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THE EFFECT OF MAGNETIC FIELDS UNDER SPECIFIC BOUNDARY DATA IN THE THEORY OF LIQUID CRYSTALS*

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Abstract. We consider the response of a magnetic field in the theory of liquid crystals. We treat the Landau-de Gennes functional with the strong anchoring condition which may be non-constant under a magnetic field which may also be non-constant. Such situation is more general than the works of Lin and Pan in 2007 and Aramaki in 2012. We show that there exist two critical points of intensity of the field such that one corresponds to the superheating fields of superconductors and the other one corresponds to stability. We also show that under some special conditions, strong field does not bring the pure nematic state which is very different response from superconductors.

Key words. Effect of magnetic fields, variational problem, liquid crystal.

AMS subject classifications. 47F05, 82D30, 35A15, 58E20, 35Q60.

1. Introduction. The purpose of this paper is to extend the results of Lin and Pan [19] and provides an improvement of the previous paper Aramaki [3] under the specific boundary data. We consider the configuration of liquid crystals under a non-constant magnetic field. Let $\mathcal{F}_N(\boldsymbol{n}, \nabla \boldsymbol{n})$ be the classical Oseen-Frank density of nematic liquid crystals. We impose the strong anchoring condition for the director field $\boldsymbol{n}$, that is, the Dirichlet boundary condition $\boldsymbol{n} = \boldsymbol{e}_0$. In the presence of an applied external magnetic field $\boldsymbol{H}$, we must add an external density $-\chi_a(\boldsymbol{H} \cdot \boldsymbol{n})^2$ to $\mathcal{F}_N(\boldsymbol{n}, \nabla \boldsymbol{n})$ (cf. de Gennes and Prost [11, p. 287]), and consider the modified energy functional:

$$\int_\Omega \{\mathcal{F}_N(\boldsymbol{n}, \nabla \boldsymbol{n}) - \chi_a(\boldsymbol{H} \cdot \boldsymbol{n})^2\} dx$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain occupied by a liquid crystal, $\boldsymbol{n} : \overline{\Omega} \to \mathbb{S}^2$ is a director field of the liquid crystal, and

$$\mathcal{F}_N(\nabla \boldsymbol{n}, \boldsymbol{n}) = K_1|\operatorname{div} \boldsymbol{n}|^2 + K_2|\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{n}|^2 + K_3|\boldsymbol{n} \times \operatorname{curl} \boldsymbol{n}|^2 + \nu[\operatorname{Tr}(\nabla \boldsymbol{n})^2 - (\operatorname{div} \boldsymbol{n})^2],$$

$K_1, K_2, K_3 > 0$ are elastic coefficients and ν is a real constant. However, it was shown by Hardt et al. [15] that the integral of the last term represents the surface integral, i.e.,

$$\mathcal{S}(\boldsymbol{e}_0) = \int_\Omega [\operatorname{Tr}(\nabla \boldsymbol{n})^2 - (\operatorname{div} \boldsymbol{n})^2] dx = \int_{\partial\Omega} [(\nabla_{\tan} \boldsymbol{n})\boldsymbol{n} - \operatorname{Tr}(\nabla_{\tan} \boldsymbol{n})\boldsymbol{n}] \cdot \boldsymbol{\nu} dS$$

where $\nabla_{\tan} \boldsymbol{n} = \nabla \boldsymbol{n} - (\nabla \boldsymbol{n})\boldsymbol{\nu} \otimes \boldsymbol{\nu}$ on $\partial\Omega$ and $\boldsymbol{\nu}$ is the outward unit normal vector of $\partial\Omega$. So the term depends only on the boundary data $\boldsymbol{e}_0$. Thus since it does not affect the problem finding the equilibrium configuration, we omit the term.

The effect of applied electric and magnetic fields on liquid crystals is an important problem in the physics of liquid crystals. It is well known that as the magnetic field

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increases passing a critical value the configuration will lose its stability. This phenomenon has been studied by many physicists and mathematicians, and the previous works related to this paper include Atkin and Stewart [6, 7], Cohen and Luskin [9], [19].

In [19] the authors considered the case where $\boldsymbol{e}_0$ and the magnetic field $\boldsymbol{H}$ are constant vectors satisfying $\boldsymbol{H} \cdot \boldsymbol{e}_0 = 0$, in this paper we treat the case where $\boldsymbol{e}_0$ and $\boldsymbol{H}$ may be non-constant vector fields.

Throughout this paper, we assume that Ω is a simply connected bounded domain with smooth boundary $\partial\Omega$. From now, for some Euclidean space $E\,(= \mathbb{R}, \mathbb{C}, \mathbb{R}^3, \mathbb{C}^3$ or the space $\mathbb{R}^9$ of real 3×3 matrices), $W^{1,2}(\Omega, E)$ denotes the usual Sobolev space of E-valued functions, and we denote $W^{1,2}(\Omega, \mathbb{R})$ by $W^{1,2}(\Omega)$.

