

A SPECTRAL REPRESENTATION FOR SPIN-WEIGHTED SPHEROIDAL WAVE OPERATORS WITH COMPLEX ASPHERICAL PARAMETER*

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Abstract. A family of spectral decompositions of the spin-weighted spheroidal wave operator is constructed for complex aspherical parameters with bounded imaginary part. As the operator is not symmetric, its spectrum is complex and Jordan chains may appear. We prove uniform upper bounds for the length of the Jordan chains and the norms of the idempotent operators mapping onto the invariant subspaces. The completeness of the spectral decomposition is proven.

Key words. Non-selfadjoint spectral problem, Sturm-Liouville equation, spin-weighted spheroidal wave operator, angular Teukolsky operator.

AMS subject classifications. 47E05, 34B05, 34L10, 34L15.

1. Introduction and Statement of Results. The spin-weighted spheroidal wave equation arises in the study of electromagnetic, gravitational and neutrino-field perturbations of rotating black holes when separating variables in the so-called Teukolsky master equation (see [2, 22] or the survey paper [7]). In the spin-weighted spheroidal wave equation, the *spin* of the wave enters as a parameter $s \in \{0, \frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$. We are mainly interested in the cases $s = 1$ of an electromagnetic and $s = 2$ of a gravitational field. If s is an integer, the spin-weighted wave equation is the eigenvalue equation

$$\mathcal{A}\Psi = \lambda\Psi, \tag{1.1}$$

where the spin-weighted spheroidal wave operator \mathcal{A} is an elliptic operator with smooth coefficients on the unit sphere S^2 . More specifically, choosing polar coordinates $\vartheta \in (0, \pi)$ and $\varphi \in [0, 2\pi)$, we have (see for example [24])

$$\mathcal{A} = -\frac{\partial}{\partial \cos \vartheta} \sin^2 \vartheta \frac{\partial}{\partial \cos \vartheta} + \frac{1}{\sin^2 \vartheta} \left(\Omega \sin^2 \vartheta + i \frac{\partial}{\partial \varphi} - s \cos \vartheta \right)^2.$$

Here $\Omega \in \mathbb{C}$ is the *aspherical parameter*. In the special case $\Omega = 0$, we obtain the spin-weighted Laplacian on the sphere, whose eigenvalues and eigenfunctions can be given explicitly [14]. In the case $s = 0$ and $\Omega \neq 0$, one gets the spheroidal wave operator ([13, 8]). Setting $\Omega = 0$ and $s = 0$, one simply obtains the Laplacian on the sphere. We consider \mathcal{A} on the Hilbert space $\mathcal{H} = L^2(S^2)$ with domain of definition $\mathcal{D}(\mathcal{A}) = C^\infty(S^2)$. We remark that \mathcal{A} clearly is an elliptic operator on the sphere. However, even in the case $s = 0$ and for real Ω , in general there is no Riemannian metric on the sphere which realizes the spheroidal wave operator as the Laplace-Beltrami operator. Thus the spheroidal wave operator cannot be identified with the Laplace-Beltrami operator on a Riemannian manifold. For general spin, this means in particular that the methods used for spin-weighted spherical harmonics in [17, Section 4.15] do not seem to generalize to the spheroidal situation.

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As the spin-weighted spheroidal wave operator is axisymmetric, we can separate out the φ -dependence with a plane wave ansatz,

$$\Psi(\vartheta, \varphi) = e^{-ik\varphi} \Theta(\vartheta) \quad \text{with } k \in \mathbb{Z}.$$

Then \mathcal{A} becomes the ordinary differential operator

$$\mathcal{A}_k := -\frac{\partial}{\partial \cos \vartheta} \sin^2 \vartheta \frac{\partial}{\partial \cos \vartheta} + \frac{1}{\sin^2 \vartheta} (\Omega \sin^2 \vartheta + k - s \cos \vartheta)^2. \quad (1.2)$$

This operator acts on the vectors in \mathcal{H} with the prescribed ϕ -dependence, which we denote by \mathcal{H}_k ,

$$\mathcal{H}_k := L^2(S^2) \cap \{e^{-ik\varphi} \Theta(\vartheta) \mid \Theta : (0, \pi) \rightarrow \mathbb{C}\}.$$

The domain of definition reduces to

$$\mathcal{D}(\mathcal{A}_k) = C^\infty(S^2) \cap \mathcal{H}_k.$$

The Hilbert space \mathcal{H}_k can be identified with

$$\mathcal{H}_k = L^2((0, \pi), \sin \vartheta d\vartheta).$$

Also, one can consider \mathcal{A}_k as an ordinary differential operator on this Hilbert space, for example with the domain of definition $C^\infty((0, \pi)) \cap L^2((0, \pi))$. However, when doing so, one still needs to specify boundary conditions at $\vartheta = 0, \pi$. As will be explained in detail in Section 2 below, the correct boundary conditions are that the limits

$$\lim_{\vartheta \rightarrow 0, \pi} \Theta(\vartheta) \quad \text{must exist.} \quad (1.3)$$

In this formulation as a pure ODE problem, the spheroidal wave equation (1.2) can also be used in the case of half-integer spin (to describe neutrino or Rarita-Schwinger fields), if k is chosen to be a half-integer. Thus in what follows, we fix the parameters s and k such that

$$2s \in \mathbb{N}_0 \quad \text{and} \quad k - s \in \mathbb{Z}. \quad (1.4)$$

We are interested in the case that Ω is *complex*. Then the potential in (1.2) is complex, so that the operator \mathcal{A}_k is not a symmetric operator on \mathcal{H}_k . As a consequence, the spectral theorem in Hilbert spaces does not apply. The spectrum will in general be complex. Moreover, the operator need not be diagonalizable, because Jordan chains may form. The main task of the present paper is to control the length of these Jordan chains to obtain a decomposition of the Hilbert space into invariant subspaces of \mathcal{A}_k . This is our main result:

THEOREM 1.1. *For any s and k in the range (1.4) and any $c > 0$, we let $U \subset \mathbb{C}$ be the strip*

$$|\operatorname{Im} \Omega| < c. \quad (1.5)$$

Then there is a positive integer N and a family of bounded linear operators $Q_n(\Omega)$ on \mathcal{H}_k defined for all $n \in \mathbb{N} \cup \{0\}$ and $\Omega \in U$ with the following properties:

- (i) *The image of the operator Q_0 is an N -dimensional invariant subspace of \mathcal{A}_k .*

(ii) For every $n \geq 1$, the image of the operator Q_n is an at most two-dimensional invariant subspace of \mathcal{A}_k .

(iii) The Q_n are uniformly bounded in $L(\mathcal{H}_k)$, i.e. for all $n \in \mathbb{N} \cup \{0\}$ and $\Omega \in U$,

$$\|Q_n\| \leq c_2 \quad (1.6)$$

for a suitable constant $c_2 = c_2(s, k, c)$ (here $\|\cdot\|$ denotes the sup-norm on \mathcal{H}_k).

(iv) The Q_n are idempotent and mutually orthogonal in the sense that

$$Q_n Q_{n'} = \delta_{n,n'} Q_n \quad \text{for all } n, n' \in \mathbb{N} \cup \{0\}.$$

(v) The Q_n are complete in the sense that for every $\Omega \in U$,

$$\sum_{n=0}^{\infty} Q_n = \mathbf{1} \quad (1.7)$$

with strong convergence of the series.

Note that the operators Q_n are in general not symmetric (i.e. $Q_n^* \neq Q_n$). This corresponds to the fact that for non-symmetric operators, the eigenvectors corresponding to different eigenvalues are in general not orthogonal.

2. Reformulation as a Sturm-Liouville Problem. We first bring the operator (1.2) to the standard Sturm-Liouville form (for more details see [12, Section 2]). To this end, we first write the operator in the variable $u = \vartheta \in (0, \pi)$,

$$\mathcal{A}_k = -\frac{1}{\sin u} \frac{d}{du} \sin u \frac{d}{du} + \frac{1}{\sin^2 u} (\Omega \sin^2 u + k - s \cos u)^2.$$

Introducing the function ϕ by

$$\phi = \sqrt{\sin u} \Theta, \quad (2.1)$$

we get the eigenvalue equation

$$H\phi = \lambda\phi, \quad (2.2)$$

where H has the form of a one-dimensional Hamiltonian

$$H = -\frac{d^2}{du^2} + W \quad (2.3)$$

where W is the complex potential

$$W = -\frac{1}{4} \frac{\cos^2 u}{\sin^2 u} - \frac{1}{2} + \frac{1}{\sin^2 u} (\Omega \sin^2 u + k - s \cos u)^2 \quad (2.4)$$

$$= \Omega^2 \sin^2 u + \left(k^2 + s^2 - \frac{1}{4}\right) \frac{1}{\sin^2 u} + 2\Omega k - s^2 - \frac{1}{4} \quad (2.5)$$

$$- 2s\Omega \cos u - 2sk \frac{\cos u}{\sin^2 u}. \quad (2.6)$$

For what follows, it is usually most convenient to write (2.2) as the the Sturm-Liouville equation

$$\left(-\frac{d^2}{du^2} + V\right) \phi = 0, \quad (2.7)$$

where V is the potential

$$V = \Omega^2 \sin^2 u + \left(k^2 + s^2 - \frac{1}{4}\right) \frac{1}{\sin^2 u} - 2s\Omega \cos u - 2sk \frac{\cos u}{\sin^2 u} - \mu, \quad (2.8)$$

and μ is the constant

$$\mu = \lambda - 2\Omega k + s^2 + \frac{1}{4}. \quad (2.9)$$

The transformation (2.1) from Θ to Y becomes a unitary transformation if the integration measure in the corresponding Hilbert spaces is transformed from $\sin u \, du$ to du . Hence the eigenvalue equation (1.1) on \mathcal{H}_k is equivalent to (2.7) on the Hilbert space $L^2((0, \pi), du)$.

If the potential V were continuous on the interval $[0, \pi]$, we would get a well-defined boundary problem by imposing Dirichlet or Neumann or more general mixed boundary values at 0 and π (for details see [3, Chapter 12]). In our situation, there is the complication that the potential (2.8) has poles at the boundary points. As a consequence, the fundamental solutions will also have singularities, so that it is no longer obvious how to introduce suitable boundary conditions. In the case when s is an integer, the correct boundary values can be determined by going back to the eigenfunctions on the sphere (1.1), as we now explain. Due to elliptic regularity theory, the eigenfunctions Ψ of the angular operator (1.1) are smooth functions on the sphere. Therefore, we obtain (1.3) as a necessary condition. In view of the transformation (2.1), this implies that the limits

$$\lim_{u \searrow 0} u^{-\frac{1}{2}} \phi(u) \quad \text{and} \quad \lim_{u \nearrow \pi} (\pi - u)^{-\frac{1}{2}} \phi(u) \quad \text{must exist.} \quad (2.10)$$

These boundary conditions can also be understood by looking at the asymptotics of the solutions of (2.7) near the boundary points. Namely, expanding the potential (2.8) near the boundary points, we obtain

$$\begin{aligned} V(u) &= \frac{1}{u^2} \left((k-s)^2 - \frac{1}{4} \right) + \mathcal{O}(u^{-1}) \\ V(u) &= \frac{1}{(\pi-u)^2} \left((k+s)^2 - \frac{1}{4} \right) + \mathcal{O}((\pi-u)^{-1}). \end{aligned}$$

If the factors $k \pm s$ are non-zero, the solutions have the asymptotics

$$\phi(u) \sim u^{\frac{1}{2} \pm |k-s|} (1 + \mathcal{O}(u)) \quad \text{and} \quad \phi(u) \sim (\pi-u)^{\frac{1}{2} \pm |k+s|} (1 + \mathcal{O}(\pi-u)). \quad (2.11)$$

If on the other hand, the factors $k \pm s$ are zero, the asymptotic solutions involve an additional logarithm (for detail see [12, Sections 7 and 8]),

$$\phi(u) = c_1 \sqrt{u} + c_2 \sqrt{u} \log u + \mathcal{O}(u) \quad \text{if } k = s \quad (2.12)$$

$$\phi(u) = c_1 \sqrt{\pi-u} + c_2 \sqrt{\pi-u} \log(\pi-u) + \mathcal{O}(\pi-u) \quad \text{if } k = -s. \quad (2.13)$$

In each case, the boundary conditions (2.10) single out one of the two fundamental solutions. In this way, the conditions (2.10) give mathematically reasonable boundary conditions. We remark that in the case (2.11), our boundary conditions are equivalent to Dirichlet boundary conditions. Alternatively, these boundary conditions could be implemented simply by demanding that the eigenfunctions must be square integrable

(note that, in view of (1.4), the parameter $k - s$ is always an integer). In the exceptional cases (2.12) and (2.13), however, both fundamental solutions satisfy Dirichlet boundary conditions and are square integrable. Thus in these cases, it is essential to state the boundary conditions in the form (2.10).

In order to bring the boundary conditions (2.10) into a more tractable form, it is convenient to work with solutions of the corresponding Riccati equation: For any solution ϕ of the Sturm-Liouville equation (2.7), the function $y := \phi'/\phi$ satisfies the corresponding Riccati equation

$$y' = V - y^2. \quad (2.14)$$

Using the results and methods in [10, 12], we can construct a solution y of the Riccati equation with rigorous error bounds. With this in mind, let us assume that a solution y of the Riccati equation is known. Then a particular solution of the corresponding Sturm-Liouville equation is obtained by integration,

$$\phi(u) = \exp\left(\int_{u_0}^u y\right). \quad (2.15)$$

The general solution can be constructed by integrating the equation for the Wronskian. Namely, if $\hat{\phi}$ is another solution of the Sturm-Liouville equation, the Wronskian

$$w(\phi, \hat{\phi}) := \phi' \hat{\phi} - \phi \hat{\phi}'$$

is a constant, and thus

$$\hat{\phi}(u) = \phi(u) \left(\frac{\hat{\phi}(u_0)}{\phi(u_0)} - \int_{u_0}^u \frac{w(\phi, \hat{\phi})}{\phi^2} \right). \quad (2.16)$$

In particular, this relation can be used to construct solutions of the Sturm-Liouville equation (2.7) which satisfy the boundary conditions (2.10). We denote these solutions by ϕ_L^D and ϕ_R^D (where the subscript D refers to ‘‘Dirichlet’’, and L/R to the left and right boundary points at $u = 0$ and $u = \pi$, respectively). To this end, we let ϕ_L and ϕ_R be generic solutions which do *not* satisfy the boundary conditions (2.10), i.e.

$$\phi_L(u) \sim \begin{cases} u^{\frac{1}{2}-|k-s|} (1 + \mathcal{O}(u)) & \text{if } k \neq s \\ \sqrt{u} \log u (1 + \mathcal{O}(u)) & \text{if } k = s \end{cases}$$

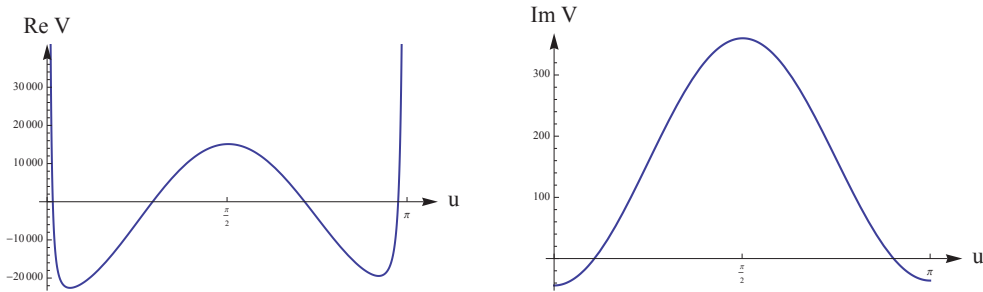
$$\phi_R(u) \sim \begin{cases} (\pi - u)^{\frac{1}{2}-|k+s|} (1 + \mathcal{O}(\pi - u)) & \text{if } k \neq -s \\ \sqrt{\pi - u} \log(\pi - u) (1 + \mathcal{O}(\pi - u)) & \text{if } k = -s. \end{cases}$$

Then, using (2.16), the solutions which do satisfy (2.10) are given (up to irrelevant prefactors) by

$$\phi_L^D(u) = \phi_L(u) \int_0^u \frac{1}{\phi_L^2} \quad \text{and} \quad \phi_R^D(u) = -\phi_R(u) \int_u^\pi \frac{1}{\phi_R^2}. \quad (2.17)$$

If ϕ is a solution of (2.7) subject to the boundary conditions (2.10), then this solution must be a multiple of both ϕ_L^D and ϕ_R^D . Hence ϕ_L^D and ϕ_R^D are linearly dependent, and their Wronskian vanishes,

$$w(\phi_L^D, \phi_R^D) = 0. \quad (2.18)$$

FIG. 1. Typical plot of the potential V .

In this way, we have reformulated the existence problem for solutions satisfying the boundary conditions (2.10) in terms of the vanishing of the Wronskian (2.18). More generally, the Wronskian can be used to describe the spectrum of the Hamiltonian. Namely, we just saw that if the Wronskian vanishes, then there is an eigensolution which satisfies the boundary conditions (2.7). Conversely, if the Wronskian is non-zero, we may introduce the *Green's function* by

$$s_\lambda(u, u') = \frac{1}{w(\phi_L^D, \phi_R^D)} \times \begin{cases} \phi_L^D(u) \phi_R^D(u') & \text{if } u \leq u' \\ \phi_L^D(u') \phi_R^D(u) & \text{if } u' < u. \end{cases} \quad (2.19)$$

By direct computation one verifies that the Green's function satisfies the equation

$$(H - \lambda) s_\lambda(u, u') = \delta(u - u').$$

Thus taking $s_\lambda(u, u')$ as the integral kernel of a corresponding operator s_λ on $\mathcal{H}_k = L^2((0, \pi), du)$, this operator is a bounded inverse of the operator $(H - \lambda)$. Thus λ is in the *resolvent set*, and s_λ is the *resolvent*. We conclude that the *spectrum* of H , defined as the complement of the resolvent set, is given as the set of all λ for which the Wronskian (2.18) vanishes for non-trivial solutions ϕ_L^D and ϕ_R^D satisfying the boundary conditions (2.10) at $u = 0$ and $u = \pi$, respectively.

3. The Qualitative Behavior of the Spectrum. We now explain qualitatively how the spectrum of the angular operator looks and how this qualitative behavior can be understood. This will also motivate and explain the statements in Theorem 1.1. Before discussing the effect of the imaginary part, we consider the situation that Ω and λ are real, so that V is real-valued. Then the spectrum can be understood most easily by considering the Sturm-Liouville equation (2.2) as a one-dimensional Schrödinger equation with Hamiltonian (2.3). As shown on the left of Figure 1, the potential looks typically like a *double-well potential*. This potential is *approximately symmetric* (because the quadratic terms in Ω are symmetric around $\pi/2$ according to (2.5), but the terms (2.6) are anti-symmetric). If instead of a double-well potential we had a single-well potential, the n^{th} eigenvalue could be computed approximately for large n by the *Bohr-Sommerfeld-Wilson quantization condition* (see [16, §48] or [19, eq. (2.5.51)])

$$\oint p dq = 2\pi n,$$

where one integrates momentum along a closed classical path of the particle. Thus, denoting the zeros of $V = W - \lambda$ by u_ℓ and u_r (with $u_\ell < u_r < \pi/2$), we obtain

$$\int_{u_\ell}^{u_r} 2\sqrt{\lambda_n - W} du = 2\pi n. \quad (3.1)$$

From this formula, the expected eigenvalue gaps can be computed by

$$\pi \approx \int_{u_\ell(n+1)}^{u_r(n+1)} \sqrt{\lambda_{n+1} - W} du - \int_{u_\ell(n)}^{u_r(n)} \sqrt{\lambda_n - W} du \approx \int_{u_\ell}^{u_r} \frac{\lambda_{n+1} - \lambda_n}{2\sqrt{\lambda_n - W}} du$$

so that

$$\lambda_{n+1} - \lambda_n \approx 2\pi \left(\int_{u_\ell}^{u_r} \frac{1}{\sqrt{\lambda_n - W}} du \right)^{-1}. \quad (3.2)$$

In particular, for the large eigenvalues we obtain Weyl's asymptotics (see [21, Section 11.6])

$$\lambda_n \simeq n^2, \quad \lambda_{n+1} - \lambda_n \simeq n \quad (\text{as } n \rightarrow \infty). \quad (3.3)$$

Another spectral region of interest is if λ lies near the minimum of the potential. Approximating W by a quadratic potential and using that $W'' \simeq \Omega^2$, in this case we obtain the scaling

$$\lambda_n \simeq n|\Omega|, \quad \lambda_{n+1} - \lambda_n \simeq |\Omega| \quad (\text{if } 1 \ll n \ll |\Omega|). \quad (3.4)$$

These formulas describe the behavior of the eigenvalues if we had a single-well potential. The eigenvalues of the double-well potential can be understood by considering two Hamiltonians with a single-well potential and by weakly coupling them together via a potential barrier (see for example [20, Section 3.3]). If our double-well potential was symmetric about $\pi/2$, the two single-well Hamiltonians would have degenerate eigenvalues. Coupling them together slightly removes the degeneracy, leading to the well-known eigenfunctions with even and odd parity (similar as considered for example in [20, Sections 3.4 and 3.5]). In this way, we would end up with pairs of eigenvalues. These pairs would be separated by spectral gaps having the behavior (3.2). Since in our situation, the double-well potential is not symmetric about $\pi/2$, we do not know a-priori whether the eigenvalues of the two single-well Hamiltonians are degenerate or not. But we can conclude that the eigenvalues of the double-well Hamiltonian can appear at most in pairs, separated by gaps which again scale according to (3.2). If λ is chosen much larger than the potential barrier, the eigenfunctions no longer see a double-well potential. Therefore, Weyl's asymptotics (3.3) should again hold for the large eigenvalues.

These simple qualitative arguments already allow us to understand the statement of Theorem 1.1 in the special case of a real potential. Namely, the operator Q_0 is the spectral projection on all the small eigenvalues, for which the Born-Sommerfeld rule is not a good approximation. The operators Q_1, Q_2, \dots are spectral projection operators corresponding to one or two eigenvalues (depending on whether there is a spectral pair or not).

Before moving on to the complex potential, we remark that in the case $k = \pm s$, the potential at the pole goes to minus infinity (for a typical example see Figure 2). However, it turns out that, using the known asymptotics of the wave functions near

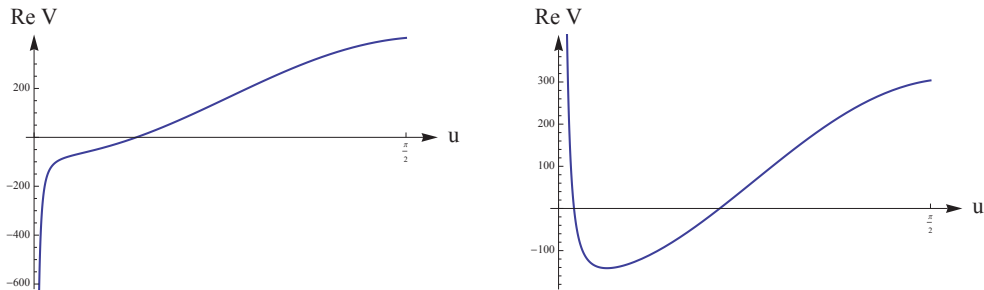


FIG. 2. The poles of $\text{Re } V$ in the case $k = s$ (left) and $k \neq s$ (right).

the pole, the above qualitative arguments still go through if we choose $u_\ell \sim |\Omega|^{-\frac{1}{2}}$ close to the inflection point of the potential.

We next discuss the situation for a complex potential. One potential method is to treat the imaginary part of W as a *slightly non-selfadjoint perturbation* (see [15, V.4.5] or as the application to the spheroidal wave operator in [8, Section 8]). For this method to be applicable, the imaginary part of the potential must be small compared to the gaps, i.e.

$$|\text{Im } W| \leq \lambda_{n+1} - \lambda_n. \quad (3.5)$$

For any fixed Ω , this condition will be satisfied for sufficiently large n in view of Weyl's asymptotics (3.3). But the inequality (3.5) cannot be satisfied uniformly in Ω , as the following argument shows: Using (1.5) in (2.4), one sees that

$$\sup_{(0, \pi)} |\text{Im } W| \gtrsim c |\Omega|$$

with a constant c which may be large. Therefore, the inequality (3.5) is in general violated if we are in the asymptotic regime (3.4). By choosing Ω large, one can arrange that this asymptotic regime includes arbitrarily many eigenvalues. We conclude that $\text{Im } W$ *cannot* in general be treated as a slightly non-selfadjoint perturbation. This means qualitatively that the imaginary part of W shifts the eigenvalues considerably on the scale of the gaps. The eigenvalues will typically move into the complex plane. Moreover, degeneracies and Jordan chains may form.

In order to locate the spectrum in the complex plane, for a complex potential whose real part has a single well one can again use the Bohr-Sommerfeld condition (3.1), which now makes a statement on both the real and imaginary parts of the integral on the left (3.1). Treating the imaginary part of $\lambda - W$ as a perturbation, we thus obtain to first order

$$\int_{u_\ell}^{u_r} \sqrt{\text{Re}(\lambda_n - W)} \, du = \pi n \quad (3.6)$$

$$\int_{u_\ell}^{u_r} \frac{\text{Im}(\lambda_n - W)}{\sqrt{\text{Re}(\lambda_n - W)}} \, du = 0. \quad (3.7)$$

We will prove that these relations really make it possible to locate the spectrum in the complex plane. Applying these relations naively, we find for the Hamiltonian with the single-well potential that the real part of the eigenvalues behaves just as discussed for the real potential. The imaginary part of the potential, however, must be adjusted

such that (3.7) holds. In particular, we again find that the spectral points form at most pairs, separated by spectral gaps which scale similar to (3.2). If the two spectral points of the pair coincide, a Jordan chain of length at most two may form. In this way, one can understand all statements of Theorem 1.1.

4. Overview of the Proof of the Main Theorem. Making the above qualitative arguments precise requires an intricate combination of different mathematical methods. In order to facilitate reading, we now give a short overview of the proof of Theorem 1.1. In Section 5, we collect general statements on Sturm-Liouville operators with a complex potential. We show that the spectrum is purely discrete, and that the Hilbert space can be decomposed into a direct sum of invariant subspaces. Moreover, idempotent operators mapping onto these invariant subspaces can be constructed using contour integral methods. In Section 6 we introduce a useful method for analyzing the oscillatory behavior of solutions of the Sturm-Liouville equation of the form (2.17). These estimates are essential for locating the spectrum and for making the Bohr-Sommerfeld condition (3.6) precise.

Our proof involves a deformation argument where we continuously deform a real potential to our complex potential (Section 16). Moreover, in our proof we will sometimes be able to treat the imaginary part of the potential as a perturbation (cf. Section 8). The starting point of these methods is to have detailed information on the spectrum and the spectral gaps for a real potential. These estimates are worked out in Section 7.

In Section 8 we employ the method of slightly non-selfadjoint perturbations to obtain the desired spectral representation provided that Ω lies in bounded set (see Proposition 8.1). Therefore, all the subsequent sections are devoted to the problem of getting estimates for large $|\Omega|$, uniformly in the spectral parameter λ .

In Section 9 we derive an a-priori estimate for the imaginary parts of all eigenvalues. The method is to evaluate an expectation value (see (9.1)) giving an equation which makes the Bohr-Sommerfeld condition (3.7) precise. This a-priori estimate is needed in order to distinguish the different cases and regions in Section 10. In Section 12 we shall return to the method and refine it considerably.

For the remaining estimates we shall construct approximate solutions of the Sturm-Liouville equation by glueing together WKB, Airy and parabolic cylinder functions as well as asymptotic solutions near the poles at $u = 0$ and $u = \pi$. Moreover, we derive rigorous error bounds. In Section 10 we give an overview of the different cases and regions and explain how to locate the spectrum. The detailed estimates are worked out in Section 11.

Section 12 gives refined integral estimates of the imaginary part of the potential (see Propositions 12.1 and 12.2). These estimates make use of the specific form of our potential and will be needed several times in the subsequent sections.

In order to show that the Jordan chains have length at most two, our method is to show that if λ_0 is an eigenvalue, then there is an annular region around λ_0 which contains at most one other eigenvalue (see Figure 7). In order to construct this annular region, we differentiate the equations with respect to λ and use an implicit function argument. The λ -derivatives are computed and estimated in Section 13. The construction of the annular regions is given in Section 14.

Section 15 is devoted to estimates of the Green's function. Here the main task is to estimate the Wronskian $w(\phi_L^P, \phi_R^P)$ of the fundamental solutions in (2.17).

In Section 16 we continuously deform the potential from a real potential to our complex potential. Combining all the results from the previous sections, we can track

the eigenvalues and control the spectral gaps. We also derive uniform norm estimates for the operators Q_n and show that their sum converges strongly to the identity.

5. General Functional Analytic Results. In order to get into the standard functional analytic framework, we consider the Sturm-Liouville operator (2.3) as an operator on the Hilbert space $L^2((0, \pi))$. As the dense domain of definition we choose those function in $C^2((0, \pi)) \cap L^2((0, \pi))$ which satisfy the boundary conditions (2.10).

LEMMA 5.1. *The spectrum of the Hamiltonian (2.3) is discrete and has no limit points.*

Proof. For Sturm-Liouville equations with a continuous potential, this is proved in [3, Chapter 12]. Since the potential (2.8) has poles at $u = 0$ and $u = \pi$, we give the proof in detail. For any $\lambda \in \mathbb{C}$, we choose non-trivial solutions ϕ_L and ϕ_R with the generic asymptotic behavior (2.17). These solutions can be chosen to depend locally holomorphically on λ in the sense that every $\lambda_0 \in \mathbb{C}$ has an open neighborhood U such that the functions ϕ_L and ϕ_R are holomorphic in $\lambda \in U$ (these holomorphic families can be constructed for example by taking the solutions of the Sturm-Liouville equation for variable λ but fixed boundary values at some $u \in (0, \pi)$). Then the functions $\phi_L^{\mathbb{D}}$ and $\phi_R^{\mathbb{D}}$ defined by (2.17) as well as their Wronskian in (2.18) are also holomorphic in $\lambda \in U$.

Let us show that the function $w(\phi_L^{\mathbb{D}}, \phi_R^{\mathbb{D}})$ does not vanish identically. If this were the case, by analytic continuation we would conclude that $w(\phi_L^{\mathbb{D}}, \phi_R^{\mathbb{D}})$ vanishes identically for all $\lambda \in \mathbb{C}$. Thus for every $\lambda \in \mathbb{C}$ there would exist a non-trivial solution ϕ satisfying the boundary conditions (2.10). On the other hand, the computation

$$\frac{d^2}{du^2} |\phi|^2 = 2 |\phi'|^2 + 2 |\phi|^2 \operatorname{Re}(W - \lambda)$$

shows that if λ is large and negative, then the absolute square of ϕ is convex away from small neighborhoods of the poles at $0, \pi$. But this convexity is incompatible with the asymptotics near the poles in (2.10), a contradiction.

The result follows because holomorphic functions which do not vanish identically have isolated zeros. \square

For a self-adjoint operator, one can construct the spectral projection operators by integrating the resolvent along a closed contour. In our non-selfadjoint setting, where the operator need not be diagonalizable, we cannot expect to obtain a spectral decomposition. But we can detect invariant subspaces:

LEMMA 5.2. *Let Γ be a closed contour which lies entirely in the resolvent set and encloses points in the spectrum with winding number one. Then the contour integral*

$$Q_{\Gamma} := -\frac{1}{2\pi i} \oint_{\Gamma} s_{\lambda} d\lambda \tag{5.1}$$

defines a bounded linear operator whose image is the invariant subspace corresponding to the spectral points enclosed by Γ .

The operator Q_{Γ} is idempotent. Moreover, the product of two operators Q_{Γ} and $Q_{\Gamma'}$ is given by

$$Q_{\Gamma} Q_{\Gamma'} = Q_{\tilde{\Gamma}}, \tag{5.2}$$

where $\tilde{\Gamma}$ is any contour which encloses precisely all the spectral points enclosed by Γ and Γ' , all with winding number one.

Proof. We first show that Q_Γ is idempotent. Multiplying the identity

$$(H - \lambda) - (H - \lambda') = \lambda' - \lambda$$

for $\lambda \neq \lambda'$ from the left by s_λ and from the right by $s_{\lambda'}$, one obtains the resolvent identity (see for example [18, Theorem VI.5])

$$s_\lambda s_{\lambda'} = \frac{1}{\lambda - \lambda'} (s_\lambda - s_{\lambda'}).$$

We let Γ' be a contour obtained by continuously deforming the contour Γ in the resolvent set such that every point of Γ is enclosed by Γ' with winding number one. Then, using the resolvent identity,

$$\begin{aligned} Q_\Gamma Q_\Gamma &= Q_\Gamma Q_{\Gamma'} = -\frac{1}{4\pi^2} \oint_\Gamma d\lambda \oint_{\Gamma'} d\lambda' \frac{1}{\lambda - \lambda'} (s_\lambda - s_{\lambda'}) \\ &= -\frac{1}{4\pi^2} \oint_\Gamma \left(\oint_{\Gamma'} \frac{d\lambda'}{\lambda - \lambda'} \right) s_\lambda d\lambda + \frac{1}{4\pi^2} \oint_{\Gamma'} \left(\oint_\Gamma \frac{d\lambda}{\lambda - \lambda'} \right) s_{\lambda'} d\lambda'. \end{aligned} \quad (5.3)$$

Carrying out the inner contour integrals with residues, the integral in the first summand gives $-2\pi i$, whereas the integral in the second summand vanishes. We conclude that $Q_\Gamma Q_\Gamma = Q_\Gamma$.

In order to prove the more general formula (5.2), it is convenient to deform the contours and to decompose each contour integral into a finite sum of integrals where each contour encloses only one spectral point. Then, in view of the idempotence of the Q_λ , it remains to prove (5.2) in the case that Γ and Γ' enclose different points of the spectrum. By continuously deforming Γ' without crossing spectral points we can again arrange that the contours Γ and Γ' do not intersect. As in (5.3), we obtain

$$Q_\Gamma Q_{\Gamma'} = -\frac{1}{4\pi^2} \oint_\Gamma \left(\oint_{\Gamma'} \frac{d\lambda'}{\lambda - \lambda'} \right) s_\lambda d\lambda + \frac{1}{4\pi^2} \oint_{\Gamma'} \left(\oint_\Gamma \frac{d\lambda}{\lambda - \lambda'} \right) s_{\lambda'} d\lambda'. \quad (5.4)$$

Since the contours enclose different points in the spectrum, no point of Γ is enclosed by Γ' and vice versa. Hence the inner integrals in (5.4) vanish, proving that $Q_\Gamma Q_{\Gamma'} = 0$.

It remains to show that the image of Q_Γ consists of the invariant subspaces corresponding to all the spectral points enclosed in Γ . Let $\Omega \subset \mathbb{C}$ be the open set enclosed by Γ . Since the spectral points are isolated, we may decompose Ω into a finite number of subsets such that the boundary of each subset is a closed contour enclosing only one spectral point. Thus it suffices to consider the situation that Γ encloses exactly one spectral point λ_0 . Since the Wronskian in (2.19) is holomorphic in λ , the resolvent s_λ at λ_0 has a pole of finite order n . Iterating the identity

$$H Q_\Gamma = -\frac{1}{2\pi i} \oint_\Gamma (\lambda s_\lambda - \mathbb{1}) d\lambda = -\frac{1}{2\pi i} \oint_\Gamma \lambda s_\lambda d\lambda,$$

we obtain

$$(H - \lambda_0)^n Q_\Gamma = -\frac{1}{2\pi i} \oint_\Gamma (\lambda - \lambda_0)^n s_\lambda d\lambda = 0,$$

where in the last step we used that the integrand is holomorphic. Hence every vector in the image of Q_Γ is contained in the invariant subspace corresponding to λ_0 .

Conversely, let ψ be a vector in this invariant subspace. Then there is $N \in \mathbb{N}$ such that $(H - \lambda_0)^N \psi = 0$, and thus

$$0 = ((H - \lambda) + (\lambda - \lambda_0))^N \psi = \sum_{k=0}^N \binom{N}{k} (\lambda - \lambda_0)^{N-k} (H - \lambda)^k \psi.$$

Multiplying by s_λ and solving for $s_\lambda \psi$, we obtain

$$s_\lambda \psi = - \sum_{k=1}^N \binom{N}{k} (\lambda - \lambda_0)^{-k} (H - \lambda)^{k-1} \psi.$$

Taking the contour integral, a computation with residues yields

$$\begin{aligned} Q_\Gamma \psi &= \frac{1}{2\pi i} \sum_{k=1}^N \binom{N}{k} \oint_\Gamma \frac{(H - \lambda)^{k-1} \psi}{(\lambda - \lambda_0)^k} d\lambda = \frac{1}{2\pi i} \sum_{k=1}^N \binom{N}{k} \oint_\Gamma \frac{\psi}{(\lambda - \lambda_0)} d\lambda \\ &= \sum_{k=1}^N \binom{N}{k} (-1)^{k-1} \psi = \psi - \sum_{k=0}^N \binom{N}{k} (-1)^k \psi = \psi - (1 - 1)^N \psi = \psi. \end{aligned}$$

Hence ψ really lies in the image of Q_λ . This concludes the proof. \square

This lemma also shows that the dimension of the invariant subspace is at most the order of the pole of the resolvent.

