \langle F_1, u-u^1\rangle \\ &= \big\langle \sigma^*, ((u-u^1)|_{\Gamma_8}, (u-u^1)|_{\Gamma_9})\big\rangle - \big(\sigma_\tau, (u-u^1)|_{\Gamma_8}\big)_{\Gamma_8} - \big(\sigma_n, (u-u^1)|_{\Gamma_9}\big)_{\Gamma_9} = 0,\end{aligned}$$

and so by (3.34)

$$\langle \sigma_1^*, (u_n|_{\Gamma_{10}}, u_n|_{\Gamma_{11}})\rangle = \big\langle \sigma_1^*, (u_n^1|_{\Gamma_{10}}, u_n^1|_{\Gamma_{11}})\big\rangle .$$

The functional σ_1^* is linear. Let us prove continuity of this functional.

Let $\overline{u}$ is the function corresponding to $\psi \in C_0^\infty(\Gamma_{10})$ by Lemma 3.2. Then, by Lemma 3.2 from (3.34) we have

$$\begin{aligned}&|\langle \sigma_1^*, (\psi, 0)\rangle| \\ &\le C\left[\|w\|_{\mathbf{V}}\|\overline{u}\|_{\mathbf{V}} + \|w\|^2_{\mathbf{V}}\|\overline{u}\|_{\mathbf{V}} + (\|\sigma_\tau\|_{L_2(\Gamma_8)}\right. \\ &\qquad \left.+\|\sigma_\tau\|_{L_2(\Gamma_9)})\|\overline{u}\| + \|F_1\|_{\mathbf{V}^*}\|\overline{u}\|_{\mathbf{V}}\right] \\ &\le C\left[\|w\|_{\mathbf{V}}\| + \|w\|^2_{\mathbf{V}} + (\|\sigma_\tau\|_{L_2(\Gamma_8)} + \|\sigma_\tau\|_{L_2(\Gamma_9)}) + \|F_1\|_{\mathbf{V}^*}\right] \cdot \|\psi\|_{H^{\frac{1}{2}}(\Gamma_{10})}.\end{aligned} \tag{3.35}$$

Also assuming that $\overline{u}$ is the function corresponding to $\psi \in C_0^\infty(\Gamma_{11})$ by Lemma 3.2, we have

$$\begin{aligned}&|\langle \sigma_1^*, (0, \psi)\rangle| \\ &\le C\left[\|w\|_{\mathbf{V}}\|\overline{u}\|_{\mathbf{V}} + \|w\|^2_{\mathbf{V}}\|\overline{u}\|_{\mathbf{V}} + (\|\sigma_\tau\|_{L_2(\Gamma_8)}\right. \\ &\qquad \left.+\|\sigma_\tau\|_{L_2(\Gamma_9)})\|\overline{u}\| + \|F_1\|_{\mathbf{V}^*}\|\overline{u}\|_{\mathbf{V}}\right] \\ &\le C\left[\|w\|_{\mathbf{V}} + \|w\|^2_{\mathbf{V}} + (\|\sigma_\tau\|_{L_2(\Gamma_8)} + \|\sigma_\tau\|_{L_2(\Gamma_9)}) + \|F_1\|_{\mathbf{V}^*}\right] \cdot \|\psi\|_{H^{\frac{1}{2}}(\Gamma_{11})}.\end{aligned} \tag{3.36}$$

Since $H_0^{1/2}(\Gamma_i) = H^{\frac{1}{2}}(\Gamma_i), i = 10, 11$, (cf. Theorem 11.1 in [42]), (3.35) and (3.36) show that the functional σ_1^* is continuous on the subspace of $H^{\frac{1}{2}}(\Gamma_{10}) \times H^{\frac{1}{2}}(\Gamma_{11})$ mentioned above. Thus, by the Hahn-Banach theorem the functional is extended as a functional on $H^{\frac{1}{2}}(\Gamma_{10}) \times H^{\frac{1}{2}}(\Gamma_{11})$.

Therefore, there exists an element $(\sigma_{+n}, \sigma_{-n}) \in H^{-1/2}(\Gamma_{10}) \times H^{-1/2}(\Gamma_{11})$ such that

$$\langle \sigma_1^*, (u|_{\Gamma_{10}}, u|_{\Gamma_{11}}) \rangle = \langle \sigma_{+n}, u|_{\Gamma_{10}} \rangle_{\Gamma_{10}} + \langle \sigma_{-n}, u|_{\Gamma_{11}} \rangle_{\Gamma_{11}} \quad \forall u \in \mathbf{V}(\Omega). \tag{3.37}$$

When $\psi \geq 0$ is such that $\psi \in C_0^\infty(\Gamma_{10})$, let $\overline{u} \in K(\Omega)$ be the function asserted in Lemma 3.2. Putting $u = w + \overline{u}$, by (3.24) we have

$$a_{01}(w, \overline{u}) + a_{11}(w, w, \overline{u}) + J(w + \overline{u}) - J(w) - \langle F_1, \overline{u} \rangle \geq 0. \tag{3.38}$$

On the other hand, by (3.34), (3.37) and property of $\overline{u}$,

$$a_{01}(w, \overline{u}) + a_{11}(w, w, \overline{u}) - \langle F_1, \overline{u} \rangle = \langle \sigma_{+n}, \psi \rangle_{\Gamma_{10}}$$

and so from (3.38) we have that

$$\langle \sigma_{+n}, \psi \rangle_{\Gamma_{10}} + J(w + \overline{u}) - J(w) \geq 0. \tag{3.39}$$

By (3.16), (3.17) and property of $\overline{u}$,

$$J(w + \overline{u}) - J(w) = \langle g_{+n}, \psi \rangle_{\Gamma_{10}},$$

and combining with (3.39) we have

$$\langle \sigma_{+n}, \psi \rangle_{\Gamma_{10}} + (g_{+n}, \psi)_{\Gamma_{10}} \geq 0,$$

that is,

$$\sigma_{+n} + g_{+n} \geq 0. \tag{3.40}$$

When $\psi \leq 0$ is such that $\psi \in C_0^\infty(\Gamma_{11})$, let $\overline{u} \in K(\Omega)$ be the function asserted in Lemma 3.2. Then, in the same way we have that

$$\langle \sigma_{-n}, -\psi \rangle_{\Gamma_{11}} - (g_{-n}, -\psi)_{\Gamma_{11}} \geq 0,$$

that is,

$$\sigma_{-n} - g_{-n} \leq 0. \tag{3.41}$$

From (3.34), (3.37), we have

$$\begin{aligned} a_{01}(w, u) + a_{11}(w, w, u) - (\sigma_\tau, u_\tau)_{\Gamma_8} - (\sigma_n, u)_{\Gamma_9} \\ - \langle \sigma_{+n}, u \rangle_{\Gamma_{10}} - \langle \sigma_{-n}, u \rangle_{\Gamma_{11}} = \langle F_1, u \rangle \quad \forall u \in \mathbf{V}(\Omega). \end{aligned} \tag{3.42}$$

Putting $u = 0$ in (3.24) and taking into account (3.42) with $u = w$, we have

$$\begin{aligned} (\sigma_\tau, w)_{\Gamma_8} + (\sigma_n, w)_{\Gamma_9} + \langle \sigma_{+n}, w_n \rangle_{\Gamma_{10}} + \langle \sigma_{-n}, w_n \rangle_{\Gamma_{11}} \\ + j_\tau(w) + j_n(w) + j_+(w) + j_-(w) \leq 0, \end{aligned}$$

that is,

$$\begin{aligned}\int_{\Gamma_8}(\sigma_\tau w_\tau + g_\tau|w_\tau|)\,ds + \int_{\Gamma_9}(\sigma_n w_n + g_n|w_n|)\,ds\\ + \langle \sigma_{+n} + g_{+n}, w_n\rangle_{\Gamma_{10}} + \langle \sigma_{-n} - g_{-n}, w_n\rangle_{\Gamma_{11}} \le 0.\end{aligned}\tag{3.43}$$

Since on Γ_8, Γ_9, Γ_{10} and Γ_{11}, respectively, $w_\tau = v_\tau$, $w_n = v_n$, $w_n = v_n \ge 0$ and $w_n = v_n \le 0$, taking into account (3.32), (3.40),(3.41), by (3.43) we have

$$\begin{aligned}&\sigma_\tau v_\tau + g_\tau|v_\tau| = 0, \quad \sigma_n v_n + g_n|v_n| = 0,\\ &\langle \sigma_{+n} + g_{+n}, v_n\rangle = 0, \quad \langle \sigma_{-n} - g_{-n}, v_n\rangle = 0.\end{aligned}\tag{3.44}$$

Therefore, by virtue of (3.27), (3.32), (3.40)-(3.42), (3.44), we come to the conclusion. □

Taking $(v\cdot\nabla)v = \operatorname{rot} v \times v + \frac{1}{2}\operatorname{grad}|v|^2$ into account and putting $v = w + U$, by (3.1), (3.2) and Assumption 3.1 we can see that smooth solutions v of problem (2.6), (2.8) satisfy the following.

$$\begin{cases} v - U = w \in K(\Omega),\\ 2\nu(\varepsilon(w), \varepsilon(u)) + \langle \operatorname{rot} w \times w, u\rangle + \langle \operatorname{rot} U \times w, u\rangle + \langle \operatorname{rot} w \times U, u\rangle\\ \qquad + 2\nu(k(x)w, u)_{\Gamma_2} + 2\nu(S\tilde{w}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)w, u)_{\Gamma_5} + \nu(k(x)w, u)_{\Gamma_7}\\ \qquad - 2(\nu\varepsilon_{n\tau}(w+U), u)_{\Gamma_8} + \big(p + \frac{1}{2}|v|^2 - 2\nu\varepsilon_{nn}(w+U), u_n\big)_{\Gamma_9\cup\Gamma_{10}\cup\Gamma_{11}}\\ \quad = -2\nu(\varepsilon(U), \varepsilon(u)) - \langle \operatorname{rot} U \times U, u\rangle - 2\nu(k(x)U, u)_{\Gamma_2} - 2\nu(S\tilde{U}, \tilde{u})_{\Gamma_3}\\ \qquad - 2(\alpha(x)U, u)_{\Gamma_5} - \nu(k(x)U, u)_{\Gamma_7} + \langle f, u\rangle\\ \qquad + \sum_{i=2,4,7}\langle \phi_i, u_n\rangle_{\Gamma_i} + \sum_{i=3,5,6}\langle \phi_i, u\rangle_{\Gamma_i} \qquad \forall u \in \mathbf{V}(\Omega),\\ |\sigma^t_\tau(v)| \le g_\tau,\ \sigma^t_\tau \cdot v_\tau + g_\tau|v_\tau| = 0 \quad \text{on } \Gamma_8,\\ |\sigma^t_n(v)| \le g_n,\ \sigma^t_n(v)v_n + g_n|v_n| = 0 \quad \text{on } \Gamma_9,\\ \sigma^t_n(v) + g_{+n} \ge 0, \quad (\sigma^t_n(v) + g_{+n})v_n = 0 \quad \text{on } \Gamma_{10},\\ \sigma^t_n(v) - g_{-n} \le 0, \quad (\sigma^t_n(v) - g_{-n})v_n = 0 \quad \text{on } \Gamma_{11}.\end{cases}\tag{3.45}$$