Let $\boldsymbol{e}_0 \in C^2(\partial\Omega, \mathbb{S}^2)$ and define an admissible space

$$W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0) = \{\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{R}^3); |\boldsymbol{n}(x)| = 1 \text{ a.e. in } \Omega, \boldsymbol{n} = \boldsymbol{e}_0 \text{ on } \partial\Omega\}.$$

We note that if Ω has a smooth boundary, then $W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0) \neq \emptyset$ (cf. [15]). According to the Landau-de Gennes theory, the phase transition of nematic states to smectic states can be described by the minimizer $(\psi, \boldsymbol{n})$ of the Landau-de Gennes energy functional

$$\begin{aligned} (1.1) \quad \mathcal{E}[\psi, \boldsymbol{n}] = \int_\Omega \Big\{ &|\nabla_{q\boldsymbol{n}}\psi|^2 + \frac{\kappa^2}{2}(1 - |\psi|^2)^2 \\ &+ K_1 |\operatorname{div} \boldsymbol{n}|^2 + K_2 |\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{n}|^2 + K_3 |\boldsymbol{n} \times \operatorname{curl} \boldsymbol{n}|^2 - \chi_a (\boldsymbol{H} \cdot \boldsymbol{n})^2 \Big\} dx \end{aligned}$$

where κ and χ_a are positive constants, and q is a real number. Here we denoted $\nabla_{q\boldsymbol{n}}\psi = \nabla\psi - iq\boldsymbol{n}\psi$. Without loss of generality, we may assume that $q \geq 0$. When $\psi \not\equiv 0$, the minimizer $(\psi, \boldsymbol{n})$ describes smectic liquid crystal and when $\psi = 0$, the minimizer describes nematic liquid crystal.

We emphasize that in [19] the boundary data and the applied field are constant vectors, but the present paper considers a more general boundary data $\boldsymbol{e}_0$ which allows a unique harmonic extension that is curl-free and orthogonal to $\boldsymbol{H}$, see (H.1) below. In the below and section 2 the author gives examples of the cases where the condition (H.1) is satisfied.

We assume that the applied field $\boldsymbol{H}$ is smooth in $\overline{\Omega}$, and is written by $\boldsymbol{H}(x) = \sigma \boldsymbol{h}(x)$, $|\boldsymbol{h}(x)| = 1$ and $\sigma > 0$ is constant.

For brevity, we write

$$(1.2) \qquad \mathcal{E}[\psi, \boldsymbol{n}] = \mathcal{G}[\psi, \boldsymbol{n}] + \mathcal{F}[\boldsymbol{n}] - \int_\Omega \chi_a (\boldsymbol{H} \cdot \boldsymbol{n})^2 dx$$

where

$$\mathcal{F}[\boldsymbol{n}] = \int_\Omega \{K_1 |\operatorname{div} \boldsymbol{n}|^2 + K_2 |\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{n}|^2 + K_3 |\boldsymbol{n} \times \operatorname{curl} \boldsymbol{n}|^2\} dx$$

is the simplified Oseen-Frank energy for nematics and

$$\mathcal{G}[\psi, \boldsymbol{n}] = \int_\Omega \Big\{ |\nabla_{q\boldsymbol{n}}\psi|^2 + \frac{\kappa^2}{2}(1 - |\psi|^2)^2 \Big\} dx$$

is the Ginzburg-Landau energy for smectics. We also write

$$\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] = \mathcal{F}[\boldsymbol{n}] - \chi_a \sigma^2 \int_\Omega (\boldsymbol{h} \cdot \boldsymbol{n})^2 dx.$$

Throughout this paper we assume that
(H.1) there exists $\boldsymbol{e} \in C^2(\overline{\Omega}, \mathbb{S}^2)$ such that $\operatorname{curl} \boldsymbol{e} = 0$, $\quad \boldsymbol{h} \cdot \boldsymbol{e} = 0$ in Ω, and $\boldsymbol{e}$ is a unique minimizer of

$$\inf_{\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)} \int_\Omega |\nabla \boldsymbol{n}|^2 dx.$$

We note that the minimizer is a harmonic map from Ω to $\mathbb{S}^2$ with boundary value $\boldsymbol{e}_0$, that is to say, $\boldsymbol{e}$ satisfies the equation

$$\begin{cases} -\Delta \boldsymbol{e} = |\nabla \boldsymbol{e}|^2 \boldsymbol{e} & \text{in } \Omega, \\ \boldsymbol{e} = \boldsymbol{e}_0 & \text{on } \partial\Omega. \end{cases}$$

Here we note that there are many situations where (H.1) holds. For example, define

$$\boldsymbol{e}(x_1, x_2, x_3) = \left(\frac{x_1 - a_1}{\sqrt{(x_1 - a_1)^2 + x_2^2}}, \frac{x_2}{\sqrt{(x_1 - a_1)^2 + x_2^2}}, 0 \right) \tag{1.3}$$

$x_1 - a_1 > 0$ for $x \in \overline{\Omega}$, $|a_1|$ large enough, $\boldsymbol{e}_0 := \boldsymbol{e}|_{\partial\Omega}$, and $\boldsymbol{h} = (0, 0, 1)$ or

$$\boldsymbol{h}(x_1, x_2, x_3) = \left(\frac{-x_2}{\sqrt{(x_1 - a_1)^2 + x_2^2}}, \frac{x_1 - a_1}{\sqrt{(x_1 - a_1)^2 + x_2^2}}, 0 \right). \tag{1.4}$$

Though it was shown by the previous paper of the author Aramaki [2] that $\boldsymbol{e}$ satisfies (H.1), we shall show this fact in section 2 briefly. There are a lot of choices of a_1. We note that such applied fields $\boldsymbol{h}$ satisfy the Maxwell equation $\operatorname{div} \boldsymbol{h} = 0$ in Ω which is not needed in this paper.