The next lemma bounds the resolvent away from the real axis.

LEMMA 5.3. *If*

$$\inf_{(0, \pi)} |\operatorname{Im} V| > 0, \tag{5.5}$$

then the Wronskian in (2.18) has no zeros. Moreover, the resolvent is bounded by

$$\|s_\lambda\| \leq \left(\inf_{(0, \pi)} |\operatorname{Im} V| \right)^{-1}$$

(where $\|\cdot\|$ denotes the sup-norm on \mathcal{H}_k).

Proof. Since $\operatorname{Im} V$ is continuous, the condition (5.5) implies that $\operatorname{Im} V$ is either always positive or always negative. We only give the proof in the first case because the second case is similar. If the Wronskian in (2.18) is zero, there is a non-trivial solution ϕ of the Sturm-Liouville equation (2.7) with the asymptotics (2.10). In the case $k \pm s \neq 0$, differentiating the asymptotics (2.11), one sees that

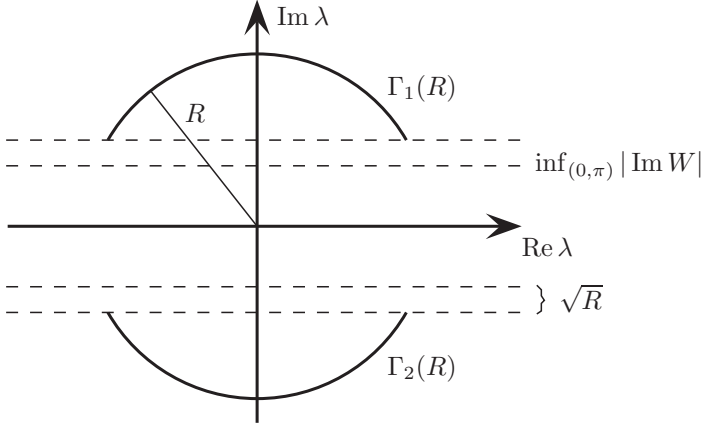
$$\phi(u) \sim u^{\frac{1}{2} + |k-s|} (1 + \mathcal{O}(u)) \quad \text{and} \quad \phi'(u) \sim u^{-\frac{1}{2} + |k-s|} (1 + \mathcal{O}(u)).$$

As a consequence, we do not get boundary terms when integrating by parts as follows,

$$0 = \langle \phi | (-\partial_u^2 + V)\phi \rangle_{L^2} = \langle \partial_u \phi | \partial_u \phi \rangle_{L^2} + \langle \phi | V\phi \rangle_{L^2}. \tag{5.6}$$

Taking the imaginary part, we conclude that

$$0 = \int_0^\pi \operatorname{Im} V |\phi|^2, \tag{5.7}$$


 FIG. 3. *The contour Γ .*

in contradiction to (5.5). In the case $k = s$, the situation is a bit more subtle because differentiating the asymptotics (2.12), one sees that

$$\phi(u) = c_1 \sqrt{u} + \mathcal{O}(u) \quad \text{and} \quad \phi'(u) = \frac{c_1}{2\sqrt{u}} + \mathcal{O}(u^0).$$

This implies that integrating by parts in (5.6) we get real-valued boundary terms, so that (5.7) again holds. The remaining case $k = -s$ is treated similarly by differentiating (2.13).

Next, setting $\psi = s_\lambda \phi$ and again integrating by parts, we obtain

$$\|\psi\| \|\phi\| \geq |\langle \psi | \phi \rangle_{L^2}| = |\langle \psi | (-\partial_u^2 + V)\psi \rangle_{L^2}| \geq |\operatorname{Im} \langle \psi | V\psi \rangle_{L^2}| \geq \|\psi\|^2 \inf_{(0,\pi)} \operatorname{Im} V,$$

implying that

$$\|\phi\| \geq \|s_\lambda \phi\| \inf_{(0,\pi)} \operatorname{Im} V.$$

Since this inequality holds for all $\phi \in \mathcal{H}_k$, the result follows. \square

We are now in the position to state a general completeness result. The method is based on an idea in [1, proof of Theorem 2.12] and was used previously in [9]. First, we write the potential (2.8) and (2.9) in the form

$$V = W - \lambda$$

with W independent of λ . For given $R > 0$, we consider the two contours Γ_1 and Γ_2 in the complex λ -plane defined by

$$\begin{aligned} \Gamma_1 &= \partial B_R(0) \cap \left\{ \operatorname{Im} \lambda < - \inf_{(0,\pi)} |\operatorname{Im} W| - \sqrt{R} \right\} \\ \Gamma_2 &= \partial B_R(0) \cap \left\{ \operatorname{Im} \lambda > \inf_{(0,\pi)} |\operatorname{Im} W| + \sqrt{R} \right\}. \end{aligned}$$

and set $\Gamma(R) = \Gamma_1 \cup \Gamma_2$ (see Figure 3).

THEOREM 5.4. For any $\phi \in C_0^\infty((0, \pi))$,

$$\phi(u) = -\frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\Gamma(R)} (s_\lambda \phi)(u) d\lambda. \quad (5.8)$$

Proof. Since the length of the contour $S_1 \cup S_2 := \partial B_R(0) \setminus \Gamma(R)$ only grows like \sqrt{R} ,

$$\left| \oint_{\partial B_R(0)} \frac{d\lambda}{\lambda} - \int_{\Gamma(R)} \frac{d\lambda}{\lambda} \right| \leq \frac{1}{R} \int_{S_1 \cup S_2} |d\lambda| \xrightarrow{R \rightarrow \infty} 0.$$

As a consequence,

$$\frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\Gamma(R)} \frac{d\lambda}{\lambda} = 1. \quad (5.9)$$

Since our contours lie in the resolvent set, we know that for every $\lambda \in \Gamma(R)$,

$$\phi = s_\lambda (-\partial_u^2 + W - \lambda) \phi.$$

Dividing by λ and integrating over $\Gamma(R)$, we can apply (5.9) to obtain

$$\begin{aligned} \phi(u) &= \frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\Gamma(R)} \frac{d\lambda}{\lambda} \left(s_\lambda (-\partial_u^2 + W - \lambda) \phi \right)(u) \\ &= -\frac{1}{2\pi i} \lim_{R \rightarrow \infty} \int_{\Gamma(R)} \left\{ (s_\lambda \phi)(u) - \frac{1}{\lambda} (s_\lambda (-\partial_u^2 + W) \phi)(u) \right\} d\lambda. \end{aligned}$$

But the second term in the curly brackets vanishes in the limit, because by Lemma 5.3,

$$\left| \int_{\Gamma(R)} (s_\lambda (-\partial_u^2 + W) \phi)(u) \frac{d\lambda}{\lambda} \right| \leq \int_{\Gamma(R)} \frac{C}{\sqrt{R}} \frac{|d\lambda|}{|\lambda|} \leq \frac{2\pi C}{\sqrt{R}}.$$

Thus (5.8) holds. \square

This theorem shows that the operators Q_Γ defined by (5.1) converge to the identity if Γ tends to a contour which encloses the whole spectrum. The advantage of this method is that it does not require a functional analytic framework, but only uses properties of the Green's function $s_\lambda(u, u')$. The drawback is that one obtains strong convergence only on a dense subspace of test functions. In order to prove strong convergence on the whole Hilbert space, we will rely on the theory of slightly non-selfadjoint perturbations (see Section 8 and Section 16.4).

6. An Osculating Circle to the ζ -Curve. In order to locate the spectrum, we need to find the zeros of the Wronskian in (2.18). The main difficulty is to understand the behavior of the integrals in (2.17). To this goal, we now develop a method referred to as the ‘‘osculating circle method.’’ For ease in notation, we only consider the solution ϕ_L^D and omit the subscript L . We denote the integral in (2.17) by

$$\zeta(u) := \int_0^u \frac{1}{\phi^2}. \quad (6.1)$$

Then

$$\phi^D = \zeta \phi, \quad (6.2)$$

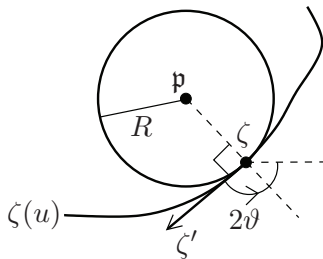


FIG. 4. The osculating circle to the curve $\zeta(u)$.

making it possible to relate the behavior of ϕ^{D} to properties of the function $\zeta(u)$.

In order to clarify the evolution of the function $\zeta(u)$ in the complex plane, it is useful to consider the osculating circle to the curve ζ at a point $\zeta(u)$ (see Figure 4). The curvature K of the curve $\zeta(u)$ and the radius R of the osculating circle are given by (see for example [4, Theorem 5.1.6])

$$K = -\frac{\text{Im}(\zeta'' \overline{\zeta'})}{|\zeta'|^3}, \quad R = \frac{1}{|K|}. \quad (6.3)$$

The center \mathbf{p} of the osculating circle is

$$\mathbf{p} = \zeta - \frac{i}{K} \frac{\zeta'}{|\zeta'|}. \quad (6.4)$$

Moreover, we introduce the angle ϑ as the argument of ϕ ,

$$\phi = |\phi| e^{i\vartheta}. \quad (6.5)$$

Then

$$\frac{\zeta'}{|\zeta'|} = \frac{|\phi|^2}{\phi^2} = e^{-2i\vartheta},$$

so that (6.4) becomes

$$\zeta = \mathbf{p} + \frac{i}{K} e^{-2i\vartheta}. \quad (6.6)$$

In order to simplify the computations, we always choose the phase and normalization of ϕ such that

$$\phi(u_0)^2 \text{Im } y(u_0) = 1 \quad (6.7)$$

for some u_0 which will be specified later. Moreover, we set

$$y_0 = y(u_0). \quad (6.8)$$

Using the definition (6.1) of ζ as well as the differential equation (2.7), we obtain useful formulas for ϑ , K , \mathbf{p} and their derivatives.

LEMMA 6.1.

$$\vartheta' = \text{Im } y. \quad (6.9)$$

Furthermore,

$$K(u) = 2|\phi|^2 \operatorname{Im} y, \quad \mathfrak{p}(u) = \zeta - \frac{i}{2\phi^2 \operatorname{Im} y} \quad (6.10)$$

$$K'(u) = 2|\phi|^2 \operatorname{Im} V, \quad \mathfrak{p}'(u) = \frac{i}{2\phi^2 \operatorname{Im}^2 y} \operatorname{Im} V. \quad (6.11)$$

and

$$R'(u) = -\frac{\operatorname{Im} V}{2|\phi|^2 \operatorname{Im} y |\operatorname{Im} y|}. \quad (6.12)$$

Before coming to the proof, we point out that for a real potential, this lemma shows that K , \mathfrak{p} and R are constant. Thus for a real potential, the curve $\zeta(u)$ lies on a fixed circle with radius R centered at \mathfrak{p} . The position of ζ on the circle is described by the angle ϑ , and its evolution is described completely by (6.9). Moreover, we point out that $|\mathfrak{p}'| = |R'|$, which can be understood from the fact that the point ζ stays on the circle as the osculating circles move.

Proof of Lemma 6.1. A direct computation yields

$$\begin{aligned} \zeta' &= \frac{1}{\phi^2}, & \zeta'' &= -2\frac{\phi'}{\phi^3} = -\frac{2y}{\phi^2} \\ \vartheta' &= -\frac{1}{2} \frac{d}{du} \operatorname{Im} \log \zeta' = -\frac{1}{2} \operatorname{Im} \frac{\zeta''}{\zeta'} = \operatorname{Im} \left(\phi^2 \frac{\phi'}{\phi^3} \right) = \operatorname{Im} y \\ \operatorname{Im}(\zeta'' \bar{\zeta}') &= -2 \frac{\operatorname{Im} y}{|\phi|^4}. \end{aligned}$$

This gives (6.9) as well as the formula for K in (6.10). Using this formula in (6.4), we obtain the second equation in (6.10).

Next, we write out the real and imaginary parts of the Riccati equation (2.14),

$$\operatorname{Re} y' = \operatorname{Re} V - \operatorname{Re}^2 y + \operatorname{Im}^2 y \quad (6.13)$$

$$\operatorname{Im} y' = \operatorname{Im} V - 2 \operatorname{Re} y \operatorname{Im} y. \quad (6.14)$$

Using (6.14), we obtain

$$K' = 2|\phi|^2 \operatorname{Im}(y') + 4 \operatorname{Re} y |\phi|^2 \operatorname{Im} y = 2 \operatorname{Im} V |\phi|^2,$$

giving the formula for K' in (6.11). Moreover,

$$\begin{aligned} \mathfrak{p}' &= \zeta' + \frac{i}{2 \operatorname{Im}^2 y} \operatorname{Im}(y') \frac{1}{\phi^2} + \frac{i}{2 \operatorname{Im} y} \frac{2}{\phi^3} \phi' \\ &= \frac{1}{\phi^2} + \frac{i}{2 \operatorname{Im}^2 y} \left(\operatorname{Im} V - 2 \operatorname{Re} y \operatorname{Im} y \right) \frac{1}{\phi^2} + \frac{i}{2 \operatorname{Im} y} \frac{2}{\phi^2} y \\ &= \frac{1}{\phi^2} + \frac{i \operatorname{Im} V}{2\phi^2 \operatorname{Im}^2 y} - \frac{i}{\phi^2 \operatorname{Im} y} \operatorname{Re} y + \frac{i}{\phi^2 \operatorname{Im} y} y = \frac{i \operatorname{Im} V}{2\phi^2 \operatorname{Im}^2 y}. \end{aligned}$$

Finally, we differentiate (6.3),

$$R' = -\frac{K K'}{|K|^{\frac{3}{2}}},$$

and using (6.10) and (6.11) gives (6.12). \square

7. Estimates for a Real Potential. In Section 16 we shall consider a homotopy of the potential which joins the potential V with its real part. In preparation for this analysis, we now derive eigenvalue estimates for a Sturm-Liouville equation with a real potential. More precisely, we replace the potential in the Sturm-Liouville equation (2.7) by its real part,

$$\left(-\frac{d^2}{du^2} + \operatorname{Re} V\right)\phi = 0, \quad (7.1)$$

where V is again given by (2.8) and (2.9). We can assume that λ is real, so that the equation can be written in the Schrödinger form (2.2) with the Hamiltonian

$$H = -\frac{d^2}{du^2} + \operatorname{Re} W. \quad (7.2)$$

This Hamiltonian has a unique self-adjoint extension, as the following consideration shows: In the case $k \neq s$, the asymptotics in (2.11) shows that one of the fundamental solutions is square-integrable near $u = 0$, whereas the other fundamental solution is not. Using Weyl's notion, the Sturm-Liouville operator is in the limit point case at $u = 0$ (see [3, Sections 9.2, 9.3]. In the case $k = s$, on the other hand, according to (2.12) both fundamental solutions are square integrable. This is the so-called limiting circle case (see [3, Sections 9.4]). In all of these cases, our boundary conditions (2.10) give rise to a unique self-adjoint extension (for details see [3, Sections 9.2, 9.3, 9.4] or [5, Chapter XIII.2]).

For ease in notation, we denote the selfadjoint extension of (7.2) again by H , and its domain of definition $\mathcal{D}(H)$. For the analysis of the spectrum, it is again useful to consider the Riccati equation corresponding to (7.1), which we write as

$$y' = \operatorname{Re} V - y^2, \quad (7.3)$$

where we again set $y := \phi'/\phi$. We consider complex-valued solutions of this equation. A direct computation (see also [8, eq. (3.8)]) shows the product $|\phi|^2 \operatorname{Im} y$ is a constant,

$$w := |\phi|^2 \operatorname{Im} y = \operatorname{const}. \quad (7.4)$$

This implies in particular that the function y cannot cross the real axis.

7.1. A Node Theorem. The classical node theorem (see for example [23, Theorem 14.10]) states that the n^{th} eigenfunction of a Sturm-Liouville operator has exactly $(n - 1)$ zeros. We now state and prove this node theorem in our setting. In the subsequent Sections 7.2 and 7.3, we will apply the node theorem to obtain eigenvalue estimates and the Weyl asymptotics. There are two reasons why we decided to give the proof of the node theorem in detail. First, due to our singular boundary conditions, the proof given in most textbooks does not apply to our problem. Second, our proof works with osculating circles and complex solutions of the corresponding Riccati equation. It can be used as an introduction to the methods needed later in this paper.

PROPOSITION 7.1. *The spectrum of the Hamiltonian (7.2) is a discrete subset of \mathbb{R} which is bounded from below. Numbering the eigenvalues in increasing order, $\lambda_0 \leq \lambda_1 \leq \dots$, the eigenfunction corresponding to λ_n has exactly n zeros on the open interval $(0, \pi)$. Moreover, choosing $\lambda = \lambda_n$, any solution y of the Riccati equation (7.3) with $\operatorname{Im} y > 0$ satisfies the relation*

$$\int_0^\pi \operatorname{Im} y = (n + 1) \pi. \quad (7.5)$$

Proof. We let ϕ_1 and ϕ_2 be two real-valued fundamental solutions of the ODE (7.1). Since their Wronskian

$$w := \phi_1' \phi_2 - \phi_1 \phi_2'$$

is a non-zero constant, the functions ϕ_1 and ϕ_2 cannot have common zeros. Hence the complex solution $\phi := \phi_1 + i\phi_2$ has no zeros. By choosing suitable fundamental solutions, we can arrange that the corresponding solution of the Riccati equation (7.3) satisfies (7.4) with $w = 1$, so that

$$|\phi|^2 \operatorname{Im} y \equiv 1. \quad (7.6)$$

The relations (6.2) and (6.1) define a solution $\phi^{\mathbb{D}}$ which satisfies the boundary condition at $u = 0$. The boundary conditions at $u = \pi$ are satisfied if and only if $\zeta(\pi) = 0$. Hence the condition for an eigenvalue can be stated as

$$\zeta(\pi) = 0. \quad (7.7)$$

In order to control the behavior of the function ζ , we again use the osculating circle method of Section 6. For a real potential, the relations (6.11) show that the center and the radius of the osculating circle are fixed. Moreover, combining the first identity in (6.10) with (7.6), one sees that $K = 2$. Hence the formula (6.6) simplifies to

$$\zeta(u) = \mathfrak{p} + \frac{i}{2} e^{-2i\vartheta}, \quad (7.8)$$

where ϑ satisfies the differential equation (6.9). As a consequence, the eigenvalue condition (7.7) can be written as

$$\int_0^\pi \operatorname{Im} y \in \pi\mathbb{Z}. \quad (7.9)$$

The above formulas are valid if we let y be any solution of the Riccati equation in the upper half plane and if we satisfy (6.7) (and consequently also (7.6)) by letting

$$\phi(u) = \frac{1}{\operatorname{Im} y(u_0)} \exp\left(\int_{u_0}^u y\right). \quad (7.10)$$

We now consider in particular a family of solutions y parametrized by $\lambda \in \mathbb{R}$ such that

$$\lim_{u \nearrow \pi} \phi^2(u) \partial_\lambda y(u) = 0 \quad \text{for all } \lambda \in \mathbb{R}. \quad (7.11)$$

Such a family exists in view of the asymptotics near $u = \pi$ as worked out in [12, Section 8] (namely, one chooses ϕ with the asymptotics as in [12, Section 8] with coefficients adjusted such that the leading asymptotics is independent of λ , implying that $\partial_\lambda y$ vanishes to leading order).

Differentiating (2.14) and (6.8) with respect to λ and using that $\partial_\lambda V = -1$ gives

$$y'_\lambda = -1 - 2yy_\lambda, \quad y_\lambda(u_0) = \partial_\lambda y_0. \quad (7.12)$$

Solving this linear ODE by integration, we obtain

$$\phi^2 y_\lambda = \phi^2(u_0) \partial_\lambda y_0 - \int_{u_0}^u \phi^2,$$

so that

$$y_\lambda(u) = \frac{\phi^2(u_0)}{\phi^2(u)} \partial_\lambda y_0 - \frac{1}{\phi^2(u)} \int_{u_0}^u \phi^2 . \quad (7.13)$$

Integrating this differential equation with respect to λ yields

$$\int_0^\pi y_\lambda = \phi^2(u_0) \partial_\lambda y_0 \zeta|_0^\pi - \int_0^\pi \left(\frac{1}{\phi^2(u)} \int_{u_0}^u \phi^2 \right) du . \quad (7.14)$$

In the last integral we perform the transformations

$$\begin{aligned} \int_{u_0}^\pi du \int_{u_0}^u dv \cdots &= \int_{u_0}^\pi dv \int_v^\pi du \\ \int_0^{u_0} du \int_u^{u_0} dv \cdots &= - \int_0^{u_0} du \int_u^{u_0} dv \cdots = - \int_0^{u_0} dv \int_0^v du \cdots \end{aligned}$$

to obtain

$$\int_0^\pi \left(\frac{1}{\phi^2(u)} \int_{u_0}^u \phi^2 \right) du = \int_{u_0}^\pi \phi^2(v) \zeta|_v^\pi dv - \int_0^{u_0} \phi^2(v) \zeta|_0^v dv .$$

Taking the limit $u_0 \nearrow \pi$ and using (7.11), the relation (7.14) simplifies to

$$\int_0^\pi \partial_\lambda y = \int_0^{u_0} \phi^2 \zeta = \int_0^{u_0} \phi \phi^D .$$

The representation for $\phi \phi^D$ derived in Lemma 7.3 below shows that the function $\phi \phi^D$ has a non-negative imaginary part, and that its imaginary part is even strictly positive on a set of positive measure. Therefore,

$$\int_0^\pi \partial_\lambda \operatorname{Im} y > 0 \quad \text{for all } \lambda \in \mathbb{R} ,$$

showing that for our family of functions y , the integral (7.9) is indeed strictly increasing in λ .

Combining this strict monotonicity of the integral (7.9) with the continuous dependence on the parameter λ , the intermediate value theorem gives rise to eigenfunctions λ_n which are uniquely characterized by their number of zeros. Since the integral (7.9) is strictly positive, converges to zero as $\lambda \rightarrow -\infty$ and tends to infinity as $\lambda \rightarrow \infty$ (using the WKB asymptotics), we conclude that there is a sequence of eigenvalues $\lambda_0 < \lambda_1 < \dots$ and that the eigenfunction corresponding to λ_n has precisely n zeros in the open interval $(0, \pi)$. Moreover, we conclude that (7.5) holds for the family of functions y satisfying (7.11).

In order to show that (7.5) holds for any smooth family of solutions y with $\operatorname{Im} y > 0$, we use the following continuity argument: For any fixed $\lambda = \lambda_n$, we denote the solution satisfying (7.11) by y_0 , and let y be any other solution with $\operatorname{Im} y > 0$. For any $\tau \in [0, 1]$, we let (y_τ) be the family of solutions of (7.3) with initial conditions

$$y_\tau\left(\frac{\pi}{2}\right) = \tau y_0\left(\frac{\pi}{2}\right) + (1 - \tau) y\left(\frac{\pi}{2}\right) .$$

Then the condition (7.9) is satisfied for any τ ,

$$\int_0^\pi \operatorname{Im} y_\tau \in \pi \mathbb{Z} .$$

By continuity, this integral is independent of τ . We conclude that (7.5) holds for any $\tau \in [0, 1]$, and in particular for $\tau = 1$. \square

REMARK 7.2. We remark that for a real potential, the eigenvalue condition (7.9) can also be understood without going through the osculating circle estimates, as we now explain. Since the real and imaginary parts of ϕ form a fundamental system, the solution ϕ^D can be represented as

$$\phi^D = c \operatorname{Im}(e^{-i\alpha}\phi) \quad (7.15)$$

for a suitable phase α and a complex prefactor c . The zeros of ϕ^D are then determined by the phase of ϕ ,

$$\phi^D = 0 \quad \iff \quad \arg \phi \in \alpha + \pi\mathbb{Z}.$$

In particular, for ϕ^D to satisfy the Dirichlet boundary conditions, it follows that

$$\arg \phi|_0^\pi \in \pi\mathbb{Z}.$$

Differentiating gives

$$\frac{d}{du} \arg \phi = \frac{d}{du} \operatorname{Im} \log \phi = \operatorname{Im} \frac{\phi'}{\phi} = \operatorname{Im} y, \quad (7.16)$$

and applying the fundamental theorem of calculus again gives (7.9). Moreover, one sees again that the integral in (7.9) gives π times the number of zeros on $(0, \pi)$ plus one. Using the osculating circle method has the advantage that with (6.2) we have an explicit formula for ϕ^D , making it unnecessary to think about how the angle α in (7.15) is to be chosen.

We append the lemma which shows that the integrand $\operatorname{Im}(\phi\phi^D)$ has a definite sign.

LEMMA 7.3. *For every solution ϕ satisfying the normalization condition (6.7) (for any $u_0 \in (0, \pi)$),*

$$\operatorname{Im}(\phi(u)\phi^D(u)) = \frac{1}{2} |\phi(u)|^2 \left(1 - \frac{\phi(u)^2}{|\phi(u)|^2} \lim_{v \searrow 0} \frac{|\phi(v)|^2}{\phi(v)^2} \right).$$

Proof. Clearly, ϕ^D is a linear combination of the fundamental solutions ϕ and $\bar{\phi}$, i.e.

$$\phi^D = \alpha\phi + \beta\bar{\phi} \quad (7.17)$$

for suitable coefficients $\alpha, \beta \in \mathbb{C}$. In order to compute these coefficients, we compute the Wronskians of ϕ^D with both ϕ and $\bar{\phi}$. First, using the ansatz (7.17), we get

$$w(\phi, \phi^D) = \beta w(\phi, \bar{\phi}), \quad w(\bar{\phi}, \phi^D) = -\alpha w(\phi, \bar{\phi}).$$

Next, using the representation (6.2) and (6.1), we obtain

$$\begin{aligned} w(\phi, \phi^D) &= \phi' \phi \zeta - \phi (\phi \zeta)' = -\phi^2 \zeta' = -1 \\ w(\bar{\phi}, \phi^D) &= \bar{\phi}' \phi \zeta - \bar{\phi} (\phi \zeta)' = -w(\phi, \bar{\phi}) \zeta - |\phi|^2 \zeta' = -w(\phi, \bar{\phi}) \zeta - \frac{|\phi|^2}{\phi^2}. \end{aligned}$$

Comparing these formulas, we can compute α and β . We obtain that for any $v \in (0, \pi)$,

$$\alpha = \zeta(v) + \frac{1}{w(\phi, \bar{\phi})} \frac{|\phi(v)|^2}{\phi(v)^2}, \quad \beta = -\frac{1}{w(\phi, \bar{\phi})}.$$

Finally, the normalization condition (6.7) implies that $w(\phi, \phi') = 2i$. Computing $\text{Im } \phi \phi^{\text{D}}$ using the above relations, taking the limit $v \searrow 0$ and using that $\lim_{v \searrow 0} \zeta(v) = 0$, we obtain the result. \square

7.2. Lower Bounds for Small Eigenvalues. In order to obtain eigenvalue estimates, we need to count the number of zeros of the function ϕ^{D} . Our method is to decompose the domain $(0, \pi)$ into subintervals on which the potential $\text{Re } V$ has a definite sign. On every interval where $\text{Re } V$ is positive, the number of zeros is a-priori bounded:

LEMMA 7.4. *If I is a closed interval with $\text{Re } V|_I \geq 0$, then ϕ^{D} has at most one zero on I .*

Proof. Assume conversely that there is more than one zero on I . We choose two neighboring zeros $u_1 < u_2$. Then, possibly by flipping the sign of ϕ we can arrange that $\phi^{\text{D}}|_{(u_1, u_2)} > 0$. As a consequence, the function ϕ^{D} is convex on $[u_1, u_2]$ (for details and other estimates using this convexity property see [8, Section 5]). This implies that ϕ^{D} is non-positive on (u_1, u_2) , a contradiction. \square

LEMMA 7.5. *Let y be any solution of the Riccati equation (7.3) with $\text{Im } y > 0$. Then the number Z of zeros of ϕ^{D} on an open interval $I \subset (0, \pi)$ is bounded by*

$$-1 + \frac{1}{\pi} \int_I \text{Im } y \leq Z < 1 + \frac{1}{\pi} \int_I \text{Im } y.$$

Proof. One method of proof is to consider the osculating circle for a real potential (7.8) and to note that the change of the phase ϑ is given by the differential equation (6.9). Finally, the representation (6.2) shows that the zeros of ϕ^{D} coincide with the zeros of ζ . An alternative method is to use the representation (7.15) with ϕ according to (7.10), and to make use of the fact that $\arg \phi$ satisfies the differential equation (7.16). \square

Combining the node theorem of Proposition 7.1 with the last two lemmas, we obtain the following corollary.

COROLLARY 7.6. *Let $I_1, \dots, I_k \subset (0, \pi)$ be open intervals such that $\text{Re } V$ is non-negative on the complement of $I_1 \cup \dots \cup I_k$. On the I_ℓ we choose any solutions y_ℓ of the Riccati equation (7.3) with $\text{Im } y_\ell > 0$. Then for the N^{th} eigenvalue λ_N ,*

$$\pi(N - 2k - 1) < \sum_{\ell=1}^k \int_{I_\ell} \text{Im } y_\ell \leq \pi(N + k).$$

We now apply this corollary to the spheroidal wave operator. We restrict attention to lower bounds for the eigenvalues, but remark that upper bounds could be derived with similar methods.

PROPOSITION 7.7. *For every constant $c_3 > 0$ and any parameters k, s , there is $N = N(c_3, k, s) \in \mathbb{N}$ such that for all Ω in the range (1.5) with $|\Omega|$ sufficiently large, the N^{th} eigenvalue is bounded from below by*

$$\lambda_N \geq c_3 |\Omega|. \tag{7.18}$$

Proof. In order to prove (7.18) we consider $\lambda \leq c_3 |\Omega|$ for large $|\Omega|$. Then, due to the summand $\Omega^2 \sin^2 u$ in (2.8), the real part of the potential is non-negative except in a neighborhood of $u = 0$ and $u = \pi$. By symmetry, it suffices to analyze the behavior in a neighborhood of $u = 0$. Then the estimate

$$\operatorname{Re} V \geq -\frac{1}{4u^2} - (c_3 + 1)|\Omega| + \frac{|\Omega|^2}{2} u^2 + |\Omega|^2 \mathcal{O}(u^3) \quad (7.19)$$

shows that the potential is positive if

$$u > u_1 := 2(c_3 + 1) |\Omega|^{-\frac{1}{2}}. \quad (7.20)$$

We begin with the case $k \neq s$. In this case, the estimate (7.19) is improved to

$$\operatorname{Re} V \geq \frac{3}{4u^2} - (c_3 + 1)|\Omega| + \frac{|\Omega|^2}{2} u^2 + |\Omega|^2 \mathcal{O}(u^3).$$

In particular, we conclude that for large $|\Omega|$,

$$\operatorname{Re} V \geq -2(c_3 + 1)|\Omega| \quad (\text{if } k \neq s). \quad (7.21)$$

We choose a (possibly empty) interval (u_-, u_+) such that $\operatorname{Re} V$ is negative inside and non-negative outside this interval. In view of (7.20) we may choose $u_-, u_+ \leq u_1$. In order to count the zeros of ϕ^{D} on the interval (u_-, u_+) we can assume that the minimum u_0 of $\operatorname{Re} V$ lies in the interval (u_-, u_+) (because otherwise the interval (u_-, u_+) is empty, and there is nothing to do). We consider the solution y of the Riccati equation (7.3) with initial conditions

$$y(u_0) = i\sqrt{|\operatorname{Re} V(u_0)|}. \quad (7.22)$$

We now apply the T-method (see [12, Theorem 3.2] or [6, Lemma 4.1]), choosing $\alpha \equiv 0$. Then $U = \operatorname{Re} V$, $\sigma \equiv 0$ and

$$\mathfrak{D} = \frac{\operatorname{Re} V'}{2}$$

(see [12, eqns. (3.3)–(3.5)]). Using that $\operatorname{Re} V$ is monotone increasing on $[u_0, u_+)$ and monotone decreasing on $(u_-, u_0]$, we obtain

$$T(u) = \left| \frac{\operatorname{Re} V(u_0)}{\operatorname{Re} V(u)} \right|^{\frac{1}{2}},$$

giving rise to an invariant disk estimate with center $m(u) = i\beta(u)$ and radius R given by

$$\begin{aligned} \beta(u) &= \frac{1}{2} \left(|\operatorname{Re} V(u_0)|^{\frac{1}{2}} + |\operatorname{Re} V(u)| |\operatorname{Re} V(u_0)|^{-\frac{1}{2}} \right) \\ R(u) &= \frac{1}{2} \left(|\operatorname{Re} V(u_0)|^{\frac{1}{2}} - |\operatorname{Re} V(u)| |\operatorname{Re} V(u_0)|^{-\frac{1}{2}} \right). \end{aligned}$$

In particular, one sees that

$$\operatorname{Im} y \leq \sqrt{|\operatorname{Re} V(u_0)|} \quad \text{on } (u_-, u_+).$$

As a consequence,

$$\int_{u_-}^{u_+} \operatorname{Im} y \leq \sqrt{|\operatorname{Re} V(u_0)|} |u_+ - u_-| \leq \sqrt{|\operatorname{Re} V(u_0)|} u_1 \leq (2(c_3 + 1))^{\frac{3}{2}},$$

where in the last step we applied (7.20) and (7.21).

In the case $k = s$, on the other hand, the function $\operatorname{Re} V$ tends to minus infinity as $u \searrow 0$. We choose $u_- = 0$ and $u_+ \leq u_1$ with $\operatorname{Re} V(u_+) = 0$. We first apply the invariant disk estimate near the pole as worked out in [12, Section 8.1]. This estimate applies up to some $\tilde{u} \lesssim |\Omega|^{-\frac{1}{2}}$. Choosing \tilde{u} such that $\operatorname{Re} V$ is monotone increasing on the interval (\tilde{u}, u_1) , on the remaining interval $[\tilde{u}, u_1]$ we can again use the T -method with $\alpha \equiv 0$. Again, this gives rise to the estimate

$$\int_{u_-}^{u_+} \operatorname{Im} y \leq C(c_3).$$

Working out similar estimates near $u = \pi$, we can apply Corollary 7.6 with $k = 2$. We conclude that we can choose N such that for all sufficiently large $|\Omega|$, the chosen λ is smaller than the N^{th} eigenvalue. This concludes the proof. \square

7.3. Weyl's Asymptotics. In the next lemma we show that our boundary conditions (2.10) give rise to the usual Weyl asymptotics.

LEMMA 7.8. *The spectrum of the Hamiltonian (7.2) with boundary conditions (2.10) lies on the real axis and consist of points $\lambda_0 < \lambda_1 < \dots$. For large n , the eigenvalues and gaps have the asymptotics*

$$\begin{aligned} \lambda_n &= n^2 + \mathcal{O}(n) \\ \lambda_{n+1} - \lambda_n &= 2n + \mathcal{O}(n^0). \end{aligned}$$

Proof. We consider the family of solutions y of the Riccati equation (7.3) with initial conditions

$$y\left(\frac{\pi}{2}\right) = i\sqrt{\lambda} \quad \text{for } \lambda \in \mathbb{R}^+.$$

Asymptotically for large λ , the potential $\operatorname{Re} V$ becomes nearly constant according to (2.8), except at the poles at $u = 0$ and $u = \pi$. In the case $k \neq s$, one can control the behavior near the poles by using the T -method similar as explained after (7.22). We thus obtain asymptotically

$$\int_0^\pi \operatorname{Im} y = \sqrt{\lambda} \pi + \mathcal{O}\left(\frac{1}{\sqrt{\lambda}}\right). \quad (7.23)$$

In the case $k = \pm s$, one can use the asymptotics of the fundamental solutions as worked out in [12, Sections 7 and 8] to again obtain (7.23).

Combining (7.23) with (7.5) gives the result. \square

8. Slightly Non-Selfadjoint Perturbations. We now prove Theorem 1.1 under the additional assumption that Ω is restricted to a bounded set:

PROPOSITION 8.1. *Let $U \subset \mathbb{C}$ be a bounded set. Then for any s and k in the range (1.4), there is a positive integer N and a family of operators $Q_n(\Omega)$ on \mathcal{H}_k*

defined for all $n \in \mathbb{N} \cup \{0\}$ and $\Omega \in U$ which has the properties (i)–(v) in the statement of Theorem 1.1.

This proposition differs from Theorem 1.1 by the fact that here the parameter N may depend on the set U , whereas in Theorem 1.1 the parameter N is to be chosen uniformly for all Ω in the unbounded strip (1.5). This uniformity in Ω is the main difficulty of the present paper; its proof will be the concern of the remaining Sections 13–16.

