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# VARIATIONS AROUND JACKSON'S QUANTUM OPERATOR*

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**Abstract.** Let $0 < q < 1$, $\omega \geq 0$, $\omega_0 := \omega/(1-q)$, and $I$ a set of real numbers. Consider the so-called quantum derivative operator, $D_{q,\omega}$, acting on functions $f : I \to \mathbb{K}$ (where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) as

$$D_{q,\omega}[f](x) := \frac{f\big(qx+\omega\big) - f(x)}{(q-1)x+\omega}\,, \quad x \in I \setminus \{\omega_0\}\,,$$

and $D_{q,\omega}[f](\omega_0) := f'(\omega_0)$ whenever $\omega_0 \in I$ and this derivative exists. This operator was introduced by W. Hahn in 1949. Its inverse operator is given in terms of the so-called Jackson-Thomae $(q,\omega)-$integral, also called Jackson-Nörlund $(q,\omega)-$integral. For $\omega = 0$ one obtains the Jackson's $q-$operator, $D_q$, whose inverse operator is given in terms of the so-called Jackson $q-$integral. In this paper we survey in an unified way most of the useful properties of the Jackson's $q-$integral and then, by establishing links between $D_{q,\omega}$ and $D_q$, as well as between the $q$ and the $(q,\omega)$ integrals, we show how to obtain the properties of $D_{q,\omega}$ and the $(q,\omega)-$integral from the corresponding ones fulfilled by $D_q$ and the $q-$integral. We also consider $(q,\omega)-$analogues of the Lebesgue spaces, denoted by $\mathscr{L}^p_{q,\omega}[a,b]$ and $L^p_{q,\omega}[a,b]$, being $a,b \in \mathbb{R}$. It is shown that the condition $a \leq \omega_0 \leq b$ ensures that these are indeed linear spaces over $\mathbb{K}$. Moreover, endowed with an appropriate norm, $L^p_{q,\omega}[a,b]$ satisfies some expected properties: it is a Banach space if $1 \leq p \leq \infty$, separable if $1 \leq p < \infty$, and reflexive if $1 < p < \infty$.

**Key words.** Jackson $q-$integral, Jackson-Nörlund $(q,\omega)-$integral, $(q,\omega)-$Lebesgue spaces, $q-$analogues.

**AMS subject classifications.** 33E20, 33E30, 40A05, 40A10.

## 1. Introduction.

Quantum calculus is, roughly speaking, the equivalent to traditional infinitesimal calculus without the notion of limits. In quantum calculus one defines the so called Jackson $q-$derivative,

$$D_q[f](x) := \frac{f(qx) - f(x)}{(q-1)x}\,,$$

and the (forward difference) $\omega-$derivative,

$$\triangle_\omega[f](x) := \frac{f(x+\omega) - f(x)}{\omega}\,,$$

and they can be treated together using the more general $(q,\omega)-$derivative operator,

$$D_{q,\omega}[f](x) := \frac{f\big(qx+\omega\big) - f(x)}{(q-1)x+\omega}\,,$$

often called the Hahn's quantum operator. (In order to simplify this introduction we omit some details related with these definitions.) The subject is old ($D_{q,\omega}$ was introduced by W. Hahn in 1949) but is receiving an increasing interest, e.g., in applications to Physics, Approximation Theory, and Optimization—see for instance the works [3, 7, 8, 9, 10, 12, 15, 16, 17, 19, 22, 23, 24, 25, 28], and references therein.

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At least at the formal level, these operators are invertible, and the corresponding inverse operators are defined by using appropriate integral analogues. Thus the so called Jackson's $q-$integral appears as inverse of the $q-$derivative operator $D_q$, and the Jackson-Thomae $(q,\omega)-$integral, also called Jackson-Nörlund $(q,\omega)-$integral, appears as the inverse of Hahn's $(q,\omega)-$derivative operator $D_{q,\omega}$ (see [2, 8, 26]).

The goal of our present contribution is twofold. First we consider the $q-$analogues of the Lebesgue function spaces and we establish precise conditions ensuring that $q-$type Minkowski and Hölder inequalities hold, and ensuring also that those become Banach, separable and reflexive spaces. Second, we establish links between $D_{q,\omega}$ and $D_q$, as well as between the $q$ and the $(q,\omega)$ integrals. We will use such connections to derive in a concise way the properties of $D_{q,\omega}$ and the $(q,\omega)-$integral from the corresponding ones fulfilled by $D_q$ and the $q-$integral. We also study $(q,\omega)-$analogues of the Lebesgue spaces.

## 2. The Jackson's $q-$integral.

**2.1. Notation.** Throughout this section $q$ denotes a real number such that $0 < q < 1$, and $I$ is a set of real numbers invariant after multiplication by $q$, i.e.,

$$qI := \{\, qx \,|\, x \in I \,\} \subset I \;.$$

Some authors (see e.g. [7]) call such a set a $q-$geometric set, due to the fact that the following property holds:

$$xq^n \in I \;, \quad \forall x \in I \;,\; \forall n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \;.$$

For a fixed pair of real numbers $a$ and $b$, we define

$$[a,b]_q := \left\{\, xq^n \mid (x,n) \in \{a,b\} \times \mathbb{N}_0 \,\right\} ,$$

which is often called a $q-$interval with extreme points $a$ and $b$. Clearly, for every real numbers $a$ and $b$, the following property holds:

$$a,b \in I \quad \Rightarrow \quad [a,b]_q \subset I \;.$$

**2.2. The Jackson's $q-$integral.** In what follows we consider functions $f : I \to \mathbb{K}$ where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. The $q-$integral in $[0,1]$ of $f$ is defined as

$$\int_0^1 f \,\mathrm{d}_q := (1-q) \sum_{n=0}^{+\infty} f(q^n) q^n \;,$$

provided $1 \in I$ and the series converges. This integral was introduced in the final of the nineteen century by Thomae [29, 30], although (as we may learn from [4, p. 485]) Fermat and even Archimedes considered some special cases. Later, in the beginning of the twenty century, Jackson [20] extended this concept to an arbitrary bounded interval $[a,b]$. In fact, Jackson first introduced