We also assume that

$$K_1 \leq \min(K_2, K_3). \tag{H.2}$$

We furthermore assume that

$$1 > c(\Omega) \max_{x \in \overline{\Omega}} |\nabla \boldsymbol{e}|^2, \tag{H.3}$$

where $c(\Omega) > 0$ is the best constant such that the following Poincaré inequality holds:

$$\int_\Omega |\boldsymbol{w}|^2 dx \leq c(\Omega) \int_\Omega |\nabla \boldsymbol{w}|^2 dx$$

for any $\boldsymbol{w} \in W_0^{1,2}(\Omega, \mathbb{R}^3)$. For the above example, since

$$|\nabla \boldsymbol{e}|^2 = \frac{1}{(x_1 - a_1)^2 + x_2^2},$$

if we choose $|a_1|$ is large enough, we see that (H.3) holds.

Moreover assume that

(H.4) For any $p \in \Omega$, the integral curve of $\boldsymbol{e}$ through p intersects with $\partial\Omega$.

We note that for the above example, (H.4) also satisfied.

Thus the examples in (1.3) and (1.4) satisfy the conditions (H.1), (H.2), (H.3) and (H.4). Our purpose is an extension of their results in [19] and [3] to the case where $\boldsymbol{e}_0$ and $\boldsymbol{h}$ are non-constant vector fields and the conditions (H.1), (H.2), (H.3) and (H.4) hold.

Since by the hypothesis (H.1), $\operatorname{curl} \boldsymbol{e} = 0$ in Ω and Ω is simply connected, there exists a unique function $\varphi \in C^3(\Omega)$ such that

$$\nabla\varphi = \boldsymbol{e} \quad \text{in } \Omega, \qquad \int_\Omega \varphi dx = 0. \tag{1.5}$$

Now we can see that the energy functional $\mathcal{E}$ has two families of critical points:

$$\psi = 0, \quad \boldsymbol{n} = \boldsymbol{n}_\sigma \tag{1.6}$$

where $\boldsymbol{n}_\sigma$ is a global minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$:

$$\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_\sigma] = \inf_{\boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)} \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}],$$

and

$$\psi = ce^{iq\varphi}, \quad \boldsymbol{n} = \boldsymbol{e} \tag{1.7}$$

where φ is the function as in (1.5) and c is an arbitrary complex number such that $|c| = 1$.

There are many article on liquid crystals without external field. For example, see Aramaki [4, 5], Bauman et al. [8], [15], Pan [20], [23]. Recently, there have been increasing interests to study the effect of applied fields. See [6, 7], [9], [19], Pan [21, 22] and Aramaki [2, 1, 3].

According to the analogies between superconductors and liquid crystals (cf. [21] and [22]), we call the family in (1.6) pure nematic states corresponding to the normal states of superconductor, and the family in (1.7) pure smectic states corresponding to the Meissner states of superconductor. We shall see that there exists a critical field $H_n(0) > 0$ such that for $0 \le \sigma < H_n(0)$, the only pure nematic state is $(0, \boldsymbol{e})$. Moreover, we shall show that there exist critical fields H_{sh} and H_s where the pure smectic states change their weak stability (local minimality) at H_{sh} and change their strong stability (global minimality) at H_s. Although the critical points have the same natures as [19], the formulas have to be modified from their formulas, because of given vector fields being non-constant. Thus the critical field H_{sh} looks like the superheating field of superconductors. We shall also show that in the special case of $K_1 = K_2 = K_3$, a liquid crystal under very strong external field may not be in a pure nematic state. On the other hand, in the theory of superconductors, the breakdown of superconductivity occurs under strong external fields. See Giorgi and Phillips [13]. Thus liquid crystals and superconductors have very different response under strong field.

The plan of this paper is as follows. In section 2, we consider the examples given by (1.3) and (1.4), and state the weak stability of a critical point of $\mathcal{E}$. In section 3, we define a critical value H_{sh} and show that when σ increases, the pure smectic states change their weak stabilities at H_{sh}. In section 4, we define a critical value H_s, and show that if $\sigma > H_s$, the global minimizers of $\mathcal{E}$ are not pure smectic states and if $\sigma < H_s$, the only global minimizers of $\mathcal{E}$ are pure smectic states. Finally in section 5, we show the instabilities in pure nematic states. In the particular case of $K_1 = K_2 = K_3$, when σ is sufficiently large, the pure nematic states are not global minimizers of $\mathcal{E}$. This phenomenon clarifies the difference between the liquid crystals and superconductors.