Define $a_{02}(\cdot,\cdot), a_{12}(\cdot,\cdot,\cdot)$ and $F_2 \in V^*$ by

$$\begin{aligned} a_{02}(w,u) &= 2\nu(\varepsilon(w), \varepsilon(u)) + \langle \operatorname{rot} U \times w, u\rangle + \langle \operatorname{rot} w \times U, u\rangle + 2\nu(k(x)w, u)_{\Gamma_2}\\ &\qquad + 2\nu(S\tilde{w}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)w, u)_{\Gamma_5} + \nu(k(x)w, u)_{\Gamma_7} \quad \forall w, u \in \mathbf{V}(\Omega),\\ a_{12}(w,u,v) &= \langle \operatorname{rot} w \times u, v\rangle \quad \forall w, u, v \in \mathbf{V}(\Omega),\\ \langle F_2, u\rangle &= -2\nu(\varepsilon(U), \varepsilon(u)) - \langle \operatorname{rot} U \times U, u\rangle - 2\nu(k(x)U, u)_{\Gamma_2} - 2\nu(S\tilde{U}, \tilde{u})_{\Gamma_3}\\ &\qquad - 2(\alpha(x)U, u)_{\Gamma_5} - \nu(k(x)U, u)_{\Gamma_7} + \langle f, u\rangle + \sum_{i=2,4,7}\langle \phi_i, u_n\rangle_{\Gamma_i}\\ &\qquad + \sum_{i=3,5,6}\langle \phi_i, u\rangle_{\Gamma_i} \quad \forall u \in \mathbf{V}(\Omega).\end{aligned}\tag{3.46}$$

Then, taking into account

$$\sigma^t_\tau(v) = 2\nu\varepsilon_{n\tau}(v), \quad \sigma^t_n(v) = -(p + \frac{1}{2}|v|^2) + 2\nu\varepsilon_{nn}(v)$$

and (3.45), we introduce the following variational formulation for problem (2.6), (2.8).

PROBLEM II-VE. Find $(v, \sigma_\tau^t, \sigma_n^t, \sigma_{+n}^t, \sigma_{-n}^t) \in \big(U + K(\Omega)\big) \times \mathbf{L}_\tau^2(\Gamma_8) \times L^2(\Gamma_9) \times H^{-\frac{1}{2}}(\Gamma_{10}) \times H^{-\frac{1}{2}}(\Gamma_{11})$ such that

$$\begin{cases} v - U = w \in K(\Omega), \\ a_{02}(w,u) + a_{12}(w,w,u) - (\sigma_\tau^t, u_\tau)_{\Gamma_8} - (\sigma_n^t, u_n)_{\Gamma_9} \\ \qquad\qquad - \left\langle \sigma_{+n}^t, u_n \right\rangle_{\Gamma_{10}} - \left\langle \sigma_{-n}^t, u_n \right\rangle_{\Gamma_{11}} = \langle F_2, u\rangle \quad \forall u \in \mathbf{V}(\Omega), \\ |\sigma_\tau^t| \le g_\tau,\ \sigma_\tau^t \cdot v_\tau + g_\tau |v_\tau| = 0 \quad \text{on } \Gamma_8, \\ |\sigma_n^t| \le g_n,\ \sigma_n^t v_n + g_n|v_n| = 0 \quad \text{on } \Gamma_9, \\ \sigma_{+n}^t + g_{+n} \ge 0, \quad \left\langle \sigma_{+n}^t + g_{+n}, v_n \right\rangle_{\Gamma_{10}} = 0 \quad \text{on } \Gamma_{10}, \\ \sigma_{-n}^t - g_{-n} \le 0, \quad \left\langle \sigma_{-n}^t - g_{-n}, v_n \right\rangle_{\Gamma_{11}} = 0 \quad \text{on } \Gamma_{11}. \end{cases} \tag{3.47}$$

In the same way as Theorem 3.1 we have

THEOREM 3.4. *Assume 1), 2) of Assumption 3.1. If a solution smooth enough* $(v \in \boldsymbol{H}^2(\Omega), f \in \boldsymbol{L}^2(\Omega))$*, then Problem II-VE is equivalent to problem* (2.6), (2.8). *In addition, if among* $\Gamma_i, i = 2, 4, 6, 7, 9-11,$ *at least one is nonempty, then* p *of problem* (2.6), (2.7) *is unique.*

Then, in the same way as Problem I we get Problem II-VI formulated by a variational inequality and can prove that the problem is equivalent to Problem II-VE.

PROBLEM II-VI. Find $v = w + U$ such that

$$\begin{aligned} &a_{02}(w, u-w) + a_{12}(w,w,u-w) + J(u) - J(w) \\ &\ge \langle F_2, u - w\rangle \quad \forall u \in \mathbf{v}(\Omega), \end{aligned} \tag{3.48}$$

where a_{02}, a_{12}, F_2 are in (3.46) and J is defined by (3.16), (3.17).

THEOREM 3.5. *If* $(v, \sigma_\tau^t, \sigma_n^t, \sigma_{+n}^t, \sigma_{-n}^t)$ *is a solution to Problem II-VE, then* v *is a solution to Problem II-VI. Inversely, if* v *is a solution to Problem II-VI, then there exist* $\sigma_\tau^t, \sigma_n^t, \sigma_{+n}^t, \sigma_{-n}^t$ *such that* $(v, \sigma_\tau^t, \sigma_n^t, \sigma_{+n}^t, \sigma_{-n}^t)$ *is a solution to Problem I-VE.*

REMARK 3.2. *Boundary condition* $\nu\frac{\partial v}{\partial n} - pn = 0$ *often called "do nothing" or "free outflow" boundary condition, results from variational principle and does not have a real physical meaning but is rather used in truncating large physical domains to smaller computational domains by assuming parallel flow (cf. [8]). The condition (7) in* (2.7)*(corresponding (7) in* (2.8)*) is rather different from "do nothing" condition. Assuming that the flow is orthogonal on* Γ_7 *and applying Theorem 2.2, we get a variational formulation, and so to convert from the variational formulation to the original problem we use such a condition. (For more detail refer to Remark 2.1 in [33].) If the flow, in addition, is parallel in a near the boundary, then condition (7) in* (2.7) *is same with "do nothing" condition. In point of view of pure mathematics, to reflect correctly "do nothing" condition in variational formulation we can use other variational formulation assuming* $\Gamma_6 = \emptyset$*. Bellow we show that.*

Now, we consider the cases that $\Gamma_6 = \varnothing$ and for convenience $h_i = 0,\ i = 4, 5, 8, 9,$ in (2.7). Let $\mathbf{V}_{\Gamma 7}(\Omega) = \{u \in \mathbf{H}^1(\Omega) : \operatorname{div} u = 0, u|_{\Gamma_1} =$

$0,\ u_\tau|_{(\Gamma_2\cup\Gamma_4\cup\Gamma_9\cup\Gamma_{10}\cup\Gamma_{11})} = 0,\ u\cdot n|_{(\Gamma_3\cup\Gamma_5\cup\Gamma_8)} = 0\}$ and $\mathbf{V}_{\Gamma 17}(\Omega) = \{u \in \mathbf{H}^1(\Omega) : \operatorname{div} u = 0,\ u_\tau|_{(\Gamma_2\cup\Gamma_4\cup\Gamma_9\cup\Gamma_{10}\cup\Gamma_{11})} = 0,\ u\cdot n|_{(\Gamma_3\cup\Gamma_5\cup\Gamma_8)} = 0\}$. By Theorem 2.1 and 2.2 for $v \in \mathbf{H}^2(\Omega)\cap \mathbf{V}_{\Gamma 17}(\Omega)$ and $u \in \mathbf{V}_{\Gamma 7}(\Omega)$

$$
\begin{aligned}
-(\Delta v, u) =&(\nabla v, \nabla u) - \left(\frac{\partial v}{\partial n}, u\right)_{\cup_{i=2}^{11}\Gamma_i}\\
=&(\nabla v, \nabla u) + (k(x)v, u)_{\Gamma_2} - (\operatorname{rot} v\times n, u)_{\Gamma_3} + (S\tilde v, \tilde u)_{\Gamma_3} - (\varepsilon_{nn}(v), u\cdot n)_{\Gamma_4}\\
&- 2(\varepsilon_{n\tau}(v), u)_{\Gamma_5} - (S\tilde v, \tilde u)|_{\Gamma_5} - \left(\frac{\partial v}{\partial n}, u\right)_{\Gamma_7} - 2(\varepsilon_{n\tau}(w), u)_{\Gamma_8}\\
&- (S\tilde v, \tilde u)_{\Gamma_8} - (\varepsilon_{nn}(v), u_n)_{\Gamma_9\cup\Gamma_{10}\cup\Gamma_{11}}.
\end{aligned}
\tag{3.49}
$$

Using (3.49), (3.2) we get a variational formulation for problem (2.6), (2.7) with $(-p\cdot n + \nu\frac{\partial v}{\partial n})|_{\Gamma_7} = \phi_7 \in \mathbf{H}^{-\frac12}(\Gamma_7)$ instead of the condition (7) of (2.7):

