

GLOBAL STABILITY OF LESLIE-TYPE PREDATOR-PREY MODEL*

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Abstract. In this paper we study the global stability of diffusive predator-prey system of Leslie type in a bounded domain $\Omega \subset R^N$ with no-flux boundary condition. By using a new approach, we establish much improved global asymptotic stability of the unique positive equilibrium solution. We also show how to extend the result to more general type of systems with non-homogeneous environment and/or wider class of kinetic terms.

Key words. Global asymptotic stability, predator-prey, positive equilibrium solution, comparison principle, reaction-diffusion.

AMS subject classifications. 34C20, 34C25, 92E20.

1. Introduction. In this paper we study the diffusive Leslie-type predator-prey system in a bounded domain $\Omega \subset R^N$ with no-flux boundary condition.

$$(I) \begin{cases} u_t = d_1 \Delta u + u(\lambda - \alpha u - \beta v), & (x, t) \in \Omega \times (0, \infty) \\ v_t = d_2 \Delta v + \mu v \left(1 - \frac{v}{u}\right) & (x, t) \in \Omega \times (0, \infty) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x, t) \in \partial \Omega \times (0, \infty) \\ u(x, 0) = u_0(x) > 0, \quad v(x, 0) = v_0(x) \geq 0 (\neq 0) & x \in \bar{\Omega}. \end{cases}$$

Here $u(x, t)$ and $v(x, t)$ are the density of prey and predator, respectively, Ω is a bounded domain with smooth boundary $\partial \Omega$, λ , μ , α , and β are positive constants. We assume throughout this paper that the two diffusion coefficients d_1 and d_2 are positive and equal, but not necessarily constants. The no-flux boundary condition is imposed to guarantee that the ecosystem is not disturbed by exterior factors which may influence population flow cross the boundary.

The system is a well established population model and is widely studied in literature, see [5, 6] and the reference therein. In particular, the following result was proved in [5] by the construction of a Lyapunov function.

THEOREM. Suppose d_1, d_2 are positive constants, and $\alpha > \beta$ or $\alpha/\beta > s_0 \in (1/5, 1/4)$, then the unique positive equilibrium point

$$(u^*, v^*) = \left(\frac{\lambda}{\alpha + \beta}, \frac{\lambda}{\alpha + \beta} \right)$$

is globally asymptotically stable in the sense that every solution to (I) satisfies

$$\lim_{t \rightarrow \infty} (u, v) = (u^*, v^*) \quad \text{uniformly in } \Omega.$$

It is interesting to ask whether (u^*, v^*) is a global attractor under all combination of the parameters $(\alpha, \beta, \lambda, \mu)$. As a matter of fact, the following open question was proposed in [6]:

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OPEN QUESTION. Is (u^*, v^*) globally asymptotically stable for all combination of α and β ?

But, it seems to us that the above result and the open problem is somewhat biased as they should not ignore the role of μ and λ in their statements, which definitely play an important part in the dynamics of solutions.

In this work, we prove a new global stability result for the positive equilibrium by using a novel comparison argument, which is different from the one used in literature such as [2].

Our main result is as follows.

THEOREM 1. *Suppose the two diffusion coefficients are strictly positive and $d_1 = d_2$, and $(\alpha, \beta, \lambda, \mu)$ are positive constants. Then, (u^*, v^*) is globally asymptotically stable if $\mu > \beta\lambda/\alpha$.*

REMARK. Our result is completely new and in a way answers the open question. Given any α and β , (u^*, v^*) is globally asymptotically stable if μ and λ are chosen suitably.

REMARK. Our result can be interpreted as saying that if μ is suitably large, in relation to (α, β, λ) , then the evolution of v in time will be adjusted sufficiently fast to any change in u so that both converge to the same equilibrium value as $t \rightarrow \infty$. It reveals more intimate relation of the various parameters to determine the large time behavior of solution than previous works.

REMARK. It will be clear from our proof that a simplified version of our approach can yield (u^*, v^*) is globally asymptotically stable if $\alpha > \beta$, without the restriction of $d_1 = d_2$.

REMARK. The method we use here is more flexible than the Lyapunov function method and the results covers more general settings such as when the Laplace operator is replaced by a uniform elliptic operator. It means the environment is non-homogeneous.