$$\int_0^a f \,\mathrm{d}_q := a(1-q) \sum_{n=0}^{+\infty} f(aq^n) q^n \;, \tag{2.1}$$

whenever $a \in I$ and the series converges. (We may allow either $a \geq 0$ or $a < 0$.) Moreover, $f$ is called $q-$integrable in $[0,a]$ if the last series is convergent. Then he defined

$$\int_a^b f \,\mathrm{d}_q := \int_0^b f \,\mathrm{d}_q - \int_0^a f \,\mathrm{d}_q \;, \tag{2.2}$$

provided $a, b \in I$ and $f$ is $q-$integrable at least in one of the two intervals $[0, a]$ and $[0, b]$ (as before, we may allow either $a \leq b$ or $a > b$), $f$ being called $q-$integrable in $[a, b]$ if it is both $q-$integrable in $[0, a]$ and in $[0, b]$.

Due to (2.1)–(2.2) we have

$$\int_a^b f \, \mathrm{d}_q = (1-q) \sum_{n=0}^{+\infty} \Big[ bf(bq^n) - af(aq^n) \Big] q^n \ , \tag{2.3}$$

provided $f$ is $q-$integrable in $[a, b]$. Thus, under this condition, the inequality

$$\left| \int_a^b f \, \mathrm{d}_q \right| \leq 2M_{a,b} \sup_{n \in \mathbb{N}_0} \Big\{ \big| f\,(aq^n) \,\big| , \big| f\,(bq^n) \,\big| \Big\}$$

holds, where $M_{a,b} := \max\{|a|, |b|\}$. (Notice that this inequality holds trivially if the supremum on the right-hand side is infinite, meaning that at least one of the sequences $(f\,(aq^n))_n$ and $(f\,(bq^n))_n$ is unbounded.) Moreover, one easily see that if $a, b \in I$ and both $\big(f(aq^n)\big)_n$ and $\big(f(bq^n)\big)_n$ are bounded sequences (this holds, for instance, when $\lim_{x \to 0^\pm} f(x)$ exist and, in particular, if $f$ is continuous at 0), then $f$ is $q-$integrable in $[a, b]$. Finally, notice that if $f$ is a continuous function on $[-M_{a,b}, M_{a,b}]$ then it is $q-$integrable and Riemann integrable in $[a, b]$, and

$$\lim_{q \to 1} \int_a^b f(x) \, \mathrm{d}_q x = \int_a^b f(x) \, \mathrm{d}x \ ,$$

hence the Jackson $q-$integral is a $q-$analogue of the Riemann integral.

Remark 2.1. *If $I$ is a set of real numbers invariant after multiplication by $q$ and $q^{-1}$, and such that $1 \in I$, Jackson also considered*

$$\int_0^{+\infty} f \, \mathrm{d}_q := (1-q) \sum_{n=-\infty}^{+\infty} f(q^n) q^n \ , \tag{2.4}$$

*whenever the series converges. Similarly,*

$$\int_{-\infty}^{+\infty} f \, \mathrm{d}_q := (1-q) \sum_{n=-\infty}^{+\infty} [f(q^n) + f(-q^n)] q^n \ , \tag{2.5}$$

*whenever $I$ is a set invariant after multiplication by $\pm q^{\pm 1}$ and such that $1 \in I$. We notice that, under appropriate conditions, the properties that will be presented in the next sections for the $q-$integral in $[a, b]$ may be extended to the integrals (2.4) and (2.5).*

## 3. The spaces $\mathscr{L}_q^p[a, b]$ and $L_q^p[a, b]$.

**3.1. The space $\mathscr{L}_q^p[a, b]$.** From (2.3) one immediately see that the equality

$$\int_a^b f \, \mathrm{d}_q = -\int_b^a f \, \mathrm{d}_q$$

holds, provided $f$ is $q-$integrable in $[a, b]$, and similar relations occur for the $q-$integrals appearing in (2.4) and (2.5). Hence, unless stated otherwise, from now on we will assume $a < b$. The following result is of fundamental importance for the theory developed hereafter.

PROPOSITION 3.1. *Let $f$ and $g$ be two $q-$integrable functions in $[a,b]$ such that $f(x) \leq g(x)$ for all $x \in [a,b]_q$. If $a \leq 0 \leq b$, then*

$$\int_a^b f \, \mathrm{d}_q \leq \int_a^b g \, \mathrm{d}_q \, . \tag{3.1}$$

*Proof.* By hypothesis the inequalities $f(bq^n) \leq g(bq^n)$ and $f(aq^n) \leq g(aq^n)$ hold for all $n = 0, 1, 2 \cdots$. Therefore, since $a \leq 0 \leq b$, we obtain

$$bf(bq^n) - af(aq^n) \leq bg(bq^n) - ag(aq^n) \, , \quad n = 0, 1, 2, \cdots \, .$$

Thus (3.1) follows from (2.3). ◻

REMARK 3.2. *If the condition $a \leq 0 \leq b$ fails then there exist functions $f$ and $g$ such that $f(x) \leq g(x)$ for all $x \in [a,b]_q$, but the inequality (3.1) does not hold. The fundamental importance of such condition has been remarked by M. E. H. Ismail [19, p.297] in the case $p = 2$.*

COROLLARY 3.3. *If $a \leq 0 \leq b$ and $|f|$ is $q-$integrable in $[a,b]$ then $f$ is $q-$integrable in $[a,b]$ and the following inequality holds:*

$$\left| \int_a^b f \, \mathrm{d}_q \right| \leq \int_a^b |f| \, \mathrm{d}_q \, .$$

*Proof.* It is an immediate consequence of Proposition 3.1, taking into account that the inequalities $-|f(x)| \leq f(x) \leq |f(x)|$ hold for all $x \in [a,b]_q$. ◻