Problem I'-VE. Find $(v, \sigma_\tau, \sigma_n, \sigma_{+n}, \sigma_{-n}) \in \big(U + K(\Omega)\big)\times \mathbf{L}^2_\tau(\Gamma_8)\times L^2(\Gamma_9)\times H^{-\frac12}(\Gamma_{10})\times H^{-\frac12}(\Gamma_{11})$ such that

$$
\begin{cases}
v|_{\Gamma_1} = h_1,\\
\nu(\nabla v, \nabla u) + (v\cdot\nabla)v + \nu(k(x)v, u)_{\Gamma_2} + \nu(S\tilde v, \tilde u)_{\Gamma_3} + 2(\alpha(x)v, u)_{\Gamma_5} - \nu(S\tilde v, \tilde u)_{\Gamma_5}\\
\qquad - \nu(S\tilde v, \tilde u)_{\Gamma_8} - (\sigma_\tau, u_\tau)_{\Gamma_8} - (\sigma_n, u_n)_{\Gamma_9} - \langle\sigma_{+n}, u_n\rangle_{\Gamma_{10}} - \langle\sigma_{-n}, u_n\rangle_{\Gamma_{11}}\\
\quad = \langle f, u\rangle + \sum_{i=2,4}\langle\phi_i, u_n\rangle_{\Gamma_i} + \sum_{i=3,5,7}\langle\phi_i, u\rangle_{\Gamma_i} \quad \forall u \in \mathbf{V}(\Omega),\\
|\sigma_\tau| \le g_\tau,\ \sigma_\tau\cdot v_\tau + g_\tau|v_\tau| = 0 \quad \text{on } \Gamma_8,\\
|\sigma_n| \le g_n,\ \sigma_n v_n + g_n|v_n| = 0 \quad \text{on } \Gamma_9,\\
\sigma_{+n} + g_{+n} \ge 0, \quad \langle\sigma_{+n} + g_{+n}, v_n\rangle_{\Gamma_{10}} = 0 \quad \text{on } \Gamma_{10},\\
\sigma_{-n} - g_{-n} \le 0, \quad \langle\sigma_{-n} - g_{-n}, v_n\rangle_{\Gamma_{11}} = 0 \quad \text{on } \Gamma_{11}.
\end{cases}
\tag{3.50}
$$

In the same way as Problem I we get the below equivalent formulations of Problem III for the Stokes equation with boundary condition (2.7).

Problem III-VE. Find $(v, \sigma_\tau, \sigma_n, \sigma_{+n}, \sigma_{-n}) \in \big(U + K(\Omega)\big)\times \mathbf{L}^2_\tau(\Gamma_8)\times L^2(\Gamma_9)\times H^{-\frac12}(\Gamma_{10})\times H^{-\frac12}(\Gamma_{11})$ such that

$$
\begin{cases}
v - U = w \in K(\Omega),\\
a_{03}(w, u) - (\sigma_\tau, u_\tau)_{\Gamma_8} - (\sigma_n, u_n)_{\Gamma_9} - \langle\sigma_{+n}, u_n\rangle_{\Gamma_{10}} - \langle\sigma_{-n}, u_n\rangle_{\Gamma_{11}}\\
\qquad = \langle F_3, u\rangle \quad \forall u \in \mathbf{V}(\Omega),\\
|\sigma_\tau| \le g_\tau,\ \sigma_\tau\cdot v_\tau + g_\tau|v_\tau| = 0 \quad \text{on } \Gamma_8,\\
|\sigma_n| \le g_n,\ \sigma_n v_n + g_n|v_n| = 0 \quad \text{on } \Gamma_9,\\
\sigma_{+n} + g_{+n} \ge 0, \quad \langle\sigma_{+n} + g_{+n}, v_n\rangle = 0 \quad \text{on } \Gamma_{10},\\
\sigma_{-n} - g_{-n} \le 0, \quad \langle\sigma_{-n} - g_{-n}, v_n\rangle = 0 \quad \text{on } \Gamma_{11},
\end{cases}
\tag{3.51}
$$

where

$$
\begin{aligned}
a_{03}(w,u) &= 2\nu(\varepsilon(w),\varepsilon(u)) + 2\nu(k(x)w,u)_{\Gamma_2} \\
&\quad + 2\nu(S\tilde{w},\tilde{u})_{\Gamma_3} + 2(\alpha(x)w,u)_{\Gamma_5} + \nu(k(x)w,u)_{\Gamma_7} \quad \forall w,u \in \mathbf{V}(\Omega), \\
\langle F_3,u\rangle &= -2\nu(\varepsilon(U),\varepsilon(u)) - 2\nu(k(x)U,u)_{\Gamma_2} - 2\nu(S\tilde{U},\tilde{u})_{\Gamma_3} - 2(\alpha(x)U,u)_{\Gamma_5} \\
&\quad - \nu(k(x)U,u)_{\Gamma_7} + \langle f,u\rangle + \sum_{i=2,4,7}\langle\phi_i,u_n\rangle_{\Gamma_i} + \sum_{i=3,5,6}\langle\phi_i,u\rangle_{\Gamma_i} \quad \forall u \in \mathbf{V}(\Omega).
\end{aligned}
\tag{3.52}
$$

PROBLEM III-VI Find v such that

$$
\begin{aligned}
&v - U = w \in K(\Omega), \\
&a_{03}(w,u-w) + J(u) - J(w) \geq \langle F_3, u-w\rangle \quad \forall u \in \mathbf{V}(\Omega),
\end{aligned}
\tag{3.53}
$$

where the functionals J is defined by (3.16), (3.17).

4. Existence, uniqueness and estimates of solutions to variational inequalities. In this section we study some variational inequalities for the problems in Section 3.

THEOREM 4.1. *Let X, X_1 be real separable Hilbert spaces such that $X \hookrightarrow\hookrightarrow X_1$, and X^* be dual space of X. Assume the followings.*

1) $J \in (X \to [0,+\infty])$ is a proper lower semi-continuous convex functional such that $J(0_X) = 0$.

2) $a_0(\cdot,\cdot) \in (X \times X \to R)$ is a bilinear form such that

$$
\begin{aligned}
&|a_0(u,v)| \leq K\|u\|_X\|v\|_X \quad \forall u,v \in X, \\
&a_0(u,u) \geq \alpha\|u\|_X^2 \quad \exists \alpha > 0, \forall u \in X.
\end{aligned}
$$

3) $a_1(\cdot,\cdot,\cdot) \in (X_1 \times X \times X \to R)$ is a triple linear functional such that

$$
\begin{aligned}
&a_1(w,u,u) = 0 \quad \forall w \in X_1, \forall u \in X, \\
&|a_1(w,u,v)| \leq K\|w\|_{X_1}\|u\|_X\|v\|_X, \quad \forall w \in X_1, \forall u,v \in X.
\end{aligned}
$$

Then for $f \in X^*$ there exists a solution to the variational inequality

$$
a_0(v,u-v) + a_1(v,v,u-v) + J(u) - J(v) \geq \langle f,u-v\rangle \quad \forall u \in X \tag{4.1}
$$

and all solutions v satisfy the estimate

$$
\|v\|_X \leq \frac{1}{\alpha}\|f\|_{X^*}. \tag{4.2}
$$

In addition to, if

$$
\frac{Kc}{\alpha^2}\|f\|_{X^*} < 1, \tag{4.3}
$$

then solution is unique, where c is a constant in $\|\cdot\|_{X_1} \leq c\|\cdot\|_X$.

Proof. Fixing $w \in X_1$, let us consider a variational inequality

$$
a_0(v,u-v) + a_1(w,v,u-v) + J(u) - J(v) \geq \langle f,u-v\rangle \quad \forall u \in X, \tag{4.4}
$$

where $f \in X^*$. There exists a unique solution to (4.4) (cf. Theorem 10.5 in [6]). Let v_1, v_2 be the solutions corresponding to f_1, f_2 instead of f. Then, under consideration of condition 2) it is easy to verify that

$$\|v_1 - v_2\|_X \le \frac{1}{\alpha}\|f_1 - f_2\|_{X^*}. \tag{4.5}$$

Now, let us consider the operator which maps w to the solution v of (4.4)

$$T \in (X_1 \to X) : w \to T(w) = v.$$

Taking into account condition 1), we can easily verify that the solution corresponding to $f = 0_{X^*}$ is 0_X. Thus, from (4.5) we have

$$\|v\|_X \le \frac{1}{\alpha}\|f\|_{X^*} \quad \forall w \in X_1. \tag{4.6}$$

Note that this estimate is independent from w.

Denote by v_1 and v_2, respectively, the solutions to (4.4) corresponding to w_1 and w_2. Then

$$\begin{aligned} &a_0(v_1, u - v_1) + a_1(w_1, v_1, u - v_1) + J(u) - J(v_1) \ge \langle f, u - v_1\rangle \quad \forall u \in X, \\ &a_0(v_2, u - v_2) + a_1(w_2, v_2, u - v_2) + J(u) - J(v_2) \ge \langle f, u - v_2\rangle \quad \forall u \in X. \end{aligned} \tag{4.7}$$

Putting $u = v_2$ and $u = v_1$, respectively, in the first formula and the second one of (4.7), and adding two formulae, we get

$$a_0(v_1 - v_2, v_2 - v_1) + a_1(w_1, v_1, v_2 - v_1) + a_1(w_2, v_2, v_1 - v_2) \ge 0. \tag{4.8}$$

From (4.8), the conditions 2), 3) of Theorem and (4.6), we get

$$\begin{aligned} \|v_2 - v_1\|_X^2 &\le \frac{1}{\alpha}|a_1(w_1, v_1, v_2 - v_1) - a_1(w_2, v_1, v_2 - v_1) \\ &\qquad\qquad + a_1(w_2, v_1, v_2 - v_1) - a_1(w_2, v_2, v_2 - v_1)| \\ &\le \frac{1}{\alpha}|a_1(w_1 - w_2, v_1, v_2 - v_1)| + \frac{1}{\alpha}|a_1(w_2, v_2 - v_1, v_2 - v_1)| \\ &\le \frac{K}{\alpha}\|w_1 - w_2\|_{X_1}\|v_1\|_X\|v_2 - v_1\|_X \\ &\le \frac{K\|f\|_{X^*}}{\alpha^2}\|w_1 - w_2\|_{X_1}\|v_2 - v_1\|_X \quad \forall w_1, w_2 \in X_1, \end{aligned}$$

which implies

$$\|v_2 - v_1\|_X \le \frac{K\|f\|_{X^*}}{\alpha^2}\|w_1 - w_2\|_{X_1} \quad \forall w_1, w_2 \in X_1. \tag{4.9}$$

By (4.6), (4.9) and Schauder fixed-point theorem(cf. Theorem 2.A in [50]) there exists a solution to (4.1). And any solution is a fixed point of operator T, and by (4.6) all solutions satisfy the estimate (4.2).

If (4.3) holds, then the operator $T : w \in X \to v \in X$ is contract, and so we come to the last conclusion. □

Let us study variational inequalities when the condition 3) of the above theorem is weakened.