Proof of Proposition 8.1. We again consider the Hamiltonian (7.2) with a real potential with boundary conditions (2.10). Choosing contours which enclose each of the eigenvalues $\lambda_0, \lambda_1, \dots$ with winding number one, the contour integral (5.1) defines idempotent operators Q_n , $n \in 0, 1, \dots$. Since H is formally self-adjoint, these operators are symmetric, implying that the Q_n are orthogonal projection operators onto mutually orthogonal subspaces. For any $\lambda \in \mathbb{C} \setminus \{\lambda_0, \lambda_1, \dots\}$, we define the resolvent of the self-adjoint problem by

$$s_\lambda = \sum_{n=0}^{\infty} \frac{1}{\lambda - \lambda_n} Q_n.$$

Here the sum converges absolutely in $L(\mathcal{H})$. Moreover, the resolvent satisfies the identities

$$\left(-\frac{d^2}{du^2} + \operatorname{Re} W - \lambda \mathbb{1} \right) s_\lambda = \mathbb{1} \quad \text{and} \quad \|Q_\lambda\| = \sup_{n \in \mathbb{N} \cup \{0\}} \frac{1}{|\lambda - \lambda_n|}.$$

Our method for treating the imaginary part of the potential is to use the theory of slightly self-adjoint perturbations (see [15, V.4.5]), similar as worked out in [8, Section 8] or [3, Chapter 12]. We first note that, since the poles in (2.5) are real-valued, the imaginary part of the potential is bounded,

$$|\operatorname{Im} W(u)| \leq C \quad \text{for all } u \in (0, \pi) \text{ and } \Omega \in U \quad (8.1)$$

(where the constant C clearly depends on U). Next, using Weyl's asymptotics of Lemma 7.8, we can choose N so large that

$$|\lambda_{n+1} - \lambda_n| \geq 4C \quad \text{for all } n \geq N.$$

We choose contours Γ_n (for $n \geq N$) as circles centered at λ_n with radius $2C$. Moreover, we choose Γ_0 as a circle which encloses the eigenvalues $\lambda_0, \dots, \lambda_{N-1}$, and whose distance to the spectrum is at least $2C$ (see Figure 5). Then for any λ on one of these contours,

$$\|s_\lambda\| \leq \frac{1}{2C}. \quad (8.2)$$

This makes it possible to define the resolvent for the Hamiltonian (2.3) with the complex potential, which we denote for clarity by a tilde, via a Neumann series,

$$\tilde{s}_\lambda := \sum_{k=0}^{\infty} (-s_\lambda \operatorname{Im} W)^k s_\lambda. \quad (8.3)$$

We now integrate this resolvent along the contours Γ_n ,

$$\tilde{Q}_n := -\frac{1}{2\pi i} \oint_{\Gamma_n} \tilde{s}_\lambda d\lambda, \quad n \in \{0, N, N+1, \dots\}. \quad (8.4)$$

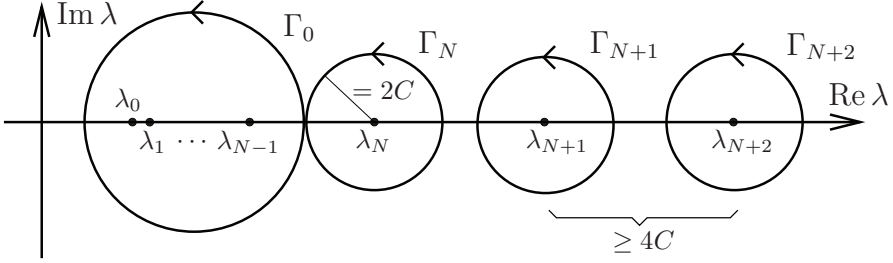


FIG. 5. Contour integrals for slightly non-selfadjoint perturbations.

As explained in [15, V.4.5], these operators are idempotent and map onto the invariant subspaces corresponding to the spectral points enclosed by the contour. Moreover, it is shown in [15, V.4.5] that the spectral projections are complete. The bound (1.6) follows immediately by estimating the contour integral (8.4) and the Neumann series (8.3) using (8.2) and (8.1). \square

9. An A-Priori Estimate for $\text{Im } V$. Assume that $\lambda \in \mathbb{C}$ is an eigenvalue. We let ϕ^{D} be a corresponding eigenfunction. This function satisfies the Dirichlet boundary conditions at $u = 0$ and $u = \pi$. Therefore, the corresponding functions ϕ_L^{D} and ϕ_R^{D} as defined by (2.17) are both multiples of ϕ^{D} . Multiplying the differential equation for ϕ^{D} by $\overline{\phi^{\text{D}}}$ and integrating, we obtain

$$0 = \int_0^\pi \overline{\phi^{\text{D}}} \left(-\frac{d^2}{du^2} + V \right) \phi^{\text{D}} \stackrel{(\star)}{=} \int_0^\pi \overline{\left(-\frac{d^2}{du^2} + \overline{V} \right)} \phi^{\text{D}} \phi^{\text{D}} = \int_0^\pi (V - \overline{V}) \overline{\phi^{\text{D}}} \phi^{\text{D}}, \quad (9.1)$$

where in (\star) we integrated by parts and used the asymptotics for the decaying solution in (2.11) and (2.12), (2.13) to conclude that the boundary terms vanish. We thus obtain the relation

$$\int_0^\pi \text{Im } V |\phi^{\text{D}}|^2 = 0. \quad (9.2)$$

This identity immediately gives rise to the following a-priori estimate.

LEMMA 9.1. *Suppose that $\lambda \in \mathbb{C}$ is an eigenvalue. Then*

$$|\text{Im } \lambda|, |\text{Im } V| \leq C |\Omega|$$

with a constant C which is independent of λ and Ω .

Proof. Using the explicit form of the potential (2.8) together with (1.5), one sees that

$$|\text{Im } V - \text{Im } \lambda| \leq 2c |\Omega| + \text{const}. \quad (9.3)$$

The integral equation (9.2) implies that the function $\text{Im } V$ must change sign on the interval $(0, \pi)$. As a consequence, the absolute value of $\text{Im } \lambda$ is bounded by the right side of (9.3). This gives the result. \square

10. Overview of the Estimates for a Complex Potential. We now enter the general estimates. Recall that our equations involve the parameters k , s , Ω and λ . We always keep k and s fixed. The parameters Ω and λ , however, may vary in a certain

parameter range to be specified later on, and we must make sure that our estimates are uniform in these parameters. In order to keep track of the dependence on Ω and λ , we adopt the convention that

all constants are independent of Ω and λ ,

but they may depend on k and s . Moreover, in order to have a compact and clear notation, we always denote constants which may be increased during our constructions by capital letters $\mathcal{C}_1, \mathcal{C}_2, \dots$. However, constants with small letters $\mathfrak{c}_1, \mathfrak{c}_2, \dots$ are determined at the beginning and are fixed throughout. We use the symbol

$$\lesssim \dots \quad \text{for} \quad \leq \mathfrak{c} \dots$$

with a constant \mathfrak{c} which is independent of the capital constants \mathcal{C}_l (and may thus be fixed right away, without the need to increase it later on).

When increasing the constants \mathcal{C}_l , we must keep track of the mutual dependences of these constants. To this end, we adopt the convention that the constant \mathcal{C}_l may depend on all previous constants $\mathcal{C}_1, \dots, \mathcal{C}_{l-1}$, but is independent of the subsequent constants \mathcal{C}_{l+1}, \dots . In particular, we may choose the capital constants such that $\mathcal{C}_1 \ll \mathcal{C}_2 \ll \dots$. This dependence of the constants implies that increasing \mathcal{C}_l may also make it necessary to increase the subsequent constants $\mathcal{C}_{l+1}, \mathcal{C}_{l+2}, \dots$. For brevity, when we write “possibly after increasing \mathcal{C}_l ” we implicitly mean that the subsequent constants $\mathcal{C}_{l+1}, \mathcal{C}_{l+2}, \dots$ are also suitably increased.

10.1. Different Cases and Regions. In view of Proposition 8.1, it suffices to consider the case that $|\Omega|$ is large. Thus in what follows we always assume that

$$|\Omega| \geq \mathcal{C}_4.$$

Since the imaginary part of Ω is bounded by (1.5), by increasing \mathcal{C}_4 we can always arrange that

$$|\operatorname{Re} \Omega|^2 \geq \frac{3}{4} |\Omega|^2.$$

Furthermore, Lemma 9.1 gives us an a-priori bound on the imaginary part of the eigenvalues,

$$\operatorname{Im} \lambda \lesssim |\Omega|. \tag{10.1}$$

Moreover, in view of Proposition 7.7, we know in the case of a real potential that by choosing N sufficiently large, it suffices to consider the case that λ is real and $\lambda \gg |\Omega|$. With this in mind, in our estimates we may restrict attention to the case

$$\operatorname{Re} \lambda \geq \mathcal{C}_5 |\Omega|. \tag{10.2}$$

This inequality will be justified a-posteriori by showing that if we deform the potential continuously starting from a real potential and ending with our complex potential V , then the inequality (10.2) will be preserved for all spectral points λ_n with $n \geq N$ (for details see Section 16.3).

For large $|\Omega|$, the real part of the potential looks qualitatively like a double-well potential (see the left of Figure 1). More quantitatively, in the region $[\frac{\pi}{3}, \frac{2\pi}{3}]$ away from the poles at $u = 0$ and $u = \pi$, according to (2.8) we have

$$\operatorname{Re} V = \operatorname{Re}(\Omega^2) \sin^2 u - \operatorname{Re} \lambda + \mathcal{O}(\Omega) \quad (10.3)$$

$$\operatorname{Re} V' = 2 \operatorname{Re}(\Omega^2) \sin u \cos u + \mathcal{O}(\Omega) \quad (10.4)$$

$$\operatorname{Re} V'' = 2 \operatorname{Re}(\Omega^2) \cos(2u) + \mathcal{O}(\Omega). \quad (10.5)$$

In particular, one sees that $\operatorname{Re} V$ has a unique local maximum at a point near $\frac{\pi}{2}$, which we denote by u_{\max} ,

$$\operatorname{Re} V'(u_{\max}) = 0 \quad \text{and} \quad u_{\max} = \frac{\pi}{2} + \mathcal{O}(|\Omega|^{-1}).$$

Moreover, the real part of the potential is concave near this maximum,

$$-2|\Omega|^2 \leq \operatorname{Re} V'' \leq -\frac{|\Omega|^2}{2} \quad \text{on} \quad \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]. \quad (10.6)$$

As the intervals $(0, u_{\max}]$ and $[u_{\max}, \pi)$ can be treated similarly, we mainly restrict attention to the interval $(0, u_{\max}]$. The value of the real part of the potential at its local maximum distinguishes different *cases*:

$$\left\{ \begin{array}{ll} \text{WKB case} & \text{if } \operatorname{Re} V(u_{\max}) < -\mathcal{C}_1 |\Omega| \\ \text{parabolic cylinder case} & \text{if } -\mathcal{C}_1 |\Omega| \leq \operatorname{Re} V(u_{\max}) < \mathcal{C}_1 |\Omega| \\ \text{Airy case} & \text{if } \operatorname{Re} V(u_{\max}) \geq \mathcal{C}_1 |\Omega|. \end{array} \right. \quad (10.7)$$

Here \mathcal{C}_1 is a new constant which later on we will choose sufficiently large.

In each of the above cases, we estimate the solution by considering different *regions*, as we now explain. First, we distinguish the *pole region* as the interval $(0, u_\ell)$ with

$$u_\ell := \frac{\mathcal{C}_1}{\sqrt{\operatorname{Re} \lambda}}. \quad (10.8)$$

To the right of the pole region, there is a (possibly empty) WKB region (u_ℓ, u_r) . The definition of u_r depends on the different cases. In the WKB case, we simply set $u_r = u_{\max}$. In the parabolic cylinder case, the fact that the function $\operatorname{Re} V$ is concave (10.6) implies that there is a unique point $u_r \in (\frac{\pi}{3}, u_{\max})$ with $\operatorname{Re} V(u_r) = -\mathcal{C}_1 |\Omega|$. In the Airy case, we make use of the following result.

LEMMA 10.1. *In the Airy case, there are unique points u_r, u_+ in the interval*

$$\underline{u} < u < \min(u_{\max}, \bar{u}) \quad (10.9)$$

with

$$\operatorname{Re} V(u_r) = -\nu, \quad \operatorname{Re} V(u_+) = \nu, \quad (10.10)$$

where \underline{u} , \bar{u} and ν are defined by

$$\underline{u} = \frac{\sqrt{\operatorname{Re} \lambda}}{2|\Omega|}, \quad \bar{u} = 4 \frac{\sqrt{\operatorname{Re} \lambda}}{|\Omega|} \quad (10.11)$$

$$\nu = \min\left(\frac{1}{4}(\mathcal{C}_5^2 |\Omega|^2 \operatorname{Re} \lambda)^{\frac{1}{3}}, \frac{1}{2}(\mathcal{C}_1^2 |\Omega|^2 \operatorname{Re} V(u_{\max}))^{\frac{1}{3}}\right). \quad (10.12)$$

Proof. We first show that $\operatorname{Re} V$ is strictly increasing on the interval $(\underline{u}, u_{\max})$. First, according to (10.4), by increasing \mathcal{C}_4 we can arrange that the function $\operatorname{Re} V$ is strictly increasing on the interval $(\frac{\pi}{8}, \frac{3\pi}{8})$. Moreover, the concavity of $\operatorname{Re} V$ implies that $\operatorname{Re} V$ is also monotone increasing on the interval $[\frac{3\pi}{8}, u_{\max})$. On the remaining interval $(\underline{u}, \frac{\pi}{8}]$, we have the estimate

$$\begin{aligned} \operatorname{Re} V' &\geq \frac{|\Omega|^2}{2} u - \frac{\mathfrak{c}}{u^3} \geq \left(\frac{|\Omega|^2}{2} - \frac{\mathfrak{c}}{u^4} \right) u \\ &\geq \left(\frac{1}{2} - \frac{16\mathfrak{c}|\Omega|^2}{\operatorname{Re}^2 \lambda} \right) |\Omega|^2 u \geq \left(\frac{1}{2} - \frac{16\mathfrak{c}}{\mathcal{C}_5^2} \right) |\Omega|^2 u \geq \frac{|\Omega|^2}{4} u > 0, \end{aligned} \quad (10.13)$$

where in the last step we possibly increased \mathcal{C}_5 . We conclude that $\operatorname{Re} V$ is strictly increasing on the whole interval $(\underline{u}, u_{\max})$.

Next, at \underline{u} we have the estimate

$$\begin{aligned} \operatorname{Re} V(\underline{u}) &\leq |\Omega|^2 \underline{u}^2 + \frac{\mathfrak{c}}{\underline{u}^2} - \operatorname{Re} \lambda = \frac{\operatorname{Re} \lambda}{4} + \frac{4\mathfrak{c}|\Omega|^2}{\operatorname{Re} \lambda} - \operatorname{Re} \lambda \\ &\leq \frac{\operatorname{Re} \lambda}{4} + \frac{4\mathfrak{c}}{\mathcal{C}_5^2} \operatorname{Re} \lambda - \operatorname{Re} \lambda \leq -\frac{\operatorname{Re} \lambda}{2}, \end{aligned}$$

where in the last step we possibly again increased \mathcal{C}_5 . Moreover, if $\bar{u} < u_{\max}$, we have the estimate

$$\begin{aligned} \operatorname{Re} V(\bar{u}) &\geq \frac{1}{4} |\Omega|^2 \bar{u}^2 - \frac{\mathfrak{c}}{\bar{u}^2} - \operatorname{Re} \lambda \geq 4 \operatorname{Re} \lambda - \frac{4\mathfrak{c}|\Omega|^2}{\operatorname{Re} \lambda} - \operatorname{Re} \lambda \\ &\geq 4 \operatorname{Re} \lambda - \frac{4\mathfrak{c}}{\mathcal{C}_5^2} \operatorname{Re} \lambda - \operatorname{Re} \lambda \geq \operatorname{Re} \lambda, \end{aligned}$$

where in the last step we possibly again increased \mathcal{C}_5 . Next, it follows from the definition of ν in (10.12) and (10.2) that $\nu < \operatorname{Re} \lambda/4$. Hence

$$\operatorname{Re} V(\underline{u}) < -\nu \quad \text{and} \quad \operatorname{Re} V(\min(\bar{u}, u_{\max})) > \nu.$$

Now the existence of solutions u_r and u_+ of (10.10) follows from the intermediate value theorem. Uniqueness is an immediate consequence of the above strict monotonicity of $\operatorname{Re} V$. \square

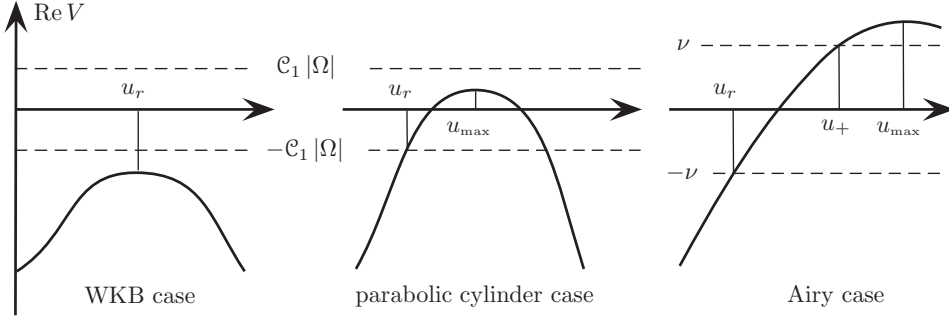
To summarize, the point u_r is defined by

$$\begin{cases} u_r = u_{\max} & \text{in the WKB case} \\ \operatorname{Re} V(u_r) = -\mathcal{C}_1 |\Omega| & \text{in the parabolic cylinder case} \\ \operatorname{Re} V(u_r) = -\nu & \text{in the Airy case.} \end{cases} \quad (10.14)$$

In the Airy case, the interval (u_+, u_{\max}) with u_+ as in (10.10) is another WKB region to the right of the zero of $\operatorname{Re} V$. We thus obtain the following regions:

$$\left\{ \begin{array}{lll} \text{pole region} & (0, u_\ell) & \text{in all cases} \\ \text{WKB region} & (u_\ell, u_r) & \text{in all cases} \\ \text{parabolic cylinder region} & (u_r, u_{\max}) & \text{in the parabolic cylinder case} \\ \text{Airy region} & (u_r, u_+) & \text{in the Airy case} \\ \text{WKB region with } \operatorname{Re} V > 0 & (u_+, u_{\max}) & \text{in the Airy case.} \end{array} \right. \quad (10.15)$$

The different cases are illustrated in Figure 6.


 FIG. 6. *The different cases.*

10.2. Locating the Eigenvalues. Our general strategy is to construct a special solution y_L of the Riccati equation (2.14) on the interval $(0, u_{\max}]$, and a special solution y_R on the interval $[u_{\max}, \pi)$. These solutions are defined by the initial conditions

$$y_L(u_0^L) = y_0^L, \quad y_R(u_0^R) = y_0^R, \quad (10.16)$$

where u_0^L and u_0^R are chosen near the poles at $u = 0$ respectively $u = \pi$ (for details see Section 11.2 and 11.3 below). We choose these special solutions in such a way that our estimates become as simple as possible. This means in particular that these solutions have no singularities. Then we introduce corresponding smooth solutions of the Sturm-Liouville equation (2.7) by integration (cf. (2.15)),

$$\phi_L(u) := \exp\left(\int^u y_L\right), \quad \phi_R(u) := \exp\left(\int^u y_R\right), \quad (10.17)$$

both normalized according to (6.7). These solutions will *not* satisfy the Dirichlet boundary conditions (2.10). By introducing the functions ϕ_L^D and ϕ_R^D again by (2.17), we obtain solutions which do satisfy the Dirichlet boundary conditions. In order to locate the eigenvalues, we must analyze the eigenvalue condition (2.18). It is most convenient to evaluate the Wronskian at u_{\max} ,

$$w(\phi_L^D, \phi_R^D)|_{u_{\max}} = 0. \quad (10.18)$$

Similar to (6.1) and (6.2) we set

$$\zeta_L(u) = \int_0^u \frac{1}{\phi_L^2}, \quad \phi_L^D = \phi_L \zeta_L, \quad (10.19)$$

$$\zeta_R(u) = -\int_u^\pi \frac{1}{\phi_R^2}, \quad \phi_R^D = \phi_R \zeta_R. \quad (10.20)$$

Differentiating these relations, we obtain

$$\frac{(\phi_L^D)'}{\phi_L^D} = \frac{\phi_L'}{\phi_L} + \frac{\zeta_L'}{\zeta_L} = y_L + \frac{1}{\phi_L^2 \zeta_L}$$

and thus

$$\begin{aligned}
w(\phi_L^{\text{D}}, \phi_R^{\text{D}}) &= \phi_L^{\text{D}} \phi_R^{\text{D}} \left(\frac{(\phi_L^{\text{D}})'}{\phi_L^{\text{D}}} - \frac{(\phi_R^{\text{D}})'}{\phi_R^{\text{D}}} \right) \\
&= \phi_L^{\text{D}} \phi_R^{\text{D}} \left(y_L + \frac{1}{\phi_L^2 \zeta_L} - y_R - \frac{1}{\phi_R^2 \zeta_R} \right) \\
&= \phi_L \zeta_L \phi_R \zeta_R \left(y_L + \frac{1}{\phi_L^2 \zeta_L} - y_R - \frac{1}{\phi_R^2 \zeta_R} \right). \tag{10.21}
\end{aligned}$$

Therefore, the eigenvalue condition (10.18) can be written alternatively as

$$(y_L - y_R) + \frac{1}{\phi_L^2 \zeta_L} - \frac{1}{\phi_R^2 \zeta_R} = 0. \tag{10.22}$$

Indeed, in this form the eigenvalue condition is most suited for our analysis. Our main task is to analyze the behavior of the functions y_L and y_R as well as the derived functions ϕ_L, ϕ_R and ζ_L, ζ_R (obtained by (10.17) and (10.19), (10.20)).

11. Estimates in Different Regions.

11.1. Estimates in the WKB Region (u_ℓ, u_r). The name ‘‘WKB region’’ suggests that in these regions the WKB solutions should be a good approximation. This really is the case, in the following sense:

PROPOSITION 11.1. *For any $\delta > 0$ and for sufficiently large \mathcal{C}_1 , the WKB conditions*

$$\frac{|V'|}{|V|^{\frac{3}{2}}}, \frac{|V''|}{|V|^2}, \frac{|V''|}{|V|^{\frac{5}{2}}} < \delta \tag{11.1}$$

hold in the WKB regions (u_ℓ, u_r) and (u_+, u_{\max}) (see (10.15)), uniformly in Ω and λ .

For the significance of the inequalities (11.1) we also refer to [12, eq. (4.1)] and the estimates in [12, Section 4].

The proof of this proposition is split up into several lemmas. The proof will be completed at the end of this section.

LEMMA 11.2. *Possibly by increasing \mathcal{C}_1 , we can arrange that in the region $u > u_\ell$, the potential and its derivatives are bounded by*

$$\begin{aligned}
|W| &\leq |\Omega|^2 u^2 + \frac{\text{Re } \lambda}{2}, & |V'| &\lesssim |\Omega|^2 u + \frac{(\text{Re } \lambda)^{\frac{3}{2}}}{\mathcal{C}_1^3} \\
|V''| &\lesssim |\Omega|^2 + \frac{(\text{Re } \lambda)^2}{\mathcal{C}_1^4}, & |V''| &\lesssim |\Omega|^2 + \frac{(\text{Re } \lambda)^{\frac{5}{2}}}{\mathcal{C}_1^5}.
\end{aligned}$$

Proof. Using the explicit form of the potentials in (2.5), (2.6) and (2.8), we obtain

$$\begin{aligned}
|W| &\leq |\Omega|^2 u^2 + \frac{\mathbf{c}}{u_\ell^2} \leq |\Omega|^2 u^2 + \frac{\mathbf{c}}{\mathcal{C}_1^2} \text{Re } \lambda \stackrel{(*)}{\leq} |\Omega|^2 u^2 + \frac{\text{Re } \lambda}{2} \\
|V'| &\leq 2|\Omega|^2 u + \frac{\mathbf{c}}{u_\ell^3} \lesssim |\Omega|^2 u + \frac{(\text{Re } \lambda)^{\frac{3}{2}}}{\mathcal{C}_1^3} \\
|V''| &\lesssim |\Omega|^2 + \frac{1}{u_\ell^4} \lesssim |\Omega|^2 + \frac{(\text{Re } \lambda)^2}{\mathcal{C}_1^4}, & |V''| &\lesssim |\Omega|^2 + \frac{\mathbf{c}}{u_\ell^5} \lesssim |\Omega|^2 + \frac{(\text{Re } \lambda)^{\frac{5}{2}}}{\mathcal{C}_1^5},
\end{aligned}$$

where in (\star) we possibly increased \mathcal{C}_1 . \square

LEMMA 11.3. *The statement of Proposition 11.1 holds in the region*

$$u_\ell < u < \underline{u}, \quad (11.2)$$

where \underline{u} is again defined by (10.11).

Proof. Using (2.8), we obtain

$$|V(u)| \geq |\operatorname{Re} \lambda| - |W| \geq \frac{\operatorname{Re} \lambda}{2} - |\Omega|^2 u^2 \geq \frac{\operatorname{Re} \lambda}{4} \gtrsim \operatorname{Re} \lambda. \quad (11.3)$$

Hence

$$\begin{aligned} |V'| &\lesssim |\Omega|^2 u + \frac{(\operatorname{Re} \lambda)^{\frac{3}{2}}}{\mathcal{C}_1^3} \lesssim |\Omega| \sqrt{\operatorname{Re} \lambda} + \frac{(\operatorname{Re} \lambda)^{\frac{3}{2}}}{\mathcal{C}_1^3} \\ \frac{|V'|}{|V|^{\frac{3}{2}}} &\lesssim \frac{|\Omega|}{\operatorname{Re} \lambda} + \frac{1}{\mathcal{C}_1^3} \lesssim \frac{1}{\mathcal{C}_5} + \frac{1}{\mathcal{C}_1^3} \\ \frac{|V''|}{|V|^2} &\lesssim \frac{|\Omega|^2}{(\operatorname{Re} \lambda)^2} + \frac{1}{\mathcal{C}_1^4} \lesssim \frac{1}{\mathcal{C}_5^2} + \frac{1}{\mathcal{C}_1^4} \\ \frac{|V'''}{|V|^{\frac{5}{2}}} &\lesssim \frac{|\Omega|^2}{(\operatorname{Re} \lambda)^{\frac{5}{2}}} + \frac{1}{\mathcal{C}_1^5} \lesssim \frac{1}{\mathcal{C}_5^{\frac{5}{2}} \mathcal{C}_4^{\frac{1}{2}}} + \frac{1}{\mathcal{C}_1^5}, \end{aligned}$$

giving the result. \square

It remains to consider the complement of the region (11.2). This complement is empty unless

$$\frac{\sqrt{\operatorname{Re} \lambda}}{2|\Omega|} \leq \frac{\pi}{2}.$$

Therefore, in what follows we can assume that

$$\operatorname{Re} \lambda \lesssim |\Omega|^2. \quad (11.4)$$

Moreover,

$$u \geq \frac{\sqrt{\operatorname{Re} \lambda}}{2|\Omega|} \geq \frac{\sqrt{\operatorname{Re} \lambda}}{2|\Omega|} \geq \frac{\sqrt{\mathcal{C}_5}}{2\sqrt{|\Omega|}} \gtrsim \frac{1}{\sqrt{|\Omega|}}. \quad (11.5)$$

LEMMA 11.4. *Under the assumptions of Proposition 11.1,*

$$\frac{|V''|}{|V|^2}, \frac{|V'''}{|V|^{\frac{5}{2}}} < \delta.$$

Proof. From (10.10), (10.12), (10.1) and (10.2) we know that

$$|V| \geq \nu \gtrsim \min(\mathcal{C}_5 |\Omega|, \mathcal{C}_1 |\Omega|) = \mathcal{C}_1 |\Omega|.$$

Combining this inequality with (11.4) and (11.5), the derivatives of the potential can be estimated by

$$\begin{aligned} |V''| &\lesssim |\Omega|^2 + \frac{1}{u^4} \lesssim |\Omega|^2 + |\Omega|^2 \lesssim |\Omega|^2 \\ |V'''} &\lesssim |\Omega|^2 + \frac{\mathfrak{c}}{u^5} \lesssim |\Omega|^2 + |\Omega|^{\frac{5}{2}} \lesssim |\Omega|^{\frac{5}{2}}. \end{aligned}$$

As a consequence,

$$\frac{|V''|}{|V|^2} \lesssim \frac{|\Omega|^2}{\mathfrak{C}_1^2 |\Omega|^2} = \frac{1}{\mathfrak{C}_1^2}, \quad \frac{|V'''}{|V|^{\frac{5}{2}}}} \lesssim \frac{|\Omega|^{\frac{5}{2}}}{\mathfrak{C}_1^{\frac{5}{2}} |\Omega|^{\frac{5}{2}}} = \frac{1}{\mathfrak{C}_1^{\frac{5}{2}}},$$

completing the proof. \square

It remains to estimate the term involving the first derivatives in (11.1).

LEMMA 11.5. *The statement of Proposition 11.1 holds in the region*

$$u \geq \bar{u},$$

where \bar{u} is again defined by (10.11).

Proof. We can bound the potential from below by

$$\operatorname{Re} V \geq \frac{|\Omega|^2}{4} u^2 - \operatorname{Re} \lambda \gtrsim \frac{|\Omega|^2}{8} u^2.$$

Hence

$$\frac{|V'|}{|V|^{\frac{3}{2}}} \lesssim \frac{|\Omega|^2 u}{|\Omega|^3 u^3} + \frac{(\operatorname{Re} \lambda)^{\frac{3}{2}}}{\mathfrak{C}_1^3 |\Omega|^3 u^3} \lesssim \frac{|\Omega|}{\operatorname{Re} \lambda} + \frac{1}{\mathfrak{C}_1^3} \leq \frac{1}{\mathfrak{C}_5} + \frac{1}{\mathfrak{C}_1^3},$$

giving the result. \square

It remains to estimate the term involving the first derivatives in (11.1) in the region

$$\underline{u} < u < \bar{u}. \quad (11.6)$$

We begin with a lemma in the Airy case.

LEMMA 11.6. *The statement of Proposition 11.1 holds in the Airy case if*

$$(\mathfrak{C}_5^2 |\Omega|^2 \operatorname{Re} \lambda)^{\frac{1}{3}} \leq 2 (\mathfrak{C}_1^2 |\Omega|^2 \operatorname{Re} V(u_{\max}))^{\frac{1}{3}}. \quad (11.7)$$

Proof. In view of (10.12), the assumptions imply that

$$\nu = \frac{1}{4} (\mathfrak{C}_5^2 |\Omega|^2 \operatorname{Re} \lambda)^{\frac{1}{3}}. \quad (11.8)$$

Moreover, according to (10.10), we know that $|V| \geq \nu$. Hence

$$\begin{aligned} |V'| &\lesssim |\Omega|^2 u + \frac{\mathfrak{c}}{u^3} \lesssim |\Omega|^2 u \left(1 + \frac{1}{|\Omega|^2 u^4} \right) \\ &\lesssim |\Omega|^2 u \left(1 + \frac{|\Omega|^2}{\operatorname{Re}^2 \lambda} \right) \stackrel{(10.2)}{\lesssim} |\Omega|^2 u \stackrel{(10.11)}{\lesssim} |\Omega| \sqrt{\operatorname{Re} \lambda} \\ \frac{|V'|}{|V|^{\frac{3}{2}}} &\lesssim \frac{|\Omega| \sqrt{\operatorname{Re} \lambda}}{\nu^{\frac{3}{2}}} \stackrel{(11.8)}{\lesssim} \frac{1}{\mathfrak{C}_5}. \end{aligned}$$

This concludes the proof. \square

Thus in the Airy case, in what follows we may assume that (11.7) is violated. Since in the WKB region under consideration, we stay away from the zeros of $\operatorname{Re} V$ in the sense that $|\operatorname{Re} V| \geq \nu$ with ν as in (10.12), it follows that

$$|\operatorname{Re} V| \geq \frac{1}{2} (\mathcal{C}_1^2 |\Omega|^2 \operatorname{Re} V(u_{\max}))^{\frac{1}{3}} \quad \text{in the Airy case.} \quad (11.9)$$

We now return to the analysis of the region (11.6), without specifying whether we are in the WKB, the Airy or the parabolic cylinder case. As a consequence of (11.6),

$$|V'| \lesssim |\Omega|^2 u + \frac{1}{u^3} \lesssim |\Omega| \sqrt{\operatorname{Re} \lambda} \lesssim |\Omega|^2,$$

where in the last step we again applied (11.4). From this inequality, we obtain the desired estimate provided that one of the following two inequalities holds:

$$|V(u)| \geq \mathcal{C}_2 |\Omega|^{\frac{4}{3}} \quad \text{or} \quad \operatorname{Re} V(u_{\max}) \geq \frac{|\Omega|^2}{\mathcal{C}_1}. \quad (11.10)$$

Namely, the first inequality implies that

$$\frac{|V'|}{|V|^{\frac{3}{2}}} \lesssim \frac{|\Omega|^2}{\mathcal{C}_2^{\frac{3}{2}} |\Omega|^2} = \frac{1}{\mathcal{C}_2^{\frac{3}{2}}}. \quad (11.11)$$

On the other hand, if the second inequality in (11.10) holds, we are in the Airy case (possibly after increasing \mathcal{C}_4), so that (11.9) yields the estimate

$$|V| \geq (\mathcal{C}_1^2 |\Omega|^2 \operatorname{Re} V(u_{\max}))^{\frac{1}{3}} \geq \mathcal{C}_1^{\frac{1}{3}} |\Omega|^{\frac{4}{3}}.$$

This implies that the first inequality in (11.10) again holds (for $\mathcal{C}_2 = \mathcal{C}_1^{\frac{1}{3}}$), making it possible to again use the estimate (11.11).

It remains to consider the case that both inequalities in (11.10) are violated, i.e.

$$|V(u)| < \mathcal{C}_2 |\Omega|^{\frac{4}{3}} \quad \text{and} \quad \operatorname{Re} V(u_{\max}) < \frac{|\Omega|^2}{\mathcal{C}_1}.$$

In this case, $|\operatorname{Re} V(u) - \operatorname{Re} V(u_{\max})| \lesssim |\Omega|^2 / \mathcal{C}_1$, implying that $u \approx u_{\max}$ (more precisely, by increasing \mathcal{C}_1 we can make $|u - u_{\max}|$ arbitrarily small, uniformly in Ω and λ). Since $\operatorname{Re} V$ is concave near u_{\max} (10.6), we may integrate this inequality to obtain

$$\begin{aligned} \frac{|\Omega|^2}{2} |u - u_{\max}| &\leq |V'(u)| \leq |\Omega|^2 |u - u_{\max}| \\ \frac{|\Omega|^2}{4} (u - u_{\max})^2 &\leq \operatorname{Re} V(u_{\max}) - \operatorname{Re} V(u) \leq \frac{|\Omega|^2}{2} (u - u_{\max})^2. \end{aligned}$$

Hence

$$|V'(u)| \lesssim |\Omega| \sqrt{\operatorname{Re} V(u_{\max}) - \operatorname{Re} V(u)} \leq |\Omega| \left(\sqrt{|\operatorname{Re} V(u_{\max})|} + \sqrt{|V(u)|} \right)$$

In the case $|\operatorname{Re} V(u_{\max})| \leq |V(u)|$, it follows that

$$\frac{|V'(u)|}{|V(u)|^{\frac{3}{2}}} \leq \frac{2|\Omega|}{|V(u)|} \lesssim \frac{|\Omega|}{\mathcal{C}_1 |\Omega|} = \frac{1}{\mathcal{C}_1},$$

giving the result. In the remaining case $|\operatorname{Re} V(u_{\max})| > |V(u)|$, we know from (10.14) and (10.7) that we are in the Airy case. Hence, again using (11.9), we obtain

$$\frac{|V'(u)|}{|V(u)|^{\frac{3}{2}}} \leq \frac{2|\Omega|\sqrt{\operatorname{Re} V(u_{\max})}}{|V(u)|^{\frac{3}{2}}} \lesssim \frac{2^{\frac{3}{2}}|\Omega|\sqrt{|\operatorname{Re} V(u_{\max})|}}{\sqrt{\mathcal{C}_1^2|\Omega|^2|\operatorname{Re} V(u_{\max})|}} = \frac{2^{\frac{3}{2}}}{\mathcal{C}_1}.$$

This concludes the proof of Proposition 11.1.