For any real number $p \geq 1$ and $a, b \in I$, we will denote by $\mathscr{L}_q^p[a,b]$ the set of functions $f : I \to \mathbb{K}$ such that $|f|^p$ is $q-$integrable in $[a,b]$, i.e.,

$$\mathscr{L}_q^p[a,b] = \left\{ f : I \to \mathbb{K} \;\Big|\; \int_a^b |f|^p \mathrm{d}_q < \infty \right\} \, .$$

We also set

$$\mathscr{L}_q^\infty[a,b] = \left\{ f : I \to \mathbb{K} \;\Big|\; \sup_{n \in \mathbb{N}_0} \Big\{ |f(aq^n)|, |f(bq^n)| \Big\} < \infty \right\} \, .$$

The next theorem shows that a $q-$type Hölder inequality holds, provided we take $a \leq 0 \leq b$ and $p > 1$. As usual, by $p'$ we denote the conjugate exponent of a real number $p \geq 1$, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$, with the convention $p' = \infty$ if $p = 1$.

THEOREM 3.4. *If $a \leq 0 \leq b$ and $1 < p < \infty$, then*

$$\int_a^b |fg| \, \mathrm{d}_q \leq \left( \int_a^b |f|^p \, \mathrm{d}_q \right)^{\frac{1}{p}} \left( \int_a^b |g|^{p'} \, \mathrm{d}_q \right)^{\frac{1}{p'}} \, , \tag{3.2}$$

*whenever $f \in \mathscr{L}_q^p[a,b]$ and $g \in \mathscr{L}_q^{p'}[a,b]$.*

*Proof.* The proof of (3.2) can be done by following the same steps as for the case of the classical Hölder inequality in the framework of the Lebesgue integral, taking into account that Proposition 3.1 holds. □

REMARK 3.5. *When $p = 1$ and $a \le 0 \le b$, it follows immediately from (2.3) and Proposition 3.1 that the inequality*

$$\int_a^b |fg| \, \mathrm{d}_q \le \sup_{n\in\mathbb{N}_0} \Big\{ |g(aq^n)|, |g(bq^n)| \Big\} \int_a^b |f| \, \mathrm{d}_q$$

*holds, provided $f \in \mathscr{L}_q^1[a,b]$ and $g \in \mathscr{L}_q^\infty[a,b]$.*

The following theorem is a $q-$type version of Minkowski inequality.

THEOREM 3.6. *If $a \le 0 \le b$ and $1 \le p < \infty$, then*

$$\left( \int_a^b |f+g|^p \, \mathrm{d}_q \right)^{\frac{1}{p}} \le \left( \int_a^b |f|^p \, \mathrm{d}_q \right)^{\frac{1}{p}} + \left( \int_a^b |g|^p \, \mathrm{d}_q \right)^{\frac{1}{p}}$$

*for all $f, g \in \mathscr{L}_q^p[a,b]$.*

*Proof.* The proof is similar to the proof for the case of the Lebesgue integral, taking into account that Proposition 3.1 holds. □

As an immediate consequence of the $q-$Minkowski inequality we may state the following important property (the case $p = \infty$ is trivial).

COROLLARY 3.7. *If $a \le 0 \le b$ and $1 \le p \le \infty$, then the set $\mathscr{L}_q^p[a,b]$, with the usual operations of addition of functions and multiplication of a function by a real number, becomes a linear space over $\mathbb{R}$ or $\mathbb{C}$.*

**3.2. The space $L_q^p[a,b]$.** For $f, g \in \mathscr{L}_q^p[a,b]$, we write $f \sim g$ if

$$f(aq^n) = g(aq^n) \quad \text{and} \quad f(bq^n) = g(bq^n) \tag{3.3}$$

holds for all $n = 0, 1, 2, \cdots$. Clearly, $\sim$ defines an equivalence relation in $\mathscr{L}_q^p[a,b]$. We will represent by $L_q^p[a,b]$ the corresponding quotient set:

$$L_q^p[a,b] := \mathscr{L}_q^p[a,b] / \sim \ .$$

The following proposition generalizes Lemma 3.1 in the paper [6] by M.H. Annaby, where the special case $[a,b] = [-1,1]$ was considered for real $p \ge 1$ (proved therein by a different method).

THEOREM 3.8. *If $a \le 0 \le b$ and $1 \le p \le \infty$, then the following holds:*
(i) *$L_q^p[a,b]$ is a normed linear space over $\mathbb{K}$, with norm*

$$\|f\|_{L_q^p[a,b]} := \begin{cases} \left( \displaystyle\int_a^b |f|^p \, \mathrm{d}_q \right)^{\frac{1}{p}} & \text{if} \quad 1 \le p < \infty \, ; \\ \displaystyle\sup_{n\in\mathbb{N}_0} \Big\{ |f(aq^n)| \, , |f(bq^n)| \Big\} & \text{if} \quad p = \infty \, . \end{cases} \tag{3.4}$$

*Moreover, with this norm $L_q^p[a,b]$ becomes a Banach space for $1 \le p \le \infty$, which is separable if $1 \le p < \infty$ and reflexive if $1 < p < \infty$.*

(ii) $L^2_q[a,b]$ *is a Hilbert space with inner product*

$$\langle f,g\rangle_q := \int_a^b f\,\overline{g}\,\mathrm{d}_q\,, \quad f,g\in L^2_q[a,b]\,.$$

Remark 3.9. *As usual, in the right-hand side of (3.4), $f$ denotes any representative (i.e., a function in $\mathscr{L}^p_q[a,b]$) of the class $f\in L^p_q[a,b]$ appearing in the norm on the left-hand side. Of course, in view of (2.3) and (3.3), the definition of the norm $\|f\|_{L^p_q[a,b]}$ is independent of the chosen representative.*

*Proof.* Consider the usual space $\ell^p$ of all real or complex sequences $x=(\xi_n)_n$ such that $\sum_{n=0}^\infty|\xi_n|^p<\infty$ if $1\le p<\infty$, or $(\xi_n)_n$ is a bounded sequence if $p=\infty$. It is well known that endowed with the norm

$$\|x\|_{\ell^p} := \begin{cases} \left(\displaystyle\sum_{n=0}^\infty |\xi_n|^p\right)^{\frac{1}{p}} & \text{if} \quad 1\le p<\infty\,; \\ \displaystyle\sup_{n\in\mathbb{N}_0}|\xi_n|\,, & \text{if} \quad p=\infty\,, \end{cases}$$