Theorem 4.2. *Let X be a real separable Hilbert space. Assume the followings.*
1) Condition 1) of Theorem 4.1 holds.
2) Condition 2) of Theorem 4.1 holds.
3) $a_1(\cdot,\cdot,\cdot) \in (X \times X \times X \to R)$ is a triple linear functional such that

$$|a_1(w,u,v)| \le K\|w\|_X\|u\|_X\|v\|_X, \quad \forall w,u,v \in X.$$

If f is small enough, then in $\mathscr{O}_M(0_X)$, where M is determined in (4.18), there exists a unique solution to the variational inequality

$$a_0(v,u-v) + a_1(v,v,u-v) + J(u) - J(v) \ge \langle f, u-v\rangle \quad \forall u \in X. \tag{4.10}$$

Proof. Fixing $w \in X$, let us consider a variational inequality

$$a_0(v,u-v) + a_1(w,w,u-v) + J(u) - J(v) \ge \langle f, u-v\rangle \quad \forall u \in X, \tag{4.11}$$

where $f \in X^*$. Defining an element $a_1(w) \in X^*$ by

$$\langle a_1(w), u\rangle = a_1(w,w,u) \quad \forall u \in X,$$

by condition 3) we have

$$\|a_1(w)\|_{X^*} \le K\|w\|_X^2 \quad \forall w \in X, \tag{4.12}$$

Then, (4.11) is rewritten as follows.

$$a_0(v,u-v) + J(u) - J(v) \ge \langle f - a_1(w), u-v\rangle \quad \forall u \in X. \tag{4.13}$$

By the same argument as Theorem 4.1, there exists a unique solution v_w to (4.13) and

$$\|v_w\| \le \frac{1}{\alpha}(\|f\|_{X^*} + \|a_1(w)\|_{X^*}) \le \frac{1}{\alpha}(\|f\|_{X^*} + K\|w\|_X^2), \tag{4.14}$$

where (4.12) was used.

Now, let us consider the operator which maps w to the solution of (4.13)

$$T \in (X \to X) : w \to T(w) = v$$

Denote by v_1 and v_2, respectively, the solutions to (4.11) corresponding to $w_1, w_2 \in \mathscr{O}_M(0_X)$, where M is determined bellow. Then

$$\|v_1 - v_2\|_X \le \frac{1}{\alpha}\|a_1(w_1) - a_1(w_2)\|_{X^*}. \tag{4.15}$$

By condition 3)

$$\|a_1(w_1) - a_1(w_2)\|_{X^*} \le K\left(\|w_2 - w_1\|_X\|w_2\|_X + \|w_1\|_X\|w_1 - w_2\|_X\right). \tag{4.16}$$

Thus, by (4.15), (4.16)

$$\begin{aligned}\|v_1 - v_2\|_X &\le \frac{K}{\alpha}\left(\|w_2 - w_1\|_X\|w_2\|_X + \|w_1\|_X\|w_1 - w_2\|_X\right) \\ &\le \frac{2KM}{\alpha}\|w_2 - w_1\|_X \quad \forall w_1, w_2 \in \mathscr{O}_M(0_X).\end{aligned} \tag{4.17}$$

Therefore, if M is taken satisfied (If α is large and $\|f\|_{X^*}$ is small enough, then such choosing is possible.)

$$\begin{cases} M = \dfrac{1}{\alpha}(\|f\|_{X^*} + KM^2), \\ \dfrac{2KM}{\alpha} < 1, \end{cases} \tag{4.18}$$

then by (4.14), (4.17) the operator T on $\mathscr{O}_M(0_X)$ is contract, and so there exists a unique solution to (4.10). □

THEOREM 4.3. *Let X be a real separable Hilbert space and X^* be its dual space. Assume that*

1) $J \in (X \to R)$ is a finite weak continuous convex functional, $J_\varepsilon \in (X \to R)$ is convex such that

$$J_\varepsilon(v) \to J(v) \quad \text{uniformly on } X \text{ as } \varepsilon \to 0,$$
$$\text{Gateaux derivative } DJ_\varepsilon \equiv A_\varepsilon \in (X \to X^*) \text{ is weak continuous and } A_\varepsilon(0_X) = 0_{X^*};$$

2) $a(\cdot,\cdot,\cdot) \in (X \times X \times X \to R)$ is a form such that

$$\begin{aligned} &\text{when } w \in X, (u,v) \to a(w;u,v) \text{ is bilinear on } X \times X, \\ &a(v,v,v) \geq \alpha\|v\|_X^2 \quad \exists \alpha > 0, \forall v \in X \text{ and} \\ &\text{when } v_m \rightharpoonup v \text{ weakly in } X,\ a(v_m,v_m,u) \to a(v,v,u)\ \forall u \in X \text{ and} \\ &\qquad\qquad\qquad \liminf_{m\to\infty} a(v_m,v_m,v_m) \geq a(v,v,v). \end{aligned}$$

Then for $f \in X^*$ there exists a solution to a variational inequality

$$a(v,v,u-v) + J(u) - J(v) \geq \langle f, u-v\rangle \quad \forall u \in X \tag{4.19}$$

satisfying an estimate

$$\|v\|_X \leq \frac{1}{\alpha}\|f\|_{X^*}. \tag{4.20}$$

Proof. First let us prove existence of a solution to a variational equation

$$a(v,v,u) + \langle A_\varepsilon(v), u\rangle = \langle f, u\rangle \quad \forall u \in X. \tag{4.21}$$

We will do it as Theorem 1.2 in ch. 4 of [25]. Let $\{w_n\}$ be a base of X and denote by X_m the subspace of X spanned by $w_1, \cdots, w_m$.

We find $v_m = \sum_{i=1}^m \nu_i w_i \in X_m$ satisfying

$$a(v_m,v_m,u) + \langle A_\varepsilon(v_m), u\rangle = \langle f, u\rangle \quad \forall u \in X_m. \tag{4.22}$$

Define $\Phi_m \in (X_m \to X_m)$ by

$$(\Phi_m(v), w_i) = a(v,v,w_i) + \langle A_\varepsilon(v), w_i\rangle - \langle f, w_i\rangle, \quad 1 \leq i \leq m. \tag{4.23}$$

Since Gateaux derivative of convex functional is monotone (cf. Lemma 4.10, ch. 3 in [23]) and $A_\varepsilon(0_X) = 0_{X^*}$,

$$\langle A_\varepsilon(u) - A_\varepsilon(0_X), u - 0_X\rangle = \langle A_\varepsilon(u), u\rangle \geq 0 \quad \forall u \in X.$$

Thus,

$$a(u,u,u)+\langle A_\varepsilon(u),u\rangle \geq \alpha\|u\|_X^2 \quad \forall u\in X. \tag{4.24}$$

From (4.23), (4.24) we get

$$(\Phi_m(v),v)\geq(\alpha\|v\|_X-\|f\|_{X^*})\|v\|_X \quad \forall v\in X_m. \tag{4.25}$$

Therefore,

$$(\Phi_m(v),v)\geq 0 \quad \forall v\in X \text{ with } \|v\|_X=\frac{\|f\|_{X^*}}{\alpha}.$$

And Φ_m is continuous in X_m by virtue of the assumption 2). Thus, there exists a solution $v_{\varepsilon m}$ to problem (4.22). By (4.25) for all solution $v_{\varepsilon m}$ to (4.22)

$$0=(\Phi_m(v_{\varepsilon m}),v_{\varepsilon m})\geq(\alpha\|v_{\varepsilon m}\|_X-\|f\|_{X^*})\|v_{\varepsilon m}\|_X,$$

which implies

$$\|v_{\varepsilon m}\|_X\leq\frac{1}{\alpha}\|f\|_{X^*}. \tag{4.26}$$

Note this estimation is independent from ε,m. Thus, from $\{v_{\varepsilon m}\}$ we can extract a subsequence $\{v_{\varepsilon m_p}\}$ such that

$$v_{\varepsilon m_p}\rightharpoonup v_\varepsilon \quad \text{weakly in } X \text{ as } p\to+\infty.$$

By the assumptions of theorem

$$a(v_{\varepsilon m_p},v_{\varepsilon m_p},u)+\langle A_\varepsilon(v_{\varepsilon m_p}),u\rangle\to a(v_\varepsilon,v_\varepsilon,u)+\langle A_\varepsilon(v_\varepsilon),u\rangle \quad \forall u\in X. \tag{4.27}$$

From (4.22), (4.27), (4.26) we know that v_ε is a solution to (4.21) and satisfies

$$\|v_\varepsilon\|_X\leq\frac{1}{\alpha}\|f\|_{X^*}. \tag{4.28}$$

Subtracting the following two formula which are got from (4.21)

$$\begin{aligned}&a(v_\varepsilon,v_\varepsilon,u)+\langle A_\varepsilon(v_\varepsilon),u\rangle=\langle f,u\rangle \quad \forall u\in X,\\&a(v_\varepsilon,v_\varepsilon,v_\varepsilon)+\langle A_\varepsilon(v_\varepsilon),v_\varepsilon\rangle=\langle f,v_\varepsilon\rangle\end{aligned}$$

and taking into account that

$$J_\varepsilon(u)-J_\varepsilon(v_\varepsilon)\geq\langle A_\varepsilon(v_\varepsilon),u-v_\varepsilon\rangle$$

which is due to convexity of J_ε, we come to the following inequality

$$a(v_\varepsilon,v_\varepsilon,u-v_\varepsilon)+J_\varepsilon(u)-J_\varepsilon(v_\varepsilon)\geq\langle f,u-v_\varepsilon\rangle \quad \forall u\in X. \tag{4.29}$$

By (4.28) we can choose $\{v_{\varepsilon_k}\}$ such that

$$v_{\varepsilon_k}\rightharpoonup v^* \quad \text{weakly in } X \text{ as } \varepsilon_k\to 0. \tag{4.30}$$

By virtue of assumption 1)

$$|J_{\varepsilon_k}(v_{\varepsilon_k})-J(v^*)|\leq|J_{\varepsilon_k}(v_{\varepsilon_k})-J(v_{\varepsilon_k})|+|J(v_{\varepsilon_k})-J(v^*)|\to 0 \quad \text{as } \varepsilon_k\to 0,$$

and so

$$J_{\varepsilon_k}(v_{\varepsilon_k}) \to J(v^*) \text{ as } \varepsilon_k \to 0. \tag{4.31}$$

Also

$$J_{\varepsilon_k}(u) \to J(u) \quad \forall u \in X \quad \text{as } \varepsilon_k \to 0. \tag{4.32}$$

By virtue of assumption 2)

$$\begin{aligned} &a(v_{\varepsilon_k}, v_{\varepsilon_k}, u) \to a(v^*, v^*, u) \quad \forall u \in X, \\ &\liminf_{k\to\infty} a(v_{\varepsilon_k}, v_{\varepsilon_k}, v_{\varepsilon_k}) \geq a(v^*, v^*, v^*). \end{aligned} \tag{4.33}$$

Taking into account (4.31)-(4.33), from (4.29) we get

$$a(v^*, v^*, u - v^*) + J(u) - J(v^*) \geq \langle f, u - v^* \rangle \quad \forall u \in X.$$

By (4.28) we have

$$\|v^*\|_X \leq \frac{1}{\alpha}\|f\|_{X^*}. \tag{4.34}$$

□

Remark 4.1. *The estimate of solutions in Theorem 4.1 is for all solutions of the problem, but one in Theorem 4.3 is for the solution guaranteed existence by the theorem.*

5. Mixed boundary value problems of the Navier-Stokes and Stokes equations. In this section relying on the results in Section 4, we are concerned with problems in Section 3.