2. Preliminaries. First we shall show that there are many situations where (H.1) hold. For example, let Ω be a smooth bounded domain in $\mathbb{R}^3$, and define a vector field

$$\boldsymbol{e}(x) = \left(\frac{x_1 - a_1}{\sqrt{(x_1 - a_1)^2 + x_2^2}}, \frac{x_2}{\sqrt{(x_1 - a_1)^2 + x_2^2}}, 0 \right) \tag{2.1}$$

where $x_1 - a_1 > 0$ for all $x \in \overline{\Omega}$. There are lots of the choices of a_1. By simple computations, we can show that $\boldsymbol{e}$ is a harmonic map from Ω into the unit sphere $\mathbb{S}^2$ and curl-free, that is to say, $\boldsymbol{e}$ satisfies

$$\begin{cases} -\Delta \boldsymbol{e} = |\nabla \boldsymbol{e}|^2 \boldsymbol{e} & \text{in } \Omega, \\ \boldsymbol{e} = \boldsymbol{e}_0 := \boldsymbol{e}|_{\partial\Omega} & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

and $\operatorname{curl} \boldsymbol{e} = 0$ in Ω. For large $|a_1|$, putting $p = (1, 0, 0)$, we can see that $\boldsymbol{e}(\overline{\Omega}) \subset B_r(p) \subset \mathbb{S}^2$ where $B_r(p) = \{q \in \mathbb{S}^2; \text{dist } (q, p) \le r\}$ and $0 < r < \pi/2$. Here $\text{dist}(q, p)$ denotes the geodesic distance on $\mathbb{S}^2$. We note that $B_r(p)$ satisfies the cut locus condition. That is to say, for any two points in $B_r(p)$, there exists a unique geodesic in $B_r(p)$ joining the two points.

According to Jäger and Kaul [17], Hildebrandt et al. [16], the harmonic map $\boldsymbol{e}$ such that $\boldsymbol{e}(\overline{\Omega}) \subset B_r(p)$ with the Dirichlet data $\boldsymbol{e}_0$ exists and is unique. We can show that $\boldsymbol{e}$ is a unique minimizing harmonic map of

$$\inf_{\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)} \int_\Omega |\nabla \boldsymbol{n}|^2 dx. \tag{2.3}$$

PROPOSITION 2.1. *Let $\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ be a minimizer of* (2.3). *Then $\boldsymbol{n}(\overline{\Omega}) \subset B_r(p)$. So we have $\boldsymbol{n} = \boldsymbol{e}$ in Ω.*

In order to prove this proposition, we need a lemma due to Jost [18]

LEMMA 2.2. *Let B_0 and B_1 be closed subsets of $\mathbb{S}^2$ with $B_0 \subset B_1$ and there exists a C^1 retraction map $\Pi : B_1 \to B_0$ such that*

$$|\nabla \Pi(x)(\boldsymbol{v})| < |\boldsymbol{v}| \quad \textit{if } \boldsymbol{v} \in T_x\mathbb{S}^2,\ x \in B_1 \setminus B_0.$$

For a given boundary data $\boldsymbol{g} : \partial\Omega \to B_0$, if $\boldsymbol{n} : \Omega \to B_1$ is a energy minimizing map of (2.3) *with the boundary data $\boldsymbol{g}$, then $\boldsymbol{n}(\Omega) \subset B_0$.*

For the proof of Proposition 2.1, see [2].

LEMMA 2.3. *Assume that* (H.1) *and* (H.2) *hold. Then the vector field $\boldsymbol{e}$ defined in* (H.1) *is also a unique minimizer of*

$$\inf_{\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)} \mathcal{F}[\boldsymbol{n}].$$

Proof. For any $\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$, it follows from (H.1) that

$$\int_\Omega |\nabla \boldsymbol{e}|^2 dx \le \int_\Omega |\nabla \boldsymbol{n}|^2 dx.$$

Since $(\operatorname{div}\boldsymbol{n})^2 + |\operatorname{curl}\boldsymbol{n}|^2 = |\nabla\boldsymbol{n}|^2 - [\operatorname{Tr}(\nabla\boldsymbol{n})^2 - (\operatorname{div}\boldsymbol{n})^2]$, we can see from (H.1) and (H.2) that

$$\begin{aligned}\mathcal{F}[\boldsymbol{e}] &= \int_\Omega K_1(\operatorname{div}\boldsymbol{e})^2 dx = \int_\Omega K_1|\nabla\boldsymbol{e}|^2 dx - K_1\mathcal{S}(\boldsymbol{e}_0)\\ &\le \int_\Omega K_1|\nabla\boldsymbol{n}|^2 dx - K_1\mathcal{S}(\boldsymbol{e}_0) \le \int_\Omega K_1\{(\operatorname{div}\boldsymbol{n})^2 + |\operatorname{curl}\boldsymbol{n}|^2\}dx \le \mathcal{F}[\boldsymbol{n}].\end{aligned}$$