Let

$$\mathfrak{L}u = \sum_{i,j=1}^N a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}$$

be a uniform elliptic operator in Ω with continuous coefficients $a_{ij}(x)$, $i, j = 1, \dots, N$. Then, we can show a result similar to Theorem 1 for the following initial-boundary value problem:

$$(II) \begin{cases} u_t = \mathfrak{L}u + u(\lambda - \alpha u - \beta v), & (x, t) \in \Omega \times (0, \infty) \\ v_t = \mathfrak{L}v + \mu v \left(1 - \frac{v}{u}\right) & (x, t) \in \Omega \times (0, \infty) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x, t) \in \partial\Omega \times (0, \infty) \\ u(x, 0) = u_0(x) > 0, v(x, 0) = v_0(x) \geq 0 (\neq 0) & x \in \bar{\Omega}. \end{cases}$$

THEOREM 2. *Suppose $(\alpha, \beta, \lambda, \mu)$ are positive constants. Then, the unique positive equilibrium (u^*, v^*) of (II) is globally asymptotically stable if $\mu > \beta\lambda/\alpha$.*

In another direction, we study the following system where a more general type of reaction-terms is considered.

$$(III) \begin{cases} u_t = d_1 \Delta u + u(\lambda - \alpha u^\sigma - \beta v), & (x, t) \in \Omega \times (0, \infty) \\ v_t = d_2 \Delta v + \mu v \left(1 - \frac{v}{u^\sigma}\right) & (x, t) \in \partial\Omega \times (0, \infty) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x, t) \in \partial\Omega \times (0, \infty) \\ u(x, 0) = u_0(x) > 0, v(x, 0) = v_0(x) \geq 0 (\neq 0) & x \in \Omega. \end{cases}$$

Here $0 < \sigma < 1$ and d_1, d_2 , and $(\alpha, \beta, \lambda, \mu)$ are as in Theorem 1.

THEOREM 3. *Suppose $d_1 = d_2 > 0$, $0 < \sigma < 1$ and $(\alpha, \beta, \lambda, \mu)$ are positive constants. Then, the unique positive equilibrium*

$$(u^*, v^*) = \left(\left(\frac{\lambda}{\alpha + \beta} \right)^{1/\sigma}, \frac{\lambda}{\alpha + \beta} \right)$$

of (III) is globally asymptotically stable if $\mu > \beta\lambda\sigma/\alpha$.

The organization of the paper is that we shall prove Theorem 1 in section 2. In section 3, we shall discuss results on the more general setting to prove Theorem 2 and Theorem 3. We end the paper with discussion on how the method can be applied to other type of models in section 4.

2. Proof of Theorem 1. Let $w = \frac{v}{u}$. It's easy to compute

$$w_t = \frac{v_t}{u} - \frac{u_t v}{u^2}, \quad \nabla w = \frac{\nabla v}{u} - \frac{\nabla u}{u^2} v,$$

$$\Delta w = \frac{\Delta v}{u} - \frac{v \Delta u}{u^2} - \frac{2 \nabla u \cdot \nabla v}{u^2} + \frac{2 |\nabla u|^2}{u^3} v.$$

The equation satisfied by w is

$$\begin{aligned} w_t - d_1 \Delta w &= \mu \frac{v}{u} \left(1 - \frac{v}{u}\right) - \frac{v}{u} (\lambda - \alpha u - \beta v) + \frac{2d_1}{u} \nabla u \cdot \nabla w \\ &= w(\mu - \lambda + \alpha u - w(\mu - \beta u)) + \frac{2d_1}{u} \nabla u \cdot \nabla w. \end{aligned} \tag{2.1}$$

LEMMA 2.1. *Suppose $\mu > \frac{\beta}{\alpha} \lambda$ and $\varepsilon_1 > 0$ small. There exists T sufficiently large such that when $t \geq T$,*

$$u \leq \bar{u}_2(\varepsilon_1) \equiv \frac{\lambda}{\alpha} \left[1 - \frac{\beta(\mu - \beta\sigma\bar{u}_1)}{(\alpha + \beta)\mu - \beta\lambda} \right] + O(\varepsilon_1), \text{ in } \Omega,$$

where $\bar{u}_1 \equiv \frac{\lambda}{\alpha}$.