Theorem 5.1. *Let Assumption 3.1 hold, the surfaces Γ_{2j}, Γ_{3j}, Γ_{7j} be convex (cf. Definition 2.1), α positive and $\|U\|_{\mathbf{H}^1(\Omega)}$ small enough. Then, when f and ϕ_i, $i = 2 \sim 7$, are small enough, there exists a unique solution to Problem I-VI for the stationary Navier-Stokes problem with mixed boundary condition* (2.7) *in a neighborhood of U in $\mathbf{H}^1(\Omega)$.*

Proof. Define a functional $J(u)$ by (3.16), (3.17). Trace operator is continuous and sum of convex functions is also convex. Thus, the functional satisfies condition 1) of Theorem 4.2.

Let $w = v - U$, U be a function in Assumption 3.1 and $a_{01}(\cdot,\cdot), a_{11}(\cdot,\cdot,\cdot)$ and $F_1 \in \mathbf{V}(\Omega)^*$ be as (3.4):

$$\begin{aligned} &a_{01}(w, u) = 2\nu(\varepsilon(w), \varepsilon(u)) + \langle (U\cdot\nabla)w, u\rangle + \langle (w\cdot\nabla)U, u\rangle + 2\nu(k(x)w, u)_{\Gamma_2} \\ &\qquad + 2\nu(S\tilde{w}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)w, u)_{\Gamma_5} + \nu(k(x)w, u)_{\Gamma_7} \quad \forall w, u \in \mathbf{V}(\Omega), \\ &a_{11}(w, u, v) = \langle (w\cdot\nabla)u, v\rangle \quad \forall w, u, v \in \mathbf{V}(\Omega), \\ &\langle F_1, u\rangle = -2\nu(\varepsilon(U), \varepsilon(u)) - \langle (U\cdot\nabla)U, u\rangle - 2\nu(k(x)U, u)_{\Gamma_2} - 2\nu(S\tilde{U}, \tilde{u})_{\Gamma_3} \\ &\qquad - 2(\alpha(x)U, u)_{\Gamma_5} - \nu(k(x)U, u)_{\Gamma_7} + \langle f, u\rangle + \sum_{i=2,4,7}\langle \phi_i, u_n\rangle_{\Gamma_i} \\ &\qquad + \sum_{i=3,5,6}\langle \phi_i, u\rangle_{\Gamma_i} \quad \forall u \in \mathbf{V}(\Omega), \end{aligned} \tag{5.1}$$

By Korn's inequality

$$2\nu(\varepsilon(w),\varepsilon(w)) \geq \delta\|w\|^2_{\mathbf{V}}. \tag{5.2}$$

On the other hand, applying Hölder inequality for $w \in \mathbf{V}(\Omega)$ we have

$$|\langle (U\cdot\nabla)w, w\rangle + \langle (w\cdot\nabla)U, w\rangle| \leq \gamma\|w\|^2_{\mathbf{V}}\cdot\|U\|_{\mathbf{H}^1(\Omega)}. \tag{5.3}$$

Therefore, if $\delta - \gamma\|U\|_{\mathbf{H}^1(\Omega)} = \beta_1 > 0$, then by (5.2), (5.3), Assumption 3.1 and Lemma 2.3 we have

$$a_{01}(u,u) \geq \beta_1\|u\|^2_{\mathbf{V}} \quad \forall u \in \mathbf{V}(\Omega). \tag{5.4}$$

It is easy to verify that

$$|a_{01}(u,v)| \leq c\|u\|_{\mathbf{V}}\|v\|_{\mathbf{V}} \quad \forall u,v \in \mathbf{V}(\Omega). \tag{5.5}$$

By (5.4) and (5.5), $a_0(u,v)$ satisfies condition 2) of Theorem 4.2.

By Hölder inequality we can see

$$|a_{11}(w,u,v)| \leq c\|w\|_{\mathbf{V}}\|u\|_{\mathbf{V}}\|v\|_{\mathbf{V}} \quad \forall w,u,v \in \mathbf{V}(\Omega). \tag{5.6}$$

which means $a_{11}(w,u,v)$ satisfies condition 3) of Theorem 4.2.

Also

$$\begin{aligned}\|F_1\|_{\mathbf{V}^*} \leq M_1\Big(&\|U\|_{\mathbf{H}^1} + \|U\|^2_{\mathbf{H}^1} + \|f\|_{\mathbf{V}^*} \\ &+ \sum_{i=2,4,7}\|\phi_i\|_{H^{-\frac{1}{2}}(\Gamma_i)} + \sum_{i=3,5,6}\|\phi_i\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_i)}\Big),\end{aligned} \tag{5.7}$$

where M_1 depends on mean curvature of Γ_7, shape operator of Γ_3, ν and α.

By Theorem 4.2, if $\|U\|_{\mathbf{H}^1}, \|f\|_{\mathbf{V}^*}, \|\phi_i\|_{H^{-\frac{1}{2}}(\Gamma_i)}, i = 2,4,7$, and $\|\phi_i\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_i)}, i = 3,5,6$, are small enough, then there exists a unique solution $w \in K(\Omega)$ to

$$\begin{aligned}&a_{01}(w,u-w) + a_{11}(w,w,u-w) + J(u) - J(w) \\ &\geq \langle F_1, u-w\rangle \quad \forall u \in K(\Omega).\end{aligned} \tag{5.8}$$

Since $v = w + U$ is solution, we come to the asserted conclusion. □

Theorem 5.2. *Let Assumption 3.1 hold, the surfaces Γ_{2j}, Γ_{3j}, Γ_{7j} be convex, α positive and $\|U\|_{\mathbf{H}^1(\Omega)}$ small enough. Then, for any f ϕ_i, $i = 2 \sim 7$, there exists a solution v to Problem II-VI for the stationary Navier-Stokes problem with mixed boundary condition* (2.8) *in a neighborhood of U in $\mathbf{H}^1(\Omega)$ and all solutions satisfy*

$$\begin{aligned}\|v-U\|_{\mathbf{H}^1} \leq \frac{M_1}{\delta - \gamma\|U\|_{\mathbf{H}^1}}\Big(&\|U\|_{\mathbf{H}^1} + \|U\|^2_{\mathbf{H}^1} + \|f\|_{\mathbf{V}^*} \\ &+ \sum_{i=2,4,7}\|\phi_i\|_{H^{-\frac{1}{2}}(\Gamma_i)} + \sum_{i=3,5,6}\|\phi_i\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_i)}\Big),\end{aligned} \tag{5.9}$$

where δ, γ, M_1 are as (5.11), (5.12), (5.20).

If $\|U\|_{\mathbf{H}^1}, \|f\|_{\mathbf{V}^*}, \|\phi_i\|_{H^{-\frac{1}{2}}(\Gamma_i)}, i = 2,4,7$, and $\|\phi_i\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_i)}, i = 3,5,6$, are small enough, then the solution is unique.

Proof. Define a functional $J(u)$ by (3.16), (3.17). Then, this functional satisfies condition 1) of Theorem 4.1.

Let $a_{02}(\cdot,\cdot), a_{12}(\cdot,\cdot,\cdot)$ and $F_2 \in V^*$ are as (3.28):

$$\begin{aligned} a_{02}(w,u) &= 2\nu(\varepsilon(w),\varepsilon(u)) + \langle \operatorname{rot} U \times w, u\rangle + \langle \operatorname{rot} w \times U, u\rangle + 2\nu(k(x)w,u)_{\Gamma_2} \\ &\quad + 2\nu(S\tilde{w},\tilde{u})_{\Gamma_3} + 2(\alpha(x)w,u)_{\Gamma_5} + \nu(k(x)w,u)_{\Gamma_7}, \\ a_{12}(w,u,v) &= \langle \operatorname{rot} w \times u, v\rangle, \\ \langle F_2, u\rangle &= -2\nu(\varepsilon(U),\varepsilon(u)) - \langle \operatorname{rot} U \times U, u\rangle - 2\nu(k(x)U,u)_{\Gamma_2} - 2\nu(S\tilde{U},\tilde{u})_{\Gamma_3} \\ &\quad - 2(\alpha(x)U,u)_{\Gamma_5} - \nu(k(x)U,u)_{\Gamma_7} + \langle f,u\rangle + \sum_{i=2,4,7}\langle \phi_i, u_n\rangle_{\Gamma_i} + \sum_{i=3,5,6}\langle \phi_i, u\rangle_{\Gamma_i}. \end{aligned} \tag{5.10}$$

By Korn's inequality

$$2\nu(\varepsilon(w),\varepsilon(w)) \geq \delta\|w\|^2_{\mathbf{V}}. \tag{5.11}$$

On the other hand, for any $w \in \mathbf{V}(\Omega)$ we have

$$\begin{aligned} &\langle \operatorname{rot} U \times w, w\rangle = 0, \\ &|\langle \operatorname{rot} w \times U, w\rangle| \leq \gamma\|w\|^2_{\mathbf{V}} \cdot \|U\|_{\mathbf{H}^1(\Omega)}. \end{aligned} \tag{5.12}$$

Therefore, if $\delta - \gamma\|U\|_{\mathbf{H}^1(\Omega)} = \beta_1 > 0$, then by (5.11), (5.12), Assumption 3.1 and Lemma 2.3 we have

$$a_{02}(u,u) \geq \beta_1\|u\|^2_{\mathbf{V}} \quad \forall u \in \mathbf{V}(\Omega). \tag{5.13}$$

It is easy to verify

$$|a_{02}(u,v)| \leq c\|u\|_{\mathbf{V}(\Omega)}\|v\|_{\mathbf{V}(\Omega)} \quad \forall u,v \in \mathbf{V}(\Omega). \tag{5.14}$$

Then, (5.13), (5.14) show that $a_{02}(u,v)$ satisfy condition 2) of Theorem 4.1.