Conversely, if $\boldsymbol{n}$ is a global minimizer of $\mathcal{F}$, we see that

$$\begin{aligned}K_1\int_\Omega |\nabla\boldsymbol{n}|^2 dx - K_1\mathcal{S}(\boldsymbol{e}_0) &\le \mathcal{F}[\boldsymbol{n}] \le \mathcal{F}[\boldsymbol{e}]\\ &= K_1\int_\Omega(\operatorname{div}\boldsymbol{e})^2 dx = K_1\int_\Omega|\nabla\boldsymbol{e}|^2 dx - K_1\mathcal{S}(\boldsymbol{e}_0).\end{aligned}$$

From (H.1), we see that $\boldsymbol{n} = \boldsymbol{e}$. □

Next we give the definition of the weak stability of critical points and a sufficient condition for weak stability for a general applied field $\boldsymbol{H}$ and a boundary data $\boldsymbol{u}_0$.

Definition 2.4. (1) *We say that* $(\psi_0, \boldsymbol{n}_0) \in W^{1,2}(\Omega, \mathbb{C}) \times W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{u}_0)$ *is a critical point of* $\mathcal{E}$, *if and only if for any* $\phi \in W^{1,2}(\Omega, \mathbb{C})$ *and any* $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$,

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{E}[\psi_t, \boldsymbol{n}_t] = 0$$

where

$$\psi_t = \psi_0 + t\phi, \quad \boldsymbol{n}_t = \frac{\boldsymbol{n}_0 + t\boldsymbol{v}}{|\boldsymbol{n}_0 + t\boldsymbol{v}|}. \tag{2.4}$$

(2) *We say that a critical point* $(\psi_0, \boldsymbol{n}_0)$ *of* $\mathcal{E}$ *is weakly stable* (*local minimizer*), *if for any* $\phi \in W^{1,2}(\Omega, \mathbb{C})$ *and any* $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$, *there exists* $T = T(\phi, \boldsymbol{v}) > 0$ *such that for any* $0 < t < T$,

$$\mathcal{E}[\psi_0, \boldsymbol{n}_0] \le \mathcal{E}[\psi_t, \boldsymbol{n}_t].$$

By computations as in [19], we can write

$$\boldsymbol{n}_t = \boldsymbol{n}_0 + t\boldsymbol{n}_1 + t^2\boldsymbol{n}_2 + O(t^3) \tag{2.5}$$

where

$$\begin{aligned}\boldsymbol{n}_1 &= \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{n}_0)\boldsymbol{n}_0,\\ \boldsymbol{n}_2 &= -(\boldsymbol{v}\cdot\boldsymbol{n}_0)\boldsymbol{v} + \frac{1}{2}[3(\boldsymbol{v}\cdot\boldsymbol{n}_0)^2 - |\boldsymbol{v}|^2]\boldsymbol{n}_0\end{aligned}$$

and

$$\nabla_{q\boldsymbol{n}_t}\psi_t = \nabla_{q\boldsymbol{n}_0}\psi_0 + t\Phi_1 + t^2\Phi_2 + O(t^3) \tag{2.6}$$

where

$$\begin{aligned}\Phi_1 &= \nabla_{q\boldsymbol{n}_0}\phi - iq\boldsymbol{n}_1\psi_0,\\ \Phi_2 &= -iq(\boldsymbol{n}_1\phi + \boldsymbol{n}_2\psi_0).\end{aligned}$$

Using these formulas, for small t, we can write

$$\begin{aligned}\mathcal{G}[\psi_t, \boldsymbol{n}_t] = \mathcal{G}[\psi_0, \boldsymbol{n}_0]& \\ &+2t\int_\Omega \big\{\Re[\overline{\nabla_{q\boldsymbol{n}_0}\phi}\cdot\nabla_{q\boldsymbol{n}_0}\psi_0 - \kappa^2\overline{\phi}(1-|\psi_0|^2)\psi_0] \\ &-q\boldsymbol{n}_1\cdot\Im(\overline{\psi}_0\nabla_{q\boldsymbol{n}_0}\psi_0)\big\}dx \\ &+t^2\int_\Omega \big\{|\Phi_1|^2 - \kappa^2(1-|\psi_0|^2)|\phi|^2 + 2\kappa^2(\Re(\overline{\phi}\psi_0))^2 \\ &-2q\Im[(\boldsymbol{n}_1\overline{\phi} + \boldsymbol{n}_2\overline{\psi}_0)\cdot\nabla_{q\boldsymbol{n}_0}\psi_0]\big\}dx + O(t^3).\end{aligned}$$

Henceforth, we denote the real part and imaginary part of a complex number z by $\Re[z]$ and $\Im[z]$, respectively.