Proof. Since $v \geq 0$, it's clear that u satisfies

$$u_t - d_1 \Delta u \leq u(\lambda - \alpha u) \text{ in } \Omega \times (0, \infty).$$

It is a well established fact that any positive solution of

$$\begin{cases} u_t - d_1 \Delta u = u(\lambda - \alpha u), & \text{in } \Omega \times (0, \infty) \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0, \infty) \end{cases}$$

converges uniformly to $\frac{\lambda}{\alpha}$ as $t \rightarrow \infty$. Therefore, $\exists t_1 > 0$ such that

$$u(x, t) < \bar{u}_1(\varepsilon_1) \equiv \frac{\lambda}{\alpha} + \frac{\varepsilon_1}{5} \text{ in } \Omega \times [t_1, \infty).$$

Then,

$$w_t - d_1 \Delta w \leq w[(\mu - \lambda + \alpha \bar{u}_1(\varepsilon_1)) - w(\mu - \beta \bar{u}_1(\varepsilon_1))] + \frac{2d}{u} \nabla u \cdot \nabla w.$$

We assume ε_1 is sufficiently small so that $\mu > \beta \bar{u}_1(\varepsilon_1)$. Hence, $w(x, t + t_1) \leq W(t)$, where $W(t)$ is a solution of

$$\begin{cases} W_t = W[(\mu - \lambda + \alpha \bar{u}_1(\varepsilon_1)) - W(\mu - \beta \bar{u}_1(\varepsilon_1))], \\ W(0) = \max_{\bar{\Omega}} W(x, t_1). \end{cases}$$

It is clear that $\exists t_2 > t_1$ such that

$$w(x, t) \leq \bar{w}_1(\varepsilon_1) \equiv \frac{\mu - \lambda + \alpha \bar{u}_1(\varepsilon_1)}{\mu - \beta \bar{u}_1(\varepsilon_1)} + \frac{\varepsilon_1}{5} \text{ in } \Omega \times [t_2, \infty).$$

Substitute the above inequality into the first equation in (I), we have

$$u_t - d \Delta u \geq u[\lambda - \alpha u - \beta \bar{w}_1(\varepsilon_1)u] \text{ in } \Omega \times [t_2, \infty).$$

This, in turn, implies there exists $t_3 > t_2$ such that

$$u \geq \underline{u}_1(\varepsilon_1) \equiv \frac{\lambda}{\alpha + \beta \bar{w}_1(\varepsilon_1)} - \frac{\varepsilon_1}{5} \text{ in } \Omega \times [t_3, \infty).$$

By using the above inequality in the second equation, one obtains that there exists $t_4 > t_3$ such that

$$v \geq \underline{v}_1(\varepsilon_1) = \frac{\lambda}{\alpha + \beta \bar{w}_1(\varepsilon_1)} - \frac{\varepsilon_1}{4} \text{ in } \Omega \times [t_4, \infty).$$

Subsequently, when the above lower bound of v is used in the first equation of (I), we obtain

$$u_t - d_1 \Delta u \leq u[\lambda - \alpha u - \beta \underline{v}_1(\varepsilon_1)] \text{ in } \Omega \times [t_4, \infty).$$

This yields there exists $t_5 > t_4$ such that

$$u \leq \bar{u}_2(\varepsilon_1) \equiv \frac{\lambda - \beta \underline{v}_1(\varepsilon_1)}{\alpha} + \frac{\varepsilon_1}{5} \text{ in } \Omega \times [t_5, \infty).$$

Simple computation shows

$$\begin{aligned} \bar{u}_2(\varepsilon_1) &= \frac{\lambda - \beta \underline{v}_1(\varepsilon_1)}{\alpha} + \frac{\varepsilon_1}{5} \\ &= \frac{\lambda}{\alpha} \left[1 - \frac{\beta}{\alpha + \beta \bar{w}_1(\varepsilon)} \right] + O(\varepsilon_1) \\ &= \frac{\lambda}{\alpha} \left[1 - \frac{\beta}{\alpha + \beta \left[\frac{\mu - \lambda + \alpha \bar{u}_1}{\mu - \beta \bar{u}_1} \right]} \right] + O(\varepsilon_1) \\ &= \frac{\lambda}{\alpha} \left[1 - \frac{\beta(\mu - \beta \bar{u}_1)}{(\alpha + \beta)\mu - \beta\lambda} \right] + O(\varepsilon_1). \end{aligned} \tag{2.2}$$