11.2. Estimates in the Pole Region in the Case $k = s$. In this section, we analyze the pole region $(0, u_\ell)$ in the case $k = s$. We consider the solution ϕ_L as defined by (10.16) and (10.17). For ease in notation, we omit all subscripts L . The parameter u_0 in (10.16) is chosen as

$$u_0 := \frac{1}{2} |\Omega|^{-\frac{1}{2}}. \quad (11.12)$$

Possibly by increasing \mathcal{C}_5 , we can arrange that $u_0 > u_\ell$ (cf. (10.8) and (10.2)), so that u_0 lies in the WKB region. We choose the initial values at u_0 in (10.16) equal to the value $\phi'_{\text{WKB}}/\phi_{\text{WKB}}$ for the WKB approximation,

$$y_0 = \sqrt{V(u_0)} - \frac{V'(y_0)}{4V(y_0)} \quad \text{and} \quad \operatorname{Im} \sqrt{V(u_0)} > 0. \quad (11.13)$$

We expand the potential near $u = 0$,

$$V(u) = -\frac{1}{4u^2} - \mu + \Omega^2 u^2 + \mathcal{O}(|\Omega|u^2) + \mathcal{O}(|\Omega|^2 u^4). \quad (11.14)$$

LEMMA 11.7. *For any $\delta > 0$, we can arrange by increasing \mathcal{C}_5 that*

$$\int_0^{u_\ell} \frac{1}{|\phi|^2} \leq \delta,$$

uniformly in Ω and λ .

Proof. Using the asymptotics as worked out in [12, Section 7.1], on the interval $(0, u_0)$ the solution ϕ has the form

$$\phi(u) \approx -c\sqrt{u} \left(K_0(\sqrt{\mu}u) + (\arg \sqrt{\mu} - \log(2) + \gamma + i) I_0(\sqrt{\mu}u) \right), \quad (11.15)$$

where c is the constant (see [12, eqn. (7.3)])

$$c = \frac{\sqrt{2\pi}}{\arg \sqrt{\mu} - \log(2) + \gamma + i}$$

(and μ is related to λ and Ω by (2.9)). As specified in [12, Section 8.1], the error in (11.15) becomes arbitrarily small for large $|\Omega|$. Note that c is bounded uniformly in μ . For small u , the function ϕ has the asymptotics (see [12, Section 7.1])

$$\phi(u) = -c(\sqrt{u} \log |\sqrt{\mu}u| + i\sqrt{u} + \mathcal{O}(\sqrt{\mu}u)). \quad (11.16)$$

In particular, using that

$$\int_0^u \frac{1}{u(1 + \log^2(\sqrt{\mu}u))} = \arctan(\log(\sqrt{\mu}u)) + \frac{\pi}{2},$$

one sees that $1/|\phi|^2$ is integrable. Hence

$$\lim_{v \searrow 0} \int_0^v \frac{1}{|\phi|^2} = 0, \quad \text{uniformly in } \mu.$$

In view of (10.8), by increasing \mathcal{C}_5 we can make $\sqrt{\mu} u_\ell$ as small as we like. This gives the result. \square

11.3. Estimates in the Pole Region in the Case $k \neq s$. We now analyze the pole region $(0, u_\ell)$ in the case $k \neq s$. We again consider the solution ϕ_L as defined by (10.16) and (10.17) and omit all subscripts L . The parameter u_0 in (10.16) is chosen as the minimum of $\operatorname{Re} V$,

$$\operatorname{Re} V'(u_0) = 0. \quad (11.17)$$

Introducing the abbreviation

$$\Lambda = (k - s)^2 - \frac{1}{4},$$

the potential near $u = 0$ has the expansion

$$V(u) = \frac{\Lambda}{u^2} - \mu + \Omega^2 u^2 + \mathcal{O}(|\Omega|u^2) + \mathcal{O}(|\Omega|^2 u^4). \quad (11.18)$$

Computing the zero of the derivative, we obtain

$$u_0 = \Lambda^{\frac{1}{4}} |\Omega|^{-\frac{1}{2}} + \mathcal{O}(|\Omega|^{-\frac{3}{2}}), \quad (11.19)$$

so that for large Ω we have the estimates

$$\frac{1}{2} \Lambda^{\frac{1}{4}} |\Omega|^{-\frac{1}{2}} \leq u_0 \leq 2\Lambda^{\frac{1}{4}} |\Omega|^{-\frac{1}{2}}. \quad (11.20)$$

Possibly by increasing the constant \mathcal{C}_5 in (10.2), we can again arrange that $u_0 > u_\ell$, so that u_0 lies in the WKB region. We again choose the initial values at u_0 in agreement with the WKB approximation (11.13).

LEMMA 11.8. *For any $\delta > 0$, we can arrange by increasing \mathcal{C}_5 that*

$$\int_0^{u_\ell} \frac{1}{|\phi|^2} \leq \delta,$$

uniformly in Ω and λ .

Proof. Using the asymptotics as worked out in [12, Section 7.2], the solution ϕ has the form

$$\phi(u) \approx c\sqrt{u} K_{|k-s|}(-\sqrt{\mu}u), \quad (11.21)$$

where c is the constant

$$c = \sqrt{-\frac{2}{\pi}}.$$

As specified in [12, Section 8.2], the error in (11.21) becomes arbitrarily small for large $|\Omega|$. For small u , the function ϕ has the asymptotics (see [12, Section 7.2])

$$\begin{aligned} \phi(u) &= c \frac{(n-1)!}{\sqrt{2}\mu^{\frac{1}{4}}} \left(-\frac{\sqrt{\mu}u}{2}\right)^{\frac{1}{2}-|k-s|} (1 + \mathcal{O}(u)) \\ &\quad - c \frac{\sqrt{2}}{|k-s|!\mu^{\frac{1}{4}}} \left(\frac{\sqrt{\mu}u}{2}\right)^{\frac{1}{2}+|k-s|} (1 + \mathcal{O}(u)). \end{aligned} \quad (11.22)$$

In particular, one sees that $|\phi|$ has a pole at $u = 0$ and is thus bounded from below near $u = 0$. Hence

$$\lim_{v \searrow 0} \int_0^v \frac{1}{|\phi|^2} = 0, \quad \text{uniformly in } \mu.$$

In view of (10.8), by increasing \mathcal{C}_5 we can make $\sqrt{\mu}u_\ell$ as small as we like. This gives the result. \square

11.4. Estimates in the Parabolic Cylinder Region. In the next proposition we estimate the Riccati solution in the parabolic cylinder region.

PROPOSITION 11.9. *Assume that in the parabolic cylinder region $[u_r, u_{\max}]$, one of the following two conditions hold:*

- (a) *The potential has a positive imaginary part, $\text{Im } V|_{[u_r, u_+]} \geq 0$.*
- (b) *The imaginary part of the potential has a zero on $[u_r^L, u_r^R]$.*

Moreover, assume that the Riccati solution begins in the upper half plane, $\text{Im } y(u_r) \geq 0$. Then there is a constant \mathcal{C}_2 (depending on \mathcal{C}_1) such that for large $|\Omega|$, the solution on the interval $[u_r, u_{\max}]$ can be estimated in terms of $y(u_r)$ by

$$\begin{aligned} |y(u)| &\leq \mathcal{C}_2 |y(u_r)| \\ \text{Im } y(u) &\geq \frac{\text{Im } y(u_r)}{\mathcal{C}_2} \\ \frac{|\phi(u_r)|}{\mathcal{C}_2} &\leq |\phi(u)| \leq \mathcal{C}_2 |\phi(u_r)|. \end{aligned}$$

Proof. We set $\nu = \mathcal{C}_1|\Omega|$. Using that the function $\text{Re } V$ is concave near u_{\max} , we obtain

$$(u_{\max} - u_r)^2 \lesssim \frac{\nu}{|\Omega|^2} \quad (11.23)$$

and thus

$$\nu (u_{\max} - u_r)^2 \lesssim \mathcal{C}_1^2.$$

Our strategy is to estimate y using the T -method as introduced in [12, Section 3.2] choosing

$$\alpha = \sqrt{2\nu} \quad \text{and} \quad \tilde{\beta} = 0. \quad (11.24)$$

Hence

$$\tilde{V} = \alpha^2 = 2\nu \quad \text{and} \quad U = \text{Re } V - \alpha^2 \leq -\nu. \quad (11.25)$$

In case (a), our method is to apply [12, Theorem 3.3] for $g \equiv 0$. The terms E_1, \dots, E_4 are estimated as follows,

$$\begin{aligned} E_2 &= E_4 = 0 \\ |E_1| &\lesssim \frac{1}{\sqrt{\nu}} |\operatorname{Re} V - \operatorname{Re} \tilde{V}| + \frac{\operatorname{Re} V'}{\nu} \lesssim \sqrt{\nu} + \frac{\operatorname{Re} V'}{\nu} \\ |E_3| &\lesssim \frac{|\operatorname{Im} V|}{\sqrt{\nu}} \stackrel{(*)}{\lesssim} \frac{|\Omega|}{\sqrt{\nu}}, \end{aligned}$$

where in $(*)$ we used (10.1) as well as the fact that

$$|\operatorname{Im}(\Omega^2)| \lesssim |\Omega| |\operatorname{Im} \Omega| \stackrel{(1.5)}{\lesssim} |\Omega|.$$

As a consequence, we can apply Lemma 11.10 to obtain

$$\begin{aligned} \int_{u_r}^{u_{\max}} (|E_1| + |E_3|) &\lesssim \sqrt{\nu} (u_{\max} - u_r) + \frac{1}{\nu} \int_{u_r}^{u_{\max}} \operatorname{Re} V' + \frac{|\Omega|}{\sqrt{\nu}} (u_{\max} - u_r) \\ &= \sqrt{\nu} (u_{\max} - u_r) + \frac{1}{\nu} (\operatorname{Re} V(u_{\max}) - \operatorname{Re} V(u_r)) + \frac{|\Omega|}{\nu} \sqrt{\nu} (u_{\max} - u_r) \\ &\lesssim \mathcal{C}_1 + 2 + \mathcal{C}_1 \frac{|\Omega|}{\nu} \leq \mathcal{C}_1 + 3. \end{aligned}$$

This concludes the proof in case (a).

In case (b), the imaginary part of V could be negative. Therefore, in order to apply [12, Theorem 3.3] we need to choose the function g positive in accordance with the inequality

$$g \geq T - 1. \quad (11.26)$$

We choose α and $\tilde{\beta}$ as in (11.24) and $g = |\Omega|^{\frac{1}{2}}$. Then the error terms E_1, E_2 and E_3 estimated just as above. Estimating $\operatorname{Im} V$ with the help of the mean value theorem by

$$|\operatorname{Im} V| \leq (u_r^+ - u_r^-) \sup_{[u_r^L, u_r^R]} |V'| \lesssim |\Omega| (u_r^R - u_r^L)$$

(where in the last step we used the explicit form of the potential (2.8)), the error term E_4 , is estimated by

$$\begin{aligned} \int_{u_r}^{u_{\max}} |E_4| &= g \int_{u_r}^{u_{\max}} \frac{|\operatorname{Im} V|}{\sqrt{|U|}} \lesssim g \int_{u_r}^{u_{\max}} \frac{|\Omega| (u_r^R - u_r^L)}{\sqrt{\nu}} \\ &\lesssim \frac{g |\Omega|}{\sqrt{\nu}} (u_r^R - u_r^L)^2 \stackrel{(11.23)}{\lesssim} \frac{g \sqrt{\nu}}{|\Omega|} = \frac{g \sqrt{\mathcal{C}_1}}{\sqrt{|\Omega|}} = \frac{\sqrt{\mathcal{C}_1}}{|\Omega|^{\frac{1}{4}}}. \end{aligned}$$

This can be made arbitrarily small by increasing $|\Omega|$, implying that the inequality (11.26) holds. This concludes the proof. \square

11.5. Estimates in the Airy Region. We proceed with estimates in the Airy region. We first recall that it remains to consider the interval (10.9). For this interval to be non-empty, we can again assume that (11.4) holds,

$$\operatorname{Re} \lambda \lesssim |\Omega|^2 \quad \text{and thus} \quad |\operatorname{Re} V| \lesssim |\Omega|^2. \quad (11.27)$$

We begin with a preparatory lemma.

LEMMA 11.10. *In the Airy region, the function $\operatorname{Re} V$ is strictly monotone. Moreover,*

$$\nu(u_+ - u_r)^2 \approx \mathcal{C}_1^2. \quad (11.28)$$

Proof. The strict monotonicity was already shown in the proof of Lemma 10.1. In preparation for the estimate (11.28), we recall that the region (u_r, u_+) is contained in the interval (\underline{u}, \bar{u}) (see Lemma 10.1), and thus

$$u_r, u_+ \approx \frac{\sqrt{\operatorname{Re} \lambda}}{|\Omega|}. \quad (11.29)$$

We consider the regions $(\underline{u}, \frac{3\pi}{8})$ and $[\frac{3\pi}{8}, u_{\max})$ separately. In the region $(\underline{u}, \frac{3\pi}{8})$, we know from (10.13) that $\operatorname{Re} V' \gtrsim |\Omega|^2 u$. Moreover, the method in (10.13) also gives the reverse inequality,

$$\begin{aligned} \operatorname{Re} V' &\leq |\Omega|^2 u + \frac{\mathfrak{c}}{u^3} \lesssim \left(1 + \frac{1}{|\Omega|^2 \underline{u}^4}\right) |\Omega|^2 u \\ &\approx \left(1 + \frac{|\Omega|^2}{\operatorname{Re}^2 \lambda}\right) |\Omega|^2 u \lesssim |\Omega|^2 u, \end{aligned}$$

where in the last step we used (11.29) and (10.1). We conclude that

$$\operatorname{Re} V'|_{(u_r, u_+)} \approx |\Omega|^2 u \approx |\Omega| \sqrt{\operatorname{Re} \lambda}.$$

As a consequence, the mean value theorems

$$(u_+ - u_r) \inf_{(u_r, u_+)} \operatorname{Re} V' \leq 2\nu \leq (u_+ - u_r) \sup_{(u_r, u_+)} \operatorname{Re} V'$$

give rise to the estimate

$$u_+ - u_r \approx \frac{\nu}{|\Omega| \sqrt{\operatorname{Re} \lambda}}$$

and thus

$$\nu(u_+ - u_r)^2 \approx \frac{\nu^3}{|\Omega|^2 \operatorname{Re} \lambda}. \quad (11.30)$$

In order to estimate this further, we need to determine the scaling of ν . Using the estimate for the second derivative in (10.5) with the fact that $\operatorname{Re} V$ has a maximum at u_{\max} and no zero on the interval $(\frac{3\pi}{8}, u_{\max})$, we conclude that

$$\operatorname{Re} V(u_{\max}) \gtrsim |\Omega|^2.$$

Combining this inequality with the first inequality in (11.27), we find that the first term in (10.12) can be bounded in terms of the second term. More precisely, we obtain the inequality

$$\nu^3 \approx \min(\mathcal{C}_5^2, \mathcal{C}_1^2) |\Omega|^2 \operatorname{Re} \lambda = \mathcal{C}_1^2 |\Omega|^2.$$

Using this inequality in (11.30) gives

$$\nu(u_+ - u_r)^2 \approx \mathcal{C}_1^2.$$

This concludes the proof on the interval $(\frac{u}{8}, \frac{3\pi}{8})$.

It remains to consider the region $(\frac{3\pi}{8}, u_{\max})$. We make use of the concavity of $\operatorname{Re} V$, (10.5). Denoting the zero of $\operatorname{Re} V$ by u_1 , we obtain

$$\operatorname{Re} V(u_{\max}) \approx |\Omega|^2 (u_{\max} - u_1)^2. \quad (11.31)$$

On the other hand, for the potential to have a zero near $\frac{\pi}{2}$, the spectral parameter must scale like $\lambda \approx |\Omega|^2$. Therefore, the second term in (10.12) can be estimated in terms of the first, so that

$$\nu^3 \approx \min(\mathcal{C}_1^2, \mathcal{C}_5^2) |\Omega|^2 \operatorname{Re} V(u_{\max}) = \mathcal{C}_1^2 |\Omega|^2 \operatorname{Re} V(u_{\max}). \quad (11.32)$$

Since $\nu < \operatorname{Re} V(u_{\max})/2$ (cf. (10.12) and (10.7)), the estimate (11.31) can be extended to

$$\operatorname{Re} V(u_{\max}) \approx |\Omega|^2 (u_{\max} - u)^2 \quad \text{for all } u \in [u_r, u_+]. \quad (11.33)$$

Moreover,

$$\operatorname{Re} V'(u) \approx |\Omega|^2 (u_{\max} - u) \quad \text{for all } u \in [u_r, u_+]. \quad (11.34)$$

We now combine the estimates (11.32), (11.33) and (11.34) to obtain

$$\begin{aligned} u_+ - u_r &\approx \frac{\nu}{\operatorname{Re} V'} \\ \nu(u_+ - u_r)^2 &\approx \frac{\nu^3}{\operatorname{Re}^2 V'} \approx \frac{\mathcal{C}_1^2 |\Omega|^2 \operatorname{Re} V(u_{\max})}{\operatorname{Re}^2 V'} \\ &\approx \frac{\mathcal{C}_1^2 |\Omega|^4 (u_{\max} - u)^2}{|\Omega|^4 (u_{\max} - u)^2} = \mathcal{C}_1^2. \end{aligned}$$

This concludes the proof. \square

We now estimate the Riccati solution in the Airy region.

PROPOSITION 11.11. *Assume that in the Airy region $[u_r, u_+]$, one of the following two conditions hold:*

- (a) *The potential has a positive imaginary part, $\operatorname{Im} V|_{[u_r, u_+]} \geq 0$.*
- (b) *The imaginary part of the potential is small in the sense that*

$$|\operatorname{Im} V| \leq |\Omega|^{1-\delta} \quad \text{for a suitable constant } \delta > 0.$$

Moreover, assume that the Riccati solution begins in the upper half plane, $\operatorname{Im} y(u_r) \geq 0$. Then there is a constant \mathcal{C}_2 (depending on \mathcal{C}_1) such that for large $|\Omega|$, the solution on the interval $[u_r, u_+]$ can be estimated in terms of $y(u_r)$ by

$$|y(u)| \leq \mathcal{C}_2 |y(u_r)| \quad (11.35)$$

$$\operatorname{Im} y(u) \geq \frac{\operatorname{Im} y(u_r)}{\mathcal{C}_2} \quad (11.36)$$

$$\frac{|\phi(u_r)|}{\mathcal{C}_2} \leq |\phi(u)| \leq \mathcal{C}_2 |\phi(u_r)|. \quad (11.37)$$

Proof. As in the proof of Proposition 11.9 we use the T -method choosing α and $\tilde{\beta}$ as in (11.24). Then V and U are again estimated by (11.25).

In case (a), we apply [12, Theorem 3.3] for $g \equiv 0$. The error terms E_1, \dots, E_4 are estimated as follows,

$$\begin{aligned} E_2 &= E_4 = 0 \\ |E_1| &\lesssim \frac{1}{\sqrt{\nu}} |\operatorname{Re} V - \operatorname{Re} \tilde{V}| + \frac{\operatorname{Re} V'}{\nu} \lesssim \sqrt{\nu} + \frac{\operatorname{Re} V'}{\nu} \\ |E_3| &\lesssim \frac{|\operatorname{Im} V|}{\sqrt{\nu}} \lesssim \frac{|\Omega|}{\sqrt{\nu}}. \end{aligned}$$

As a consequence, we can apply Lemma 11.10 to obtain

$$\begin{aligned} \int_{u_r}^{u_+} (|E_1| + |E_3|) &\lesssim \sqrt{\nu} (u_+ - u_r) + \frac{1}{\nu} \int_{u_r}^{u_+} \operatorname{Re} V' + \frac{|\Omega|}{\sqrt{\nu}} (u_+ - u_r) \\ &= \sqrt{\nu} (u_+ - u_r) + \frac{1}{\nu} \left(\operatorname{Re} V(u_+) - \operatorname{Re} V(u_r) \right) + \frac{|\Omega|}{\nu} \sqrt{\nu} (u_+ - u_r) \\ &\lesssim \mathfrak{C}_1 + 2 + \mathfrak{C}_1 \frac{|\Omega|}{\nu}. \end{aligned}$$

Finally, by combining (10.12) with (10.1), (10.2) and (10.7), we conclude that $\nu \gtrsim |\Omega|$. This concludes the proof in case (a).

In the remaining case (b), we choose g as

$$g = |\Omega|^{\frac{\delta}{2}}.$$

Then the error term E_4 is estimated by

$$\int_{u_r}^{u_{\max}} |E_4| = g \int_{u_r}^{u_+} \frac{|\operatorname{Im} V|}{\sqrt{|U|}} \lesssim g |\Omega|^{1-\delta} \frac{(u_+ - u_r)}{\sqrt{\nu}} \stackrel{(11.28)}{\lesssim} g |\Omega|^{1-\delta} \frac{\sqrt{\mathfrak{C}_1}}{\nu} \lesssim \frac{|\Omega|^{-\frac{\delta}{2}}}{\sqrt{\mathfrak{C}_1}}.$$

This concludes the proof. \square

In the next lemma we compare the imaginary part of the potential on the two Airy regions $[u_r^L, u_+^L]$ and $[u_r^R, u_+^R]$.

LEMMA 11.12. *In the Airy case, one of the following three statements holds:*

- (i) $\operatorname{Im} V|_{[u_r^L, u_+^L]} \geq 0$ and $\operatorname{Im} V|_{[u_r^R, u_+^R]} \geq 0$
- (ii) $\operatorname{Im} V|_{[u_r^L, u_+^L]} \leq 0$ and $\operatorname{Im} V|_{[u_r^R, u_+^R]} \leq 0$
- (iii) $|\operatorname{Im} V|_{[u_r^L, u_+^L]} \lesssim \sqrt{|\Omega|}$ and $|\operatorname{Im} V|_{[u_r^R, u_+^R]} \lesssim \sqrt{|\Omega|}$.

Proof. We first consider the case that the Airy regions are near the poles in the sense that $u_+^L, (\pi - u_+^R) \lesssim |\Omega|^{-\frac{1}{4}}$. Then the factor $\sin^2 u$ in (2.8) is bounded by $|\Omega|^{-\frac{1}{2}}$, implying that $\operatorname{Im}(V + \lambda)$ is bounded by $\sqrt{|\Omega|}$. Therefore, depending on the value of $\operatorname{Im} \lambda$, we are in one of the above cases (i)–(iii).

It remains to consider the case that the Airy regions are away from the poles. Then the factors $1/(\sin^2 u)$ in (2.8) is bounded by $\sqrt{|\Omega|}$. As a consequence, the imaginary part of V can be related to its real part by

$$\operatorname{Im} V = \frac{2 \operatorname{Im} \Omega}{\operatorname{Re} \Omega} \operatorname{Re} V + (\operatorname{const}) + \mathcal{O}(\sqrt{|\Omega|}).$$

The function $\operatorname{Re} V$ has a zero $v_{L/R}$ in each of the intervals $[u_r^L, u_+^L]$ and $[u_r^R, u_+^R]$. It follows that at these zeros, the imaginary part of the potential has the form

$$\operatorname{Im} V(v_{L/R}) = (\text{const}) + \mathcal{O}(\sqrt{|\Omega|}).$$

It remains to show that on each of the intervals $[u_r^L, u_+^L]$ and $[u_r^R, u_+^R]$, the total variation of the function $\operatorname{Im} V$ is $\lesssim \sqrt{|\Omega|}$. By symmetry, it suffices to consider the interval $[u_r^L, u_+^L]$. For ease in notation, we omit then index L . We then obtain

$$\begin{aligned} \operatorname{TV}_{[u_r, u_+]} \operatorname{Im} V &\leq (u_+^L - u_r^L) \sup_{[u_r^L, u_+^L]} |\operatorname{Im} V'| \\ &\lesssim \sqrt{\nu} (u_+^L - u_r^L) \frac{|\Omega|}{\sqrt{\nu}} \stackrel{(11.28)}{\lesssim} \frac{|\Omega|}{\sqrt{\nu}} \lesssim \frac{\sqrt{|\Omega|}}{\mathcal{C}_1}. \end{aligned}$$

This concludes the proof. \square

11.6. Estimates on the Interval $[u_+, u_{\max}]$. It remains to estimate the solution in the Airy case on the interval $[u_+, u_{\max}]$ (see (10.15) and (10.10)).

LEMMA 11.13. *For any $\varepsilon > 0$, by increasing \mathcal{C}_1 one can arrange that*

$$\int_{u_+}^{u_{\max}} \frac{1}{|\phi|^2} \leq \varepsilon.$$

Moreover,

$$|\phi(u)| \gtrsim \frac{1}{(\operatorname{Re} V(u))^{\frac{1}{4}}} e^{\int_{u_+}^u \operatorname{Re} \sqrt{V}}. \quad (11.38)$$

Proof. In Proposition 11.1 it was shown that the WKB conditions (11.1) are satisfied on the interval $[u_+, u_{\max}]$. Thus the solution is well-approximated by the WKB solution

$$\phi \approx \frac{1}{(\operatorname{Re} V)^{\frac{1}{4}}} \left(C_1 e^{\int_{u_+}^u \sqrt{V}} + C_2 e^{-\int_{u_+}^u \sqrt{V}} \right), \quad (11.39)$$

with error terms which are under control in view of the estimates in [12]. Note that one of the fundamental solutions in (11.39) is exponentially increasing, whereas the other is exponentially decaying.

Combining the estimate of Proposition 11.1 at $u = u_r$ with the estimates (11.35) and (11.36) on the interval $[u_r, u_+]$ and taking into account the normalization (6.7), one sees that the coefficient of the exponentially increasing fundamental solution in (11.39) is bounded away from zero, and that $|\phi(u_+)| \approx |\operatorname{Re} V(u_+)|^{-\frac{1}{4}}$. This gives (11.38). Next, we increase \mathcal{C}_5 to a new constant \mathcal{C}_3 . Denoting the corresponding boundary of the Airy region by \tilde{u}_+ , we obtain

$$|\phi| \gtrsim \frac{1}{(\operatorname{Re} V)^{\frac{1}{4}}} e^{\int_{u_+}^{\tilde{u}_+} \operatorname{Re} \sqrt{V}} e^{\int_{\tilde{u}_+}^u \operatorname{Re} \sqrt{V}}.$$

As a consequence,

$$\begin{aligned} \int_{\tilde{u}_+}^{u_{\max}} \frac{1}{|\phi|^2} &\lesssim -e^{-2 \int_{u_+}^{\tilde{u}_+} \operatorname{Re} \sqrt{V}} \int_{\tilde{u}_+}^{u_{\max}} \frac{d}{du} \left(e^{-2 \int_{\tilde{u}_+}^u \operatorname{Re} \sqrt{V}} \right) \\ &= e^{-2 \int_{u_+}^{\tilde{u}_+} \operatorname{Re} \sqrt{V}} \left(1 - e^{-2 \int_{\tilde{u}_+}^{u_{\max}} \operatorname{Re} \sqrt{V}} \right) \leq e^{-2 \int_{u_+}^{\tilde{u}_+} \operatorname{Re} \sqrt{V}}. \end{aligned} \quad (11.40)$$

The integral in the last exponent can be estimated from above by $\sqrt{\nu}(\tilde{u}_+ - u_+)$. Applying Lemma 11.10, this term can be made arbitrarily large by increasing \mathcal{C}_1 (and consequently \mathcal{C}_5). As a consequence, the last exponent in (11.40) can be made arbitrarily small. This gives the result. \square

12. Integral Estimates of $\text{Im } V$. In this section we shall derive the following estimates.

PROPOSITION 12.1. *For all eigenvalues λ , the following inequality holds,*

$$\int_{u_\ell}^{u_r} \frac{|\text{Im } V|}{\sqrt{|V|}} \lesssim 1.$$

PROPOSITION 12.2. *For any $\delta > 0$, by increasing \mathcal{C}_1 one can arrange that that for all eigenvalues λ in the WKB case and the parabolic cylinder case the following inequality holds:*

$$\left| \int_{u_\ell}^{u_r} \frac{\text{Im } V}{\sqrt{|V|}} \right| < \delta. \quad (12.1)$$

12.1. Elementary Estimates of the Potential. We begin with integral estimates of our potential.

LEMMA 12.3. *The function $\text{Re } V$ is monotone increasing on the interval $[u_0, u_r]$. Moreover,*

$$\int_{u_0}^{u_r} \frac{1}{|\text{Re } V|^{\frac{3}{2}}} \lesssim \frac{1}{\mathcal{C}_1 |\Omega|^2}. \quad (12.2)$$

Proof. In order to prove the monotonicity of $\text{Re } V$, we first recall that in the proof of Lemma 10.1 we already showed that $\text{Re } V$ is monotone increasing on the interval $(\underline{u}, u_{\max})$. On the remaining interval $[u_0, \underline{u}]$, we need to consider the cases $k = s$ and $k \neq s$ separately. In the case $k = s$, the monotonicity of $\text{Re } V$ is obvious from (11.14). In the case $k \neq s$, we see from (11.18) that $\text{Re } V$ is convex. Combining this with the fact that u_0 is chosen as a minimum of $\text{Re } V$ (see (11.17)), we conclude again that $\text{Re } V$ is monotone increasing on the interval $[u_0, \underline{u}]$.

For the integral estimate (12.2), we first consider the interval (u_ℓ, u_2) with

$$u_2 = \min(u_r, \sqrt{\mathcal{C}_1} |\Omega|^{-\frac{1}{2}}). \quad (12.3)$$

Then the desired estimate is obtained by using that the integration range scales like $|\Omega|^{-\frac{1}{2}}$. On the remaining interval (u_2, u_r) we consider the regions $(u_2, \frac{\pi}{6})$ and $(\frac{\pi}{6}, u_r)$ separately. In the first region, we approximate $\text{Re } V$ by the quadratic potential

$$|\text{Re } V| \geq a - c(u - u_2)^2, \quad (12.4)$$

with parameters $a, c > 0$ to be specified below. Applying Lemma 9.1, it follows by explicit computation that

$$\begin{aligned} \int_{u_2}^{\min(u_r, \frac{\pi}{6})} \frac{1}{|V|^{\frac{3}{2}}} &\leq \int_{u_2}^{\min(u_r, \frac{\pi}{6})} \frac{1}{(a - c(u - u_2)^2)^{\frac{3}{2}}} \\ &= \frac{u - u_2}{a \sqrt{a - c(u - u_2)^2}} \Big|_{\min(u_r, \frac{\pi}{6})} \leq \frac{u - u_2}{a \sqrt{\mathcal{C}_1} |\Omega|} \Big|_{\min(u_r, \frac{\pi}{6})}, \end{aligned}$$

where in the last step we used that we are in the WKB region (see (10.15), (10.14) and (10.12)). Since the quadratic polynomial $a - c(u - u_2)^2$ is positive, we know that

$$u - u_2 \leq \sqrt{\frac{a}{c}},$$

implying that

$$\int_{u_2}^{\min(u_r, \frac{\pi}{8})} \frac{1}{|V|^{\frac{3}{2}}} \leq \frac{1}{\sqrt{\mathcal{C}_1 a c} |\Omega|}.$$

Using that $a \geq \mathcal{C}_1 |\Omega|$ and $c \approx |\Omega|^2$, we obtain the desired estimate.

On the remaining interval $[\frac{\pi}{8}, u_r]$, we estimate $\operatorname{Re} V$ on the interval $[u_0, u_r]$ by the quadratic polynomial

$$|\operatorname{Re} V| \geq a + b(u_r - u) + c(u_r - u)^2, \quad (12.5)$$

where the values of the positive coefficients a , b and c will be estimated below. It follows by explicit computation that

$$\begin{aligned} \int_{u_0}^{u_r} \frac{1}{|\operatorname{Re} V|^{\frac{3}{2}}} &\leq \int_{u_0}^{u_r} \left(a + b(u_r - u) + c(u_r - u)^2 \right)^{-\frac{3}{2}} \\ &\leq \int_{-\infty}^{u_r} \left(a + b(u_r - u) + c(u_r - u)^2 \right)^{-\frac{3}{2}} = \frac{2}{\sqrt{a} b + 2a\sqrt{c}}. \end{aligned}$$

It remains to analyze the coefficients a , b and c . In view of (10.12), (10.14) and (10.12), we can choose $a \geq \mathcal{C}_5 |\Omega|$. Moreover, the expansions (10.3)–(10.5) show that we can choose either $b \approx |\Omega|^2$ or $c \approx |\Omega|^2$. This concludes the proof. \square

LEMMA 12.4. *For any $\delta > 0$ there is a constant $C = C(\delta)$ such that the following inequality holds,*

$$\int_{u_\ell}^{\min(u_r, \frac{\pi}{2} - \delta)} \frac{1}{\sqrt{|V|}} \leq \frac{C}{|\Omega|}. \quad (12.6)$$

Moreover, there is a constant C such that

$$\int_{u_\ell}^{u_r} \frac{1}{\sqrt{|V|}} \leq C \frac{\log |\Omega|}{|\Omega|}. \quad (12.7)$$

Proof. As in the previous lemma, we first consider the interval (u_ℓ, u_2) with u_2 according to (12.3). Then, according to (10.7) and (10.15), (10.14), (10.12) and (11.3), we know that

$$|\operatorname{Re} V| \Big|_{(u_\ell, u_r)} \gtrsim \mathcal{C}_1 |\Omega|$$

and thus

$$\int_{u_0}^{u_2} \frac{1}{\sqrt{|V|}} \lesssim \frac{u_2 - u_0}{\sqrt{\mathcal{C}_1} |\Omega|}.$$

The desired estimate is obtained by using that the integration range scales like $|\Omega|^{-\frac{1}{2}}$.

It remains to consider the interval (u_2, u_r) . We consider the regions $(u_2, \frac{\pi}{6})$ and $(\frac{\pi}{6}, u_r)$ separately. In the first region, the first and second derivatives of $\operatorname{Re} V$ are non-negative. We again estimate $\operatorname{Re} V$ by the quadratic polynomial (12.4), where the values of the positive coefficients a and c will be estimated below. It follows by explicit computation that

$$\begin{aligned} \int_{u_2}^{\min(u_r, \frac{\pi}{6})} \frac{1}{\sqrt{|V|}} &\leq \int_{u_2}^{\min(u_r, \frac{\pi}{6})} \frac{1}{\sqrt{a - c(u - u_2)^2}} \\ &= \frac{1}{\sqrt{c}} \arctan \left(\frac{\sqrt{c}(u - u_2)}{\sqrt{a - c(u - u_2)^2}} \right) \Big|_{\min(u_r, \frac{\pi}{6})}. \end{aligned}$$

Since the arctan is bounded, it suffices to note that $c \approx |\Omega|^2$ to obtain the desired $1/|\Omega|$ behavior.

On the remaining interval $(\frac{\pi}{6}, u_r)$, we again estimate $\operatorname{Re} V$ by the quadratic polynomial (12.5), where the values of the positive coefficients a , b and c will be estimated below. An explicit computation yields

$$\begin{aligned} \int_{u_2}^{u_r} \frac{1}{\sqrt{|V|}} &\leq \int_{u_2}^{u_r} \frac{1}{\sqrt{a + b(u_r - u) + c(u_r - u)^2}} \\ &= \frac{1}{\sqrt{c}} \log \left(b + 2c(u_r - u) + 2\sqrt{c} \sqrt{a + b(u_r - u) + c(u_r - u)^2} \right) \Big|_{u_r}^{u_2}. \end{aligned} \quad (12.8)$$

It remains to analyze the argument of the logarithm. From the explicit form of the potential (2.8) we know that we may choose $b, c \approx |\Omega|^2$. If $a \gg |\Omega|^2$ is large, the logarithm in (12.8) is given approximately by $\log(2\sqrt{c}\sqrt{a})$. As a consequence, the difference of the logarithms at the upper and lower boundary points is uniformly bounded. Using that $c \approx |\Omega|^2$, we obtain the desired estimate.

It remains to consider the case that $a \lesssim |\Omega|^2$. Then the arguments of the logarithm scale like $|\Omega|^2$, both at the upper and lower boundary point. As a consequence, the logarithm is again uniformly bounded, giving the desired estimate. This concludes the proof of the inequality (12.6).

In order to prove (12.7), we first note that the estimate (12.6) fails to hold in general if $\delta = 0$. The problem is that in this case, the parameter b in (12.8) could be small, leading to a factor $\log |\Omega|$. This gives (12.7). \square

Applying Lemma 9.1 to (12.7), we obtain the estimate

$$\int_{u_\ell}^{u_r} \frac{|\operatorname{Im} V|}{\sqrt{|V|}} \lesssim \log |\Omega|.$$

Unfortunately, the factor $\log |\Omega|$ is not good enough for our purposes. The next lemma shows that, if $\operatorname{Im} V$ vanishes at the right boundary point, then we get an estimate without such a logarithmic factor.

LEMMA 12.5. *Assume that $\tilde{u} \in (u_\ell, u_r)$ is a point where $\operatorname{Im} V(\tilde{u}) = 0$. Then*

$$\int_{u_\ell}^{\tilde{u}} \frac{|\operatorname{Im} V|}{\sqrt{|V|}} \lesssim 1.$$

Proof. Combining Lemma 9.1 with Lemma 12.4, it remains to consider the case $u_r > \frac{\pi}{2} - \delta$. Moreover, it remains to estimate the integral over the interval $(\frac{\pi}{6}, u_r)$.