By a property of mixed product,

$$a_{12}(w,u,u) = \langle \operatorname{rot} w \times u, u\rangle = 0 \quad \forall w \in \mathbf{V}^{\frac{2}{3}}(\Omega), \forall u \in \mathbf{V}(\Omega), \tag{5.15}$$

where $\mathbf{V}^{\frac{2}{3}}(\Omega) = \{u \in \mathbf{H}^{\frac{2}{3}}(\Omega) : \operatorname{div} u = 0, u|_{\Gamma_1} = 0, u_\tau|_{(\Gamma_2\cup\Gamma_4\cup\Gamma_7\cup\Gamma_9)} = 0, u \cdot n|_{(\Gamma_3\cup\Gamma_5\cup\Gamma_8)} = 0\}$. On the other hand, by density argument we get

$$a_{12}(w,u,v) = \langle \operatorname{rot} w \times u, v\rangle = -\langle \operatorname{rot} w, v \times u\rangle. \tag{5.16}$$

When $u, v \in \mathbf{V}(\Omega)$, $v \times u \in \mathbf{H}^{\frac{1}{2}}(\Omega)$ and

$$\|v \times u\|_{\mathbf{H}^{\frac{1}{3}}(\Omega)} \leq c_1\|v \times u\|_{\mathbf{H}^{\frac{1}{2}}(\Omega)} \leq c\|v\|_{\mathbf{V}(\Omega)}\|u\|_{\mathbf{V}(\Omega)}. \tag{5.17}$$

(cf. Theorem 1.4.4.2 in [26].) Also, if $w \in \mathbf{V}^{\frac{2}{3}}(\Omega)$, then $\operatorname{rot} w \in \mathbf{H}^{-\frac{1}{3}}(\Omega)$ and

$$\|\operatorname{rot} w\|_{\mathbf{H}^{-\frac{1}{3}}(\Omega)} \leq c\|w\|_{\mathbf{H}^{\frac{2}{3}}(\Omega)}. \tag{5.18}$$

(cf. Proposition 12.1, ch. 1 in [42].) Since $H_0^{\frac{1}{3}}(\Omega) = H^{\frac{1}{3}}(\Omega)$(cf. Theorem 11.1, ch. 1 in [42]), by (5.16)-(5.18) we get

$$|a_{12}(w,u,v)| \leq K\|w\|_{\mathbf{V}^{\frac{2}{3}}(\Omega)}\|u\|_{\mathbf{V}(\Omega)}\|v\|_{\mathbf{V}(\Omega)} \quad \forall w \in \mathbf{V}^{\frac{2}{3}}(\Omega), \forall u,v \in \mathbf{V}(\Omega). \tag{5.19}$$

Since $\mathbf{V}(\Omega) \hookrightarrow\hookrightarrow \mathbf{V}^{\frac{2}{3}}(\Omega)$, setting $X = \mathbf{V}(\Omega)$, $X_1 = \mathbf{V}^{\frac{2}{3}}(\Omega)$ by (5.15), (5.19) $a_{11}(w,u,v)$ satisfies condition 3) of Theorem 4.1.

Also, we have

$$\|F_2\|_{\mathbf{V}^*} \leq M_1\Big(\|U\|_{\mathbf{H}^1} + \|U\|^2_{\mathbf{H}^1} + \|f\|_{\mathbf{V}^*} + \sum_{i=2,4,7} \|\phi_i\|_{H^{-\frac{1}{2}}(\Gamma_i)} + \sum_{i=3,5,6} \|\phi_i\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_i)}\Big), \tag{5.20}$$

where M_1 depends on mean curvature, shape operator, ν and α.

Therefore, by Theorem 4.1, we have existence and an estimate of solutions to

$$a_{02}(w, u-w) + a_{12}(w,w,u-w) + J(u) - J(w) \geq \langle F_2, u-w\rangle \quad \forall u \in \mathbf{V}(\Omega).$$

Since $v = w + U$ is solution to the given problem, we have existence of solutions and the estimate (5.9).

If $\|U\|_{\mathbf{H}^1}, \|f\|_{\mathbf{V}^*}$, $\|\phi_i\|_{H^{-\frac{1}{2}}(\Gamma_i)}, i = 2,4,7$, and $\|\phi_i\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_i)}, i = 3,5,6$, are small enough, then the solution is unique. □

Let us consider a special case of the Navier-Stokes problem with boundary condition (2.7) in which there is not any flux across boundary except Γ_1, Γ_8.

THEOREM 5.3. *Let Assumption 3.1 hold,* $\Gamma_i = \emptyset (i = 2,4,6,7,9-11)$, *the surfaces* Γ_{3j} *be convex,* α *positive and* $\|U\|_{\mathbf{H}^1(\Omega)}$ *small enough. Then, for any* f *and* ϕ_i, $i = 3,5$ *there exists a solution* v *to Problem I-VI for the stationary Navier-Stokes problem with mixed boundary condition* (2.7) *and all solutions satisfy*

$$\|v - U\|_{\mathbf{H}^1} \leq \frac{M_1}{\delta - \gamma\|U\|_{\mathbf{H}^1}}\Big(\|U\|^2_{\mathbf{H}^1} + \|f\|_{\mathbf{V}^*} + \sum_{i=3,5} \|\phi_i\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_i)}\Big), \tag{5.21}$$

where δ, γ, M_1 *are as* (5.2), (5.3), (5.7).

In addition, if $\|f\|_{\mathbf{V}^*}$, $\|\phi_i\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_i)}, i = 3,5$, *are small enough, then the solution is unique.*

Proof. Define a functional $J(u) = j_\tau(u)$ by (3.16), (3.17). Then, the functional satisfies condition 1) of Theorem 4.2.

Let $w = v - U$, U be a function in Assumption 3.1 and $a_{01}(\cdot,\cdot), a_{11}(\cdot,\cdot,\cdot)$ and $F_1 \in \mathbf{V}(\Omega)^*$ be as (3.4):

$$\begin{aligned} a_{01}(w,u) &= 2\nu(\varepsilon(w), \varepsilon(u)) + \langle (U\cdot\nabla)w, u\rangle + \langle (w\cdot\nabla)U, u\rangle + 2\nu(S\tilde{w}, \tilde{u})_{\Gamma_3} \\ &\quad + 2(\alpha(x)w, u)_{\Gamma_5} \quad \forall w, u \in \mathbf{V}(\Omega), \\ a_{11}(w,u,v) &= \langle (w\cdot\nabla)u, v\rangle \quad \forall w,u,v \in \mathbf{V}(\Omega), \\ \langle F_1, u\rangle &= -2\nu(\varepsilon(U), \varepsilon(u)) - \langle (U\cdot\nabla)U, u\rangle - 2\nu(S\tilde{U}, \tilde{u})_{\Gamma_3} - 2(\alpha(x)U, u)_{\Gamma_5} \\ &\quad + \langle f, u\rangle + \sum_{i=3,5} \langle \phi_i, u\rangle_{\Gamma_i} \quad \forall u \in \mathbf{V}(\Omega), \end{aligned}$$

We can see that the condition 2) in Theorem 4.1 are satisfied(cf. proof of Theorem 5.1).

By the condition of theorem,

$$a_{11}(w,u,u) = \langle (w\cdot\nabla)u, u\rangle = 0 \quad \forall w \in \mathbf{V}^{\frac{2}{3}}(\Omega), \forall u \in \mathbf{V}(\Omega). \tag{5.22}$$

By Hölder inequality we can see

$$|a_{11}(w,u,v)| \le K\|w\|_{\mathbf{V}^{\frac{2}{3}}(\Omega)}\|u\|_{\mathbf{V}(\Omega)}\|v\|_{\mathbf{V}(\Omega)} \quad \forall w\in \mathbf{V}^{\frac{2}{3}}(\Omega), \forall u,v\in \mathbf{V}(\Omega). \tag{5.23}$$

By (5.22), (5.23), $a_{11}(w,u,v)$ satisfies condition 3) of Theorem 4.1.

Applying Theorem 4.1 to

$$a_{01}(w,u-w)+a_{11}(w,w,u-w)+J(u)-J(w)\ge \langle F_1,u-w\rangle \quad \forall u\in K(\Omega),$$

we come to the asserted conclusion. □

Remark 5.1. *Assumption* $\Gamma_i=\varnothing, i=2,4,6,7,9-11,$ *is only used to get* (5.22).

Relying on Theorem 4.3, again let us study the problem concerned in Theorem 5.3. This is generalization of methods used in previous papers relying on smooth approximation of functional in variational inequalities(cf. [40]).

Lemma 5.4. *Let* X,Y *be reflex Banach spaces, an operator* $i\in(X\to Y)$ *be completely linear continuous,* $j\in(Y\to R)$ *be convex and Gateaux derivative* $Dj(y)=a(y)$ *for* $y\in Y$*. Then,* $J(v)\equiv j(iv)\in(X\to R)$ *is convex,* $DJ(v)\equiv A(v)=i^*a(iv)$*, where* i^* *is the operator adjoint to* i*, and* $A\in(X\to X^*)$ *is weak continuous.*

Proof. It is easy to verify convexity of J.

$$\begin{aligned}\langle A(v),u\rangle_X &= \lim_{t\to 0}\frac{J(v+tu)-J(u)}{t}=\lim_{t\to 0}\frac{j(i(v+tu))-j(iu)}{t}\\ &=\langle a(iv),iu\rangle_Y=\langle i^*a(iv),u\rangle_X \qquad \forall v,u\in X,\end{aligned}$$

which means $A(v)=i^*a(iv)$.

Let $v_n\rightharpoonup v$ weakly in X. Since Gateaux derivative of a finite convex functional is monotone and demi-continuous(cf. Lemmas 4.10, 4.12, ch. 3 in [23]) and $iv_n\to iv$ in Y,

$$\langle A(v_n),u\rangle_X=\langle i^*a(iv_n),u\rangle_X=\langle a(iv_n),iu\rangle_Y\to\langle a(iv),iu\rangle_Y=\langle i^*a(iv),u\rangle_X \qquad \forall u\in X,$$

that is, $DJ=A\in(X\to X^*)$ is weak continuous. □

Theorem 5.5. *Let Assumption 3.1 hold,* $\Gamma_i=\emptyset (i=2,4,6,7,9-11)$*, the surfaces* Γ_{3j} *be convex,* α *positive and* $\|U\|_{\mathbf{H}^1(\Omega)}$ *small enough. Then, for any* f *and* ϕ_i*,* $i=3,5$*, there exists a solution* v *to Problem I-VI for the stationary Navier-Stokes problem with mixed boundary condition* (2.7) *and the solution satisfies the estimate* (5.21).