This proves the lemma. \square

By repeating the above procedure, for any positive integer n , there exists t_M sufficiently large such that when $t \geq t_M$,

$$u \leq \bar{u}_{n+1}(\varepsilon_1) \equiv \frac{\lambda - \beta \underline{u}_n(\varepsilon_1)}{\alpha} + O(\varepsilon_1),$$

$$u \geq \underline{u}_n(\varepsilon_1) \equiv \frac{\lambda(\mu - \beta \bar{u}_n)}{(\alpha + \beta)\mu - \lambda\beta} + O(\varepsilon_1),$$

uniformly in Ω . Let $\varepsilon_1 = 0$, we have

$$\bar{u}_{n+1} = \frac{\lambda - \beta \underline{u}_n}{\alpha}, \quad \underline{u}_n = \frac{\lambda[\mu - \beta \bar{u}_n]}{(\alpha + \beta)\mu - \lambda\beta}, \quad n = 1, 2, \dots$$

with $\bar{u}_1 > \bar{u}_2 > u^*$, $\underline{u}_1 < u^*$. It's easy to see that $\{\bar{u}_n\}$ is a decreasing sequence with $\bar{u}_n > u^*$, $\forall n \geq 1$ and $\{\underline{u}_n\}$ is an increasing sequence with $\underline{u}_n < u^*$, $\forall n \geq 1$. Suppose

$$\lim_{n \rightarrow \infty} \bar{u}_n = \bar{u}^* \quad \text{and} \quad \lim_{n \rightarrow \infty} \underline{u}_n = \underline{u}^*,$$

then

$$\bar{u}^* = \frac{\lambda - \beta \underline{u}^*}{\alpha},$$

$$\underline{u}^* = \frac{\lambda[\mu - \beta \bar{u}^*]}{(\alpha + \beta)\mu - \lambda\beta}$$

$$= \frac{\lambda}{\alpha + \beta} + \frac{\lambda\beta}{(\alpha + \beta)\mu - \lambda\beta} \left(\frac{\lambda}{\alpha + \beta} - \beta \bar{u}^* \right).$$

The combination of the two yields

$$\underline{u}^* = \frac{\lambda}{\alpha + \beta} + \frac{\lambda\beta}{(\alpha + \beta)\mu - \lambda\beta} \left(\frac{\lambda}{\alpha + \beta} - \frac{\lambda}{\alpha} + \frac{\beta \underline{u}^*}{\alpha} \right),$$

which has a unique solution $\underline{u}^* = u^*$. Consequently, $\bar{u}^* = u^*$. This shows

$$\lim_{n \rightarrow \infty} \bar{u}_n = u^* \quad \text{and} \quad \lim_{n \rightarrow \infty} \underline{u}_n = u^*.$$

Similarly,

$$\bar{v}_n(\varepsilon_1) = \bar{u}_n(\varepsilon_1)\bar{w}_n(\varepsilon_1) + O(\varepsilon_1),$$

$$\underline{v}_n(\varepsilon_1) = \underline{u}_n(\varepsilon_1) + O(\varepsilon_1)$$

with

$$\bar{w}_n(\varepsilon_1) = \frac{\mu - \lambda + \alpha \bar{u}_n(\varepsilon_1)}{\mu - \beta \bar{u}_n(\varepsilon_1)} + O(\varepsilon_1).$$

Setting $\varepsilon_1 = 0$, we have

$$\bar{w}_n = \frac{\mu - \lambda + \alpha \bar{u}_n}{\mu - \beta \bar{u}_n}, \quad \bar{v}_n = \bar{u}_n \bar{w}_n, \quad \underline{v}_n = \underline{u}_n.$$

$$\lim_{n \rightarrow \infty} \bar{w}_n = 1 \text{ and } \lim_{n \rightarrow \infty} \underline{v}_n = v^*.$$

Now, we show $\lim_{t \rightarrow \infty} (u, v) = (u^*, v^*)$ uniformly in Ω .