$\ell^p$ is a Banach space for $1\le p\le\infty$, which is separable if $1\le p<\infty$ and reflexive if $1<p<\infty$. We will use these facts to prove the theorem. Assume $1\le p<\infty$. Let $\ell^p_q$ be the space of all sequences $x=(\xi_n)_n$ such that

$$\sum_{n=0}^\infty q^n|\xi_n|^p<\infty\,.$$

Then, endowed with the norm

$$\|x\|_{\ell^p_q} := \left(\sum_{n=0}^\infty q^n|\xi_n|^p\right)^{\frac{1}{p}}\,,$$

$\ell^p_q$ is a Banach space if $1\le p<\infty$, separable if $1\le p<\infty$ and reflexive if $1<p<\infty$. This may be easily justified taking into account the following fact: for any sequence $x=(\xi_n)_n$, we have $x\in\ell^p_q$ if and only if $x_q\in\ell^p$, where $x_q:=(q^{n/p}\xi_n)_n$, and, in such a case, the equality

$$\|x\|_{\ell^p_q}=\|x_q\|_{\ell^p}$$

holds. As a matter of fact, the mapping $T_q:\ell^p_q\to\ell^p$ $(x\mapsto x_q)$ is an isometric isomorphism. Moreover, taking into account (2.3), we see that $f\in L^p_q[a,b]$ if and only if $(f(aq^n))_n\in\ell^p_q$ and $(f(bq^n))_n\in\ell^p_q$, and, in such a case, the equality

$$\|f\|_{L^p_q[a,b]} = \Big\{(1-q)\Big(b\,\|\,(f(bq^n))_n\,\|^p_{\ell^p_q} - a\,\|\,(f(aq^n))_n\,\|^p_{\ell^p_q}\Big)\Big\}^{1/p} \tag{3.5}$$

holds. Now, consider the product space $\ell^p_q\times\ell^p_q$ endowed with the norm

$$\|(x,y)\|_{\ell^p_q\times\ell^p_q} := \Big\{(1-q)\Big(b\,\|y\|^p_{\ell^p_q} - a\,\|x\|^p_{\ell^p_q}\Big)\Big\}^{1/p}\,.$$

Then $\ell_q^p \times \ell_q^p$ is a Banach space if $1 \leq p < \infty$, separable if $1 \leq p < \infty$ and reflexive if $1 < p < \infty$. Moreover, the mapping $T : L_q^p[a,b] \to \ell_q^p \times \ell_q^p$ defined by

$$Tf := \Big( \big(f(aq^n)\big)_n , \big(f(bq^n)\big)_n \Big)$$

is an isometric isomorphism. Indeed, it is clear that $T$ is linear and, by (3.5), it is an isometry. To prove that $T$ is onto, take any pair of sequences $(x,y) \in \ell_q^p \times \ell_q^p$. Set $x = (\xi_n)_n$ and $y = (\eta_n)_n$ and take $f : I \to \mathbb{K}$ such that $f|[a,b]_q$ is defined by

$$f(t) := \begin{cases} \xi_n \, , & \text{if} \quad t = aq^n \, , \; n = 0,1,2,\cdots \\ \eta_n \, , & \text{if} \quad t = bq^n \, , \; n = 0,1,2,\cdots \, . \end{cases}$$

(It doesn't matter how one defines the function $f(t)$ for $t \neq aq^n$ and $t \neq bq^n$.) Since $x = (f(aq^n))_n \in \ell_q^p$ and $y = (f(bq^n))_n \in \ell_q^p$ then, from (3.5), we derive that $f \in L_q^p[a,b]$, and, of course, $Tf = (x,y)$. This completes the proof of (i) for $1 \leq p < \infty$. The proof for $p = \infty$ is now easy, as well as the proof of (ii). □

In [12, 14, 15], basic-Fourier expansions in a $q-$linear lattice were established in terms of an orthogonal system, which was derived from the inner product (ii) of Theorem 3.8 with $a = -1$ and $b = 1$. The starting point was the paper [13] by J. Bustoz and S. Suslov, where basic-Fourier expansions in a $q-$quadratic lattice were presented. For a nice survey on this issue we refer to the book [27] by S. Suslov. Those $q-$expansion results proved to be very useful in $q-$versions of the famous classical sampling theorem of Whittaker, Kotel'nikov and Shannon, as well as the Kramer's analytic theorem (see [1, 5, 6, 18]).

**4. Basic properties of Jackson's $q-$integral.** In this section we review some basic properties of the Jackson's $q-$integral. These properties are in the basis of the results appearing in section 7 for the $(q,\omega)-$integral. The results presented in bellow are contained (or they are slight generalizations of the results contained) in references [4, 6, 12, 13, 19, 20, 21, 22, 27]. We include proofs for the sake of completeness.

**4.1. The Jackson $q-$derivative operator.** Fix $q \neq 1$, and let $I$ be a set of real numbers invariant after multiplication by $q$. The Jackson $q-$derivative operator, $D_q$, is defined for a function $f : I \to \mathbb{K}$ as

$$D_q[f](x) := \begin{cases} \dfrac{f(qx) - f(x)}{(q-1)x} & \text{if} \quad x \neq 0 \, , \\ f'(0) & \text{if} \quad x = 0 \, . \end{cases}$$

Here, $D_q[f](0)$ is defined whenever $f$ is well defined in a neighborhood of the origin and $f'(0)$ exists. Notice that if $I$ is an interval and $f$ is differentiable at a point $x \in I$, then

$$\lim_{q \to 1} D_q[f](x) = f'(x) \, ,$$

hence $D_q$ is a $q-$analogue of the standard derivative operator. The $q-$derivative fulfils many properties which may be regarded as $q-$analogues of the corresponding properties for the usual derivative. For instance,

$$D_q[\alpha f + \beta g](x) = \alpha D_q[f](x) + \beta D_q[g](x) \, , \tag{4.1}$$

where $\alpha$ and $\beta$ are any real or complex numbers, and

$$D_q[f\cdot g](x)=D_q[f](x)\cdot g(x)+f(qx)\cdot D_q[g](x)\ , \tag{4.2}$$

which is often referred to as the $q-$product rule. These equalities hold for all $x\neq 0$, and also for $x=0$ whenever $f'(0)$ and $g'(0)$ exist.