We again estimate $\operatorname{Re} V$ by the quadratic polynomial (12.5) with positive coefficients a , b and c . Moreover, we estimate $\operatorname{Im} V$ by

$$|\operatorname{Im} V(u)| \lesssim |\Omega| (\tilde{u} - u).$$

We thus obtain

$$\begin{aligned} \int_{u_2}^{\tilde{u}} \frac{|\operatorname{Im} V|}{\sqrt{|V|}} &\lesssim \int_{u_2}^{\tilde{u}} \frac{|\Omega| (\tilde{u} - u)}{\sqrt{a + b(u_r - u) + c(u_r - u)^2}} \leq \int_{u_2}^{u_r} \frac{|\Omega| (u_r - u)}{\sqrt{a + b(u_r - u) + c(u_r - u)^2}} \\ &\leq \frac{\Omega}{2c} \int_{u_2}^{u_r} \frac{2c(u_r - u) + b}{\sqrt{a + b(u_r - u) + c(u_r - u)^2}} = \frac{\Omega}{c} \sqrt{a + b(u_r - u) + c(u_r - u)^2} \Big|_{u_2}^{u_r}. \end{aligned}$$

The result follows because $c \approx |\Omega|^2$ and the argument of the square root is bounded by $\lesssim |\Omega|^2$. \square

12.2. Estimates on the Interval $(0, u_\ell)$. LEMMA 12.6. *For any $\tilde{\delta} > 0$, we can arrange by increasing \mathcal{C}_5 that*

$$\int_0^{u_\ell} |\operatorname{Im} V| |\phi_L^{\mathbb{D}}|^2 \leq \tilde{\delta},$$

uniformly in Ω and λ .

Proof. We begin with the case $k = s$. Using the asymptotics as in the proof of Lemma 11.7, we find that on the interval $(0, u_\ell)$,

$$\begin{aligned} |\phi_L(u)| &\leq 2|c| \sqrt{u} \log(\sqrt{\mu}u) \\ |\zeta_L(u)| &\leq \frac{1}{|c|^2} \left(\arctan(\log(\sqrt{\mu}u)) + \frac{\pi}{2} \right) \lesssim \frac{1}{\log(\sqrt{\mu}u)}, \end{aligned}$$

uniformly in μ . As a consequence,

$$|\phi_L^{\mathbb{D}}|^2 \lesssim u \quad \text{uniformly in } \mu.$$

Applying Lemma 9.1, we obtain

$$\int_0^{u_\ell} |\operatorname{Im} V| |\phi_L^{\mathbb{D}}|^2 \lesssim |\Omega| u_\ell^2 \stackrel{(10.8)}{\leq} \frac{\mathcal{C}_1^2 |\Omega|}{\operatorname{Re} \lambda} \stackrel{(10.2)}{\leq} \frac{\mathcal{C}_1^2}{\mathcal{C}_5}.$$

This gives the result.

In the case $k \neq s$, we work similarly with the asymptotics (11.22),

$$\begin{aligned} |\phi| &\approx |\mu|^{-\frac{L}{2}} u^{\frac{1}{2}-L} \\ \int_0^u \frac{1}{|\phi|^2} &\lesssim |\mu|^L u^{2L} \\ |\phi_L^{\mathbb{D}}| &\lesssim |\mu|^{\frac{L}{2}} u^{L+\frac{1}{2}} \\ \int_0^{u_\ell} |\operatorname{Im} V| |\phi_L^{\mathbb{D}}|^2 &\lesssim |\Omega| |\mu|^L u_\ell^{2L+2} \stackrel{(10.8)}{\lesssim} \frac{\mathcal{C}_1^{2L+2} |\Omega|}{|\operatorname{Re} \lambda|} \stackrel{(10.2)}{\leq} \frac{\mathcal{C}_1^{2L+2}}{\mathcal{C}_5}. \end{aligned}$$

This concludes the proof. \square

12.3. WKB Representation of ϕ_L^D . We again let ϕ_L^D be the solutions (2.17) with ϕ_L defined by (10.17) and (10.16)). In this section, we shall approximate ϕ_L^D on the interval (u_ℓ, u_r) by a suitable WKB wave function. Our starting point is the WKB approximation of ϕ_L

$$\phi_L \approx \phi_{\text{WKB}} = \frac{c_{\text{WKB}}}{\sqrt[4]{V}} e^{\int_{u_0}^u \sqrt{V}}. \quad (12.9)$$

In order to comply with the initial conditions (6.7), we must choose

$$c_{\text{WKB}}^2 = \frac{\sqrt{V(u_0)}}{\text{Im} \sqrt{V(u_0)}}.$$

In the next proposition, we compute what this approximation means for the osculating circles as introduced in Section 6.

PROPOSITION 12.7. *On the interval $[u_\ell, u_r]$, the radius and center of the osculating circle are given by*

$$\mathfrak{p} \approx \zeta(u_0) + \frac{1}{2c_{\text{WKB}}^2} \quad \text{and} \quad R \approx \frac{1}{2} e^{-2 \int_{u_0}^u \text{Re} \sqrt{V}}, \quad (12.10)$$

where the error can be made arbitrarily small by increasing \mathcal{C}_1 .

Proof. Using the WKB approximation (12.9), we obtain

$$y = \frac{\phi'}{\phi} \approx \sqrt{V} - \frac{V'}{4V}$$

and thus

$$\begin{aligned} \zeta(u) - \zeta(u_0) &= \int_{u_0}^u \frac{1}{\phi^2} \approx \frac{1}{c_{\text{WKB}}^2} \int_{u_0}^u \sqrt{V(\tau)} e^{-2 \int_{u_0}^\tau \sqrt{V}} d\tau \\ &= -\frac{1}{2c_{\text{WKB}}^2} \int_{u_0}^u \frac{d}{d\tau} e^{-2 \int_{u_0}^\tau \sqrt{V}} d\tau = -\frac{1}{2c_{\text{WKB}}^2} \left(e^{-2 \int_{u_0}^u \sqrt{V}} - 1 \right). \end{aligned} \quad (12.11)$$

It is remarkable that the integral can be carried out explicitly, giving a simple expression for $\zeta(u)$. \square

We next compute ϕ_L^D . Using (12.9) and (12.11), we obtain

$$\begin{aligned} \phi_L^D(u) &= \phi_L(u) \zeta_L(u) = \phi_L \left(\int_0^{u_0} \frac{1}{\phi^2} + (\zeta(u) - \zeta(u_0)) \right) \\ &\approx \frac{c_{\text{WKB}}}{\sqrt[4]{V}} e^{\int_{u_0}^u \sqrt{V}} \left(\int_0^{u_0} \frac{1}{\phi^2} - \frac{1}{2c_{\text{WKB}}^2} \left(e^{-2 \int_{u_0}^u \sqrt{V}} - 1 \right) \right). \end{aligned}$$

Hence

$$\phi_L^D(u) \approx \frac{1}{2c_{\text{WKB}}} \left(\frac{\alpha}{\sqrt[4]{V}} e^{\int_{u_0}^u \sqrt{V}} - \frac{1}{\sqrt[4]{V}} e^{-\int_{u_0}^u \sqrt{V}} \right), \quad (12.12)$$

where α is the constant

$$\alpha = 1 + 2 c_{\text{WKB}}^2 \int_0^{u_0} \frac{1}{\phi^2}.$$

LEMMA 12.8. *By choosing \mathcal{C}_5 sufficiently large, we can make the expression*

$$|\alpha| - 1$$

arbitrarily small.

Proof. Choosing \mathcal{C}_5 sufficiently large, we can arrange that the potential at u_0 is approximately real and negative (cf. (11.12), (11.14) and (11.19), (11.18)). Hence

$$c_{\text{WKB}}^2 \approx i,$$

implying that

$$|\alpha| \approx \left| 1 + 2i \int_0^{u_0} \frac{1}{\phi^2} \right| = |1 + 2i\zeta(u_0)|, \quad (12.13)$$

with an arbitrarily small error. Next, using the initial conditions (6.7) in (6.10), we obtain

$$\mathbf{p}(u_0) = \zeta(u_0) - \frac{i}{2} \quad \text{and} \quad R(u_0) = \frac{1}{2}.$$

Solving for $\zeta(u_0)$ and substituting into (12.13), one finds that $|\alpha| \approx 2|\mathbf{p}(u_0)|$, and thus

$$|\alpha| - 1 \leq 2 \left| |\mathbf{p}(u_0)| - R(u_0) \right|.$$

Since $\zeta(0) = 0$, we know that $|\mathbf{p}(0)| = R(0)$. Hence

$$\left| |\mathbf{p}(u_0)| - R(u_0) \right| \leq \int_0^{u_0} (|\mathbf{p}'| + |R'|).$$

In view of (6.10) and (6.12), we know that $|\mathbf{p}'| = |R'|$. Hence our task is to show that the total variation of R on the interval $(0, u_0)$ is arbitrarily small. Since $R(u_0) = \frac{1}{2}$, it suffices to show that the total variation of $\log R$ is small. Thus, according to (6.12), it remains to estimate the integral

$$\int_0^{u_0} \frac{|R'|}{R} = \int_0^{u_0} \frac{|\text{Im } V|}{\text{Im } y}.$$

In view of the estimates in Sections 11.2 and 11.3, we know that

$$\text{Im } y \gtrsim \text{Re } \mu.$$

Hence

$$\int_0^{u_0} \frac{|\text{Im } V|}{\text{Im } y} \lesssim \frac{u_0}{\text{Re } \mu} \lesssim \frac{|\Omega|}{\text{Re } \mu}.$$

By choosing \mathcal{C}_5 sufficiently large, we can make this expression arbitrarily small. \square

Using this lemma in formula (12.12), we obtain the estimate

$$|\phi_L^{\text{D}}(u)|^2 \lesssim \frac{1}{\sqrt{|V|}} \cosh \left(2 \int_{u_0}^u \text{Re } \sqrt{V} \right). \quad (12.14)$$

12.4. Integral Estimates of WKB Solutions.

LEMMA 12.9. *Assume that $\tilde{u} \in (u_\ell, u_r)$ is a point where $\text{Im } V(\tilde{u}) = 0$. Then*

$$\int_{u_\ell}^{\tilde{u}} |\text{Im } V| |\phi_L^{\text{D}}|^2 \lesssim 1.$$

Proof. We can work with the WKB approximation (12.14). According to Lemma 12.5, we know that

$$\left| \int_{u_0}^{\tilde{u}} \text{Re } \sqrt{V} \right| \lesssim \int_{u_r}^{\tilde{u}} \frac{|\text{Im } V|}{\sqrt{|V|}} \lesssim 1,$$

giving uniform control of the absolute value of the hyperbolic cosine in (12.14). As a consequence,

$$\int_{u_\ell}^{\tilde{u}} |\text{Im } V| |\phi_L^{\text{D}}|^2 \lesssim \int_{u_\ell}^{\tilde{u}} \frac{|\text{Im } V|}{\sqrt{|V|}} \lesssim 1,$$

where in the last step we again applied Lemma 12.5. This concludes the proof. \square

Next, we take the absolute square of the WKB approximation (12.12),

$$|\phi_L^{\text{D}}(u)|^2 \approx \frac{|\alpha|^2}{4|c_{\text{WKB}}|^2} \frac{1}{\sqrt{|V|}} e^{2 \int_{u_0}^u \text{Re } \sqrt{V}} + \frac{1}{4|c_{\text{WKB}}|^2} \frac{1}{\sqrt{|V|}} e^{-2 \int_{u_0}^u \text{Re } \sqrt{V}} \quad (12.15)$$

$$+ \frac{1}{2|c_{\text{WKB}}|^2} \frac{1}{\sqrt{|V|}} \text{Re} \left(\alpha e^{2i \int_{u_0}^u \text{Im } \sqrt{V}} \right). \quad (12.16)$$

The integrand of the last term is oscillatory. As a consequence, the resulting integral is small, as quantified in the next lemma.

LEMMA 12.10. *For any $\delta > 0$, by increasing \mathcal{C}_1 one can arrange that for all eigenvalues λ the following inequality holds:*

$$\int_{u_\ell}^{u_r} \frac{|\text{Im } V|}{\sqrt{|V|}} e^{2i \int_{u_0}^u \text{Im } \sqrt{V}} < \delta.$$

Proof. Since on the interval $(0, \frac{\pi}{2})$, the function $\text{Im } V$ changes signs only once, it suffices to show that for any $\tilde{u}_1, \tilde{u}_2 \in [u_\ell, u_r]$,

$$\left| \int_{\tilde{u}_1}^{\tilde{u}_2} \frac{\text{Im } V}{\sqrt{|V|}} e^{2i \int_{u_0}^u \text{Im } \sqrt{V}} \right| < \delta.$$

Integrating by parts,

$$\begin{aligned} \int_{\tilde{u}_1}^{\tilde{u}_2} \frac{\text{Im } V}{\sqrt{|V|}} e^{2i \int_{u_0}^u \text{Im } \sqrt{V}} &= \int_{\tilde{u}_1}^{\tilde{u}_2} \frac{\text{Im } V}{\sqrt{|V|}} \frac{1}{2i \text{Im } \sqrt{V}} \frac{d}{du} e^{2i \int_{u_0}^u \text{Im } \sqrt{V}} \\ &= \frac{\text{Im } V}{\sqrt{|V|}} \frac{1}{2i \text{Im } \sqrt{V}} e^{2i \int_{u_0}^u \text{Im } \sqrt{V}} \Big|_{\tilde{u}_1}^{\tilde{u}_2} - \int_{\tilde{u}_1}^{\tilde{u}_2} \frac{d}{du} \left(\frac{\text{Im } V}{\sqrt{|V|}} \frac{1}{2i \text{Im } \sqrt{V}} \right) e^{2i \int_{u_0}^u \text{Im } \sqrt{V}}, \end{aligned}$$

we obtain the estimate

$$\begin{aligned}
 & \left| \int_{\tilde{u}_1}^{\tilde{u}_2} \frac{\operatorname{Im} V}{\sqrt{|V|}} e^{2i \int_{u_0}^u \operatorname{Im} \sqrt{V}} \right| \\
 & \leq \frac{|\operatorname{Im} V|}{2|V|} \Big|_{\tilde{u}_1} + \frac{|\operatorname{Im} V|}{2|V|} \Big|_{\tilde{u}_2} + \int_{\tilde{u}_1}^{\tilde{u}_2} \left| \frac{d}{du} \left(\frac{\operatorname{Im} V}{2\sqrt{|V|} \operatorname{Im} \sqrt{V}} \right) \right| \\
 & \lesssim \frac{|\operatorname{Im} V|}{|\operatorname{Re} V|} \Big|_{\tilde{u}_1} + \frac{|\operatorname{Im} V|}{|\operatorname{Re} V|} \Big|_{\tilde{u}_2} + \int_{\tilde{u}_1}^{\tilde{u}_2} \frac{|\operatorname{Im} V'|}{|\operatorname{Re} V|} + \int_{\tilde{u}_1}^{\tilde{u}_2} \frac{|\operatorname{Im} V'|^2}{|\operatorname{Re} V|^2} + \int_{\tilde{u}_1}^{\tilde{u}_2} \frac{|\operatorname{Re} V'| |\operatorname{Im} V|}{|\operatorname{Re} V|^2}.
 \end{aligned}$$

All the terms except for the last summand can immediately be estimated in the desired way using the explicit form of our potential. For the last term we use the monotonicity of $\operatorname{Re} V$ (see Lemma 12.3) to obtain

$$\int_{\tilde{u}_1}^{\tilde{u}_2} \frac{|\operatorname{Re} V'| |\operatorname{Im} V|}{|\operatorname{Re} V|^2} \lesssim |\Omega| \int_{\tilde{u}_1}^{\tilde{u}_2} \frac{|\operatorname{Re} V'|}{|\operatorname{Re} V|^2} \leq |\Omega| \sup_{[\tilde{u}_1, \tilde{u}_2]} |V|^{-1}.$$

This gives the result. \square

Keeping track of the constants, we now write ϕ^{D} as

$$\phi^{\text{D}}(u) = \begin{cases} c_L \phi_L^{\text{D}} & \text{if } u \leq \frac{\pi}{2} \\ c_R \phi_R^{\text{D}} & \text{if } u > \frac{\pi}{2}, \end{cases} \quad (12.17)$$

where c_L and c_R are non-zero complex numbers (and ϕ_L^{D} and ϕ_R^{D} are again the solutions (2.17) with ϕ_L and ϕ_R defined by (10.17) and (10.16)).

LEMMA 12.11. *If $|c_L/c_R| \leq 1$, then*

$$\int_{\frac{\pi}{2}}^{\pi} |\operatorname{Im} V| |\phi^{\text{D}}|^2 \lesssim |c_R|^2. \quad (12.18)$$

Proof. We again denote the zeros of $\operatorname{Im} V$ by \tilde{u}_L and \tilde{u}_R . Lemma 12.6 and Lemma 12.9 imply that

$$\left(\int_0^{\tilde{u}_L} + \int_{\tilde{u}_R}^{\frac{\pi}{2}} \right) |\operatorname{Im} V| |\phi^{\text{D}}|^2 \lesssim |c_R|^2.$$

Moreover, using the representation (12.17) in (9.2), we obtain

$$\int_{\frac{\pi}{2}}^{\tilde{u}_R} |\operatorname{Im} V| |\phi^{\text{D}}|^2 \lesssim |c_R|^2.$$

This gives the result. \square

LEMMA 12.12. *If $|c_L/c_R| \leq 1$, then*

$$\int_{u_0}^{u_r} \frac{|\operatorname{Im} V|}{\sqrt{|V|}} \lesssim 1.$$

Proof. The strategy is to combine Lemma 12.11 with the fact that the potential is approximately symmetric with respect to reflections at $u = \frac{\pi}{2}$. In order to make this approximate symmetry precise, we consider the homotopy

$$V_\tau(u) := \tau V(u) + (1 - \tau) V(\pi - u) \quad \text{for} \quad \tau \in [0, 1]. \quad (12.19)$$

Then the mean value theorem implies that

$$|\operatorname{Re} V_1(u) - \operatorname{Re} V_2(u)| \leq \sup_{\tau \in [0, 1]} \left| \frac{d}{d\tau} \operatorname{Re} V_\tau(u) \right|,$$

and similarly for the imaginary part. Using that the function $\sin^2 u$ in (2.8) is reflection symmetric, one finds that

$$|\operatorname{Re} V_1(u) - \operatorname{Re} V_2(u)| \lesssim |\Omega| \quad \text{and} \quad |\operatorname{Im} V_1(u) - \operatorname{Im} V_2(u)| \lesssim 1. \quad (12.20)$$

This implies that the WKB approximation holds on the “reflected WKB-region” $[\pi - u_r, \pi - u_0]$. Using the WKB approximation (12.15) and (12.16) in (12.18), the oscillatory contribution (12.16) was estimated in Lemma 12.10. Noting that one of the factors $\exp(\pm 2 \int_{u_0}^u \operatorname{Re} \sqrt{V})$ in (12.15) is greater than one, Lemma 12.11 implies that

$$\int_{\pi - u_r}^{\pi - u_0} \frac{|\operatorname{Im} V|}{\sqrt{|V|}} \lesssim 1.$$

Again applying the reflection argument and the mean value theorem, it remains to show that

$$\int_{u_0}^{u_r} \left| \frac{d}{d\tau} \left(\frac{\operatorname{Im} V_\tau}{\sqrt{V_\tau}} \right) \right| \lesssim 1.$$

Again using the explicit form of the potential (2.8),

$$\left| \frac{d}{d\tau} \left(\frac{\operatorname{Im} V_\tau}{\sqrt{V_\tau}} \right) \right| = \left| \frac{\operatorname{Im} \partial_\tau V_\tau}{\sqrt{V_\tau}} \right| + \left| \frac{(\partial_\tau V_\tau) \operatorname{Im} V_\tau}{V_\tau^{\frac{3}{2}}} \right| \lesssim \frac{1}{\sqrt{|\Omega|}} + \frac{|\Omega|^2}{|V|^{\frac{3}{2}}}.$$

Integrating this inequality from u_0 to u_r , we can apply Lemma 12.3 to obtain the result. \square

Proof of Proposition 12.1. It suffices to consider the case $|c_L/c_R| \leq 1$ and to show that

$$\int_{u_r^L}^{u_r^L} \frac{|\operatorname{Im} V|}{\sqrt{|V|}} + \int_{u_r^R}^{u_r^R} \frac{|\operatorname{Im} V|}{\sqrt{|V|}} \lesssim 1.$$

Then case $|c_L/c_R| > 1$ can be treated similarly by exchanging the left and right subintervals with the reflection $u \leftrightarrow \pi - u$.

On the interval $[u_r^R, u_l^R]$, we argue as in the proof of Lemma 12.11 to obtain

$$\int_{u_r^R}^{u_l^R} \frac{|\operatorname{Im} V|}{\sqrt{|V|}} \lesssim 1.$$

On the interval $[u_l^L, u_r^L]$ we consider different subintervals: The region from u_l to u_0 is estimated in Lemma 12.4. The region from u_0 to u_r , on the other hand, is estimated in Lemma 12.12. This concludes the proof. \square

12.5. Estimates in the WKB and Parabolic Cylinder Cases. We now give the proof of Proposition 12.2. Our strategy is to refine the method of Section 9 and to combine it with the “reflection argument” which was already used in the proof of Lemma 12.12. Another ingredient is Proposition 12.1 (whose proof was completed in the previous section).

We will apply Proposition 12.1 in the following way. In the WKB region, we know from (10.1), (10.2) and (10.14) that

$$|\operatorname{Im} V| \lesssim |\Omega| \quad \text{and} \quad \operatorname{Re} V \lesssim -\mathfrak{C}_1 |\Omega|$$

and thus

$$\frac{|\operatorname{Im} V|}{|\operatorname{Re} V|} \lesssim \frac{1}{\mathfrak{C}_1} \quad \text{in the WKB region.}$$

As a consequence, we may expand the square root of the potential as

$$\sqrt{V} = \sqrt{\operatorname{Re} V + i \operatorname{Im} V} = i \sqrt{|\operatorname{Re} V| - i \operatorname{Im} V} = i \sqrt{|\operatorname{Re} V|} + \frac{\operatorname{Im} V}{\sqrt{|\operatorname{Re} V|}} + \dots,$$

showing that

$$\operatorname{Re} \sqrt{V} = \frac{\operatorname{Im} V}{\sqrt{|\operatorname{Re} V|}} \left(1 + \mathcal{O}\left(\frac{1}{\mathfrak{C}_1}\right) \right). \quad (12.21)$$

In particular,

$$|\operatorname{Re} \sqrt{V}| \lesssim \frac{|\operatorname{Im} V|}{\sqrt{|\operatorname{Re} V|}},$$

and applying Proposition 12.1, we conclude that

$$\int_{u_\ell}^{u_r} |\operatorname{Re} \sqrt{V}| \lesssim 1. \quad (12.22)$$

This shows that the exponentials and hyperbolic cosine in the WKB approximation (see (12.9), (12.12), (12.14) and (12.15), (12.16)) are uniformly bounded.

We again assume that $\lambda \in \mathbb{C}$ is an eigenvalue and $\phi^{\mathfrak{D}}$ the corresponding eigenfunction. Moreover, assume that we are in the WKB case or the parabolic cylinder case (but not in the Airy case, which is excluded in Proposition 12.2). Using the representation (12.17) in (9.2), we obtain the identity

$$|c_L|^2 \int_0^{\frac{\pi}{2}} \operatorname{Im} V |\phi_L^{\mathfrak{D}}|^2 + |c_R|^2 \int_{\frac{\pi}{2}}^{\pi} \operatorname{Im} V |\phi_R^{\mathfrak{D}}|^2 = 0. \quad (12.23)$$

Denoting the integrands by f ,

$$f(u) := \begin{cases} \operatorname{Im} V |\phi_L^{\mathfrak{D}}|^2 & \text{if } u \leq \frac{\pi}{2} \\ \operatorname{Im} V |\phi_R^{\mathfrak{D}}|^2 & \text{if } u > \frac{\pi}{2} \end{cases},$$

we decompose f into its even and odd parts,

$$f = f_+ + f_- \quad \text{with} \quad f_{\pm}(u) := \frac{1}{2} \left(f(u) \pm f(\pi - u) \right). \quad (12.24)$$

Then we can rewrite (12.23) as

$$\begin{aligned}
0 &= |c_L|^2 \int_0^{\frac{\pi}{2}} (f_+ + f_-) + |c_R|^2 \int_{\frac{\pi}{2}}^{\pi} (f_+ + f_-) \\
&= |c_L|^2 \int_0^{\frac{\pi}{2}} (f_+ + f_-) + |c_R|^2 \int_0^{\frac{\pi}{2}} (f_+ - f_-) \\
&= \left(|c_L|^2 + |c_R|^2\right) \int_0^{\frac{\pi}{2}} f_+ + \left(|c_L|^2 - |c_R|^2\right) \int_0^{\frac{\pi}{2}} f_-.
\end{aligned}$$

Since the constants c_L and c_R are non-zero, we obtain the inequality

$$\left| \int_0^{\frac{\pi}{2}} f_+ \right| \leq \left| \int_0^{\frac{\pi}{2}} f_- \right|.$$

Combining this estimate with (12.24), we obtain the inequality

$$\left| \int_0^{\frac{\pi}{2}} \operatorname{Im} V |\phi_L^{\mathbb{D}}|^2 \right| = \left| \int_0^{\frac{\pi}{2}} (f_+ + f_-) \right| \leq 2 \left| \int_0^{\frac{\pi}{2}} f_- \right|. \quad (12.25)$$

We now estimate the right side of this inequality obtain the following result.

LEMMA 12.13. *For any $\delta > 0$, by increasing \mathcal{C}_1 one can arrange that that for all eigenvalues λ in the WKB case and the parabolic cylinder case the following inequality holds:*

$$\left| \int_{u_\ell}^{u_r} \operatorname{Im} V |\phi_L^{\mathbb{D}}|^2 \right| < \delta.$$

Proof. We introduce the “parity transformation” P which reflects at the point $\frac{\pi}{2}$,

$$P : (0, \pi) \rightarrow (0, \pi), \quad Pu = \pi - u.$$

Then (12.25) can be written as

$$\left| \int_0^{\frac{\pi}{2}} \operatorname{Im} V |\phi_L^{\mathbb{D}}|^2 \right| \leq \left| \int_0^{\frac{\pi}{2}} \left(\operatorname{Im} V |\phi_L^{\mathbb{D}}|^2 - \operatorname{Im}(V \circ P) |\phi_R^{\mathbb{D}} \circ P|^2 \right) \right|.$$

On the interval $(0, u_\ell)$, this integral can be estimated by Lemma 12.6, where we choose $\tilde{\delta} = \delta/16$. Moreover, in view of the estimates of Lemma 11.9, in the parabolic cylinder case we may estimate the functions $\phi^{\mathbb{D}}$ and ζ at the point u_{\max} in terms of their values at u_r . This shows that the integral over $[u_r, \frac{\pi}{2}]$ can be made smaller than $\delta/8$. We conclude that

$$\left| \int_0^{\frac{\pi}{2}} \operatorname{Im} V |\phi_L^{\mathbb{D}}|^2 \right| \leq \left| \int_{u_\ell}^{u_r} \left(\operatorname{Im} V |\phi_L^{\mathbb{D}}|^2 - \operatorname{Im}(V \circ P) |\phi_R^{\mathbb{D}} \circ P|^2 \right) \right| + \frac{\delta}{4}. \quad (12.26)$$

We next specify the wave functions $\phi_L^{\mathbb{D}}$ and $\phi_R^{\mathbb{D}}$. Using (12.15) and (12.16) together with Lemma 12.8 and Lemma 12.10, we know that in the integral on the right side of (12.26), the factor $|\phi_L^{\mathbb{D}}|^2$ may be replaced by the function

$$\frac{1}{2\sqrt{|\bar{V}|}} \cosh \left(2 \int_{u_0}^u \operatorname{Re} \sqrt{V} \right),$$

making an error which can be made arbitrarily small by increasing \mathcal{C}_1 (here we use Proposition 12.1 to conclude that a small pointwise error gives rise to a small error of the integral). Moreover, using (12.21), we may replace $\operatorname{Re} \sqrt{V}$ by $\operatorname{Im} V / \sqrt{|V|}$, again making an arbitrarily small error. Therefore, setting

$$\rho_L^{\mathbb{D}}(u) := \frac{1}{2\sqrt{|V|}} \cosh \left(2 \int_{u_0}^u \frac{\operatorname{Im} V}{\sqrt{|V|}} \right), \quad (12.27)$$

we can arrange that

$$\left| \int_{u_\ell}^{u_r} \operatorname{Im} V |\phi_L^{\mathbb{D}}|^2 - \int_{u_\ell}^{u_r} \operatorname{Im} V \rho_L^{\mathbb{D}} \right| \leq \frac{\delta}{8}. \quad (12.28)$$

Using the same argument on the interval $[\frac{\pi}{2}, \pi]$, we conclude that

$$\begin{aligned} \left| \int_0^{\frac{\pi}{2}} \operatorname{Im} V |\phi_L^{\mathbb{D}}|^2 \right| &\leq \left| \int_{u_\ell}^{u_r} \left(\operatorname{Im} V \rho_L^{\mathbb{D}} - \operatorname{Im}(V \circ P) (\rho_R^{\mathbb{D}} \circ P) \right) \right| + \frac{\delta}{2} \\ &\leq \left| \int_{u_\ell}^{u_r} \operatorname{Im} (V - V \circ P) \rho_L^{\mathbb{D}} \right| \end{aligned} \quad (12.29)$$

$$+ \left| \int_{u_\ell}^{u_r} \operatorname{Im}(V \circ P) \left(\rho_L^{\mathbb{D}} - (\rho_R^{\mathbb{D}} \circ P) \right) \right| + \frac{\delta}{2}. \quad (12.30)$$

We next estimate the integrals in (12.29) and (12.30) after each other. In order to estimate (12.29), we first note that, from the explicit form of the potential (2.8), it is obvious that

$$|\operatorname{Im} (V - V \circ P)| \lesssim 1.$$

As a consequence,

$$\int_{u_\ell}^{u_r} |\operatorname{Im} (V - V \circ P)| |\rho_L^{\mathbb{D}}| \lesssim \int_{u_\ell}^{u_r} |\rho_L^{\mathbb{D}}| \lesssim \int_{u_\ell}^{u_r} \frac{1}{\sqrt{|V|}},$$

where in the last step we again used Proposition 12.1 to conclude that the hyperbolic cosine in (12.27) is uniformly bounded. Using the estimate (12.7) in Lemma 12.4, we conclude that

$$\int_{u_\ell}^{u_r} |\operatorname{Im} (V - V \circ P)| |\rho_L^{\mathbb{D}}| \lesssim \frac{\log |\Omega|}{|\Omega|},$$

which tends to zero for large $|\Omega|$ and can thus be made smaller than $\delta/4$.

In order to estimate (12.30), we again use the homotopy (12.19). Setting

$$\begin{aligned} u_0(\tau) &= \tau u_0^L + (1 - \tau) u_0^R \\ \rho_\tau^{\mathbb{D}} &= \frac{1}{2\sqrt{|V_\tau|}} \cosh \left(2 \int_{u_0(\tau)}^u \frac{\operatorname{Im} V_\tau}{\sqrt{|V_\tau|}} \right), \end{aligned}$$

we again use the mean value theorem to obtain

$$\begin{aligned} |\rho_L^{\mathbb{D}} - (\rho_R^{\mathbb{D}} \circ P)| &= |\rho_1^{\mathbb{D}} - \rho_0^{\mathbb{D}}| \leq \sup_{\tau \in [0,1]} \left| \frac{d}{d\tau} \rho_\tau^{\mathbb{D}} \right| \\ &\lesssim \sup_{\tau} \frac{1}{\sqrt{|V_\tau|}} \left(\frac{|\partial_\tau V_\tau|}{|V_\tau|} + \frac{|\operatorname{Im} V_\tau|}{\sqrt{|V_\tau|}} |\partial_\tau u_0(\tau)| + \int_{u_0(\tau)}^u \left(\frac{|\partial_\tau \operatorname{Im} V_\tau|}{\sqrt{|V_\tau|}} + \frac{|\partial_\tau \operatorname{Im} V_\tau \partial_\tau V_\tau|}{|V_\tau|^{\frac{3}{2}}} \right) \right) \end{aligned}$$

(where we again used (12.22) to conclude that the hyperbolic cosine is uniformly bounded). Using that (see also (12.20))

$$|\partial_\tau \operatorname{Re} V| \lesssim |\Omega|, \quad |\partial_\tau \operatorname{Im} V| \lesssim 1 \quad \text{and} \quad |\partial_\tau u_0(\tau)| \lesssim \frac{1}{\sqrt{|\Omega|}},$$

a straightforward computation shows that for any $\tilde{\delta} > 0$, we can arrange that

$$|\rho_L^{\mathbb{D}} - (\rho_R^{\mathbb{D}} \circ P)| \leq \frac{\tilde{\delta}}{\sqrt{|V|}}.$$

Hence

$$\left| \int_{u_\ell}^{u_r} \operatorname{Im}(V \circ P) \left(\rho_L^{\mathbb{D}} - (\rho_R^{\mathbb{D}} \circ P) \right) \right| \leq \tilde{\delta} \int_{u_\ell}^{u_r} \frac{|\operatorname{Im}(V \circ P)|}{\sqrt{|V|}} \lesssim \tilde{\delta},$$

where in the last step we again applied Proposition 12.1. By choosing $\tilde{\delta}$ sufficiently small, we can arrange that

$$\left| \int_{u_\ell}^{u_r} \operatorname{Im}(V \circ P) \left(\rho_L^{\mathbb{D}} - (\rho_R^{\mathbb{D}} \circ P) \right) \right| \leq \frac{\delta}{4}.$$

This concludes the proof. \square

Proof of Proposition 12.2. The remaining task is to estimate the integral in (12.1) from above by the integral in the statement of Lemma 12.13. Applying (12.28), we obtain

$$\left| \int_{u_\ell}^{u_r} \operatorname{Im} V |\phi_L^{\mathbb{D}}|^2 \right| \geq \left| \int_{u_\ell}^{u_r} \operatorname{Im} V \rho_L^{\mathbb{D}} \right| - \delta.$$

Moreover, using (12.27), we obtain

$$\begin{aligned} \int_{u_\ell}^{u_r} \operatorname{Im} V \rho_L^{\mathbb{D}} &= \int_{u_\ell}^{u_r} \frac{\operatorname{Im} V}{2\sqrt{|V|}} \cosh \left(2 \int_{u_0}^u \frac{\operatorname{Im} V}{\sqrt{|V|}} \right) = \frac{1}{4} \int_{u_\ell}^{u_r} \frac{d}{du} \sinh \left(2 \int_{u_0}^u \frac{\operatorname{Im} V}{\sqrt{|V|}} \right) \\ &= \frac{1}{4} \sinh \left(2 \int_{u_0}^{u_r} \frac{\operatorname{Im} V}{\sqrt{|V|}} \right) - \frac{1}{4} \sinh \left(2 \int_{u_0}^{u_\ell} \frac{\operatorname{Im} V}{\sqrt{|V|}} \right). \end{aligned}$$

In the last summand we can use the estimate

$$\int_{u_0}^{u_\ell} \frac{|\operatorname{Im} V|}{\sqrt{|V|}} \lesssim \frac{|\Omega|}{\sqrt{\mathfrak{C}_1 |\Omega|}} (u_0 - u_\ell) \lesssim \frac{1}{\sqrt{\mathfrak{C}_1}}$$

to conclude that this summand can be made arbitrarily small. In the first summand, on the other hand, we apply the inequality $|\sinh x| \geq |x|$. This gives the result. \square

13. The λ -Dependence of the Osculating Circles. In view of the result of Proposition 8.1, it remains to consider the situation for large $|\Omega|$. In this regime, Weyl's asymptotics as worked out in Lemma 7.8 is of no use, because we have no control of how the error terms $\mathcal{O}(n)$ and $\mathcal{O}(n^0)$ depend on Ω . In particular, we cannot expect that the gaps between the eigenvalues for a real potential are so large that the imaginary part of the potential can be treated as a slightly non-selfadjoint perturbation. As a consequence, we must analyze the spectrum of the Hamiltonian

with the complex potential (2.3) in detail. As a technical tool, we will again work with the osculating circle estimates as developed in Section 6. As a refinement, we need to analyze in detail how the osculating circles depend on the spectral parameter λ . In view of (2.8) and (2.9), we know that

$$V_\lambda := \partial_\lambda V \equiv -1. \quad (13.1)$$

Moreover, we choose u_0 in (6.8) independent of λ (where y_0 clearly depends on λ).

In the following computations, we treat $\operatorname{Re} \lambda$ and $\operatorname{Im} \lambda$ as two independent real variables. Then for any function $f(\lambda)$ which is complex differentiable, we have

$$\frac{\partial}{\partial \operatorname{Re} \lambda} f = f_\lambda, \quad \frac{\partial}{\partial \operatorname{Im} \lambda} f = \lim_{h \searrow 0} \frac{f(\lambda + ih) - f(\lambda)}{h} = if_\lambda.$$

In the next lemma we compute the λ -derivatives of ϕ .