Proof of Theorem 1. $\forall \varepsilon > 0$, there exists $n_0 > 1$ such that when $n \geq n_0$,

$$|\bar{u}_n - u^*| + |\underline{u}_n - u^*| < \frac{\varepsilon}{4}. \tag{2.3}$$

Choose $\varepsilon_1 > 0$ sufficiently small such that

$$|\bar{u}_{n_0}(\varepsilon_1) - \bar{u}_{n_0}| + |\underline{u}_{n_0}(\varepsilon_1) - \underline{u}_{n_0}| < \frac{\varepsilon}{4} \tag{2.4}$$

and the same to $\underline{v}_n(\varepsilon_1)$, \underline{v}_n , $\bar{v}_n(\varepsilon_1)$, \bar{v}_n and v^* . Furthermore, there exists $t_M \gg 1$ such that when $t \geq t_M$,

$$\underline{u}_{n_0}(\varepsilon_1) \leq u(x, t) \leq \bar{u}_{n_0}(\varepsilon_1) \text{ in } \Omega.$$

Hence, by (2.3) and (2.4), when $t \geq t_M$,

$$|u(x, t) - u^*| < \varepsilon \text{ in } \Omega.$$

This proves $\lim_{t \rightarrow \infty} u(x, t) = u^*$ uniformly in Ω . Similarly, $\lim_{t \rightarrow \infty} v(x, t) = v^*$ uniformly in Ω . \square

REMARK. It is clear from the proof of Lemma 2.1 that if $\beta < \alpha$, using $u \leq \tilde{u}_1 = \lambda/\alpha$ in the second equation of (I) we get, ignoring ε_1 , $v \leq \tilde{v}_1 = \lambda/\alpha$, which in turn, when used in the first equation gives

$$u \geq \tilde{u}_1 = \frac{\lambda}{\alpha} - \frac{\beta}{a} \tilde{u}_1.$$

This will enable us to obtain, from the second equation, $v \geq \tilde{u}_1$ and subsequently, from the first equation,

$$u \leq \tilde{u}_2 = \frac{\lambda}{\alpha} - \frac{\beta}{\alpha} \tilde{u}_1.$$

An iteration of the above procedure resulted in two sequences

$$\tilde{u}_{n+1} = \frac{\lambda}{\alpha} - \frac{\beta}{a} \tilde{u}_n, \quad \tilde{u}_n = \frac{\lambda}{\alpha} - \frac{\beta}{\alpha} \tilde{u}_{n-1}$$

with $\tilde{u}_n > u^*$, $\tilde{u}_n < u^*$ and both converge to u^* . This recovers the result of [5] when $\alpha > \beta$ but without the restriction that $d_1, d_2 > 0$ must be constants.

3. More general settings. In this section, we prove Theorem 2 and Theorem 3. We start by looking into (III).

Let $w = \frac{v}{u^\sigma}$. It's easy to compute

$$w_t = \frac{v_t}{u^\sigma} - \frac{\sigma u_t v}{u^{\sigma+1}}, \quad \nabla w = \frac{\nabla v}{u^\sigma} - \frac{\sigma \nabla u}{u^{\sigma+1}} v,$$

$$\Delta w = \frac{\Delta v}{u^\sigma} - \frac{\sigma v \Delta u}{u^{\sigma+1}} - \frac{2\sigma \nabla u \cdot \nabla v}{u^{\sigma+1}} + \frac{\sigma(\sigma+1)|\nabla u|^2}{u^{\sigma+2}} v.$$

The equation satisfied by w is

$$w_t - d_1 \Delta w = w[(\mu - \sigma\lambda + \sigma\alpha u^\sigma) - w(\mu - \beta\sigma u^\sigma)] + \frac{2d_1\sigma \nabla u \cdot \nabla w}{u} + \frac{d_1\sigma(\sigma - 1)|\nabla u|^2 w}{u^2}.$$

LEMMA 3.1. *Suppose $\mu > \frac{\beta\sigma\lambda}{\alpha}$, $0 < \sigma < 1$ and $\varepsilon_1 > 0$ small. There exists T sufficiently large such that when $t \geq T$,*

$$u^\sigma \leq \bar{u}_2^\sigma(\varepsilon_1) \equiv \frac{\lambda}{\alpha} \left(1 - \frac{\beta(\mu - \beta\sigma\bar{u}_1^\sigma)}{\alpha\mu + \beta\mu - \beta\sigma\lambda} \right) + O(\varepsilon_1), \quad \text{in } \Omega,$$

where $\bar{u}_1^\sigma \equiv \frac{\lambda}{\alpha}$.