**4.2. The fundamental theorem of $q-$calculus.** The next proposition is a $q-$analogue of the fundamental theorem of calculus for the Riemmann integral.

THEOREM 4.1. *Let $0<q<1$, and $I$ a set of real numbers such that $qI\subset I$. Fix $a,b\in I$, and let $f:I\to\mathbb{K}$ be a function such that $D_q[f]\in\mathscr{L}_q^1[a,b]$. Then:*

(i) *The equality*

$$\int_a^b D_q[f]\,\mathrm{d}_q=\Big[f(s)-\lim_{n\to+\infty}f(sq^n)\Big]_{s=a}^b \tag{4.3}$$

*holds, provided the involved limits exist.*

(ii) *In addition, if $f$ is continuous at $0$, then*

$$\int_a^b D_q[f]\,\mathrm{d}_q=f(b)-f(a)\ .$$

*Proof.* From (2.3), we may write

$$\int_a^b D_q[f]\,\mathrm{d}_q=(1-q)\sum_{n=0}^{\infty}\{\,bD_q[f](bq^n)-aD_q[f](aq^n)\,\}\,q^n\ .$$

Now, since the equality

$$sD_q[f](sq^n)=\frac{f(sq^n)-f(sq^{n+1})}{(1-q)q^n}$$

holds for $s\in\{a,b\}$, we deduce

$$\int_a^b D_q[f]\,\mathrm{d}_q=\sum_{n=0}^{\infty}\big\{\big(f(bq^n)-f(aq^n)\big)-\big(f(bq^{n+1})-f(aq^{n+1})\big)\big\}\ .$$

Therefore, since by hypothesis the limits $\lim_{n\to+\infty}f(sq^n)$ exist for $s\in\{a,b\}$ then, by the telescoping property, (4.3) follows immediately. ∎

**4.3. The $q-$integration by parts formula.** There is a $q-$analogue of the integration by parts formula:

THEOREM 4.2. *Assume $0<q<1$, and let $I$ be a set of real numbers such that $qI\subset I$. Let $f:I\to\mathbb{K}$ and $g:I\to\mathbb{K}$. For $a,b\in I$, the equality*

$$\int_a^b D_q[f]\cdot g\,\mathrm{d}_q=\Big[(f\cdot g)(s)-\lim_{n\to+\infty}(f\cdot g)(sq^n)\Big]_{s=a}^b-\int_a^b h_q[f]\cdot D_q[g]\,\mathrm{d}_q$$

*holds, provided $f,g\in\mathscr{L}_q^1[a,b]$, $D_q[f]$ and $D_q[g]$ are bounded in $[a,b]_q$, and the limits exist, where $h_q[f](x):=f(qx)$. If, in addition, $f$ and $g$ are continuous at $0$, then*

$$\int_a^b D_q[f]\cdot g\,\mathrm{d}_q=\Big[f\cdot g\Big]_a^b-\int_a^b h_q[f]\cdot D_q[g]\,\mathrm{d}_q\ .$$

*Proof.* By the $q-$product rule (4.2) one has

$$g(x)D_q[f](x) = D_q[f \cdot g](x) - h_q[f](x) \cdot D_q[g](x) ,$$

hence, integrating both sides of this equality over the interval $[a, b]$ and taking into account Theorem 4.1, the result follows. (Notice that the above equality and the hypothesis on $f$ and $g$ give $D_q[f \cdot g] = g \cdot D_q[f] + h_q[f] \cdot D_q[g] \in \mathscr{L}_q^1[a, b]$, hence the hypothesis of Theorem 4.1 are fulfilled.) □

**4.4. The $q-$antiderivative.** Let $I$ be an interval and $f : I \to \mathbb{K}$. We say that $F(\cdot; q)$ is a $q-$antiderivative of $f$ in $I$ if

$$D_q[F(\cdot; q)](x) = f(x) , \quad \forall x \in I .$$

THEOREM 4.3. *Assume $0 < q < 1$. Let $I$ be an interval containing the origin and take $a \in I$. If $f : I \to \mathbb{K}$ is $q-$integrable on $[a, x]$ for every $x \in I$, and if it is bounded in a neighborhood of the origin, then the function $F(\cdot; q)$ given by*

$$F(x; q) := \int_a^x f \, \mathrm{d}_q , \quad x \in I$$

*is a $q-$antiderivative of $f$ in $I$.*

*Proof.* In fact, from the definition of $D_q$ and (2.3), we deduce, for every $x \in I$,

$$\begin{aligned} D_q[F(\cdot; q)](x) &= \frac{F(qx; q) - F(x; q)}{(q-1)x} = \frac{1}{(q-1)x} \int_x^{qx} f \, \mathrm{d}_q \\ &= \sum_{n=0}^{+\infty} \Big( f(xq^n)q^n - f(xq^{n+1})q^{n+1} \Big) = f(x) - \lim_{n \to +\infty} q^{1+n} f\left(xq^{1+n}\right) \\ &= f(x) , \end{aligned}$$

where the last equality is justified by the given assumptions on $f$. □

## 5. The Jackson-Thomae-Nörlund integral revisited.

**5.1. Notation.** Most of the notation in this section is taken from [10]. Hereafter, $(q, \omega)$ denotes a pair of real numbers such that

$$0 < q < 1 , \quad \omega \ge 0 .$$

Once the pair $(q, \omega)$ is fixed, we set

$$\omega_0 := \frac{\omega}{1-q} , \quad \sigma(x) \equiv \sigma_{q,\omega}(x) := qx + \omega = \omega_0 + (x - \omega_0)q .$$