LEMMA 13.1. *Choosing the initial conditions (6.7),*

$$\begin{aligned} \frac{\partial}{\partial \operatorname{Re} \lambda} \log \phi(u) &= -\frac{\operatorname{Im} y_\lambda(u_0)}{2 \operatorname{Im} y(u_0)} + \int_{u_0}^u y_\lambda \\ \frac{\partial}{\partial \operatorname{Im} \lambda} \log \phi(u) &= -\frac{\operatorname{Re} y_\lambda(u_0)}{2 \operatorname{Im} y(u_0)} + i \int_{u_0}^u y_\lambda. \end{aligned}$$

Proof. Differentiating (6.7) and (2.15) with respect to λ gives

$$\begin{aligned} \frac{\partial}{\partial \operatorname{Re} \lambda} \log \phi(u_0) &= -\frac{1}{2} \frac{\operatorname{Im} y_\lambda(u_0)}{\operatorname{Im} y(u_0)} \\ \frac{\partial}{\partial \operatorname{Im} \lambda} \log \phi(u_0) &= -\frac{1}{2} \frac{\operatorname{Re} y_\lambda(u_0)}{\operatorname{Im} y(u_0)} \\ \frac{\partial}{\partial u} \frac{\partial}{\partial \operatorname{Re} \lambda} \log \phi(u) &= \frac{\partial}{\partial \operatorname{Re} \lambda} y(u) = y_\lambda(u) \\ \frac{\partial}{\partial u} \frac{\partial}{\partial \operatorname{Im} \lambda} \log \phi(u) &= \frac{\partial}{\partial \operatorname{Im} \lambda} y(u) = iy_\lambda(u). \end{aligned}$$

Integrating the last two differential equations from u_0 to u gives the result. \square

We next compute the second mixed derivatives of K and p .

LEMMA 13.2. *Choosing the initial conditions (6.7),*

$$\frac{\partial}{\partial \operatorname{Re} \lambda} \frac{\partial}{\partial u} \log K(u, \lambda) = -\frac{\operatorname{Im} V \operatorname{Im} y_\lambda}{\operatorname{Im}^2 y} \quad (13.2)$$

$$\frac{\partial}{\partial \operatorname{Re} \lambda} \frac{\partial}{\partial u} p(u, \lambda) = -i \frac{\operatorname{Im} V}{\phi^2 \operatorname{Im}^2 y} \left(\frac{\operatorname{Im} y_\lambda}{\operatorname{Im} y} - \frac{\operatorname{Im} y_\lambda(u_0)}{2 \operatorname{Im} y(u_0)} + \int_{u_0}^u y_\lambda \right) \quad (13.3)$$

$$\frac{\partial}{\partial \operatorname{Im} \lambda} \frac{\partial}{\partial u} \log K(u, \lambda) = -\frac{1}{\operatorname{Im} y} - \frac{\operatorname{Im} V \operatorname{Re} y_\lambda}{\operatorname{Im}^2 y} \quad (13.4)$$

$$\begin{aligned} \frac{\partial}{\partial \operatorname{Im} \lambda} \frac{\partial}{\partial u} p(u, \lambda) &= -\frac{i}{2 \phi^2 \operatorname{Im}^2 y} \\ &\quad - i \frac{\operatorname{Im} V}{\phi^2 \operatorname{Im}^2 y} \left(\frac{\operatorname{Re} y_\lambda}{\operatorname{Im} y} - \frac{\operatorname{Re} y_\lambda(u_0)}{2 \operatorname{Im} y(u_0)} + i \int_{u_0}^u y_\lambda \right). \end{aligned} \quad (13.5)$$

Proof. Comparing (6.13) and (6.14), one sees that

$$\frac{\partial}{\partial u} \log K(u, \lambda) = \frac{\operatorname{Im} V}{\operatorname{Im} y}.$$

Differentiating with respect to λ gives

$$\begin{aligned} \frac{\partial}{\partial \operatorname{Re} \lambda} \frac{\partial}{\partial u} \log K(u, \lambda) &= \frac{\partial}{\partial \operatorname{Re} \lambda} \frac{\operatorname{Im} V}{\operatorname{Im} y} \\ \frac{\partial}{\partial \operatorname{Im} \lambda} \frac{\partial}{\partial u} \log K(u, \lambda) &= \frac{\partial}{\partial \operatorname{Im} \lambda} \frac{\operatorname{Im} V}{\operatorname{Im} y}, \end{aligned}$$

and using (13.1) gives (13.2) and (13.4). In order to derive (13.3) and (13.5), we first apply (6.10) to obtain

$$\begin{aligned} K \frac{\partial}{\partial \operatorname{Re} \lambda} \frac{\partial}{\partial u} p(u, \lambda) &= K \frac{\partial}{\partial \operatorname{Re} \lambda} \left(\frac{i \operatorname{Im} V}{2 \phi^2 \operatorname{Im}^2 y} \right) \\ &= i e^{-2i\vartheta} \frac{\operatorname{Im} V}{\operatorname{Im} y} \left(-2 \frac{\partial}{\partial \operatorname{Re} \lambda} \log \phi - 2 \frac{\operatorname{Im} y_\lambda}{\operatorname{Im} y} \right) \\ K \frac{\partial}{\partial \operatorname{Im} \lambda} \frac{\partial}{\partial u} p(u, \lambda) &= K \frac{\partial}{\partial \operatorname{Im} \lambda} \left(\frac{i \operatorname{Im} V}{2 \phi^2 \operatorname{Im}^2 y} \right) \\ &= i e^{-2i\vartheta} \frac{1}{\operatorname{Im} y} \left(-1 - 2 \operatorname{Im} V \frac{\partial}{\partial \operatorname{Im} \lambda} \log \phi - 2 \operatorname{Im} V \frac{\operatorname{Re} y_\lambda}{\operatorname{Im} y} \right). \end{aligned}$$

Applying Lemma 13.1 gives

$$\begin{aligned} K \frac{\partial}{\partial \operatorname{Re} \lambda} \frac{\partial}{\partial u} p(u, \lambda) &= -2i e^{-2i\vartheta} \frac{\operatorname{Im} V}{\operatorname{Im} y} \left(\frac{\operatorname{Im} y_\lambda}{\operatorname{Im} y} - \frac{\operatorname{Im} y_\lambda(u_0)}{2 \operatorname{Im} y(u_0)} + \int_{u_0}^u y_\lambda \right) \\ K \frac{\partial}{\partial \operatorname{Im} \lambda} \frac{\partial}{\partial u} p(u, \lambda) &= -\frac{i e^{-2i\vartheta}}{\operatorname{Im} y} - 2i e^{-2i\vartheta} \frac{\operatorname{Im} V}{\operatorname{Im} y} \left(\frac{\operatorname{Re} y_\lambda}{\operatorname{Im} y} - \frac{\operatorname{Re} y_\lambda(u_0)}{2 \operatorname{Im} y(u_0)} + i \int_{u_0}^u y_\lambda \right). \end{aligned}$$

Using (6.10) and (6.5) gives the result. \square

13.1. General Estimates in the WKB Region. In the previous section, we derived general formulas for the λ -derivatives of the center and radius of the osculating circles. We want to use these formulas in order to control the behavior of the osculating circles in the WKB region (u_ℓ, u_r) . We now work out the corresponding estimates in general, stating the assumptions needed for the estimates to work (see eqs (13.6), (13.7), (13.8) and (13.9) below). We also derive explicit formulas for the error terms (see eqs (13.11)–(13.13) below). The remaining task will be to justify the above assumptions and to control the error terms. This will be done subsequently in the different cases in the following Sections 13.2–13.3.

Consider an interval $I = [u_a, u_b] \subset (0, \frac{\pi}{2}]$. We assume that on I the following inequalities hold for suitable constants $C, c > 1$,

$$\int_{u_a}^{u_b} \frac{|\operatorname{Im} V|}{\operatorname{Im} y} \leq c \tag{13.6}$$

$$\frac{\sqrt{|\operatorname{Re} V|}}{C} \leq \operatorname{Im} y \leq C \sqrt{|\operatorname{Re} V|}. \tag{13.7}$$

LEMMA 13.3. *Assume that $u_0 \in I$ and that (13.6) and (13.7) hold. Then the function $K(u)$ (cf. (6.10)) is bounded on the interval I by*

$$\frac{1}{\tilde{c}} \leq K(u) \leq \tilde{c},$$

where $\tilde{c} = e^c$.

Proof. Comparing (6.10) with (6.7), we know that $K(u_0) = 2$. Moreover, from (6.10) and (6.11) we obtain

$$\frac{d}{du} \log K = \frac{K'}{K} = \frac{\operatorname{Im} V}{\operatorname{Im} y}.$$

Integrating and using (13.6) gives the result. \square

We now estimate the terms appearing in Lemma 13.2. We again consider an interval $I = [u_a, u_b]$. We assume that on I the inequalities (13.6) and (13.7) hold and that the following inequalities hold for suitable constants $C, c > 1$,

$$\int_{u_a}^u \frac{|\operatorname{Re} y|}{\sqrt{|\operatorname{Re} V|}} \leq \frac{c}{\sqrt{|\operatorname{Re} V(u)|}} \quad (13.8)$$

$$|\operatorname{Re} V(u)| \leq C^2 \inf_{[u_a, u]} |\operatorname{Re} V|. \quad (13.9)$$

LEMMA 13.4. *Under the assumptions (13.6)–(13.9), there is a constant $c_1 = c_1(c, C)$ such that on I ,*

$$|y_\lambda(u)| \leq c_1 \left(\frac{1}{\sqrt{|\operatorname{Re} V(u)|}} + |\partial_\lambda y_0| \right).$$

More precisely,

$$\begin{aligned} \left| y_\lambda(u) - \frac{i}{2 \operatorname{Im} y(u)} \right| &\leq c_1 \left(\frac{1}{\sqrt{|\operatorname{Re} V_0|}} + |\partial_\lambda y_0| \right) \frac{\sqrt{|\operatorname{Re} V|}}{\sqrt{|\operatorname{Re} V_0|}} \\ &\quad + c_1 \sqrt{|\operatorname{Re} V(u)|} \int_{u_a}^u \frac{1}{|\operatorname{Re} V|} \left(\frac{|\operatorname{Im} V|}{\operatorname{Im} y} + |\operatorname{Re} y| \right). \end{aligned} \quad (13.10)$$

Proof. We integrate by parts to obtain

$$\begin{aligned} \int_{u_a}^u \phi^2 &= \int_{u_a}^u \frac{|\phi|^2}{2i \operatorname{Im} y(u)} \frac{d}{du} \left(\frac{\phi^2}{|\phi|^2} \right) \\ &= \frac{\phi^2}{2i \operatorname{Im} y(u)} \Big|_{u_a}^u - \int_{u_a}^u \frac{d}{du} \left(\frac{|\phi|^2}{2i \operatorname{Im} y(u)} \right) \frac{\phi^2}{|\phi|^2} \\ &= \frac{\phi^2}{2i \operatorname{Im} y} \Big|_{u_a}^u + 2i \int_{u_a}^u \frac{\phi^2}{\operatorname{Im} y} \left(\operatorname{Re} y - \frac{\operatorname{Im} V}{4 \operatorname{Im} y} \right). \end{aligned}$$

Using this relation in (7.13) gives

$$\begin{aligned} y_\lambda(u) &= \frac{\phi^2(u_a)}{\phi^2(u)} \left(\partial_\lambda y_0 + \frac{1}{2i \operatorname{Im} y(u_a)} \right) \\ &\quad - \frac{1}{2i \operatorname{Im} y(u)} - \frac{2i}{\phi^2(u)} \int_{u_a}^u \frac{\phi^2}{\operatorname{Im} y} \left(\operatorname{Re} y - \frac{\operatorname{Im} V}{4 \operatorname{Im} y} \right). \end{aligned}$$

We now estimate the resulting terms:

$$\begin{aligned} \left| \frac{\phi^2(u_a)}{\phi^2(u)} \partial_\lambda y_0 \right| &\leq \tilde{c} \frac{\operatorname{Im} y(u)}{\operatorname{Im} y(u_a)} |\partial_\lambda y_0| \leq C^2 \tilde{c} \frac{|\partial_\lambda y_0|}{\sqrt{|\operatorname{Re} V_0|}} \sqrt{|\operatorname{Re} V(u)|} \leq C^3 \tilde{c} |\partial_\lambda y_0| \\ \left| \frac{\phi^2(u_a)}{\phi^2(u)} \frac{1}{2i \operatorname{Im} y(u_a)} \right| &\leq \tilde{c} \frac{\operatorname{Im} y(u)}{\operatorname{Im}^2 y(u_a)} \leq \frac{C^3 \tilde{c}}{|\operatorname{Re} V_0|} \sqrt{|\operatorname{Re} V(u)|} \leq \frac{C^5 \tilde{c}}{\sqrt{|\operatorname{Re} V(u)|}} \\ \left| \frac{1}{2y(u)} \right| &\leq \frac{C}{2} \frac{1}{\sqrt{|\operatorname{Re} V(u)|}}. \end{aligned}$$

Moreover,

$$\begin{aligned} \left| \frac{2}{\phi^2(u)} \int_{u_a}^u \frac{\phi^2}{\operatorname{Im} y} \left(\operatorname{Re} y - \frac{\operatorname{Im} V}{4 \operatorname{Im} y} \right) \right| &\leq 2\tilde{c}^2 \operatorname{Im} y \int_{u_a}^u \frac{1}{\operatorname{Im}^2 y} \left| \operatorname{Re} y - \frac{\operatorname{Im} V}{4 \operatorname{Im} y} \right| \\ &\leq 2C^3 \tilde{c}^2 \sqrt{|\operatorname{Re} V(u)|} \int_{u_a}^u \frac{1}{|\operatorname{Re} V|} \left| \operatorname{Re} y - \frac{\operatorname{Im} V}{4 \operatorname{Im} y} \right| \\ &\leq \frac{5}{2} c C^3 \tilde{c}^2 \frac{\sqrt{|\operatorname{Re} V(u)|}}{\inf_I |\operatorname{Re} V|} \leq \frac{5}{2} \frac{c C^5 \tilde{c}^2}{\sqrt{|\operatorname{Re} V(u)|}}. \end{aligned}$$

Combining all the terms gives the results. \square

PROPOSITION 13.5. *Under the assumptions (13.6)–(13.9), the following inequalities hold,*

$$\begin{aligned} \left| \frac{\partial}{\partial \operatorname{Im} \lambda} \log K(u, \lambda) \Big|_{u_a}^{u_b} + \int_{u_a}^{u_b} \frac{1}{\operatorname{Im} y} \right| &\leq \mathcal{E} \\ \left| \frac{\partial}{\partial \operatorname{Re} \lambda} \log K(u, \lambda) \Big|_{u_a}^{u_b} \right| &\leq \mathcal{E} \\ \left| \frac{\partial \mathfrak{p}(u, \lambda)}{\partial \operatorname{Re} \lambda} \Big|_{u_a}^{u_b} \right| + \left| \frac{\partial \mathfrak{p}(u, \lambda)}{\partial \operatorname{Im} \lambda} \Big|_{u_a}^{u_b} \right| &\leq \mathcal{E}, \end{aligned}$$

where the error term \mathcal{E} is given by

$$\mathcal{E} = c_2 \int_{u_a}^{u_b} \left(\frac{|\operatorname{Im} V|}{|\operatorname{Re} V|} + \frac{|\operatorname{Im} V(u_b)|}{|\operatorname{Re} V(u_b)|} \right) \left(\frac{1}{\sqrt{|\operatorname{Re} V(u)|}} + |\partial_\lambda y_0| \right) \quad (13.11)$$

$$+ \frac{c_2}{|\operatorname{Re} V(u_b)|} + c_2 \int_{u_a}^{u_b} \frac{|\operatorname{Re} y|}{|\operatorname{Re} V|} \quad (13.12)$$

$$\begin{aligned} + c_2 \int_{u_a}^{u_b} \left\{ \frac{\operatorname{Im}^2 V}{|\operatorname{Re} V|^{\frac{3}{2}}} + \frac{|\operatorname{Im} V'| + |\operatorname{Re} y| |\operatorname{Im} V|}{|\operatorname{Re} V|} \right\} \\ \times \left[\int_{u_a}^u \left(\frac{1}{\sqrt{|\operatorname{Re} V|}} + |\partial_\lambda y_0| \right) \right] du \quad (13.13) \end{aligned}$$

with a constant $c_2 = c_2(c, C)$.

Proof. We integrate the formulas of Lemma 13.2 from u_a to u_b . Possibly after integrating by parts, we apply (13.7)–(13.9) and Lemma 13.4. More precisely, we

estimate the individual terms as follows:

$$\begin{aligned}
& \int_{u_a}^{u_b} \left| \frac{\operatorname{Im} V \operatorname{Im} y_\lambda}{\operatorname{Im}^2 y} \right| \leq C^2 c_1 \int_{u_a}^{u_b} \frac{|\operatorname{Im} V|}{|\operatorname{Re} V|} \left(\frac{1}{\sqrt{|\operatorname{Re} V(u)|}} + |\partial_\lambda y_0| \right) \\
& \int_{u_a}^{u_b} \left| \frac{\operatorname{Im} V}{\phi^2 \operatorname{Im}^2 y} \left(\frac{\operatorname{Im} y_\lambda}{\operatorname{Im} y} - \frac{\operatorname{Im} y_\lambda(u_a)}{2 \operatorname{Im} y(u_a)} \right) \right| \\
& \leq \tilde{c} \int_{u_a}^{u_b} \left| \frac{\operatorname{Im} V}{\operatorname{Im} y} \left(\frac{\operatorname{Im} y_\lambda}{\operatorname{Im} y} - \frac{\operatorname{Im} y_\lambda(u_a)}{2 \operatorname{Im} y(u_a)} \right) \right| \leq C^2 \tilde{c} c_1 \int_{u_a}^{u_b} \frac{|\operatorname{Im} V|}{|\operatorname{Re} V|} \left(\frac{1}{\sqrt{|\operatorname{Re} V|}} + |\partial_\lambda y_0| \right) \\
& \int_{u_a}^{u_b} \left| \frac{\operatorname{Im} V}{\phi^2 \operatorname{Im}^2 y} \left(\frac{\operatorname{Re} y_\lambda}{\operatorname{Im} y} - \frac{\operatorname{Re} y_\lambda(u_a)}{2 \operatorname{Im} y(u_a)} \right) \right| \leq C^2 \tilde{c} c_1 \int_{u_a}^{u_b} \frac{|\operatorname{Im} V|}{|\operatorname{Re} V|} \left(\frac{1}{\sqrt{|\operatorname{Re} V|}} + |\partial_\lambda y_0| \right) \\
& \int_{u_a}^{u_b} \frac{1}{\phi^2 \operatorname{Im}^2 y} = \int_{u_a}^{u_b} \frac{1}{|\phi|^2 \operatorname{Im}^2 y} \frac{|\phi|^2}{\phi^2} = \int_{u_a}^{u_b} \frac{i}{2|\phi|^2 \operatorname{Im}^3 y} \frac{d}{du} \left(\frac{|\phi|^2}{\phi^2} \right) \\
& = \frac{i}{2\phi^2 \operatorname{Im}^3 y} \Big|_{u_a}^{u_b} - \frac{i}{2} \int_{u_a}^{u_b} \frac{d}{du} \left(\frac{1}{|\phi|^2 \operatorname{Im}^3 y} \right) \frac{|\phi|^2}{\phi^2} \\
& = \frac{i}{2\phi^2 \operatorname{Im}^3 y} \Big|_{u_a}^{u_b} + \frac{3i}{2} \int_{u_a}^{u_b} \frac{1}{\phi^2 \operatorname{Im}^3 y} \left(\frac{\operatorname{Im} V}{\operatorname{Im} y} - \frac{4}{3} \operatorname{Re} y \right) \\
\Rightarrow & \left| \int_{u_a}^{u_b} \frac{1}{\phi^2 \operatorname{Im}^2 y} \right| \leq \frac{C^2 \tilde{c} (1 + C^2)}{2|\operatorname{Re} V(u_b)|} + 2C^2 \tilde{c} \int_{u_a}^{u_b} \frac{1}{|\operatorname{Re} V|} \left(\frac{|\operatorname{Im} V|}{\operatorname{Im} y} + |\operatorname{Re} y| \right) \\
& \int_{u_a}^{u_b} \frac{\operatorname{Im} V}{\phi^2 \operatorname{Im}^2 y} \int_{u_a}^u y_\lambda = \int_{u_a}^{u_b} \frac{|\phi|^2}{\phi^2} \frac{\operatorname{Im} V}{|\phi|^2 \operatorname{Im}^2 y} \int_{u_a}^u y_\lambda \\
& = \int_{u_a}^{u_b} \frac{d}{du} \left(\frac{|\phi|^2}{\phi^2} \right) \frac{i \operatorname{Im} V}{2|\phi|^2 \operatorname{Im}^3 y} \int_{u_a}^u y_\lambda \\
& = \frac{i \operatorname{Im} V(u_b)}{2\phi^2(u_b) \operatorname{Im}^3 y(u_b)} \int_{u_a}^{u_b} y_\lambda - \int_{u_a}^{u_b} \frac{|\phi|^2}{\phi^2} \frac{d}{du} \left(\frac{i \operatorname{Im} V}{2|\phi|^2 \operatorname{Im}^3 y} \int_{u_a}^u y_\lambda \right) \\
& = -\frac{i}{2} \int_{u_a}^{u_b} \frac{\operatorname{Im} V \operatorname{Im} y_\lambda}{\phi^2 \operatorname{Im}^3 y} \\
& \quad + \frac{i}{2} \int_{u_a}^{u_b} \frac{3 \operatorname{Im}^2 V - \operatorname{Im} y \operatorname{Im} V' - 4 \operatorname{Im} V \operatorname{Im} y \operatorname{Re} y}{\phi^2 \operatorname{Im}^4 y} \int_{u_a}^u y_\lambda \\
\Rightarrow & \left| \int_{u_a}^{u_b} \frac{\operatorname{Im} V}{\phi^2 \operatorname{Im}^2 y} \int_{u_a}^u y_\lambda \right| \leq \frac{c_1 C^2 \tilde{c}}{2} \frac{|\operatorname{Im} V(u_b)|}{|\operatorname{Re} V(u_b)|} \int_{u_a}^{u_b} \left(\frac{1}{\sqrt{|\operatorname{Re} V(u)|}} + |\partial_\lambda y_0| \right) \\
& \quad + \frac{c_1 C^2 \tilde{c}}{2} \int_{u_a}^{u_b} \frac{|\operatorname{Im} V|}{|\operatorname{Re} V|} \left(\frac{1}{\sqrt{|\operatorname{Re} V(u)|}} + |\partial_\lambda y_0| \right) \\
& \quad + \frac{c_1 C^2 \tilde{c}}{2} \int_{u_a}^{u_b} du \left(\frac{3C \operatorname{Im}^2 V}{|\operatorname{Re} V|^{\frac{3}{2}}} + \frac{|\operatorname{Im} V'|}{|\operatorname{Re} V|} + \frac{4|\operatorname{Im} V| |\operatorname{Re} y|}{|\operatorname{Re} V|} \right) \\
& \quad \quad \times \int_{u_a}^u \left(\frac{1}{\sqrt{|\operatorname{Re} V(u)|}} + |\partial_\lambda y_0| \right).
\end{aligned}$$

Combining all the terms gives the result. \square

13.2. Estimates on the Interval $[u_0, u_r]$. We shall now apply the estimates of Section 13.1 on the interval $[u_0, u_r]$ (see (10.14), (11.12) and (11.17)). Our task is to show that the inequalities (13.6), (13.7), (13.8) and (13.9) hold. Moreover, we need to estimate the error terms \mathcal{E} in (13.11)–(13.13). We begin by collecting a few properties of our potential.

LEMMA 13.6. *At the points u_0 and \underline{u} (see (11.12), (11.19) and (10.11)), for sufficiently large \mathcal{C}_5 the potential satisfies the inequalities*

$$|\operatorname{Re} V(u_0) - \operatorname{Re} \lambda| \lesssim |\Omega| \quad (13.14)$$

$$|V'(u_0)| \lesssim |\Omega|^{\frac{3}{2}}, \quad |V''(u_0)| \lesssim |\Omega|^2 \quad (13.15)$$

$$\operatorname{Re} V(u_0) \leq \frac{9}{8} \operatorname{Re} V(\underline{u}) < 0. \quad (13.16)$$

Proof. The inequalities (13.14) and (13.15) follow immediately from (11.12), (11.14) and (11.19), (11.18). Combining (13.14) with (10.2), we find that for sufficiently large \mathcal{C}_5 ,

$$\operatorname{Re} V(u_0) \leq -\frac{15}{16} \operatorname{Re} \lambda.$$

Moreover, similar as in the proof of Lemma 10.1, again for large \mathcal{C}_5 one obtains

$$\begin{aligned} \operatorname{Re} V(\underline{u}) &\geq \frac{16}{17} |\Omega|^2 \underline{u}^2 - \frac{\mathfrak{c}}{\underline{u}^2} - \operatorname{Re} \lambda = \frac{16}{17} \frac{\operatorname{Re} \lambda}{4} - \frac{4\mathfrak{c}|\Omega|^2}{\operatorname{Re} \lambda} - \operatorname{Re} \lambda \\ &= -\frac{13}{17} \frac{\operatorname{Re} \lambda}{4} - \frac{4\mathfrak{c}|\Omega|^2}{\operatorname{Re} \lambda} \geq -\frac{14}{17} \operatorname{Re} \lambda. \end{aligned}$$

Combining these inequalities gives (13.16). \square

LEMMA 13.7. *Assume that the condition (13.6) holds. Then the conditions (13.7), (13.8) and (13.9) are satisfied on the interval $[u_0, u_r]$. Moreover, choosing c_1 and c_2 sufficiently large, we can arrange that the error terms \mathcal{E} (13.11)–(13.13) are bounded by*

$$\mathcal{E} \leq \frac{1}{20} \int_{u_0}^{u_r} \frac{1}{\sqrt{|\operatorname{Re} V|}}. \quad (13.17)$$

Proof. The inequality (13.9) follows immediately from Lemma 12.3. Since the WKB conditions are satisfied by Proposition 11.1, we know that

$$y \approx \sqrt{V} - \frac{V'}{4V}. \quad (13.18)$$

The results of the analysis in [12, 11] gives us rigorous bounds for the error of this approximation. This implies (13.7) for sufficiently large C .

In preparation for proving the other inequalities, we need a few estimates for the potential. First, from the explicit form of the potential (2.8), it is obvious that

$$|V''|, |V'''| \lesssim |\Omega|^2 \quad \text{and} \quad |\operatorname{Im} V'| \lesssim |\Omega|. \quad (13.19)$$

Estimating the potential without derivatives is a bit more subtle because the constant λ comes into play. We first note that by construction of u_r (see (10.14), (10.12), (10.7) and (10.2)), we know that

$$\operatorname{Re} V \leq -\mathcal{C}_1 |\Omega|. \quad (13.20)$$

Moreover, using (10.1) in (9.3), the imaginary part of the potential is uniformly bounded,

$$|\operatorname{Im} V| \lesssim |\Omega|. \quad (13.21)$$

Therefore, by increasing \mathfrak{C}_1 we can arrange that the real part of the potential dominates its imaginary part in the sense that

$$\frac{|\operatorname{Re} V|}{|\operatorname{Im} V|} \gtrsim \mathfrak{C}_1. \quad (13.22)$$

We now come to the proof of (13.8). We first need to estimate the real part of y . Using (13.22), we may express (13.18) in terms of the real and imaginary parts of V to obtain

$$y \approx i\sqrt{-\operatorname{Re} V - i\operatorname{Im} V} - \frac{V'}{4V} \approx i\sqrt{-\operatorname{Re} V} + \frac{\operatorname{Im} V}{\sqrt{-\operatorname{Re} V}} - \frac{V'}{4V}.$$

Hence

$$|\operatorname{Re} y| \lesssim \frac{|\operatorname{Im} V|}{\sqrt{-\operatorname{Re} V}} + \frac{|V'|}{|V|} \quad (13.23)$$

(where the errors are again under control in view of (13.22) and the fact that the WKB conditions are satisfied). The last estimate implies that

$$\int_{u_0}^u \frac{|\operatorname{Re} y|}{\sqrt{|\operatorname{Re} V|}} \lesssim \int_{u_0}^u \frac{|\operatorname{Im} V|}{|\operatorname{Re} V|} + \int_{u_0}^u \frac{|V'|}{|V|^{\frac{3}{2}}}. \quad (13.24)$$

In the first integral on the right, we apply (13.9) to obtain

$$\int_{u_0}^u \frac{|\operatorname{Im} V|}{|\operatorname{Re} V|} \leq \frac{C}{\sqrt{|\operatorname{Re} V(u)|}} \int_{u_0}^u \frac{|\operatorname{Im} V|}{\sqrt{|\operatorname{Re} V|}} \lesssim \frac{1}{\sqrt{|\operatorname{Re} V(u)|}},$$

where in the last step we again used the WKB approximation (13.18) together with the inequality (13.6). The second integral in (13.24) can be estimated by

$$\begin{aligned} \int_{u_0}^u \frac{|V'|}{|V|^{\frac{3}{2}}} &\lesssim \int_{u_0}^u \frac{|V'|}{|\operatorname{Re} V|^{\frac{3}{2}}} \leq \int_{u_0}^u \frac{\operatorname{Re} V'}{|\operatorname{Re} V|^{\frac{3}{2}}} + \int_{u_0}^u \frac{|\operatorname{Im} V'|}{|\operatorname{Re} V|^{\frac{3}{2}}} \\ &\stackrel{(13.19)}{\lesssim} \int_{u_0}^u \left(\frac{1}{\sqrt{|\operatorname{Re} V|}} \right)' + \frac{|\Omega|}{|\operatorname{Re} V(u)|^{\frac{3}{2}}} \lesssim \frac{1}{\sqrt{|\operatorname{Re} V(u)|}} + \frac{|\Omega|}{|\Omega| \sqrt{|\operatorname{Re} V(u)|}}, \end{aligned}$$

where in the last line we also used again the monotonicity statement of Lemma 12.3 together with (13.20). Combining the obtained inequalities gives (13.8).

In order to prove (13.17), we first estimate the term $\partial_\lambda y_0$. Differentiating (11.13) and using the monotonicity of $\operatorname{Re} V$ as well as the results of Lemma 13.6, we obtain

$$|\partial_\lambda y_0| \lesssim \frac{1}{\sqrt{|\operatorname{Re} V(u)|}}.$$

For this reason, we may disregard the term $\partial_\lambda y_0$ in (13.11) and (13.13). In order to estimate the integral (13.11), we simply bound the first bracket in the integrand by

$$\frac{|\operatorname{Im} V|}{|\operatorname{Re} V|} \stackrel{(13.22)}{\lesssim} \frac{1}{\mathfrak{C}_1}. \quad (13.25)$$

The first summand in (13.12) can be estimated with the help the monotonicity of $\operatorname{Re} V$ by

$$\begin{aligned} \frac{1}{|\operatorname{Re} V(u_r)|} &\leq \frac{9}{|\operatorname{Re} V(u_r)|} - \frac{8}{|\operatorname{Re} V(\underline{u})|} \stackrel{(13.16)}{\leq} \frac{9}{|\operatorname{Re} V(u_r)|} - \frac{9}{|\operatorname{Re} V(u_0)|} \\ &= 9 \int_{u_0}^{u_r} \left(\frac{1}{\operatorname{Re} V} \right)' = 9 \int_{u_0}^{u_r} \frac{\operatorname{Re} V'}{\operatorname{Re} V^2} \leq \int_{u_0}^{u_r} \frac{10}{\sqrt{|\operatorname{Re} V|}} \frac{|V'|}{|V|^{\frac{3}{2}}}, \end{aligned}$$

where in the last step we used again that the real part of V dominates the imaginary part and that $\operatorname{Re} V$ is monotone. Now the factor $|V'|/|V|^{\frac{3}{2}}$ is small in view of the WKB property established in Proposition 11.1.

To estimate the integrand in (13.12), we first use (13.23),

$$\frac{|\operatorname{Re} y|}{|\operatorname{Re} V|} \lesssim \frac{|\operatorname{Im} V|}{|\operatorname{Re} V|^{\frac{3}{2}}} + \frac{|V'|}{|\operatorname{Re} V| |V|} \lesssim \frac{2}{\sqrt{|\operatorname{Re} V|}} \left[\frac{|\operatorname{Im} V|}{|\operatorname{Re} V|} + \frac{|V'|}{|V|^{\frac{3}{2}}} \right].$$

Now the square bracket can be made arbitrarily small by applying again (13.25) and Proposition 11.1.

It remains to estimate the nested integral (13.13). We first exchange the orders of integration,

$$\begin{aligned} &\int_{u_0}^{u_r} du \left\{ \frac{\operatorname{Im}^2 V}{|\operatorname{Re} V|^{\frac{3}{2}}} + \frac{|\operatorname{Im} V'| + |\operatorname{Re} y| |\operatorname{Im} V|}{|\operatorname{Re} V|} \right\} \int_{u_0}^u \frac{d\tau}{\sqrt{|\operatorname{Re} V(\tau)|}} \\ &= \int_{u_0}^{u_r} \frac{d\tau}{\sqrt{|\operatorname{Re} V(\tau)|}} \int_{\tau}^{u_r} \left\{ \frac{\operatorname{Im}^2 V}{|\operatorname{Re} V|^{\frac{3}{2}}} + \frac{|\operatorname{Im} V'| + |\operatorname{Re} y| |\operatorname{Im} V|}{|\operatorname{Re} V|} \right\} du. \quad (13.26) \end{aligned}$$

Now we estimate the inner integral term by term. The first term can be estimated with the help of (13.21) and Lemma 12.3,

$$\int_{\tau}^{u_r} \frac{\operatorname{Im}^2 V}{|\operatorname{Re} V|^{\frac{3}{2}}} \lesssim \int_{\tau}^{u_r} \frac{|\Omega|}{|\operatorname{Re} V|^{\frac{3}{2}}} \lesssim \frac{1}{\mathfrak{C}_1}.$$

Next, using (13.19) and (13.20), we find that

$$\int_{\tau}^{u_r} \frac{|\operatorname{Im} V'|}{|\operatorname{Re} V|} \lesssim \frac{1}{\mathfrak{C}_1} (u_r - \tau) \lesssim \frac{1}{\mathfrak{C}_1}.$$

In order to estimate the remaining term involving $\operatorname{Re} y$ in (13.26), we first apply (13.23). Then the first summand on the right of (13.23) gives rise to a contribution which is precisely of the form of the first summand in the curly brackets in (13.13). Hence it remains to consider the term

$$\int_{\tau}^{u_r} \frac{|\operatorname{Im} V|}{|\operatorname{Re} V|} \frac{|V'|}{|V|} \leq \int_{\tau}^{u_r} |\operatorname{Im} V| \frac{|\operatorname{Im} V'| + \operatorname{Re} V'}{|V|^2},$$

where in the last step we applied Lemma 12.3. The first summand on the right can be estimated with the help of (13.21) and (13.19) by

$$\int_{\tau}^{u_r} |\operatorname{Im} V| \frac{|\operatorname{Im} V'|}{|V|^2} \lesssim \int_{\tau}^{u_r} \frac{|\Omega|^2}{|V|^2} \stackrel{(13.20)}{\lesssim} \frac{|\Omega|^{\frac{3}{2}}}{\sqrt{\mathfrak{C}_1}} \int_{\tau}^{u_r} \frac{1}{|V|^{\frac{3}{2}}} \lesssim \frac{1}{\sqrt{|\Omega| \mathfrak{C}_1^3}},$$

where in the last step we again applied Lemma 12.3. The remaining second summand on the right is estimated as follows,

$$\int_{\tau}^{u_r} |\operatorname{Im} V| \frac{\operatorname{Re} V'}{|V|^2} \lesssim |\Omega| \int_{\tau}^{u_r} \frac{\operatorname{Re} V'}{(\operatorname{Re} V)^2} \leq \frac{|\Omega|}{|\operatorname{Re} V(u_r)|} \lesssim \frac{1}{\mathfrak{C}_1},$$

where we again applied Lemma 12.3. This concludes the proof. \square

13.3. Estimates on the Interval $[u_\ell, u_0]$. In preparation, we note that, using (10.1) in (9.3), we again obtain the following uniform bound for $\operatorname{Im} V$,

$$|\operatorname{Im} V| \lesssim |\Omega|. \quad (13.27)$$

We treat the cases $k = s$ and $k \neq s$ separately.