Proof. Since the proof follows the same process as that of Lemma 2.1, we shall be brief. First, u satisfies

$$u_t - d_1 \Delta u \leq u(\lambda - \alpha u^\sigma) \quad \text{in } \Omega \times (0, \infty),$$

from which we derive that $\exists t_1 > 0$ such that

$$u^\sigma(x, t) < \bar{u}_1^\sigma(\varepsilon_1) \equiv \frac{\lambda}{\alpha} + \frac{\varepsilon_1}{5} \quad \text{in } \Omega \times [t_1, \infty).$$

This is because any positive solution of

$$\begin{cases} u_t - d_1 \Delta u = u(\lambda - \alpha u^\sigma), & \text{in } \Omega \times (0, \infty) \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0, \infty) \end{cases}$$

converges uniformly to $(\frac{\lambda}{\alpha})^{1/\sigma}$ as $t \rightarrow \infty$. Then,

$$w_t - d_1 \Delta w \leq w[(\mu - \lambda\sigma + \alpha\sigma\bar{u}_1^\sigma(\varepsilon_1)) - w(\mu - \beta\sigma\bar{u}_1^\sigma(\varepsilon_1))] + \frac{2d_1\sigma \nabla u \cdot \nabla w}{u}.$$

We assume ε_1 is sufficiently small so that $\mu > \beta\bar{u}_1^\sigma(\varepsilon_1)$. Hence, $w(x, t + t_1) \leq W(t)$, where $W(t)$ is a solution of

$$\begin{cases} W_t = W[(\mu - \lambda\sigma + \alpha\sigma\bar{u}_1^\sigma(\varepsilon_1)) - W(\mu - \beta\sigma\bar{u}_1^\sigma(\varepsilon_1))], \\ W(0) = \max_{\bar{\Omega}} W(x, t_1). \end{cases}$$

It is clear that $\exists t_2 > t_1$ such that

$$w(x, t) \leq \bar{w}_1(\varepsilon_1) \equiv \frac{\mu - \sigma\lambda + \alpha\sigma\bar{u}_1^\sigma(\varepsilon_1)}{\mu - \beta\sigma\bar{u}_1^\sigma(\varepsilon_1)} + \frac{\varepsilon_1}{5} \quad \text{in } \Omega \times [t_2, \infty).$$

Substitute the above inequality into the first equation in (III), we have

$$u_t - d_1 \Delta u \geq u[\lambda - \alpha u^\sigma - \beta\bar{w}_1(\varepsilon_1)u^\sigma] \quad \text{in } \Omega \times [t_2, \infty),$$

and there exists $t_3 > t_2$ such that

$$u^\sigma \geq \underline{u}_1^\sigma(\varepsilon_1) \equiv \frac{\lambda}{\alpha + \beta\bar{w}_1(\varepsilon_1)} - \frac{\varepsilon_1}{5} \quad \text{in } \Omega \times [t_3, \infty).$$

By using the above inequality in the second equation, one obtains that there exists $t_4 > t_3$ such that

$$v \geq \underline{v}_1(\varepsilon_1) = \frac{\lambda}{\alpha + \beta \underline{w}_1(\varepsilon_1)} - \frac{\varepsilon_1}{4} \text{ in } \Omega \times [t_4, \infty).$$

Subsequently, when the above lower bound of v is used in the first equation of (II), we obtain

$$u_t - d_1 \Delta u \leq u[\lambda - \alpha u^\sigma - \beta \underline{v}_1(\varepsilon_1)] \text{ in } \Omega \times [t_4, \infty).$$

This yields there exists $t_5 > t_4$ such that

$$u^\sigma \leq \bar{u}_2^\sigma(\varepsilon_1) \equiv \frac{\lambda - \beta \underline{v}_1(\varepsilon_1)}{\alpha} + \frac{\varepsilon_1}{5} \text{ in } \Omega \times [t_5, \infty).$$

Simple computation shows

$$\begin{aligned} \bar{u}_2^\sigma(\varepsilon_1) &= \frac{\lambda - \beta \underline{v}_1(\varepsilon_1)}{\alpha} + \frac{\varepsilon_1}{5} \\ &= \frac{\lambda}{\alpha} \left(1 - \frac{\beta}{\alpha + \beta \bar{w}_1(\varepsilon_1)} \right) + O(\varepsilon_1) \\ &= \frac{\lambda}{\alpha} \left(1 - \frac{\beta(\mu - \beta \sigma \bar{u}_1^\sigma)}{\alpha \mu + \beta \mu - \beta \sigma \lambda} \right) + O(\varepsilon_1). \end{aligned} \tag{3.5}$$