Proof. Define an operator $i\in\big(\mathbf{V}(\Omega)\to\mathbf{L}^2_\tau(\Gamma_8)\big)$ by $iu=u|_{\Gamma_8}$ and a functional $J\in(\mathbf{V}(\Omega)\to R)$ by $J(v)\equiv j_\tau(iv)$, where j_τ is as (3.16). Since the trace operator $(\mathbf{V}(\Omega)\to H^{\frac12}(\partial\Omega))$ is continuous and $H^{\frac12}(\partial\Omega)\hookrightarrow\hookrightarrow L^2(\partial\Omega)$, the operator i is compact, and by Lemma 5.4 $J\in(\mathbf{V}(\Omega)\to R)$ is weak continuous and convex.

Define a functional $J_\varepsilon\in(\mathbf{V}(\Omega)\to R)$ by

$$\begin{aligned}&J_\varepsilon(v)=j_{\tau\varepsilon}(iv),\\ &j_{\tau\varepsilon}(\eta)=\int_{\Gamma_8}g_\tau\rho_\varepsilon(\eta)\,ds,\\ &\rho_\varepsilon(\eta)=\begin{cases}|\eta|-\varepsilon/2 & |\eta|>\varepsilon,\\ |\eta|^2/2\varepsilon & |\eta|\le\varepsilon\,.\end{cases}\end{aligned}\tag{5.24}$$

Since

$$|j_{\tau\varepsilon}(\eta) - j_\tau(\eta)| \le \frac{\varepsilon}{2}|g_\tau| \quad \forall \eta \in \mathbf{L}^2_\tau(\Gamma_8)$$

(cf. Lemma 2.1 in [40]), we have

$$|J_\varepsilon(v) - J(v)| \le \frac{\varepsilon}{2}|g_\tau| \quad \forall v \in \mathbf{V}(\Omega). \tag{5.25}$$

Also, $j_{\tau\varepsilon}$ is convex, and so its Gateaux derivative is demi-continuous. Thus, by Lemma 5.4 $DJ_\varepsilon \equiv A_\varepsilon \in (\mathbf{V}(\Omega) \to \mathbf{V}(\Omega)^*)$ is weak continuous. By this fact together (5.25), condition 1) of Theorem 4.3 is satisfied.

Under Assumption of theorem $a_{01}(\cdot,\cdot), a_{11}(\cdot,\cdot,\cdot)$ and $F_1 \in V^*$ of (3.4) are as follows.

$$\begin{aligned}
&a_{01}(u,v) = 2\nu(\varepsilon(u),\varepsilon(v)) + \langle (U\cdot\nabla)u, v\rangle + \langle (u\cdot\nabla)U, v\rangle \\
&\qquad\qquad + 2\nu(S\tilde{u},\tilde{v})_{\Gamma_3} + 2(\alpha(x)u,v)_{\Gamma_5} \quad \forall u,v \in \mathbf{V}(\Omega),\\
&a_{11}(w,u,v) = \langle (w\cdot\nabla)u, v\rangle \quad \forall w,u,v \in \mathbf{V}(\Omega),\\
&\langle F_1, u\rangle = -2\nu(\varepsilon(U),\varepsilon(u)) - \langle (U\cdot\nabla)U, u\rangle - 2\nu(S\tilde{U},\tilde{u})_{\Gamma_3}\\
&\qquad\qquad - 2(\alpha(x)U,u)_{\Gamma_5} + \langle f,u\rangle + \sum_{i=3,5}\langle \phi_i, u\rangle_{\Gamma_i} \quad \forall u \in \mathbf{V}(\Omega),
\end{aligned} \tag{5.26}$$

By Korn's inequality

$$2\nu(\varepsilon(u),\varepsilon(u)) \ge \delta\|u\|^2_{\mathbf{V}}. \tag{5.27}$$

On the other hand, for any $w \in \mathbf{V}(\Omega)$ we have

$$|\langle (U\cdot\nabla)u, u\rangle + \langle (u\cdot\nabla)U, u\rangle| \le \gamma\|u\|^2_{\mathbf{V}} \cdot \|U\|_{\mathbf{H}^1(\Omega)}. \tag{5.28}$$

Therefore, if $\delta - \gamma\|U\|_{\mathbf{H}^1(\Omega)} = \beta_1 > 0$, then by (5.27), (5.28), Assumption 3.1 and Lemma 2.3 we have

$$a_{01}(u,u) \ge \beta_1\|u\|^2_{\mathbf{V}} \quad \forall u \in \mathbf{V}(\Omega). \tag{5.29}$$

Under condition $\Gamma_i = \emptyset, i = 2,4,6,7,9,10,11$, it is easy to verify that

$$a_{11}(v,v,v) = 0 \quad \forall v \in \mathbf{V}(\Omega). \tag{5.30}$$

Let

$$a(w,u,v) = a_{01}(u,v) + a_{11}(w,u,v).$$

Then, by (5.29), (5.30) we have

$$a(v,v,v) \ge \beta_1\|u\|^2_{\mathbf{V}} \quad \forall v \in \mathbf{V}(\Omega). \tag{5.31}$$

Let us prove that when $v_m \rightharpoonup v$ weakly in $\mathbf{V}(\Omega)$, for a subsequence $\{v_{m_p}\}$

$$a(v_{m_p}, v_{m_p}, u) \to a(v;v,u) \quad \forall u \in \mathbf{V}(\Omega). \tag{5.32}$$

To this end, first let us prove that when $v_m \rightharpoonup v$ weakly in $\mathbf{V}(\Omega)$, for a subsequence $\{v_{m_p}\}$

$$a_{01}(v_m, u) \to a_{01}(v_m, u) \quad \forall u \in \mathbf{V}(\Omega). \tag{5.33}$$

Since $U_i u_j \in L^2(\Omega), i,j = 1-3$, and $\partial_i v_m \rightharpoonup v$ in $\mathrm{L}(\Omega)^2$, we have

$$\langle (U\cdot\nabla)v_m, u\rangle \to \langle (U\cdot\nabla)v, u\rangle \quad \text{as } m\to\infty. \tag{5.34}$$

By Hölder inequalities

$$|\langle((v_m - v)\cdot\nabla)U, u\rangle| \le c\|v_m - v\|_{\mathbf{L}^3(\Omega)}\|\nabla U\|_{\mathbf{L}^2(\Omega)}\|u\|_{\mathbf{L}^6(\Omega)}.$$

Since $H^1(\Omega) \hookrightarrow\hookrightarrow L^3(\Omega)$, we can choose a subsequence $\{v_{m_p}\}$ such that $v_{mp} \to v$ in $\mathbf{L}^3(\Omega)$. Then, we have

$$\langle (v_{m_p}\cdot\nabla)U, u\rangle \to \langle (v\cdot\nabla)U, u\rangle \quad \text{as } m_p\to\infty. \tag{5.35}$$

It is easy to verify convergence of other terms. Thus, using (5.34), (5.35), we have (5.33).

Using Hölder inequality and $a_{11}(v,u,w) = -a_{11}(v,w,u)$, we have

$$\begin{aligned}
&|a_{11}(v_m,v_m,u) - a_{11}(v,v,u)| \\
&\le |a_{11}(v_m,v_m,u) - a_{11}(v,v_m,u)| + |a_{11}(v,v_m,u) - a_{11}(v,v,u)| \\
&\le c\big(\|v_m - v\|_{\mathbf{L}^3(\Omega)}\|\nabla v_m\|_{\mathbf{L}^2(\Omega)}\|u\|_{\mathbf{L}^6(\Omega)} \\
&\qquad + \|v\|_{\mathbf{L}^6(\Omega)}\|\nabla u\|_{\mathbf{L}^2(\Omega)}\|v_m - v\|_{\mathbf{L}^3(\Omega)}\big) \quad \forall u \in \mathbf{V}(\Omega).
\end{aligned}$$

Thus, we have

$$a_{11}(v_{m_p}, v_{m_p}, u) \to a_{11}(v,v,u) \quad \forall u \in \mathbf{V}(\Omega) \quad \text{as } m_p \to \infty. \tag{5.36}$$

From (5.33), (5.36) we get (5.32).

Let us prove that

$$\liminf_{m\to\infty} a(v_{m_p}, v_{m_p}, v_{m_p}) \ge a(v,v,v). \tag{5.37}$$

By lower semi-continuity of norm

$$\liminf_{m\to\infty} 2\nu(\varepsilon(v_m),\varepsilon(v_m)) \ge 2\nu(\varepsilon(v),\varepsilon(v)) \quad \text{as } v_m \rightharpoonup v \text{ in } \mathbf{V}(\Omega). \tag{5.38}$$

It is easy to prove that

$$2\nu(S\tilde{v}_m, \tilde{u})_{\Gamma_3} + 2(\alpha(x)v_m, u)_{\Gamma_5} \to 2\nu(S\tilde{v}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)v, u)_{\Gamma_5} \quad \forall u \in \mathbf{V}(\Omega). \tag{5.39}$$

Using Hölder inequality and $a_{11}(v, v_m, u) = -a_{11}(v, u, v_m)$, we have

$$\begin{aligned}
&|a_{11}(v_m,v_m,v_m) - a_{11}(v,v,v)| \\
&\quad \le |a_{11}(v_m,v_m,v_m) - a_{11}(v,v_m,v_m)| + |a_{11}(v,v_m,v_m) - a_{11}(v,v_m,v)| \\
&\qquad\quad + |a_{11}(v,v_m,v) - a_{11}(v,v,v)| \\
&\quad \le c\big(\|v_m - v\|_{\mathbf{L}^3(\Omega)}\|\nabla v_m\|_{\mathbf{L}^2(\Omega)}\|v_m\|_{\mathbf{L}^6(\Omega)} + \|v\|_{\mathbf{L}^6(\Omega)}\|\nabla v_m\|_{\mathbf{L}^2(\Omega)}\|v_m - v\|_{\mathbf{L}^3(\Omega)} \\
&\qquad\quad + \|v\|_{\mathbf{L}^6(\Omega)}\|\nabla v\|_{\mathbf{L}^2(\Omega)}\|v_m - v\|_{\mathbf{L}^3(\Omega)}\big),
\end{aligned}$$

which implies

$$a_{11}(v_{m_p}, v_{m_p}, v_{m_p}) \to a_{11}(v,v,v) \quad \text{as } m_p \to \infty. \tag{5.40}$$

From (5.37)-(5.40), we have (5.37).