$$\begin{aligned}\mathcal{F}[\boldsymbol{n}_t] = \mathcal{F}[\boldsymbol{n}_0]& \\ &+2t\int_\Omega \big\{K_1(\operatorname{div}\boldsymbol{n}_0)(\operatorname{div}\boldsymbol{n}_1) \\ &+K_2(\boldsymbol{n}_0\cdot\operatorname{curl}\boldsymbol{n}_0)(\boldsymbol{n}_1\cdot\operatorname{curl}\boldsymbol{n}_0 + \boldsymbol{n}_0\cdot\operatorname{curl}\boldsymbol{n}_1) \\ &+K_3(\boldsymbol{n}_0\times\operatorname{curl}\boldsymbol{n}_0)\cdot(\boldsymbol{n}_1\times\operatorname{curl}\boldsymbol{n}_0 + \boldsymbol{n}_0\times\operatorname{curl}\boldsymbol{n}_1)\}dx \\ &+t^2\int_\Omega \big\{K_1\{(\operatorname{div}\boldsymbol{n}_1)^2 + 2(\operatorname{div}\boldsymbol{n}_0)(\operatorname{div}\boldsymbol{n}_2)\} \\ &+K_2\{(\boldsymbol{n}_1\cdot\operatorname{curl}\boldsymbol{n}_0 + \boldsymbol{n}_0\cdot\operatorname{curl}\boldsymbol{n}_1)^2 \\ &+2(\boldsymbol{n}_0\cdot\operatorname{curl}\boldsymbol{n}_0)(\boldsymbol{n}_2\cdot\operatorname{curl}\boldsymbol{n}_0 + \boldsymbol{n}_1\cdot\operatorname{curl}\boldsymbol{n}_1 + \boldsymbol{n}_0\cdot\operatorname{curl}\boldsymbol{n}_2)\} \\ &+K_3\{|\boldsymbol{n}_1\times\operatorname{curl}\boldsymbol{n}_0 + \boldsymbol{n}_0\times\operatorname{curl}\boldsymbol{n}_1|^2 \\ &+2(\boldsymbol{n}_0\times\operatorname{curl}\boldsymbol{n}_0)\cdot(\boldsymbol{n}_2\times\operatorname{curl}\boldsymbol{n}_0 + \boldsymbol{n}_1\times\operatorname{curl}\boldsymbol{n}_1 \\ &+\boldsymbol{n}_0\times\operatorname{curl}\boldsymbol{n}_2)\}dx + O(t^3).\end{aligned}$$

$$\begin{aligned}\int_\Omega(\boldsymbol{H}\cdot\boldsymbol{n}_t)^2dx = \int_\Omega(\boldsymbol{H}\cdot\boldsymbol{n}_0)^2dx& \\ &+2t\int_\Omega(\boldsymbol{H}\cdot\boldsymbol{n}_0)(\boldsymbol{H}\cdot\boldsymbol{n}_1)dx \\ &+t^2\int_\Omega\{(\boldsymbol{H}\cdot\boldsymbol{n}_1)^2 + 2(\boldsymbol{H}\cdot\boldsymbol{n}_0)(\boldsymbol{H}\cdot\boldsymbol{n}_2)\}dx + O(t^3).\end{aligned}$$

Therefore, we can write

$$\begin{aligned}\mathcal{E}[\psi_t, \boldsymbol{n}_t] = \mathcal{E}[\psi_0, \boldsymbol{n}_0]& \\ &+ 2t\left\{\mathcal{A}(\psi_0, \boldsymbol{n}_0; \phi, \boldsymbol{v}) - \chi_a\int_\Omega(\boldsymbol{H}\cdot\boldsymbol{n}_0)(\boldsymbol{H}\cdot\boldsymbol{n}_1)dx\right\} \\ &+ t^2\bigg\{\mathcal{B}(\psi_0, \boldsymbol{n}_0; \phi, \boldsymbol{v}) \\ &- \chi_a\int_\Omega\{(\boldsymbol{H}\cdot\boldsymbol{n}_1)^2 + 2(\boldsymbol{H}\cdot\boldsymbol{n}_0)(\boldsymbol{H}\cdot\boldsymbol{n}_2)\}dx\bigg\} \\ &+ O(t^3)\end{aligned} \tag{2.7}$$

where

$$\begin{aligned}(2.8)\qquad \mathcal{A}(\psi_0,\boldsymbol{n}_0;\phi,\boldsymbol{v}) = \int_\Omega \{ & \Re[\overline{\nabla_{q\boldsymbol{n}_0}\phi}\cdot\nabla_{q\boldsymbol{n}_0}\psi_0 - \kappa^2\overline{\phi}(1-|\psi_0|^2)\psi_0] \\ & - q\boldsymbol{n}_1\cdot\Im(\overline{\psi}_0\nabla_{q\boldsymbol{n}_0}\psi_0) + K_1(\operatorname{div}\boldsymbol{n}_0)(\operatorname{div}\boldsymbol{n}_1) \\ & + K_2(\boldsymbol{n}_0\cdot\operatorname{curl}\boldsymbol{n}_0)(\boldsymbol{n}_1\cdot\operatorname{curl}\boldsymbol{n}_0+\boldsymbol{n}_0\cdot\operatorname{curl}\boldsymbol{n}_1) \\ & + K_3(\boldsymbol{n}_0\times\operatorname{curl}\boldsymbol{n}_0)\cdot(\boldsymbol{n}_1\times\operatorname{curl}\boldsymbol{n}_0+\boldsymbol{n}_0\times\operatorname{curl}\boldsymbol{n}_1)\}dx,\end{aligned}$$