Notice that, setting $\sigma^n := \sigma \circ \sigma \circ \cdots \circ \sigma$, we may write

$$\sigma^n(x) = xq^n + \omega[n]_q = \omega_0 + (x - \omega_0)q^n , \quad n = 0, 1, 2, \cdots , \tag{5.1}$$

where, as usual,

$$[0]_q := 1 , \quad [\alpha]_q := \frac{1 - q^\alpha}{1 - q} , \quad \alpha \in \mathbb{C} \setminus \{0\} .$$

Throughout this section, $I$ denotes a set of real numbers invariant under $\sigma$, i.e.,

$$\sigma(I) \subset I \, .$$

Thus, taking into account (5.1), the following property holds:

$$xq^n + \omega[n]_q \in I \, , \quad \forall x \in I \, , \quad \forall n \in \mathbb{N}_0 \, .$$

For any real numbers $a$ and $b$, we define

$$[a,b]_{q,\omega} := \left\{ sq^n + \omega[n]_q \mid s \in \{a,b\} \, , \, n \in \mathbb{N}_0 \right\} ,$$

which is often called a $(q,\omega)-$interval. Notice that

$$a,b \in I \quad \Rightarrow \quad [a,b]_{q,\omega} \subset I \, .$$

Throughout this section we denote by $\tau_v$ and $h_s$ $(s > 0)$ the *translation* operator $(f \mapsto \tau_v[f])$ and the *dilation* operator $(f \mapsto h_s[f])$, acting on functions $f$ belonging to appropriate function spaces, as

$$\tau_v[f](x) := f(x+v) \, , \quad h_s[f](x) := f(sx) \, .$$

Recall that these operators fulfill the following useful properties

$$\tau_v[f \cdot g] = \tau_v[f] \cdot \tau_v[g] \, , \quad h_s[f \cdot g] = h_s[f] \cdot h_s[g] \, ,$$

and (on appropriate function spaces) they are invertible linear operators, being

$$\tau_v^{-1} = \tau_{-v} \, , \quad h_s^{-1} = h_{s^{-1}} \, .$$

**5.2. The Jackson-Thomae-Nörlund integral.** Let $a, b \in I$ and $f : I \to \mathbb{K}$. The Jackson-Thomae-Nörlund integral of $f$ on the interval $[a,b]$ is defined by

$$\int_a^b f \, \mathrm{d}_{q,\omega} := \int_{\omega_0}^b f \, \mathrm{d}_{q,\omega} - \int_{\omega_0}^a f \, \mathrm{d}_{q,\omega} \, ,$$

where, for every $x \in I$,

$$\int_{\omega_0}^x f \, \mathrm{d}_{q,\omega} := \big(x - \sigma_{q,\omega}(x)\big) \sum_{n=0}^{\infty} q^n f(\sigma_{q,\omega}^n(x)) \, ,$$

or, more explicitly,

$$\int_{\omega_0}^x f \, \mathrm{d}_{q,\omega} := \big(x(1-q) - \omega\big) \sum_{n=0}^{\infty} q^n f\left(xq^n + \omega[n]_q\right) \, .$$

If the last series is convergent then $f$ is called $(q,\omega)-$integrable in $[\omega_0, x]$. $f$ is called $(q,\omega)-$integrable in $[a,b]$ if it is both $(q,\omega)-$integrable in $[\omega_0, a]$ and in $[\omega_0, b]$. The $(q,\omega)-$integral is called the Jackson-Thomae integral in [22], and it is called the Jackson-Nörlund integral in [2, 8, 10] (this last denomination is due to the fact that it generalizes, in the limit, the so-called Nörlund sum—see [21]).

**5.3. Connection between the $q$ and $(q,\omega)$ integrals.** Many facts concerning the $(q,\omega)-$integral have been stated recently. The proofs are usually based on the above definition, involving manipulations on series. The next proposition shows that the Jackson-Thomae-Nörlund integral may be expressed in terms of a Jackson integral. This property allow us to derive the properties of the former integral from the corresponding properties of the latter one.

THEOREM 5.1. *Let $a,b \in I$, and $f : I \to \mathbb{K}$. Then $f$ is $(q,\omega)-$integrable in $[a,b]$ if and only if $\tau_{\omega_0} f$ is $q-$integrable in $[a-\omega_0, b-\omega_0]$. In such case,*

$$\int_a^b f \,\mathrm{d}_{q,\omega} = \int_{a-\omega_0}^{b-\omega_0} \tau_{\omega_0} f \,\mathrm{d}_q \, . \tag{5.2}$$

*Proof.* The first sentence is easy to state. To prove (5.2), notice first that for every $x \in I$, we have

$$\begin{aligned} \int_{\omega_0}^x f \,\mathrm{d}_{q,\omega} &= \big(x(1-q)-\omega\big) \sum_{n=0}^{\infty} q^n f\big(\sigma^n(x)\big) \\ &= (x-\omega_0)(1-q) \sum_{n=0}^{\infty} q^n f\big(\omega_0 + (x-\omega_0)q^n\big) \\ &= \int_0^{x-\omega_0} \tau_{\omega_0} f \,\mathrm{d}_q \, , \end{aligned}$$

hence we deduce

$$\begin{aligned} \int_a^b f \,\mathrm{d}_{q,\omega} &= \int_{\omega_0}^b f \,\mathrm{d}_{q,\omega} - \int_{\omega_0}^a f \,\mathrm{d}_{q,\omega} \\ &= \int_0^{b-\omega_0} \tau_{\omega_0} f \,\mathrm{d}_q - \int_0^{a-\omega_0} \tau_{\omega_0} f \,\mathrm{d}_q \\ &= \int_{a-\omega_0}^{b-\omega_0} \tau_{\omega_0} f \,\mathrm{d}_q \, . \end{aligned}$$