□

By repeating the above procedure, for any positive integer n , there exists t sufficiently large such that

$$\begin{aligned} u^\sigma &\leq \bar{u}_{n+1}^\sigma(\varepsilon_1) \equiv \frac{\lambda - \beta \underline{u}_n^\sigma(\varepsilon_1)}{\alpha} + O(\varepsilon_1), \\ u^\sigma &\geq \underline{u}_n^\sigma(\varepsilon_1) \equiv \frac{\lambda(\mu - \beta \sigma \bar{u}_n^\sigma)}{(\alpha + \beta)\mu - \lambda \beta \sigma} + O(\varepsilon_1), \end{aligned}$$

uniformly in Ω . Let $\varepsilon_1 = 0$, we have

$$\bar{u}_{n+1}^\sigma = \frac{\lambda - \beta \sigma \underline{u}_n^\sigma}{\alpha}, \quad \underline{u}_n^\sigma = \frac{\lambda[\mu - \beta \sigma \bar{u}_n^\sigma]}{(\alpha + \beta)\mu - \lambda \beta \sigma}, \quad n = 1, 2, \dots$$

with $\bar{u}_1^\sigma > \bar{u}_2^\sigma > (u^*)^\sigma$, $\underline{u}_1^\sigma < (u^*)^\sigma$. It's easy to see that $\{\bar{u}_n^\sigma\}$ is a decreasing sequence and $\{\underline{u}_n^\sigma\}$ is an increasing sequence with

$$\lim_{n \rightarrow \infty} \bar{u}_n^\sigma = \lim_{n \rightarrow \infty} \underline{u}_n^\sigma = (u^*)^\sigma.$$

Similarly,

$$\bar{v}_n(\varepsilon_1) = \bar{u}_n^\sigma(\varepsilon_1) \bar{w}_n(\varepsilon_1) + O(\varepsilon_1),$$

$$\underline{v}_n(\varepsilon_1) = \underline{u}_n^\sigma(\varepsilon_1) + O(\varepsilon_1)$$

with

$$\bar{w}_n(\varepsilon_1) = \frac{\mu - \sigma\lambda + \alpha\sigma\bar{u}_n^\sigma(\varepsilon_1)}{\mu - \beta\sigma\bar{u}_n^\sigma(\varepsilon_1)} + O(\varepsilon_1).$$

Setting $\varepsilon_1 = 0$, we have

$$\bar{w}_n = \frac{\mu - \lambda\sigma + \alpha\sigma\bar{u}_n^\sigma}{\mu - \beta\sigma\bar{u}_n^\sigma}, \quad \bar{v}_n = \bar{u}_n^\sigma\bar{w}_n, \quad \underline{v}_n = \underline{u}_n^\sigma.$$

$$\lim_{n \rightarrow \infty} \bar{w}_n = 1 \quad \text{and} \quad \lim_{n \rightarrow \infty} \underline{v}_n = v_\sigma^*.$$

Now, we show $\lim_{t \rightarrow \infty} (u, v) = (u_\sigma^*, v_\sigma^*)$ uniformly in Ω .

Proof of Theorem 3. It follows readily from the same steps, using the result of Lemma 3.1, as in the proof of Theorem 1, and we omit the details. \square

The proof of Theorem 2 is exactly the same as in the proof of Theorem 1 and we shall not repeat it here.

4. Discussion. It is clear that we can combine the features of (II) and (III) to get a more complex model and still the same result as in Theorem 3. But, for simplicity, we will not write it down.

The method we develop in this work is new and can be applied to many interesting reaction-diffusion type models where the stability of a unique positive equilibrium solution is a key issue to be studied. For example, the Holling-Tanner predator-prey model,

$$\begin{cases} u_t = d_1 \Delta u + au - u^2 - \frac{uv}{m+u}, & (x, t) \in \Omega \times (0, \infty) \\ v_t = d_2 \Delta v + bv - \frac{v^2}{\gamma u} & (x, t) \in \Omega \times (0, \infty) \\ \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0, & (x, t) \in \partial\Omega \times (0, \infty) \\ u(x, 0) = u_0(x) > 0, \quad v(x, 0) = v_0(x) \geq 0 (\neq 0) & x \in \bar{\Omega}, \end{cases}$$

which was studied in [2], [3] and [10], will be investigated in a forthcoming work [11], where we can derive new and improved global stability result.

It will be interesting to see how can we incorporate other interesting features such as time delay into our scheme.

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