$$\begin{aligned}(2.9)\qquad \mathcal{B}(\psi_0,\boldsymbol{n}_0;\phi,\boldsymbol{v}) = \int_\Omega \{ & |\nabla_{q\boldsymbol{n}_0}\phi - iq\boldsymbol{n}_1\psi_0|^2 - \kappa^2(1-|\psi_0|^2)|\phi|^2 \\ & + 2\kappa^2(\Re(\overline{\phi}\psi_0))^2 - 2q\Im[(\boldsymbol{n}_1\overline{\phi}+\boldsymbol{n}_2\overline{\psi_0})\nabla_{q\boldsymbol{n}_0}\psi_0] \\ & + K_1\{(\operatorname{div}\boldsymbol{n}_1)^2 + 2(\operatorname{div}\boldsymbol{n}_0)(\operatorname{div}\boldsymbol{n}_2)\} \\ & + K_2\{(\boldsymbol{n}_1\cdot\operatorname{curl}\boldsymbol{n}_0+\boldsymbol{n}_0\cdot\operatorname{curl}\boldsymbol{n}_1)^2 \\ & + 2(\boldsymbol{n}_0\cdot\operatorname{curl}\boldsymbol{n}_0)(\boldsymbol{n}_2\cdot\operatorname{curl}\boldsymbol{n}_0+\boldsymbol{n}_1\cdot\operatorname{curl}\boldsymbol{n}_1 \\ & + \boldsymbol{n}_0\cdot\operatorname{curl}\boldsymbol{n}_2)\} + K_3\{|\boldsymbol{n}_1\times\operatorname{curl}\boldsymbol{n}_0+\boldsymbol{n}_0\times\operatorname{curl}\boldsymbol{n}_1|^2 \\ & + 2(\boldsymbol{n}_0\times\operatorname{curl}\boldsymbol{n}_0)\cdot(\boldsymbol{n}_2\times\operatorname{curl}\boldsymbol{n}_0+\boldsymbol{n}_1\times\operatorname{curl}\boldsymbol{n}_1 \\ & + \boldsymbol{n}_0\times\operatorname{curl}\boldsymbol{n}_2)\}dx.\end{aligned}$$

Therefore, we have the following lemma.

LEMMA 2.5. (i) $(\psi_0,\boldsymbol{n}_0)\in W^{1,2}(\Omega,\mathbb{C})\times W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{u}_0)$ *is a critical point of* $\mathcal{E}$ *if and only if for any* $\phi\in W^{1,2}(\Omega,\mathbb{C})$ *and any* $\boldsymbol{v}\in W_0^{1,2}(\Omega,\mathbb{R}^3)\cap L^\infty(\Omega,\mathbb{R}^3)$,

$$\mathcal{A}(\psi_0,\boldsymbol{n}_0;\phi,\boldsymbol{v}) - \chi_a\int_\Omega(\boldsymbol{H}\cdot\boldsymbol{n}_0)(\boldsymbol{H}\cdot\boldsymbol{n}_1)dx = 0.$$

(ii) *If a critical point* $(\psi_0,\boldsymbol{n}_0)\in W^{1,2}(\Omega,\mathbb{C})\times W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{u}_0)$ *is weakly stable, then for any* $\phi\in W^{1,2}(\Omega,\mathbb{C})$ *and any* $\boldsymbol{v}\in W_0^{1,2}(\Omega,\mathbb{R}^3)\cap L^\infty(\Omega,\mathbb{R}^3)$,

$$\mathcal{B}(\psi_0,\boldsymbol{n}_0;\phi,\boldsymbol{v}) \geq \chi_a\int_\Omega\{(\boldsymbol{H}\cdot\boldsymbol{n}_1)^2 + 2(\boldsymbol{H}\cdot\boldsymbol{n}_0)(\boldsymbol{H}\cdot\boldsymbol{n}_2)\}dx.$$

(iii) *Let* $(\psi_0,\boldsymbol{n}_0)\in W^{1,2}(\Omega,\mathbb{C})\times W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{u}_0)$ *be a critical point of* $\mathcal{E}$. *If for any* $\phi\in W^{1,2}(\Omega,\mathbb{C})$ *and any* $\boldsymbol{v}\in W_0^{1,2}(\Omega,\mathbb{R}^3)\cap L^\infty(\Omega,\mathbb{R}^3)$ *which is not parallel to* $\boldsymbol{n}_0$ *on a set of positive measure,*

$$\mathcal{B}(\psi_0,\boldsymbol{n}_0;\phi,\boldsymbol{v}) > \chi_a\int_\Omega\{(\boldsymbol{H}\cdot\boldsymbol{n}_1)^2 + 2(\boldsymbol{H}\cdot\boldsymbol{n}_0)(\boldsymbol{H}\cdot\boldsymbol{n}_2)\}dx,$$