□

REMARK 5.2. *Links between the $q$ and $(q,\omega)-$integrals appear on Appendix A in [26]. The relation between both integrals will be used in the next section to study $(q,\omega) - L^p-$type spaces, including Hölder and Minkowski-type inequalities for the $(q,\omega)-$integral.*

**6. The spaces $\mathscr{L}^p_{q,\omega}[a,b]$ and $L^p_{q,\omega}[a,b]$.** Let us denote by $\mathscr{L}^p_{q,\omega}[a,b]$ the set of all functions $f : I \to \mathbb{R}$ such that $|f|^p$ is $(q,\omega)-$integrable in $[a,b]$, i.e.,

$$\mathscr{L}^p_{q,\omega}[a,b] := \left\{ f : I \to \mathbb{R} \;\middle|\; \int_a^b |f|^p \,\mathrm{d}_{q,\omega} < +\infty \right\} .$$

For $f,g \in \mathscr{L}^p_{q,\omega}[a,b]$, we write $f \sim g$ if

$$f\big(\sigma^n(a)\big) = g\big(\sigma^n(a)\big) \, , \quad f\big(\sigma^n(b)\big) = g\big(\sigma^n(b)\big) \, , \quad \forall n \in \mathbb{N}_0 \, .$$

Then $\sim$ defines an equivalence relation on $\mathscr{L}^p_{q,\omega}[a,b]$. The corresponding quotient set will be denoted by $L^p_{q,\omega}[a,b]$, i.e.,

$$L^p_{q,\omega}[a,b] := \mathscr{L}^p_{q,\omega}[a,b]/\sim \ .$$

The following proposition is of fundamental importance.

PROPOSITION 6.1. *Let* $f$ *and* $g$ *be two* $(q,\omega)-$*integrable functions in* $[a,b]$ *such that* $f(x) \le g(x)$ *for all* $x \in [a,b]_{q,\omega}$ . *If* $a \le \omega_0 \le b$ , *then*

$$\int_a^b f \, \mathrm{d}_{q,\omega} \le \int_a^b g \, \mathrm{d}_{q,\omega} \ . \tag{6.1}$$

*As a consequence, under the same condition* $a \le \omega_0 \le b$ , *then*

$$\left| \int_a^b f \, \mathrm{d}_{q,\omega} \right| \le \int_a^b |f| \, \mathrm{d}_{q,\omega} \ . \tag{6.2}$$

*Proof.* The result follows immediately from Proposition 3.1 and Theorem 5.1, by noticing that the following property holds:

$$f \le g \quad \text{in } [a,b]_{q,\omega} \quad \Rightarrow \quad \tau_{\omega_0} f \le \tau_{\omega_0} g \quad \text{in } [a-\omega_0, b-\omega_0]_q \ .$$

□

REMARK 6.2. *If the condition* $a \le \omega_0 \le b$ *fails then inequalities (6.1) and (6.2) may not occur. A counter-example is given in [2, 8]. Proposition 6.1 is similar but not equivalent to [8, Lemma 4.3]. Each one shows that the other one gives only a sufficient condition in order to ensure that inequalities (6.1) and (6.2) hold.*

It follows from Proposition 6.1 that under the condition $a \le \omega_0 \le b$ , the Hölder-type and Minkowski-type inequalities hold for the $(q,\omega)-$integral. As a consequence, under the same condition, $\mathscr{L}^p_{q,\omega}[a,b]$ is a linear space over $\mathbb{K}$. Moreover, the following theorem holds.

THEOREM 6.3. *If* $a \le \omega_0 \le b$ *and* $1 \le p \le \infty$, *then the following holds:*
(i) $\mathscr{L}^p_{q,\omega}[a,b]$ *and* $L^p_{q,\omega}[a,b]$ *are linear spaces over* $\mathbb{K}$.
(ii) *Endowed with the norm*

$$\|f\|_{L^p_{q,\omega}} := \begin{cases} \left( \displaystyle\int_a^b |f|^p \, \mathrm{d}_{q,\omega} \right)^{\frac{1}{p}} & \text{if} \quad 1 \le p < \infty \, ; \\ \displaystyle\sup_{(n,s) \in \mathbb{N}_0 \times \{a,b\}} |\tau_{\omega_0} h_{s-\omega_0} f(q^n)| & \text{if} \quad p = \infty \, , \end{cases}$$

$L^p_{q,\omega}[a,b]$ *is a Banach space if* $1 \le p \le \infty$, *which is separable if* $1 \le p < \infty$ *and reflexive if* $1 < p < \infty$.
(iii) $L^2_{q,\omega}[a,b]$ *is a Hilbert space with inner product*

$$\langle f, g \rangle_{q,\omega} := \int_a^b f \, \overline{g} \, \mathrm{d}_{q,\omega} \ , \quad f, g \in L^2_{q,\omega}[a,b] \ .$$

*Proof.* The proof is similar to the corresponding one of Theorem 3.8. Indeed, $L^p_{q,\omega}[a,b]$ with its norm $\|\cdot\|_{L^p_{q,\omega}}$ is isometrically isomorphic to $\ell^p_q \times \ell^p_q$ endowed with the norm (for $1 \le p < \infty$)

$$|||(x,y)|||_{\ell^p_q\times\ell^p_q} := \Big\{(1-q)\Big((\omega_0-a)\,\|x\|^p_{\ell^p_q} + (b-\omega_0)\,\|y\|^p_{\ell^p_q}\Big)\Big\}^{1/p} \ ,$$

the mapping $T : L^p_{q,\omega}[a,b] \to \ell^p_q \times \ell^p_q$ defined by

$$Tf := \Big(\big(f(\sigma^n(a))\big)_n, \big(f(\sigma^n(b))\big)_n\Big)$$

being an isometric isomorphism. □

Remark 6.4. *We notice that for the so-called $\alpha,\beta-$symmetric Nörlund sums, $q-$type Hölder and Minkowski inequalities were established in [11]. These properties for the Nörlund sums can be obtained from the $(q,\omega)-$integrals by taking the limit $q \to 1^-$.*

**7. Basic properties of the $(q,\omega)-$integral.** We conclude this paper by stating some known properties for the $(q,\omega)-$integral that may be founded essentially in the works [2, 8, 10, 22]. Our proofs are based on the link between the $q-$integral and the $(q,\omega)-$integral given by Theorem 5.1.