By virtue of (5.31), (5.32) and (5.37), condition 2) of Theorem 4.3 is satisfied. Therefore, by Theorem 4.3 we have existence of a solution $w \in \mathbf{V}(\Omega)$ to

$$\begin{aligned} &a_{01}(w, u-w) + a_{11}(w, w, u-w) + j_\tau(u) - j_\tau(w) \\ &\geq \langle F_1, u-w\rangle \quad \forall u \in \mathbf{V}(\Omega) \end{aligned} \tag{5.41}$$

and an estimate. Since $v = w + U$ is a solution, we come to the asserted conclusion. ☐

REMARK 5.2. *The estimate of solution of Theorem 5.5 is not for all solutions, and so Theorem 5.5 is weaker than Theorem 5.3.*

Let us consider Problem III for the Stokes system.

THEOREM 5.6. *Let Assumption 3.1 hold, the surfaces Γ_{2j}, Γ_{3j}, Γ_{7j} be convex and α positive. Then, for any f ϕ_i, $i = 2 \sim 7$, there exists a unique solution v to Problem III-VI for the stationary Stokes problem with mixed boundary condition* (2.7) *and*

$$\begin{aligned} \|v-U\|_{\mathbf{H}^1} \leq \frac{M_1}{\delta}\Big(&\|U\|_{\mathbf{H}^1} + \|f\|_{\mathbf{V}^*} + \sum_{i=2,4,7} \|\phi_i\|_{H^{-\frac{1}{2}}(\Gamma_i)} \\ &+ \sum_{i=3,5,6} \|\phi_i\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_i)}\Big), \end{aligned} \tag{5.42}$$

where δ, M_1 are as (5.11), (5.20)*(for F_3 instead of F_2).*

If v_1, v_2 are solutions, respectively, to Problem-III-VI with $g_{\tau 1}, g_{n1}, g_{+n1}, g_{-n1}, f_1, h_i^1, \phi_i^1$ and $g_{\tau 2}, g_{n2}, g_{+n2}, g_{-n2}$, f_2, h_i^1, ϕ_i^2, then

$$\begin{aligned} \|v_1 - v_2\|_{\mathbf{H}^1} \leq \frac{M_1}{\delta}\Big(&\|U_1 - U_2\|_{\mathbf{H}^1} + \|f_1 - f_2\|_{\mathbf{V}^*} + \|g_{\tau 1} - g_{\tau 2}\|_{L^2_\tau(\Gamma_8)} \\ &+ \|g_{n1} - g_{n2}\|_{L^2(\Gamma_9)} + \|g_{+n1} - g_{+n2}\|_{L^2(\Gamma_{10})} \\ &+ \|g_{-n1} - g_{-n2}\|_{L^2(\Gamma_{10})} + \sum_{i=2,4,7} \|\phi_i^1 - \phi_i^2\|_{H^{-\frac{1}{2}}(\Gamma_i)} \\ &+ \sum_{i=3,5,6} \|\phi_i^1 - \phi_i^2\|_{\mathbf{H}^{-\frac{1}{2}}(\Gamma_i)}\Big) + \|U_1 - U_2\|_{\mathbf{H}^1}, \end{aligned} \tag{5.43}$$

where $U_j, j = 1, 2,$ are the functions in Assumption 3.1 with h_i^j instead h_i.

Proof. By arguments similar to proof of Theorem 4.2 we can apply the well known result for variational inequality

$$a_{03}(w, u-w) + J(u) - J(w) \geq \langle F_3, u-w\rangle \quad \forall u \in X, \tag{5.44}$$

where $J(u)$ is defined by (3.16), (3.17) and $a_{03}(v, u), F_3$ are as (3.34):

$$\begin{aligned} a_{03}(w, u) &= 2\nu(\varepsilon(w), \varepsilon(u)) + 2\nu(k(x)w, u)_{\Gamma_2} \\ &\qquad + 2\nu(S\tilde{w}, \tilde{u})_{\Gamma_3} + 2(\alpha(x)w, u)_{\Gamma_5} + \nu(k(x)w, u)_{\Gamma_7} \quad \forall w, u \in \mathbf{V}(\Omega), \\ \langle F_3, u\rangle &= -2\nu(\varepsilon(U), \varepsilon(u)) - 2\nu(k(x)U, u)_{\Gamma_2} - 2\nu(S\tilde{U}, \tilde{u})_{\Gamma_3} - 2(\alpha(x)U, u)_{\Gamma_5} \\ &\qquad - \nu(k(x)U, u)_{\Gamma_7} + \langle f, u\rangle + \sum_{i=2,4,7} \langle \phi_i, u_n\rangle_{\Gamma_i} + \sum_{i=3,5,6} \langle \phi_i, u\rangle_{\Gamma_i} \quad \forall u \in \mathbf{V}(\Omega). \end{aligned}$$

Thus, we have a unique existence of solution and estimate (5.42).

If $v_1 = w_1 + U_1, v_2 = w_2 + U_2$ are solutions corresponding to the given data, we get

$$\begin{aligned} &a_{03}(w_1, u - w_1) + J_1(u) - J_1(w_1) \geq \langle F_3^1, u - w_1 \rangle, \\ &a_{03}(w_2, u - w_2) + J_2(u) - J_2(w_2) \geq \langle F_3^2, u - w_2 \rangle \quad \forall u \in \mathbf{V}(\Omega), \end{aligned} \tag{5.45}$$

where $J_j(u), F_3^j, j = 1, 2$, are one corresponding to $U_j, g_{\tau j}, g_{nj}, g_{+nj}, g_{-nj}, f_j, h_i^j, \phi_i^j$. Putting $u = w_2, u = w_1$, respectively, in the first and second one in (5.45) and adding those, we have

$$\begin{aligned} &a_{03}(w_1 - w_2, w_2 - w_1) + J_1(w_2) - J_1(w_1) + J_2(w_1) - J_2(w_2) \\ &\geq \langle F_3^1 - F_3^2, w_2 - w_1 \rangle. \end{aligned} \tag{5.46}$$

By Korn's inequality and Lemma 2.3 we have

$$a_{03}(w_1 - w_2, w_1 - w_2) \geq \delta \|w_1 - w_2\|^2_{\mathbf{V}}. \tag{5.47}$$

From (5.46), (5.47) we have

$$\begin{aligned} &\|w_1 - w_2\|^2_{\mathbf{V}} \\ &\leq \frac{1}{\delta}\Big(|\langle F_3^1 - F_3^2, w_2 - w_1 \rangle| + |J_1(w_2) - J_1(w_1) + J_2(w_1) - J_2(w_2)| \Big). \end{aligned} \tag{5.48}$$

Since $w_1, w_2 \in K(\Omega)$,

$$\begin{aligned} J_1(w_2) - J_1(w_1) = &\int_{\Gamma_8} g_{\tau 1}(|w_{2\tau}| - |w_{1\tau}|)\, ds + \int_{\Gamma_9} g_{n1}(|w_{2n}| - |w_{1n}|)\, ds \\ &+ \int_{\Gamma_{10}} g_{+n1}(w_{2n} - w_{1n})\, ds - \int_{\Gamma_{11}} g_{-n1}(w_{2n} - w_{1n})\, ds, \\ J_2(w_2) - J_2(w_1) = &\int_{\Gamma_8} g_{\tau 2}(|w_{2\tau}| - |w_{1\tau}|)\, ds + \int_{\Gamma_9} g_{n2}(|w_{2n}| - |w_{1n}|)\, ds \\ &+ \int_{\Gamma_{10}} g_{+n2}(w_{2n} - w_{1n})\, ds - \int_{\Gamma_{11}} g_{-n2}(w_{2n} - w_{1n})\, ds. \end{aligned} \tag{5.49}$$

Subtracting two formulae in (5.49), we have

$$\begin{aligned} &|J_1(w_2) - J_1(w_1) + J_2(w_1) - J_2(w_2)| \\ &\leq \|g_{\tau 1} - g_{\tau 2}\|_{\mathbf{L}^2_\tau(\Gamma_8)} \|w_{2\tau} - w_{1\tau}\|_{\mathbf{L}^2_\tau(\Gamma_8)} + \|g_{n1} - g_{n2}\|_{L^2(\Gamma_9)} \|w_{2n} - w_{1n}\|_{L^2(\Gamma_9)} \\ &\quad + \|g_{+n1} - g_{+n2}\|_{L^2(\Gamma_{10})} \|w_{n2} - w_{n1}\|_{L^2(\Gamma_{10})} \\ &\quad + \|g_{-n1} - g_{-n2}\|_{L^2(\Gamma_{11})} \|w_{2n} - w_{1n}\|_{L^2(\Gamma_{11})} \\ &\leq M\big(\|g_{\tau 1} - g_{\tau 2}\|_{\mathbf{L}^2_\tau(\Gamma_8)} + \|g_{n1} - g_{n2}\|_{L^2(\Gamma_9)} + \|g_{+n1} - g_{+n2}\|_{L^2(\Gamma_{10})} \\ &\quad + \|g_{-n1} - g_{-n2}\|_{L^2(\Gamma_{11})} \big) \|w_2 - w_1\|_{\mathbf{V}(\Omega)}. \end{aligned} \tag{5.50}$$

By (5.48), (5.50) we have

$$\begin{aligned} \|w_1 - w_2\|_{\mathbf{V}} \leq \frac{M}{\delta}\Big(&\|F_3^1 - F_3^2\|_{\mathbf{V}(\Omega)^*} + \|g_{\tau 1} - g_{\tau 2}\|_{\mathbf{L}^2_\tau(\Gamma_8)} + \|g_{n1} - g_{n2}\|_{L^2(\Gamma_9)} \\ &+ \|g_{+n1} - g_{+n2}\|_{L^2(\Gamma_{10})} + \|g_{-n1} - g_{-n2}\|_{L^2(\Gamma_{11})} \Big), \end{aligned}$$

from which we get (5.43). □

REMARK 5.3. *The estimates of solutions* (5.9), (5.21), (5.42) *are independent from thresholds* $g_\tau, g_n, g_{+n}, g_{-n}$. *(cf. (8) in [3], (25) in [40].)*

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