LEMMA 13.8. (Case $k = s$) *Assume that $k = s$. On the interval $[u_\ell, u_0]$, the conditions (13.6)–(13.9) are satisfied. Moreover, by choosing \mathfrak{C}_1 sufficiently large, we can arrange that the error terms \mathcal{E} (13.11)–(13.13) are bounded by*

$$\mathcal{E} \leq \frac{1}{20} \int_{u_\ell}^{u_0} \frac{1}{\sqrt{|V|}}.$$

Proof. Near the pole, the potential has the following asymptotic expansion (cf. (11.12) and (11.14)),

$$\operatorname{Re} V \leq -\frac{3}{4} |\Omega| - \operatorname{Re} \mu + \mathcal{O}(1) \leq -\frac{|\Omega|}{2} - \operatorname{Re} \mu \leq -\operatorname{Re} \mu \lesssim \operatorname{Re} \lambda \quad (13.28)$$

$$V'(u) = \frac{1}{2u^3} + 2\Omega^2 u + \mathcal{O}(|\Omega|u) + \mathcal{O}(|\Omega|^2 u^3). \quad (13.29)$$

Since (13.29) has a positive real part, we know that $\operatorname{Re} V$ is monotone increasing. This implies (13.9) if we also keep in mind that the imaginary part of V is dominated by the real part in view of (13.28) and (13.19). Moreover, using the results of the analysis in [12, 11], we know that (13.18) holds with rigorous error bounds. This implies (13.7) for sufficiently large C . The inequality (13.6) follows from the estimate

$$\begin{aligned} \int_{u_\ell}^{u_0} \frac{|\operatorname{Im} V|}{\operatorname{Im} y} &\lesssim \int_{u_\ell}^{u_0} \frac{|\operatorname{Im} V|}{\sqrt{|\operatorname{Re} V|}} \\ &\stackrel{(13.28)}{\lesssim} \frac{1}{\sqrt{\operatorname{Re} \mu}} \int_{u_\ell}^{u_0} |\operatorname{Im} V| \stackrel{(13.27)}{\lesssim} \frac{u_0 |\Omega|}{|\operatorname{Re} \lambda|} \lesssim \frac{1}{\sqrt{\mathfrak{C}_5}}, \end{aligned} \quad (13.30)$$

where in the last step we used (10.2) and (11.12). In order to prove (13.8), we again apply (13.24) and estimate the two resulting integrals by

$$\begin{aligned} \int_{u_\ell}^u \frac{|\operatorname{Im} V|}{|\operatorname{Re} V|} &\leq \frac{1}{\sqrt{|\operatorname{Re} V(u)|}} \int_{u_\ell}^u \frac{|\operatorname{Im} V|}{\sqrt{|\operatorname{Re} V|}} \stackrel{(13.30)}{\leq} \frac{1}{\sqrt{\mathfrak{C}_5} |\operatorname{Re} V(u)|} \\ \int_{u_\ell}^u \frac{|V'|}{|V|^{\frac{3}{2}}} &\leq \int_{u_\ell}^u \frac{\operatorname{Re} V'}{(-\operatorname{Re} V)^{\frac{3}{2}}} + \int_{u_\ell}^u \frac{|\operatorname{Im} V'|}{|\operatorname{Re} V|^{\frac{3}{2}}} \\ &\leq 2 \int_{u_\ell}^u \frac{d}{du} \left(\frac{1}{\sqrt{-\operatorname{Re} V}} \right) + \frac{1}{\sqrt{|\operatorname{Re} V(u)|}} \int_{u_\ell}^u \frac{|\operatorname{Im} V'|}{|\operatorname{Re} V|} \\ &\lesssim \frac{1}{\sqrt{-\operatorname{Re} V(u)}} + \frac{1}{\sqrt{|\operatorname{Re} V(u)|}} \frac{u_0 |\Omega|}{\operatorname{Re} \lambda} \lesssim \frac{1}{\sqrt{|V(u)|}} + \frac{1}{\sqrt{|V(u)|}} \frac{1}{\mathfrak{C}_5 \sqrt{|\Omega|}}. \end{aligned}$$

In order to estimate \mathcal{E} , we proceed exactly as in the proof of Lemma 13.7, using the following estimates:

$$\begin{aligned} \int_{u_\ell}^{u_0} \frac{|\operatorname{Im} V|^2}{|\operatorname{Re} V|^{\frac{3}{2}}} &\stackrel{(13.28)}{\lesssim} \frac{u_0 |\Omega|^2}{(\operatorname{Re} \lambda)^{\frac{3}{2}}} \lesssim \frac{1}{(\mathcal{C}_5)^{\frac{3}{2}}} \\ \int_{u_\ell}^{u_0} \frac{|\operatorname{Im} V'|}{|V|} &\stackrel{(13.28)}{\lesssim} \frac{u_0 |\Omega|}{\operatorname{Re} \lambda} \lesssim \frac{1}{\mathcal{C}_5 \sqrt{|\Omega|}} \\ |V'| &\stackrel{(13.29)}{\lesssim} \frac{1}{u_\ell^3} + |\Omega|^2 u_0 \stackrel{(10.8)}{\lesssim} \frac{(\operatorname{Re} \lambda)^{\frac{3}{2}}}{\mathcal{C}_1^3} + |\Omega|^{\frac{3}{2}} \stackrel{(10.2)}{\lesssim} \frac{(\operatorname{Re} \lambda)^{\frac{3}{2}}}{\mathcal{C}_1^3} \\ \int_{u_\ell}^{u_0} \frac{|\operatorname{Im} V| |V'|}{|V|^2} &\stackrel{(13.28)}{\lesssim} \frac{u_0 |\Omega|}{(\operatorname{Re} \lambda)^2} \frac{(\operatorname{Re} \lambda)^{\frac{3}{2}}}{\mathcal{C}_1^3} \stackrel{(10.2)}{\lesssim} \frac{1}{\mathcal{C}_1^3 \sqrt{\mathcal{C}_5}}. \end{aligned}$$

This concludes the proof. \square

In the case $k \neq s$, the function $\operatorname{Re} V$ is monotone *decreasing* on the interval $[u_\ell, u_0]$. Therefore, when applying the estimates in Section 13.1 we need to proceed backwards in u , starting from u_0 . Therefore, the conditions (13.8) and (13.9) need to be replaced by

$$\int_u^{u_0} \frac{|\operatorname{Re} y|}{\sqrt{|V|}} \leq \frac{c}{\sqrt{|V(u)|}} \quad (13.31)$$

$$|\operatorname{Re} V(u)| \leq C^2 \inf_{[u, u_0]} |\operatorname{Re} V|. \quad (13.32)$$

LEMMA 13.9. (Case $k \neq s$) *Assume that $k \neq s$. Then on the interval $[u_\ell, u_0]$, the conditions (13.6), (13.7) and (13.31), (13.32) are satisfied. Moreover, by choosing \mathcal{C}_1 sufficiently large, we can arrange that the error terms \mathcal{E} (13.11)–(13.13) are bounded by*

$$\mathcal{E} \leq \frac{1}{20} \int_{u_\ell}^{u_0} \frac{1}{\sqrt{|V|}}.$$

Proof. Near the pole, the potential has the following asymptotic expansion (cf. (11.20) and (11.18)),

$$\operatorname{Re} V \leq -\frac{\operatorname{Re} \mu}{2} + 8\sqrt{\Lambda} |\Omega| \leq -\frac{\operatorname{Re} \mu}{4} \lesssim -\operatorname{Re} \lambda \quad (13.33)$$

$$V'(u) = -\frac{2\Lambda}{u^3} + 2\Omega^2 u + \mathcal{O}(|\Omega|u) + \mathcal{O}(|\Omega|^2 u^3). \quad (13.34)$$

Since $\operatorname{Re} V$ is convex on $(0, u_0]$ and has its minimum at u_0 , we know that the function $\operatorname{Re} V$ is monotone decreasing on the interval $(0, u_0]$. This implies (13.32). Moreover, using the results of the analysis in [12, 11], we know that (13.18) holds with rigorous error bounds. This implies (13.7) for sufficiently large C . The inequality (13.6) follows from the estimate

$$\int_{u_\ell}^{u_0} \frac{|\operatorname{Im} V|}{\operatorname{Im} y} \lesssim \int_{u_\ell}^{u_0} \frac{|\operatorname{Im} V|}{\sqrt{|\operatorname{Re} V|}} \leq \frac{u_0 |\Omega|}{\sqrt{\operatorname{Re} \lambda}} \lesssim \frac{1}{\sqrt{\mathcal{C}_5}}, \quad (13.35)$$

where we used (13.33), (13.27) and (11.20). In order to prove (13.31), we again apply (13.24) and estimate the two resulting integrals by

$$\begin{aligned}
\int_u^{u_0} \frac{|\operatorname{Im} V|}{|\operatorname{Re} V|} &\leq \frac{1}{\sqrt{|\operatorname{Re} V(u)|}} \int_u^{u_0} \frac{|\operatorname{Im} V|}{\sqrt{|\operatorname{Re} V|}} \stackrel{(13.35)}{\lesssim} \frac{1}{\sqrt{\mathcal{C}_5 |V(u)|}} \\
\int_u^{u_0} \frac{|V'|}{|V|^{\frac{3}{2}}} &\leq \int_u^{u_0} \frac{(-\operatorname{Re} V')}{(-\operatorname{Re} V)^{\frac{3}{2}}} + \int_u^{u_0} \frac{|\operatorname{Im} V'|}{|\operatorname{Re} V|^{\frac{3}{2}}} \\
&\leq 2 \int_u^{u_0} \frac{d}{du} \left(\frac{1}{\sqrt{-\operatorname{Re} V}} \right) + \frac{1}{\sqrt{|\operatorname{Re} V(u)|}} \int_u^{u_0} \frac{|\operatorname{Im} V'|}{|\operatorname{Re} V|} \\
&\lesssim \frac{1}{\sqrt{-\operatorname{Re} V(u)}} + \frac{1}{\sqrt{|\operatorname{Re} V(u)|}} \frac{u_0 |\Omega|}{|\operatorname{Re} V(u)|} \lesssim \frac{1}{\sqrt{|V(u)|}},
\end{aligned}$$

where in the last step we applied (11.20), (13.33) and (10.2).

In order to estimate \mathcal{E} , we proceed exactly as in the proof of Lemma 13.7, using the following estimates:

$$\begin{aligned}
\int_{u_\ell}^{u_0} \frac{|\operatorname{Im} V|^2}{|V|^{\frac{3}{2}}} &\stackrel{(13.33)}{\lesssim} \frac{u_0 |\Omega|^2}{(\operatorname{Re} \lambda)^{\frac{3}{2}}} \lesssim \frac{1}{(\mathcal{C}_5)^{\frac{3}{2}}} \\
\int_{u_\ell}^{u_0} \frac{|\operatorname{Im} V'|}{|V|} &\stackrel{(13.33)}{\lesssim} \frac{u_0 |\Omega|}{\operatorname{Re} \lambda} \lesssim \frac{1}{\mathcal{C}_5 \sqrt{|\Omega|}} \\
|V'| &\stackrel{(13.34)}{\lesssim} \frac{1}{u_\ell^3} + |\Omega|^2 u_0 \stackrel{(10.8)}{\lesssim} \frac{(\operatorname{Re} \lambda)^{\frac{3}{2}}}{\mathcal{C}_1^3} + |\Omega|^{\frac{3}{2}} \stackrel{(10.2)}{\lesssim} \frac{(\operatorname{Re} \lambda)^{\frac{3}{2}}}{\mathcal{C}_1^3} \\
\int_{u_\ell}^{u_0} \frac{|\operatorname{Im} V| |V'|}{|V|^2} &\stackrel{(13.33)}{\lesssim} \frac{u_0 |\Omega|}{(\operatorname{Re} \lambda)^2} \frac{(\operatorname{Re} \lambda)^{\frac{3}{2}}}{\mathcal{C}_1^3} \stackrel{(10.2)}{\lesssim} \frac{1}{\mathcal{C}_1^3 \sqrt{\mathcal{C}_5}}.
\end{aligned}$$

This concludes the proof. \square

14. Annular Regions where $|\zeta_L|$ is Bounded Below. We now construct regions in the complex λ -plane in which $|\zeta_L|$ is bounded below. For given λ_0 and $\Delta\lambda > 0$ we introduce the *annular region*

$$\begin{aligned}
A_L(\lambda_0) := \left\{ \lambda \in \mathbb{C} \mid |\operatorname{Re}(\lambda - \lambda_0)| \in (\Delta\lambda, 5\Delta\lambda) \right. \\
\left. \text{and } |\operatorname{Im}(\lambda - \lambda_0)| \in (\Delta\lambda, 5\Delta\lambda) \right\}
\end{aligned} \tag{14.1}$$

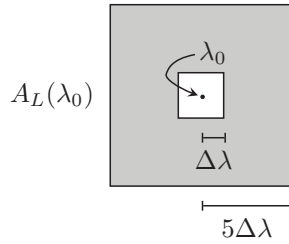
(see Figure 7). We again choose u_ℓ and u_r as the boundaries of the WKB regions (see (10.8), (10.14)). With these choices, we have the following result.

PROPOSITION 14.1. *For sufficiently large $|\Omega|$, the following statement holds: Choose any $\varepsilon < 1/(1000)$. Suppose that for a given λ_0 ,*

$$|\zeta_L(u_{\max})| \leq \varepsilon. \tag{14.2}$$

Then, choosing

$$\Delta\lambda = 75\varepsilon \left(\int_{u_\ell}^{u_r} \frac{1}{\sqrt{|V|}} \Big|_{\lambda_0} \right)^{-1}, \tag{14.3}$$

FIG. 7. *Annular regions.*

it follows that

$$\varepsilon < |\zeta_L(u_{\max})| < 1000\varepsilon \quad \text{for all } \lambda \in A_L(\lambda_0).$$

The remainder of this section is devoted to the proof of this proposition.

14.1. Estimates in the WKB Region. We again consider the family of solutions of the Riccati equation (2.14) with initial conditions (10.16) and (11.13) (where u_0 is given by (11.12) or (11.17)). We again let ϕ be the corresponding solution of the Sturm-Liouville equation normalized according to (6.7).

PROPOSITION 14.2. *Assume that*

$$\int_{u_\ell}^{u_r} \sqrt{|V|} \geq \mathcal{C}_3. \quad (14.4)$$

Moreover, assume that on $[u_\ell, u_r]$ the WKB conditions in Proposition 11.1 as well as the inequalities (13.6)–(13.9) are satisfied. Then by choosing the constants $\mathcal{C}_1, \dots, \mathcal{C}_3$ sufficiently large, we can arrange that the following statement holds. Choosing $\Delta\lambda$ according to (14.3), for every $\lambda \in A_L(\lambda_0)$,

$$\begin{aligned} 35\varepsilon &\leq \left| \vartheta(\lambda)|_{u_\ell}^{u_r} - \vartheta(\lambda_0)|_{u_\ell}^{u_r} \right| \leq 190\varepsilon \\ 70\varepsilon &\leq \left| \log K(\lambda)|_{u_\ell}^{u_r} - \log K(\lambda_0)|_{u_\ell}^{u_r} \right| \leq 380\varepsilon, \end{aligned}$$

with an arbitrarily small error.

Proof. We analyze what the condition (14.4) means. Combining the estimate

$$\mathcal{C}_3 \leq \int_{u_\ell}^{u_r} \sqrt{|V|} = \int_{u_\ell}^{u_r} \frac{|V|}{\sqrt{|V|}} \leq \sup_{[u_\ell, u_r]} |V| \int_{u_\ell}^{u_r} \frac{1}{\sqrt{|V|}}$$

with (14.3), we conclude that

$$\Delta\lambda \leq \frac{64\varepsilon}{\mathcal{C}_3} \sup_{(u_\ell, u_r)} |V|. \quad (14.5)$$

This shows that varying λ on the scale $\Delta\lambda$ keeps the form of the potential V unchanged, up to an arbitrarily small error.

By choosing the constants c_1 and c_2 sufficiently large, we can arrange that the assumptions of Lemma 13.4 and Proposition 13.5 hold. Possibly by further increasing c_1 and c_2 , we can arrange that on the interval $[u_0, u_r]$, the WKB approximation

$$y \approx \sqrt{V} - \frac{V'}{4V}$$

holds with an arbitrarily small error. Moreover, we can make the error term \mathcal{E} in Proposition 13.5 as well as the right side in (13.10) as small as we like. Hence on the interval (u_0, u_r) and at $\lambda = \lambda_0$,

$$\begin{aligned} \frac{\partial}{\partial \operatorname{Re} \lambda} \vartheta \Big|_{u_0}^{u_r} &= \int_{u_0}^{u_r} \operatorname{Im} y_\lambda \approx \int_{u_0}^{u_r} \frac{1}{2 \operatorname{Im} y} \approx \int_{u_0}^{u_r} \frac{1}{2 \operatorname{Im} \sqrt{V}} \\ \frac{\partial}{\partial \operatorname{Im} \lambda} \vartheta \Big|_{u_0}^{u_r} &\approx 0 \\ \frac{\partial}{\partial \operatorname{Re} \lambda} \log K(u, \lambda) \Big|_{u_0}^{u_r} &\approx 0 \\ \frac{\partial}{\partial \operatorname{Im} \lambda} \log K(u, \lambda) \Big|_{u_0}^{u_r} &\approx - \int_{u_0}^{u_r} \frac{1}{\operatorname{Im} y} \approx - \int_{u_0}^{u_r} \frac{1}{\operatorname{Im} \sqrt{V}} \\ \frac{\partial \mathfrak{p}(u, \lambda)}{\partial \operatorname{Re} \lambda} \Big|_{u_0}^{u_r}, \frac{\partial \mathfrak{p}(u, \lambda)}{\partial \operatorname{Im} \lambda} \Big|_{u_0}^{u_r} &\approx 0, \end{aligned}$$

where \approx means “up to an arbitrarily small error.”

Since according to (14.5), the form of the potential is nearly constant on the scale $\Delta\lambda$, it follows that for any $\lambda \in A_L(\lambda_0)$,

$$\begin{aligned} \vartheta(\lambda) \Big|_{u_0}^{u_r} - \vartheta(\lambda_0) \Big|_{u_0}^{u_r} &\approx \operatorname{Re}(\lambda - \lambda_0) \int_{u_0}^u \frac{1}{2 \operatorname{Im} \sqrt{V}} \Big|_{\lambda_0} \approx \frac{\operatorname{Re}(\lambda - \lambda_0)}{32 \Delta\lambda} \\ \log K(\lambda) \Big|_{u_0}^{u_r} - \log K(\lambda_0) \Big|_{u_0}^{u_r} &\approx - \operatorname{Im}(\lambda - \lambda_0) \int_{u_0}^u \frac{1}{\operatorname{Im} \sqrt{V}} \Big|_{\lambda_0} \approx - \frac{\operatorname{Im}(\lambda - \lambda_0)}{16 \Delta\lambda}. \end{aligned}$$

Using the form of the annular region, we obtain the result. \square

14.2. Estimates of $\partial_\lambda \zeta_L$ in the Airy and Parabolic Cylinder Regions.

Before obtaining the estimates, we explain how we can arrange that the imaginary part of the potential satisfies the assumptions (a) or (b) in Proposition 11.9 and Proposition 11.11. To this end, we make use of the fact that taking the complex conjugate of the Sturm-Liouville equation (2.7) is again of Sturm-Liouville form,

$$\left(-\frac{d^2}{du^2} + \overline{V} \right) \overline{\phi} = 0, \tag{14.6}$$

but now with the opposite sign of $\operatorname{Im} V$. For the construction of the resolvent, we are free work either with the original equation or with the complex conjugate equation because if the resolvent of (14.6) has been constructed, the corresponding resolvent of (2.7) is obtained simply by complex conjugation, preserving all our estimates. With this in mind, we can proceed as follows: In the parabolic cylinder region, the function $\operatorname{Im} V$ either has a zero, or it is everywhere positive or negative. If it has a zero, we are in case (b) of Proposition 11.9. If it is everywhere positive, we are in case (a) of Proposition 11.9. If it is everywhere negative, we work with the complex conjugate equation and are again in case (a). If we are in the Airy case, Lemma 11.12 gives three possible cases. In case (i), we may apply Proposition 11.11 in case (a). In case (ii), we work with the complex conjugate equation and again apply case (a) in Proposition 11.11. Finally, in case (iii) the assumption (b) in Proposition 11.11 are satisfied for $\delta < \frac{1}{2}$. We conclude that with this procedure, the assumptions (a) or (b) in Proposition 11.9 and Proposition (11.11) can always be satisfied. In what follows, we can take them for granted.

We consider the interval $[u_r, v]$, where (cf. (10.15))

$$v = \begin{cases} u_+ & \text{in the Airy case} \\ u_{\max} & \text{in the parabolic cylinder case} \\ u_{\max} & \text{in the WKB case.} \end{cases} \quad (14.7)$$

The following lemma is trivial in the WKB case because in this case the interval $[u_r, v]$ reduces to a single point (cf. (10.14) and (14.7)). But in the Airy and parabolic cylinder cases, the next lemma gives control of the region near the zero of $\operatorname{Re} V$.

LEMMA 14.3. *For any $u \in [u_r, v]$,*

$$\begin{aligned} & |\partial_\lambda \zeta_L(u) - \partial_\lambda \zeta_L(u_r)| \\ & \leq \frac{2\mathfrak{C}_2^2}{|\phi(u_r)|^2} \left(|\partial_\lambda \log \phi(u_r)| (u - u_r) + \frac{\mathfrak{C}_2^2}{2} |\partial_\lambda y(u_r)| (u - u_r)^2 + \frac{\mathfrak{C}_2^4}{6} (u - u_r)^3 \right). \end{aligned}$$

Proof. We first integrate the differential equation (7.12) from u_r to u to obtain

$$\phi^2 \partial_\lambda y|_{u_r}^u = \int_{u_r}^u \phi^2.$$

In the parabolic cylinder case, we now apply Proposition 11.9. Similarly, in the Airy case, we apply Proposition 11.11. This gives

$$|\phi^2(u) \partial_\lambda y(u)| \leq |\phi^2(u_r) \partial_\lambda y(u_r)| + \mathfrak{C}_2^2 |\phi(u_r)|^2 (u - u_r) \quad (14.8)$$

Next, we want to estimate $\partial_\lambda \log \phi$. Differentiating the relation $\partial_u \log \phi = y$ with respect to λ , we obtain

$$\frac{\partial}{\partial u} \frac{\partial}{\partial \lambda} \log \phi = \partial_\lambda y.$$

Integrating from u_r to u gives

$$\partial_\lambda \log \phi|_{u_r}^u = \int_{u_r}^u \partial_\lambda y.$$

Now we can apply (14.8) and again Proposition 11.9, respectively Proposition 11.11 to obtain

$$\begin{aligned} & |\partial_\lambda \log \phi(u)| - |\partial_\lambda \log \phi(u_r)| \leq \frac{\mathfrak{C}_2^2}{|\phi(u_r)|^2} \int_{u_r}^u |\phi^2 \partial_\lambda y| \\ & \leq \frac{\mathfrak{C}_2^2}{|\phi(u_r)|^2} \left(|\phi^2(u_r) \partial_\lambda y(u_r)| (u - u_r) + \frac{\mathfrak{C}_2^2}{2} |\phi(u_r)|^2 (u - u_r)^2 \right) \\ & = \mathfrak{C}_2^2 |\partial_\lambda y(u_r)| (u - u_r) + \frac{\mathfrak{C}_2^4}{2} (u - u_r)^2. \end{aligned}$$

Finally, we compute the λ -derivative of ζ_L ,

$$\partial_\lambda \zeta_L|_{u_r}^u = \int_{u_r}^u \frac{1}{\phi^2} = -2 \int_{u_r}^u \frac{1}{\phi^2} \partial_\lambda \log \phi$$

Using the above estimate for $\partial_\lambda \log \phi(u)$, we obtain

$$\begin{aligned} |\partial_\lambda \zeta_L(u) - \partial_\lambda \zeta_L(u_r)| &\leq \frac{2\mathcal{C}_2^2}{|\phi(u_r)|^2} \int_{u_r}^u |\partial_\lambda \log \phi| \\ &\leq \frac{2\mathcal{C}_2^2}{|\phi(u_r)|^2} \int_{u^-}^u \left(|\partial_\lambda \log \phi(u_r)| + \mathcal{C}_2^2 |\partial_\lambda y(u_r)| (u - u_r) + \frac{\mathcal{C}_2^4}{2} (u - u_r)^2 \right). \end{aligned}$$

Carrying out the integral gives the result. \square

14.3. Proof of the Lower Bound for $|\zeta_L|$. According to the estimates near the poles (Lemmas 11.7 and 11.8), by suitably increasing \mathcal{C}_5 we can arrange that

$$|\zeta_L(u_\ell)| < \delta. \quad (14.9)$$

As is obvious from the expansion of the potential in (11.14) and (11.18), by further increasing \mathcal{C}_5 we can arrange that on $[u_\ell, u_0]$,

$$-\operatorname{Re} V \geq \frac{\operatorname{Re} \lambda}{2} \geq \frac{\mathcal{C}_5}{2} |\Omega|.$$

Using furthermore that $\operatorname{Im} V$ is bounded by $\lesssim |\Omega|$ (see again (10.1) and (9.3)), possibly by again increasing c_3 we can again arrange that the real part of the potential dominates the imaginary part. Hence

$$\int_{u_\ell}^{u_0} \operatorname{Re} \sqrt{V} \leq 2 \int_{u_\ell}^{u_0} \frac{\operatorname{Im} V}{\sqrt{|\operatorname{Re} V|}} \lesssim (u_0 - u_\ell) \frac{|\Omega|}{\sqrt{\mathcal{C}_5 |\Omega|}} \lesssim \frac{u_0}{\sqrt{\mathcal{C}_5}} |\Omega|^{\frac{1}{2}}. \quad (14.10)$$

In view of the value of u_0 as given in (11.12) and (11.19), one concludes that by further increasing \mathcal{C}_5 , we can make the left side in (14.10) as small as we like. In view of the formula for the radius of the osculating circle in the WKB region (12.10), this means that that R is constant on the interval $[u_\ell, u_0]$, up to an arbitrarily small error. Since $R(u_0) = \frac{1}{2}$ (as is again obvious from (12.10)), we can thus arrange by choosing δ sufficiently small that

$$\left| R(u_\ell) - \frac{1}{2} \right| \leq \varepsilon. \quad (14.11)$$

Combining (14.9) with (14.11) and using that ζ_L lies on the osculating circle with center \mathbf{p} , the triangle inequality gives

$$\left| |\mathbf{p}(u_\ell)| - \frac{1}{2} \right| \leq 2\varepsilon.$$

Moreover, we know from (12.10) that \mathbf{p} is nearly constant in the WKB region. Thus we can arrange that

$$\left| |\mathbf{p}| - \frac{1}{2} \right| \leq \frac{5}{2} \varepsilon \quad \text{on } [u_\ell, u_r].$$

Combining (14.2) with the estimates of Lemma 11.13, we obtain the following estimates,

$$\begin{aligned} |\zeta_L(\lambda)| \Big|_{u_{\max}} &\geq \left| \zeta_L(\lambda) \Big|_{u_+} - \zeta_L(\lambda_0) \Big|_{u_+} \right| - 3\varepsilon \\ &\geq \left| \zeta_L(\lambda) \Big|_{u_r} - \zeta_L(\lambda_0) \Big|_{u_r} \right| - \left| \zeta_L(\lambda) \Big|_{u_r}^{u_+} - \zeta_L(\lambda_0) \Big|_{u_r}^{u_+} \right| - 3\varepsilon \\ |\zeta_L(\lambda)| \Big|_{u_{\max}} &\leq \left| \zeta_L(\lambda) \Big|_{u_r} - \zeta_L(\lambda_0) \Big|_{u_r} \right| + \left| \zeta_L(\lambda) \Big|_{u_r}^{u_+} - \zeta_L(\lambda_0) \Big|_{u_r}^{u_+} \right| + 3\varepsilon. \end{aligned}$$

In order to estimate $|\zeta_L(\lambda)|_{u_r} - \zeta_L(\lambda_0)|_{u_r}$, we go back to the osculating circle estimates of Proposition 14.2. Knowing that at u_ℓ , the function ζ_L is small (14.9) and the osculating circle has a radius close to one half (14.11), the mean value theorem yields

$$\begin{aligned} \left| \log K(\lambda)|_{u_\ell} - \log K(\lambda_0)|_{u_\ell} \right| &= \left| \log R(\lambda)|_{u_\ell} - \log R(\lambda_0)|_{u_\ell} \right| = \frac{1}{R} \left| R(\lambda)|_{u_\ell} - R(\lambda_0)|_{u_\ell} \right| \\ &\leq \left(\frac{1}{2} - \varepsilon \right)^{-\frac{1}{2}} 2\varepsilon \leq \frac{5\varepsilon}{2}. \end{aligned}$$

Moreover, elementary trigonometry shows that the angles satisfy the inequality

$$\left(\frac{1}{2} - \varepsilon \right) 2 \sin \left| \frac{\vartheta(\lambda)|_{u_\ell} - \vartheta(\lambda_0)|_{u_\ell}}{2} \right| \leq \left| \zeta(\lambda)|_{u_\ell} - \zeta(\lambda_0)|_{u_\ell} \right| \leq 2\varepsilon$$

and thus

$$\left| \vartheta(\lambda)|_{u_\ell} - \vartheta(\lambda_0)|_{u_\ell} \right| \leq 2 \arcsin \left(\left(\frac{1}{2} - \varepsilon \right)^{-1} \varepsilon \right) \leq 5\varepsilon.$$

In order to control the behavior on the interval $[u_\ell, u_r]$, we apply Proposition 14.2 (note that the condition (13.6) is satisfied in view of Proposition 12.1). It follows that one of the following two inequalities holds:

$$\begin{aligned} 30\varepsilon &\leq \left| \vartheta(\lambda)|_{u_r} - \vartheta(\lambda_0)|_{u_r} \right| \leq 200\varepsilon \\ 60\varepsilon &\leq \left| \log K(\lambda)|_{u_r} - \log K(\lambda_0)|_{u_r} \right| \leq 400\varepsilon. \end{aligned}$$

This in turn implies that the change of $|\zeta_L|$ can be estimated from below and above by

$$\begin{aligned} \left| \zeta_L(\lambda)|_{u_r} - \zeta_L(\lambda_0)|_{u_r} \right| &\geq 10\varepsilon R(\lambda_0)|_{u_r} - \left| \mathfrak{p}(\lambda)|_{u_r} - \mathfrak{p}(\lambda_0)|_{u_r} \right| \geq 4\varepsilon \\ \left| \zeta_L(\lambda)|_{u_r} - \zeta_L(\lambda_0)|_{u_r} \right| &\leq 400\varepsilon R(\lambda_0)|_{u_r} + \left| \mathfrak{p}(\lambda)|_{u_r} - \mathfrak{p}(\lambda_0)|_{u_r} \right| \leq 500\varepsilon. \end{aligned}$$

Finally, the estimate of Lemma 14.3 shows that the change of ζ_L in the Airy region or the parabolic cylinder region is much smaller than the change of ζ_L in the WKB region. In particular, by choosing $|\Omega|$ sufficiently large (and noting that the size of the parabolic cylinder and Airy regions tends to zero as $|\Omega| \rightarrow \infty$), we can arrange that

$$\left| \zeta_L(\lambda)|_{u_r}^{u+} - \zeta_L(\lambda_0)|_{u_r}^{u+} \right| \leq \varepsilon.$$

This concludes the proof of Proposition 14.1.

15. The Green's Function for a Double-Well Potential. The goal of this section is to derive pointwise estimates of the Green's function. For the statement of the result, we need the parameter \hat{u} defined as follows. If $\operatorname{Re} V(u_{\max}) \leq 0$, we simply set $\hat{u} = u_{\max}$ (this case includes the WBK case and part of the parabolic cylinder case). If conversely $\operatorname{Re} V(u_{\max}) > 0$, we denote the zeros of the real part of the potential in the parabolic cylinder or Airy regions by $v_{L/R}$,

$$\operatorname{Re} V(v_L) = 0 = \operatorname{Re} V(v_R) \quad \text{and} \quad v_L < u_{\max} < v_R.$$

Then \hat{u} is defined by the requirement that

$$\int_{v_L}^{\hat{u}} \operatorname{Re} \sqrt{V} = \int_{\hat{u}}^{v_R} \operatorname{Re} \sqrt{V}. \quad (15.1)$$

Since our potential is almost symmetric around $\frac{\pi}{2}$, the points \hat{u} and u_{\max} are all close to $\frac{\pi}{2}$. Working with our definitions has the advantage that we do not need to quantify how close these points are.

Here is the main result of this section.

PROPOSITION 15.1. *Assume that for suitable constants $C, \delta > 0$, the following inequalities hold,*

$$|\zeta_L(\hat{u}) \phi_L^2(\hat{u})|, |\zeta_R(\hat{u}) \phi_R^2(\hat{u})| \geq \frac{4}{|y_L(\hat{u}) - y_R(\hat{u})|} \quad (15.2)$$

$$|\zeta_L| \leq C \quad \text{on } (0, \hat{u}], \quad |\zeta_R| \leq C \quad \text{on } [\hat{u}, \pi) \quad (15.3)$$

$$|\zeta_L(\hat{u})|, |\zeta_R(\hat{u})| \geq \delta. \quad (15.4)$$

Then, setting

$$\Sigma = \min \left(|\phi_L(\hat{u})|^2, |\phi_R(\hat{u})|^2 \right) |(y_L - y_R)(\hat{u})|, \quad (15.5)$$

the kernel of the Green's function $s_\lambda(u, u')$ for $0 < u \leq u' < \pi$ is bounded by

$$|s_\lambda(u, u')| \lesssim \left\{ \begin{array}{ll} \frac{1}{\delta^2} \frac{|\phi_L^D(u) \phi_R^D(u')|}{\Sigma} & \text{if } u < \hat{u} \text{ and } u' > \hat{u} \\ \left(\frac{C}{\delta} + \frac{C}{\delta^2 \Sigma} \right) |\phi_R(u) \phi_R^D(u')| & \text{if } u, u' > u_+^R \\ \left(\frac{C}{\delta} + \frac{C}{\delta^2 \Sigma} \right) |\phi_L^D(u) \phi_L(u')| & \text{if } u, u' < u_+^L \\ \left\{ \frac{1}{\delta \Sigma} + \left(\frac{1}{\delta} + \frac{1}{\delta^2 \Sigma} \right) \int_{\hat{u}}^u \frac{1}{|\phi_R|^2} \right\} |\phi_R(u) \phi_R^D(u')| & \text{if } \hat{u} \leq u \leq u_+^R \\ \left\{ \frac{1}{\delta \Sigma} + \left(\frac{1}{\delta} + \frac{1}{\delta^2 \Sigma} \right) \int_{u'}^{\hat{u}} \frac{1}{|\phi_L|^2} \right\} |\phi_L^D(u) \phi_L(u')| & \text{if } u_+^L \leq u' \leq \hat{u}. \end{array} \right.$$

We begin by estimating the Wronskian from below.

LEMMA 15.2. *Assume that (15.2) holds. Then*

$$|w(\phi_L^D, \phi_R^D)| \geq \frac{1}{2} \left| \zeta_L(\hat{u}) \zeta_R(\hat{u}) w(\phi_L, \phi_R) \right|.$$

Proof. We evaluate (10.21) at $u = \hat{u}$, take the absolute value and use the inequalities (15.2). This gives

$$|w(\phi_L^D, \phi_R^D)| \geq \frac{1}{2} |\phi_L \zeta_L \phi_R \zeta_R (y_L - y_R)|(\hat{u}) = \frac{1}{2} |\zeta_L \zeta_R w(\phi_L, \phi_R)|(\hat{u}),$$

concluding the proof. \square

In order to estimate the Green's function, we need to control both $\phi_L^{\mathbb{D}}$ and $\phi_R^{\mathbb{D}}$ on the whole interval $(0, \pi)$. The estimates so far, however, only give us control of $\phi_L^{\mathbb{D}}$ on the interval $(0, \frac{\pi}{2})$ (and similarly of $\phi_R^{\mathbb{D}}$ on the interval $[\frac{\pi}{2}, \pi]$). The following lemma gives a formula for $\phi_L^{\mathbb{D}}$ on the remaining interval $[\frac{\pi}{2}, \pi]$.

LEMMA 15.3. *For any $u \in [\frac{\pi}{2}, \pi]$,*

$$\phi_L^{\mathbb{D}}(u) = \phi_R(u) \left(\zeta_L(\hat{u}) \frac{\phi_L(\hat{u})}{\phi_R(\hat{u})} + \zeta_R \Big|_{\hat{u}}^u \lim_{v \searrow 0} \frac{\phi_R(v)}{\phi_L(v)} \right), \quad (15.6)$$

where

$$\lim_{v \searrow 0} \frac{\phi_R(v)}{\phi_L(v)} = \frac{\phi_R(\hat{u})}{\phi_L(\hat{u})} + w(\phi_L, \phi_R) \zeta_L(\hat{u}). \quad (15.7)$$

Proof. First,

$$\phi_L^{\mathbb{D}}(u) = \phi_R(u) \frac{\phi_L^{\mathbb{D}}(u)}{\phi_R(u)} = \phi_R(u) \frac{\phi_L^{\mathbb{D}}(\hat{u})}{\phi_R(\hat{u})} + \phi_R(u) \int_{\hat{u}}^u \left(\frac{\phi_L^{\mathbb{D}}}{\phi_R} \right)'(\tau) d\tau.$$

Computing the derivative on the right gives

$$\left(\frac{\phi_L^{\mathbb{D}}}{\phi_R} \right)' = \frac{(\phi_L^{\mathbb{D}})' \phi_R - \phi_L^{\mathbb{D}} (\phi_R)'}{\phi_R^2} = \frac{w(\phi_L^{\mathbb{D}}, \phi_R)}{\phi_R^2}.$$

Integrating this equation from \hat{u} to u , we obtain

$$\phi_L^{\mathbb{D}}(u) = \phi_R(u) \frac{\phi_L^{\mathbb{D}}(\hat{u})}{\phi_R(\hat{u})} + w(\phi_L^{\mathbb{D}}, \phi_R) \phi_R(u) \int_{\hat{u}}^u \frac{1}{\phi_R^2}.$$

The Wronskian appearing here is most conveniently computed asymptotically at the origin,

$$w(\phi_L^{\mathbb{D}}, \phi_R) = \lim_{v \searrow 0} ((\phi_L^{\mathbb{D}})'(v) \phi_R(v) - \phi_L^{\mathbb{D}}(v) (\phi_R)'(v)) = \lim_{v \searrow 0} \frac{\phi_R(v)}{\phi_L(v)},$$

where in the last step we differentiated (2.17) and used the asymptotics of the fundamental solutions as stated in (2.11), (2.12) and (2.13). Applying (6.2) gives (15.6).