**7.1. Hahn's difference operator.** The importance of the $(q,\omega)-$integral relies on the fact that it gives the inverse of the Hahn's $(q,\omega)-$difference operator, $D_{q,\omega}$, defined for every $f : I \to \mathbb{K}$ as

$$D_{q,\omega}[f](x) := \begin{cases} \dfrac{f(qx+\omega)-f(x)}{(q-1)x+\omega} & \text{if} \quad x \neq \omega_0 \\ f'(\omega_0) & \text{if} \quad x = \omega_0\ . \end{cases}$$

Here, $D_{q,\omega}[f](\omega_0)$ is defined whenever $f$ is defined in a neighborhood of $\omega_0$, and $f'(\omega_0)$ exists (as a complex number). Notice that we may write

$$D_{q,\omega}[f](x) = \frac{f(\sigma(x))-f(x)}{\sigma(x)-x}\ , \quad x \in I \setminus \{\omega_0\}\ .$$

It is easy to see that the Hahn's $(q,\omega)-$difference operator and the Jackson's $q-$difference operator satisfy the relations

$$D_{q,\omega} = \tau_{\omega_0}^{-1} D_q \tau_{\omega_0}\ , \quad D_q = \tau_{\omega_0} D_{q,\omega} \tau_{\omega_0}^{-1}\ . \tag{7.1}$$

Thus any property fulfilled by each of these operators may be translated to a corresponding property satisfied by the other one. For instance, using (4.1), (4.2) and (7.1), or by direct inspection, we deduce

$$D_{q,\omega}[\alpha f + \beta g](x) = \alpha D_{q,\omega}[f](x) + \beta D_{q,\omega}[g](x)$$

for every complex numbers $\alpha$ and $\beta$, and

$$D_{q,\omega}[f\cdot g](x) = D_{q,\omega}[f](x)\cdot g(x) + f(qx+\omega)\cdot D_{q,\omega}[g](x)\ ,$$

which is the $(q,\omega)-$product rule. These equalities hold for all $x \in I \setminus \{\omega_0\}$, and also for $x = \omega_0$ whenever $f'(\omega_0)$ and $g'(\omega_0)$ exist.

**7.2. The fundamental theorem of the $(q,\omega)-$calculus.** From Theorems 4.1 and 5.1, and taking into account relations (7.1), we may easily deduce the following $(q,\omega)-$analogue of the fundamental theorem of calculus:

THEOREM 7.1. *Let $I$ be a set of real numbers such that $\sigma(I)\subset I$. Fix $a,b\in I$, and let $f:I\to\mathbb{K}$ be such that $D_q[f]\in\mathscr{L}^1_{q,\omega}[a,b]$. Then:*

(i) *The equality*

$$\int_a^b D_{q,\omega}[f]\,\mathrm{d}_{q,\omega}=\Big[f(s)-\lim_{n\to+\infty}f(\sigma^n(s))\Big]_{s=a}^{b}$$

*holds, provided the involved limits exist.*

(ii) *In addition, if $f$ is continuous at $\omega_0$, then*

$$\int_a^b D_{q,\omega}[f]\,\mathrm{d}_{q,\omega}=f(b)-f(a)\,.$$

**7.3. The $(q,\omega)-$integration by parts formula.** As a consequence of Theorems 5.1 and 4.2, and using relations (7.1), we may state the following well known $(q,\omega)-$integration by parts formula.

THEOREM 7.2. *Let $I$ be a set of real numbers invariant under $\sigma$. Take $a,b\in I$ and let $f:I\to\mathbb{R}$ and $g:I\to\mathbb{R}$. Then the equality*

$$\int_a^b f\cdot D_{q,\omega}[g]\,\mathrm{d}_{q,\omega}=\Big[f\cdot g-\lim_{n\to\infty}(f\cdot g)\circ\sigma^n\Big]_a^b-\int_a^b D_{q,\omega}[f]\cdot(g\circ\sigma)\;\mathrm{d}_{q,\omega}$$

*holds, provided $f,g\in\mathscr{L}^1_{q,\omega}[a,b]$, $D_{q,\omega}[f]$ and $D_{q,\omega}[g]$ are bounded in $[a,b]_{q,\omega}$, and the limits exist. In particular, if $f$ and $g$ are continuous at $\omega_0$ then*

$$\int_a^b f\cdot D_{q,\omega}[g]\,\mathrm{d}_{q,\omega}=\Big[f\cdot g\Big]_a^b-\int_a^b D_{q,\omega}[f]\cdot(g\circ\sigma)\;\mathrm{d}_{q,\omega}\,.$$

**7.4. The $(q,\omega)-$antiderivative.** Let $I$ be an interval and $f:I\to\mathbb{K}$. We say that $F(\cdot;q,\omega)$ is a $(q,\omega)-$antiderivative of $f$ in $I$ if

$$D_{q,\omega}[F(\cdot;q,\omega)](x)=f(x)\,,\quad\forall x\in I\,.$$

From Theorems 5.1 and 4.3, as well the first relation in (7.1), we may recover the following known property, showing that, indeed, the $(q,\omega)-$integral gives an inverse of the Hahn's $(q,\omega)-$difference operator.

THEOREM 7.3. *Let $I$ be an interval containing $\omega_0$ and take $a\in I$. If $f:I\to\mathbb{K}$ is $(q,\omega)-$integrable on $[a,x]$ for every $x\in I$, and if it is bounded in a neighborhood of $\omega_0$, then the function $F(\cdot;q,\omega)$ given by*

$$F(x;q,\omega):=\int_a^x f\,\mathrm{d}_{q,\omega}\,,\quad x\in I$$

*is a $(q,\omega)-$antiderivative of $f$ in $I$.*

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