In order to prove (15.7), we begin with the computation

$$\begin{aligned} \phi_R(v) &= \phi_L(v) \frac{\phi_R(v)}{\phi_L(v)} = \phi_L(v) \frac{\phi_R(\hat{u})}{\phi_L(\hat{u})} - \phi_L(v) \int_v^{\hat{u}} \left(\frac{\phi_R}{\phi_L} \right)' \\ &= \phi_L(v) \frac{\phi_R(\hat{u})}{\phi_L(\hat{u})} + \phi_L(v) \int_v^{\hat{u}} \frac{w(\phi_L, \phi_R)}{\phi_L^2}. \end{aligned}$$

We now divide by $\phi_L(v)$ and take the limit $v \searrow 0$. This concludes the proof. \square

Proof of Proposition 15.1. Applying Lemma 15.2, our task is to estimate the absolute value of the expression

$$E := \frac{\phi_L^{\mathbb{D}}(u) \phi_R^{\mathbb{D}}(u')}{\zeta_L(\hat{u}) \zeta_R(\hat{u}) w(\phi_L, \phi_R)} \quad \text{for all } 0 < u \leq u' \leq \pi. \quad (15.8)$$

We consider the different cases after each other. In the case $u < \hat{u}$ and $u' > \hat{u}$, we apply (15.4) to (15.8) to obtain

$$|E| \leq \frac{1}{\delta^2} \frac{|\phi_L^{\text{D}}(u) \phi_R^{\text{D}}(u')|}{w(\phi_L, \phi_R)}.$$

Using that

$$|w(\phi_L, \phi_R)| = |\phi_L(\hat{u}) \phi_R(\hat{u}) (y_L - y_R)(\hat{u})| \geq \Sigma,$$

we obtain the desired estimate.

Using the symmetry under reflections at $\frac{\pi}{2}$, it remains to consider the case $u, u' \geq \hat{u}$. Applying Lemma 15.3, a straightforward computation yields

$$E = -\frac{\zeta_R(\hat{u}) - \zeta_R(u)}{\zeta_R(\hat{u})} \phi_R(u) \phi_R(u') \zeta_R(u') \quad (15.9)$$

$$- \frac{1}{y_L(\hat{u}) - y_R(\hat{u})} \frac{\zeta_R(\hat{u}) - \zeta_R(u)}{\zeta_L(\hat{u}) \phi_L(\hat{u})^2 \zeta_R(\hat{u})} \phi_R(u) \phi_R(u') \zeta_R(u') \quad (15.10)$$

$$- \frac{1}{y_L(\hat{u}) - y_R(\hat{u})} \frac{1}{\zeta_R(\hat{u}) \phi_R(\hat{u})^2} \phi_R(u) \phi_R(u') \zeta_R(u'). \quad (15.11)$$

In the case $u > u_+^R$, we estimate these terms by

$$|(15.9)| \leq \frac{C}{\delta} |\phi_R(u) \phi_R^{\text{D}}(u')|$$

$$|(15.10)| \leq \frac{C}{\delta^2 \Sigma} |\phi_R(u) \phi_R^{\text{D}}(u')|$$

$$|(15.11)| \leq \frac{1}{\delta \Sigma} |\phi_R(u) \phi_R^{\text{D}}(u')|$$

giving the desired estimate.

In the remaining case $\hat{u} \leq u \leq u_+^R$, we use the identity

$$|\zeta_R(\hat{u}) - \zeta_R(u)| = \left| \int_{\hat{u}}^u \frac{1}{\phi_R^2} \right|$$

to obtain

$$|(15.9)| \leq \frac{1}{\delta} |\phi_R(u) \phi_R^{\text{D}}(u')| \left| \int_{\hat{u}}^u \frac{1}{\phi_R^2} \right|$$

$$|(15.10)| \leq \frac{1}{\delta^2 \Sigma} |\phi_R(u) \phi_R^{\text{D}}(u')| \left| \int_{\hat{u}}^u \frac{1}{\phi_R^2} \right|$$

$$|(15.11)| \leq \frac{1}{\delta \Sigma} |\phi_R(u) \phi_R^{\text{D}}(u')|.$$

This gives the result. \square

16. Deforming the Potential. In the following estimates, we distinguish the cases that the real part of the potential is positive and large near $u = u_{\max}$ or that it is negative or small there. We refer to these cases as the **double-well case** and the **single-well case**, respectively. Qualitatively speaking, in the double-well case the potential looks like in the Airy case, whereas the single-well case comprises the

WKB and parabolic cylinder cases. However, these regions are not exactly the same, making it necessary to use a new notation. The important difference is that the single- and double-well cases are defined without referring to the constants $\mathcal{C}_1, \mathcal{C}_4, \dots$. Instead, we work with the integrals (15.1) and introduce a new constant $\mathcal{K} > 0$ (which will be specified Section 16.1 below). More precisely, if $\operatorname{Re} V(u_{\max}) = 0$, we are by definition in the single-well case. If $\operatorname{Re} V(u_{\max}) > 0$ (so that \hat{u} is defined by (15.1)), we distinguish the

$$\left\{ \begin{array}{l} \text{single-well case :} \\ \text{double-well case :} \end{array} \right. \quad \begin{array}{l} \int_{v_L}^{v_R} \operatorname{Re} \sqrt{V} \leq \mathcal{K} \\ \int_{v_L}^{v_R} \operatorname{Re} \sqrt{V} > \mathcal{K} . \end{array} \quad (16.1)$$

16.1. Estimates in the Double-Well Case.

LEMMA 16.1. *By choosing \mathcal{K} sufficiently large, we can arrange that in the double-well case the following estimates hold:*

$$|\phi_L(\hat{u})|, |\phi_R(\hat{u})| \gtrsim \frac{1}{\sqrt[4]{|V|}} e^{\int_{v_L}^{\hat{u}} \operatorname{Re} \sqrt{V}} \quad (16.2)$$

$$|y_L(\hat{u}) - y_R(\hat{u})| \geq \sqrt{|V(\hat{u})|} . \quad (16.3)$$

Moreover, the function Σ defined by (15.5) is bounded by

$$\Sigma \gtrsim \exp \left(2 \int_{v_L}^{\hat{u}} \operatorname{Re} \sqrt{V} \right) . \quad (16.4)$$

Proof. If \mathcal{K} is chosen sufficiently large, we know from the estimates in Section 11.6 that the function $|\phi_L|$ (and similarly $|\phi_R|$) can be approximated by the WKB wave function (11.39). Moreover, as stated after (11.39), the coefficient of the exponentially increasing fundamental solution is non-zero. This implies (16.2). Moreover, differentiating (11.39), one finds that $y_L(\hat{u}) \approx \sqrt{|V(\hat{u})|}$. Similarly, $y_R(\hat{u}) \approx -\sqrt{|V(\hat{u})|}$, proving (16.3).

Finally, the estimate (16.4) follows immediately by using (16.2) and (16.3) in (15.5). \square

From now on, we choose \mathcal{K} so large that the the statement of this lemma applies.

LEMMA 16.2. *For any $\varepsilon > 0$, by increasing \mathcal{K} we can arrange that for every eigenvalue λ_0 ,*

$$|\zeta_L(\hat{u})| < \frac{\varepsilon}{2} \quad \text{or} \quad |\zeta_R(\hat{u})| < \frac{\varepsilon}{2} . \quad (16.5)$$

Proof. Let λ_0 be an eigenvalue. Then the eigenvalue condition (10.22) is satisfied at $u = \hat{u}$, implying that

$$\frac{1}{|\phi_L^2 \zeta_L|} + \frac{1}{|\phi_R^2 \zeta_R|} \geq |y_L - y_R| \quad \text{at } \hat{u} .$$

Applying Lemma 16.1, we obtain

$$\frac{1}{|\zeta_L(\hat{u})|} + \frac{1}{|\zeta_R(\hat{u})|} \gtrsim \exp \left(2 \int_{v_L}^{\hat{u}} \operatorname{Re} \sqrt{V} \right) \geq e^{2\mathcal{K}} .$$

Applying again the argument after (11.40), by increasing \mathcal{C}_1 we can make the exponential factors as small as we like. Hence, for any given $\varepsilon > 0$ we can arrange that

$$\frac{1}{|\zeta_L(\hat{u})|} + \frac{1}{|\zeta_R(\hat{u})|} \geq \frac{2}{\varepsilon}.$$

This gives the claim. \square

We now apply Proposition 14.1. In order to combine estimates for ζ_L and ζ_R , we modify (14.3) according to

$$\Delta\lambda := 75\varepsilon \min \left\{ \left(\int_{u_\ell^L}^{u_r^L} \frac{1}{\sqrt{|V|}} \Big|_{\lambda_0} \right)^{-1}, \int_{u_r^R}^{u_\ell^R} \frac{1}{\sqrt{|V|}} \Big|_{\lambda_0} \right\}^{-1}. \quad (16.6)$$

Thus we choose $\Delta\lambda$ so small that we can work with the same $\Delta\lambda$ both for ζ_L and ζ_R . Decreasing $\Delta\lambda$ in this way can be described equivalently by making the parameter ε in Proposition (14.1) smaller. The reason why this procedure is unproblematic is that the parameter ε is changed at most by a uniform constant:

LEMMA 16.3.

$$\int_{u_\ell^L}^{u_r^L} \frac{1}{\sqrt{|V|}} \Big|_{\lambda_0} \simeq \int_{u_r^R}^{u_\ell^R} \frac{1}{\sqrt{|V|}} \Big|_{\lambda_0}.$$

Proof. Similar as in the proof of Lemma 12.12 and Section 12.5, we make use of the fact that the potential is approximately reflection symmetric around $u = \frac{\pi}{2}$. We thus obtain

$$\left| \int_{u_\ell^L}^{u_r^L} \frac{1}{\sqrt{|V|}} \Big|_{\lambda_0} - \int_{u_r^R}^{u_\ell^R} \frac{1}{\sqrt{|V|}} \Big|_{\lambda_0} \right| \lesssim \int_{u_\ell^L}^{u_r^L} \frac{|V(u) - V(\pi - u)|}{|V|^{\frac{3}{2}}} \lesssim \frac{|\Omega|}{\mathcal{C}_1 |\Omega|} \int_{u_\ell^L}^{u_r^L} \frac{1}{\sqrt{|V|}}.$$

This gives the result. \square

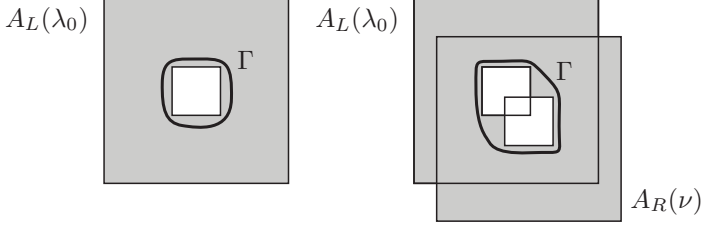
LEMMA 16.4. *Let λ_0 be an eigenvalue. Then, under the assumptions of the above Lemmas 16.1 and 16.2, there is a closed contour Γ of length at most $20\Delta\lambda$ such that*

$$|\zeta_L|, |\zeta_R| \geq \varepsilon \quad \text{along } \Gamma.$$

Proof. Let λ_0 be an eigenvalue. Using Lemma 16.2, we can arrange that (16.5) holds. Without loss of generality we may assume that

$$|\zeta_L(\hat{u})| < \frac{\varepsilon}{2},$$

because otherwise we repeat the proof with L and R interchanged. Applying Proposition 14.1, we know that $|\zeta_L(\hat{u})| > \varepsilon$ inside the annular region $A_L(\lambda_0)$. We choose a contour Γ inside $A_L(\lambda_0)$ close to the inner boundary (see the left of Figure 8). If the inequality $|\zeta_R(\hat{u})| > \varepsilon$ holds along the contour Γ , there is nothing left to prove. Otherwise, there is a point ν on Γ such that $|\zeta_R(\hat{u})| \leq \varepsilon$. Again applying Proposition 14.1, it follows that $|\zeta_R(\hat{u})| > \varepsilon$ inside the annular region $A_R(\nu)$. This makes it possible to choose the contour Γ as shown on the right of Figure 8. \square

FIG. 8. Choice of the closed contour Γ .

PROPOSITION 16.5. *The resolvent s_λ exists along Γ and is bounded by*

$$|s_\lambda(u, u')| \lesssim \begin{cases} e^{-\int_{v_L}^{v_R} \operatorname{Re} \sqrt{V}} |\phi_L^D(u) \phi_R^D(u')| & \text{if } u < \hat{u} \text{ and } u' > \hat{u} \\ |\phi_R(u) \phi_R^D(u')| & \text{if } u, u' > u_+^R \\ |\phi_L^D(u) \phi_L(u')| & \text{if } u, u' < u_+^L \\ e^{-2 \int_u^{v_R} \operatorname{Re} \sqrt{V}} |\phi_R(u) \phi_R^D(u')| & \text{if } \hat{u} \leq u \leq u_+^R \\ e^{-2 \int_{v_L}^{u'} \operatorname{Re} \sqrt{V}} |\phi_L^D(u) \phi_L(u')| & \text{if } u_+^L \leq u' \leq \hat{u}. \end{cases}$$

Proof. We apply Proposition 15.1. By symmetry, in the estimates it suffices to consider the index L .

We begin by estimating $|\zeta_L|$. In the WKB region, the total variation of the radius R of the osculating circle is under uniform control by Proposition 12.1. In the region $(0, u_\ell)$ near the pole, on the other hand, this total variation is even small by Lemma 12.6. In the Airy region, Proposition 11.11 provides uniform estimates. Finally, Lemma 11.13 controls the total variation of ζ_L on the interval $[u_+, \hat{u}]$. Combining these results, we conclude that

$$|\zeta_L| \lesssim 1 \quad \text{on } (0, \hat{u}].$$

This gives (15.3). Moreover, we know from Lemma 16.4 that (15.4) holds for $\delta = \varepsilon$. We conclude that the assumptions (15.3) and (15.4) in Proposition 15.1 are satisfied.

Let us verify the assumption (15.2). From (16.2) and (16.3) we know that

$$|\zeta_L(\hat{u}) \phi_L^2(\hat{u})| \gtrsim \frac{1}{\sqrt{|\bar{V}|}} \exp\left(2 \int_{v_L}^{\hat{u}} \operatorname{Re} \sqrt{V}\right) \gtrsim \frac{e^{2\mathcal{K}}}{|y_L(\hat{u}) - y_R(\hat{u})|}.$$

Thus by further increasing \mathcal{K} we can arrange (15.2). Therefore, Proposition 15.1 applies.

We estimate the integrals over $1/\phi_R^2$ by

$$\begin{aligned} \int_{\hat{u}}^u \frac{1}{|\phi_R|^2} &\lesssim \int_{\hat{u}}^u \operatorname{Re} \sqrt{V} e^{-2 \int_v^{u_0^R} \operatorname{Re} \sqrt{V}} dv = \frac{1}{2} \int_{\hat{u}}^u \frac{d}{dv} e^{-2 \int_v^{u_0^R} \operatorname{Re} \sqrt{V}} \\ &\lesssim \left(e^{-2 \int_u^{u_0^R} \operatorname{Re} \sqrt{V}} - e^{-2 \int_{\hat{u}}^{u_0^R} \operatorname{Re} \sqrt{V}} \right) \lesssim e^{-2 \int_u^{v_R} \operatorname{Re} \sqrt{V}}. \end{aligned}$$

Using the lower bound for Σ in (16.4) as well as (15.1) gives the desired estimate. \square

16.2. Estimates in the Single-Well Case. In the previous section we chose the parameter \mathcal{K} . We now proceed with estimates in the single-well case for this fixed value of \mathcal{K} .

PROPOSITION 16.6. *There is a suitable choice of $\varepsilon < 1/(1000)$ such that for sufficiently large \mathcal{C}_1 the following statement holds. If λ_0 is eigenvalue in the single-well case, then the resolvent exists inside the annulus $A_L(\lambda_0)$ (defined by (14.1)) and is bounded by*

$$|s_\lambda(u, u')| \lesssim \mathfrak{c} |\phi_L^{\text{D}}(u) \phi_R^{\text{D}}(u')| \quad \text{if } u \leq u'$$

(where the constant \mathfrak{c} may depend on \mathcal{K} and ε).

Proof. Let λ_0 be an eigenvalue. Then the eigenvalue condition (10.22) holds for any $u \in (0, \pi)$. We now choose a specific value \check{u} where the analysis of the expression in (10.22) is particularly simple. To this end, we consider the osculating circle corresponding to ϕ_L on the interval $(0, u_r)$. According to Lemma 12.6, we can make $|\zeta(u_\ell)|$ arbitrarily small. Thus for any $\delta > 0$ by increasing \mathcal{C}_1 we can arrange that

$$|R(u_\ell) - |\mathfrak{p}(u_\ell)|| < \delta.$$

Next, in the WKB region (u_ℓ, u_r) , we know from Proposition 12.7 that the center \mathfrak{p} of the osculating circle is approximately fixed, whereas the change of the radius R is given explicitly in terms of the integral of $\text{Re} \sqrt{V}$. Using the estimate (12.21) together with Proposition 12.2, for any given $\delta > 0$ we can arrange that that

$$|R(u_r) - |\mathfrak{p}(u_r)|| < \delta.$$

If we start at $u = u_r$ and decrease u , the function $\zeta(u)$ will move along the osculating circle with an angular velocity as given by (6.9). We choose \check{u} as the largest value of u where the angle ϑ is such that ζ is close to the origin, i.e.

$$|\zeta_L(\check{u})| < 2\delta. \tag{16.7}$$

In what follows, we evaluate all functions at the point \check{u} . For ease in notation, we shall omit the arguments of these functions. From the estimates in Section 11 we know that

$$|\phi_L|^2, |\phi_R|^2 \simeq \frac{1}{\sqrt{|V|}} \tag{16.8}$$

(where the constants in the upper and lower bounds may depend on \mathcal{K}). Moreover,

$$|y_L - y_R| \lesssim \sqrt{|V|}. \tag{16.9}$$

Therefore, the eigenvalue condition (10.22) gives rise to the estimate

$$\frac{1}{|\zeta_R|} \gtrsim \frac{1}{|\zeta_L|} - \frac{|y_L - y_R|}{\sqrt{|V|}} \gtrsim \frac{1}{\delta},$$

implying that

$$|\zeta_R| \lesssim \delta. \tag{16.10}$$

Combining (16.7) and (16.10) we conclude that for any $\tilde{\delta} > 0$, we can arrange by increasing \mathcal{C}_1 that at \check{u} ,

$$|\zeta_L|, |\zeta_R| < \tilde{\delta}.$$

We now vary the spectral parameter λ (for fixed \check{u} and \mathcal{K}). Our goal is to analyze how the left side of (10.22) depends on λ . To this end, it is most convenient to include the phase of the factor ϕ^2 into the corresponding factor ζ . Thus, using the notation in (6.5), we set

$$z = e^{2i\vartheta} \zeta \quad \text{so that} \quad \phi^2 \zeta = |\phi|^2 z.$$

Then, according to (6.6),

$$z = \mathfrak{p} e^{2i\vartheta} + \frac{i}{K}.$$

Hence (10.21) can be written as

$$\frac{w(\phi_L^{\mathfrak{D}}, \phi_R^{\mathfrak{D}})}{\phi_L^{\mathfrak{D}} \phi_R^{\mathfrak{D}}} = y_L - y_R + \frac{1}{|\phi_L|^2 z_L} - \frac{1}{|\phi_R|^2 z_R} = y_L - y_R + \frac{|\phi_R|^2 z_R - |\phi_L|^2 z_L}{|\phi_L|^2 z_L |\phi_R|^2 z_R}.$$

It follows that

$$\left| \frac{w(\phi_L^{\mathfrak{D}}, \phi_R^{\mathfrak{D}})}{\phi_L^{\mathfrak{D}} \phi_R^{\mathfrak{D}}} \right| \geq \left| \frac{|\phi_R|^2 z_R - |\phi_L|^2 z_L}{|\phi_L|^2 z_L |\phi_R|^2 z_R} \right| - |y_L - y_R|.$$

Now we can vary the angle ϑ and the radius $R = |K|^{-1}$ of the osculating circles using the formulas in Proposition 13.5. Keeping in mind that the variation of ϑ_R and K_R involves a minus sign (because we consider the differential equations backwards in u), one sees that the variations of $\operatorname{Re} z_L$ and $\operatorname{Re} z_R$ (and similarly of $\operatorname{Im} z_L$ and $\operatorname{Im} z_R$) have opposite signs.

We now choose λ in the annulus $A_L(\lambda_0)$ as defined by (14.1). The resulting variations of z_L and z_R are obtained by integrating the infinitesimal variations, exactly as explained in the proof of Proposition 14.1. Given $\varepsilon > 0$, by choosing $\tilde{\delta}$ sufficiently small we can arrange the variation is much larger than z at $\lambda = \lambda_0$ (see the formulas of Proposition 14.2). Thus the above consideration for the sign of infinitesimal variations implies that $\operatorname{Re} z_L$ and $\operatorname{Re} z_R$ (and similarly $\operatorname{Im} z_L$ and $\operatorname{Im} z_R$) have opposite signs. This gives rise to the inequality

$$||\phi_R|^2 z_R - |\phi_L|^2 z_L| \geq |\phi_L^2 z_L|.$$

It follows that inside the annulus $A_L(\lambda_0)$,

$$\left| \frac{w(\phi_L^{\mathfrak{D}}, \phi_R^{\mathfrak{D}})}{\phi_L^{\mathfrak{D}} \phi_R^{\mathfrak{D}}} \right| \geq \frac{1}{|\phi_R^2 \zeta_R|} - |y_L - y_R|.$$

Using (16.8) and (16.9) together with the fact that $|\zeta_L|$ and $|\zeta_R|$ are uniformly bounded from above, we conclude after choosing ε sufficiently small, the first summand majorizes the second summand, i.e.

$$\left| \frac{w(\phi_L^{\mathfrak{D}}, \phi_R^{\mathfrak{D}})}{\phi_L^{\mathfrak{D}} \phi_R^{\mathfrak{D}}} \right| \gtrsim \frac{1}{|\phi_R^2 \zeta_R|}.$$

Multiplying this inequality by $|\phi_L^D| = |\zeta_L \phi_L|$ and $|\phi_R^D|$, we obtain

$$|w(\phi_L^D, \phi_R^D)| \gtrsim \frac{|\phi_L \zeta_L|}{|\phi_R|}.$$

Using again that $|\zeta_L| = |z_L|$ is bounded from below, we conclude that

$$|w(\phi_L^D, \phi_R^D)| \gtrsim 1.$$

Finally, we use this estimate in (2.19). This concludes the proof. \square

16.3. Tracking the Eigenvalues. We are now in the position to complete the proof of Theorem 1.1. In order to locate the eigenvalues, we use the following deformation argument. We consider for $\tau \in [0, 1]$ the homotopy

$$V_\tau = \operatorname{Re} V + \tau \operatorname{Im} V$$

with V as in (2.7). Then at $\tau = 0$, the potential is real. As a consequence, the Sturm-Liouville operator is self-adjoint. It has a purely discrete spectrum with real eigenvalues. If $\tau > 0$, on the other hand, the potential is complex. Our method is to track each eigenvalue as τ is increased.

At $\tau = 0$, we choose the contours as shown in Figure 5, with N chosen as follows. First, we choose N at least as large as in Proposition 7.7 for c_4 so large that the eigenvalue λ_N satisfies the inequality (10.2). The next lemma makes it possible to arrange a spectral gap.

LEMMA 16.7. *By increasing N at most by four, we can arrange that for a suitable constant $\mathfrak{c} > 0$ and large $|\Omega|$,*

$$\lambda_N - \lambda_{N-1} \geq \mathfrak{c} |\Omega|. \quad (16.11)$$

Proof. In preparation, we want to show that there is a constant $\mathfrak{c}_1 > 0$ such that for sufficiently large $|\Omega|$, the eigenvalues of the Hamiltonian (7.2) satisfy the inequality

$$\lambda_{N+4} - \lambda_N \geq \mathfrak{c}_1 |\Omega|. \quad (16.12)$$

To this end, we first note that the analysis in the proof of Proposition 7.7 shows that the solution ϕ^D has no zeros on the interval $(0, u_-^L)$ near the pole at $u = 0$. Similarly, the solution ϕ^D has no zeros on the interval (u_-^R, π) near the pole at $u = \pi$. Moreover, on the interval (u_+^L, u_+^R) where the potential is non-negative, we know from Lemma 7.4 that ϕ^D has at most one zero. Therefore, counting the number of zeros of ϕ^D on the intervals (u_-^L, u_+^L) and (u_-^L, u_+^L) , this number differs at λ_{N+4} and λ_N at least by three. As a consequence, on one of the intervals (u_-^L, u_+^L) or (u_-^L, u_+^L) , the number of zeros of ϕ^D differs at least by two. By symmetry, we may assume without loss of generality that this is the case on the interval (u_-^L, u_+^L) . It follows from Lemma 7.5 that

$$\int_{u_-^L}^{u_+^L} \left(\operatorname{Im} y|_{\lambda_{N+4}} - \operatorname{Im} y|_{\lambda_N} \right) \geq \pi.$$

Applying the mean-value theorem in the parameter λ , we obtain

$$(\lambda_{N+4} - \lambda_N) \sup_{\lambda \in [\lambda_N, \lambda_{N+4}]} \int_{u_-^L}^{u_+^L} |\partial_\lambda y| \geq \pi.$$

In order to estimate $\partial_\lambda y$, we use the formula (7.13). Inserting the asymptotic expansions near the poles (see (11.15) and (11.22)) and using the rigorous estimates in [12, Section 5 and 8], a straightforward computation gives the inequality (16.12).

The inequality (16.12) shows that for any Ω , one can choose N such that (16.11) holds (for $\mathfrak{c} = \mathfrak{c}_1/4$). In order to show that N can be chosen uniformly in Ω (for large $|\Omega|$), we make use of the fact that the leading powers in Ω in the potential (2.8) are symmetric around $u = \frac{\pi}{2}$. This implies the small eigenvalues have the same asymptotics as $\operatorname{Re} \Omega \rightarrow \pm\infty$ (with the corresponding eigenfunctions transforming as $\phi_n(u) \rightarrow \phi_n(\pi - u)$). As a consequence, we can choose N such that (16.12) holds for all Ω with $|\Omega|$ sufficiently large. \square

For the chosen N , we choose the contour Γ_0 such that it encloses the lowest N eigenvalues, where N is chosen at least as large as in Proposition 7.7. The other contours enclose one eigenvalue (see Figure 5). Then all eigenvalues *not* enclosed by Γ_0 satisfy the inequality (10.2).

Now we continuously change the parameter τ and follow each of the eigenvalues. We also continuously deform the contours $\Gamma_0, \Gamma_1, \dots$ such that they enclose the corresponding points in the spectrum. If λ_0 is an eigenvalue in the single-well case, according to Proposition 16.6 the resolvent is well-defined and bounded along a closed contour Γ in the annular region $A_L(\lambda_0)$ (which can be chosen for example as on the left of Figure 8). For an eigenvalue λ_0 in the double-well case, on the other hand, there are two possible subcases. Either the eigenvalue is isolated in the sense that we can again choose the contour as on the left of Figure 7. Or else the eigenvalue can be close to another point in the spectrum, in which case we choose the contour as on the right of Figure 8. In both subcases, the resolvent estimate of Proposition 16.5 applies. Finally, we enclose the remaining N lowest spectral points again by a contour Γ_0 . Defining the corresponding operators Q_n by

$$Q_n := -\frac{1}{2\pi i} \oint_{\Gamma_n} s_\lambda d\lambda, \quad n \in \mathbb{N}_0,$$

we obtain a family of operators.

For clarity, we point out that spectral points may move from the single-well case to the double-well case and vice versa. Moreover, the contours Γ_n do not need to be chosen continuously in τ . Indeed, if two eigenvalues come close together, the corresponding contours must be changed discontinuously because the two contours enclosing the two eigenvalues (as on the left of Figure 8) must be joined to form one contour (as on the right of Figure 8). As a consequence, the operators Q_n will in general not depend continuously in τ .

It remains to verify a-posteriori that the inequality (10.2) holds for all $\tau \in [0, 1]$ and for all spectral points *not* enclosed by Γ_0 . To this end, we first point out that the eigenvalues may change considerably compared to the size of the gaps between neighboring eigenvalues (this is why the theory of slightly self-adjoint perturbations does not apply here). But the previous methods tell us how the eigenvalues change in τ (in particular see Section 6, Section 10.2 and Section 12.3). In simple terms, these results show that the eigenvalues must satisfy the complex Bohr-Sommerfeld condition (3.6) with well-defined errors. Combining this formula with the mean value theorem

$$\left| \sqrt{V_1} - \sqrt{V_0} \right| \leq \sup_{\tau \in [0, 1]} \frac{|\operatorname{Im} V|}{\sqrt{|V_\tau|}} \quad (16.13)$$

and using that on the interval (u_ℓ, u_r) the absolute value of the potential is larger than $\mathfrak{C}_1 |\Omega|$, whereas $|\operatorname{Im} V| \lesssim |\Omega|$, one sees that the eigenvalues change in such a way that (10.2) remains valid.

16.4. Uniform Boundedness of the Q_n and Completeness. It remains to verify that the constructed operators Q_0, Q_1, \dots at $\tau = 1$ have all the required properties. The idempotence and mutual orthogonality of these operators follows immediately from Lemma 5.2. We now proceed by estimating the sup-norm of the operators Q_n . For the operator Q_0 we again apply the theory of slightly non-selfadjoint perturbations:

PROPOSITION 16.8. *There is a constant \mathfrak{c}_2 such that for all sufficiently large $|\Omega|$,*

$$\|Q_0\| \leq \mathfrak{c}_2 .$$

Proof. In view of the gap estimate (16.11), by increasing $|\Omega|$ we can arrange that $|\operatorname{Im} V|$ is much smaller than the gap. This makes it possible to find a contour Γ_0 enclosing the first N spectral point whose distance to the spectrum is much larger than $|\operatorname{Im} V|$. This makes it possible to estimate the corresponding contour integral in (8.4) for $n = 0$ by estimating the Neumann series (8.3). This gives the result. \square

We next estimate the operators Q_ℓ with $\ell \geq 1$. We begin with a preparatory lemma.

LEMMA 16.9. *Along the contours $\Gamma_1, \Gamma_2, \dots$, the Hilbert-Schmidt norm of the resolvent is bounded by*

$$\|s_\lambda\|_{\text{HS}} \lesssim \int_{u_\ell^L}^{u_r^L} \frac{1}{\sqrt{|V|}} \Big|_{\lambda_0} + \int_{u_r^R}^{u_\ell^R} \frac{1}{\sqrt{|V|}} \Big|_{\lambda_0} .$$

(with λ_0 as in (16.6)).

Proof. The Hilbert-Schmidt norm of the resolvent can be expressed in terms of its kernel by

$$\|s_\lambda\|_{\text{HS}}^2 = \int_0^\pi du \int_0^\pi du' |s_\lambda(u, u')|^2 .$$

We begin with the single-well case. Proposition 16.6 gives the estimate

$$\begin{aligned} \|s_\lambda\|_{\text{HS}}^2 &\leq 2 \int_0^\pi du \int_u^\pi du' |\phi_L^{\text{D}}(u) \phi_R^{\text{D}}(u')|^2 \\ &= 2 \int_0^\pi du \int_u^\pi du' |\zeta_L(u) \phi_L(u) \zeta_R(u') \phi_R^{\text{D}}(u')|^2 . \end{aligned} \tag{16.14}$$

Near the pole at $u = 0$, the functions ϕ_L and ϕ_R may have a pole (see (11.22)). On the other hand, the function $\zeta_L(u)$ vanishes at $u = 0$. As a result, the integrand in (16.14) is bounded near the poles, as the following argument shows. We introduce the functions ρ_L and ρ_R by

$$\rho_L(u) = \sup_{(0, u]} |\zeta_L| \quad \text{and} \quad \rho_R(u) = \sup_{[u, \pi)} |\zeta_R| .$$

Then the integrand in (16.14) can be bounded by

$$|\zeta_L(u) \phi_L(u) \zeta_R(u') \phi_R^{\text{D}}(u')|^2 \leq \rho_L(u) |\phi_L(u)|^2 \rho_R(u) \rho_L(u) |\phi_R(u')|^2 \rho_R(u) ,$$

giving rise to the estimate

$$\|s_\lambda\|_{\text{HS}}^2 \lesssim \prod_{c=L,R} \int_0^\pi \rho_L(u) |\phi_c(u)|^2 \rho_R(u). \tag{16.15}$$

In order to estimate the obtained integrand near $u = 0$, we first note that ζ_R is uniformly bounded. Considering the asymptotics near $u = 0$ (see (11.16) and (11.22)), one sees that the function $|\phi_c|^2$ may have a pole at $u = 0$. But in this case, inserting the asymptotics into (6.1), one finds that $|\zeta_L|$ (and therefore also ρ_L) tends to zero at the inverse rate. We thus conclude that the integrand in (16.15) is indeed bounded near the poles.

Away from the poles, we can use the asymptotics (16.8) to obtain the result. This concludes the proof in the single-well case.

In the double-well case, we work similarly with the resolvent estimate of Proposition 16.5. The behavior near the poles is estimated just as in the single-well case. The only additional issue is the behavior on the interval $[u_+^L, u_+^R]$ in the Airy case. In this case, combining the exponential factor $\exp(-2 \int_u^{v_R} \text{Re} \sqrt{V})$ with the WKB asymptotics gives rise to the estimate

$$e^{-2 \int_u^{v_R} \text{Re} \sqrt{V}} |\phi_R^D(u) \phi_R(u')| \lesssim \frac{1}{\sqrt[4]{|V(u)| |V(u')|}} e^{-2 \int_u^{\min(u', v_R)} \text{Re} \sqrt{V}}.$$

A straightforward computation using the exponentially decaying factor on the right gives the result. \square

PROPOSITION 16.10. *There is a constant c_2 such that the operators Q_1, Q_2, \dots are bounded by*

$$\|Q_n\| \leq c_2,$$

uniformly in Ω .

Proof. We estimate the contour integrals by

$$\|Q_n\| \leq L(\Gamma_n) \sup_{\lambda \in \Gamma_n} \|s_\lambda\|,$$

where $L(\Gamma_n)$ denotes the length of the contour. The length of the contour is bounded by $20 \Delta\lambda$ with $\Delta\lambda$ as given by (16.6) (this is obvious in the single-well case, whereas in the double-well case it was proven in Lemma 16.4). Applying Lemma 16.9 and Lemma 16.3 and using that the Hilbert-Schmidt norm majorizes the sup-norm, the result follows. \square

It remains to prove completeness in the sense of (1.7) with strong convergence of the series. Since completeness is a statement for fixed Ω , we can rely on the theory of slightly non-selfadjoint perturbations. Namely, Proposition 8.1 yields that there is \tilde{N} (which might be much larger than the parameter N in the statement of Theorem 1.1) a family of operators $\tilde{Q}_0, \tilde{Q}_1, \dots$ with the completeness property

$$\sum_{\tilde{n}=0}^\infty \tilde{Q}_{\tilde{n}} = \mathbf{1} \quad \text{with strong convergence.}$$

The spectral decompositions (Q_n) and $(\tilde{Q}_{\tilde{n}})$ are related to each other simply by forming finite sums, i.e.

$$\sum_{n \in \Lambda} Q_n = \sum_{\tilde{n} \in \tilde{\Lambda}} \tilde{Q}_{\tilde{n}} \quad (\text{with } \Lambda, \tilde{\Lambda} \subset \mathbb{N}_0)$$

whenever the operators on the left and right describe the same spectral points. In particular, for large n , the operators Q_n are sums of one or two of the operators $\tilde{Q}_{\tilde{n}}$. This implies that the series in (1.7) also converges strongly. Since every the spectral point is taken into account in exactly one of the operators Q_n , it follows that

$$\sum_{n=0}^{\infty} Q_n = \sum_{\tilde{n}=0}^{\infty} \tilde{Q}_{\tilde{n}} \quad \text{with strong convergence.}$$

This concludes the proof of Theorem 1.1.

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