then $(\psi_0,\boldsymbol{n}_0)$ *is weakly stable.*

(iv) *Let* $(\psi_0,\boldsymbol{n}_0)\in W^{1,2}(\Omega,\mathbb{C})\times W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{u}_0)$ *be a critical point of* $\mathcal{E}$. *If there exist* $\phi\in W^{1,2}(\Omega,\mathbb{C})$ *and* $\boldsymbol{v}\in W_0^{1,2}(\Omega,\mathbb{R}^3)\cap L^\infty(\Omega,\mathbb{R}^3)$ *such that*

$$\mathcal{B}(\psi_0,\boldsymbol{n}_0;\phi,\boldsymbol{v}) < \chi_a\int_\Omega\{(\boldsymbol{H}\cdot\boldsymbol{n}_1)^2 + 2(\boldsymbol{H}\cdot\boldsymbol{n}_0)(\boldsymbol{H}\cdot\boldsymbol{n}_2)\}dx,$$

then $(\psi_0,\boldsymbol{n}_0)$ *is not weakly stable.*

For the proof, see [19] for (i) and (ii), and see [9] for (iii) and (iv).

REMARK 2.6. (i) *If $(\psi, \boldsymbol{n})$ is a critical point of $\mathcal{E}$, then the Euler-Lagrange equation for ψ is the following.*

$$\begin{cases} -\nabla_{q\boldsymbol{n}}^2\psi = \kappa^2(1-|\psi|^2)\psi & \text{in } \Omega, \\ \nabla_{q\boldsymbol{n}}\psi\cdot\boldsymbol{\nu} = 0 & \text{on } \partial\Omega \end{cases}$$

where $\boldsymbol{\nu}$ denotes the unit outer normal vector to $\partial\Omega$.

(ii) *We note that under the hypothesis* (H.1), $(\psi_0, \boldsymbol{n}_0) = (ce^{iq\varphi}, \boldsymbol{e})$ *where φ is the function as in* (1.5) *is a critical point of $\mathcal{E}$. In fact, since $\nabla_{q\boldsymbol{n}_0}\psi_0 = 0$ and $|\psi_0| = 1$, for any $\phi \in W^{1,2}(\Omega, \mathbb{C})$ and any $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$, it follows from* (H.1) *that*

$$\begin{aligned} \mathcal{A}(\psi_0, \boldsymbol{n}_0; \phi, \boldsymbol{v}) - \chi_a\sigma^2\int_\Omega(\boldsymbol{h}\cdot\boldsymbol{e})(\boldsymbol{h}\cdot\boldsymbol{n}_0)dx &= K_1\int_\Omega(\operatorname{div}\boldsymbol{e})(\operatorname{div}\boldsymbol{n}_1)dx \\ &= K_1\int_\Omega -\nabla(\operatorname{div}\boldsymbol{e})\cdot(\boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e})dx. \end{aligned}$$

Since $\operatorname{curl}\boldsymbol{e} = 0$, *it follows from the formula*

$$\operatorname{curl}^2\boldsymbol{e} = -\Delta\boldsymbol{e} + \nabla(\operatorname{div}\boldsymbol{e}) \tag{2.10}$$

that the last line of the above equality is equal to

$$-K_1\int_\Omega \Delta\boldsymbol{e}\cdot(\boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e})dx = K_1\int_\Omega |\nabla\boldsymbol{e}|^2\boldsymbol{e}\cdot(\boldsymbol{v}-(\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e})dx = 0$$

from (H.1). *Thus $(\psi_0, \boldsymbol{n}_0)$ is a critical point of $\mathcal{E}$.*

3. Loss of local minimality of pure smectic states. In this section we shall examine weak stability (local minimality) of pure smectic state $(\psi_0, \boldsymbol{n}_0) = (ce^{iq\varphi}, \boldsymbol{e})$ where $c \in \mathbb{C}$ and $|c| = 1$ and φ is the function as in (1.5).

For any $\phi \in W^{1,2}(\Omega, \mathbb{C})$ and any $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$, define ψ_t and $\boldsymbol{n}_t$ as in (2.4) with $\boldsymbol{n}_0 = \boldsymbol{e}$. Then it follows from $\boldsymbol{h}(x)\cdot\boldsymbol{e}(x) = 0$ and $\boldsymbol{n}_1 = \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{e})\boldsymbol{e}$ that $\sigma\boldsymbol{h}\cdot\boldsymbol{n}_1 = \sigma(\boldsymbol{v}\cdot\boldsymbol{h})$. Thus if the critical point $(\psi_0, \boldsymbol{e})$ is weakly stable, then we see from Lemma 2.5 that

$$\mathcal{B}(\psi_0, \boldsymbol{e}; \phi, \boldsymbol{v}) \geq \chi_a\sigma^2\int_\Omega(\boldsymbol{v}\cdot\boldsymbol{h})^2dx. \tag{3.1}$$

Since $W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$ is dense in $W_0^{1,2}(\Omega, \mathbb{R}^3)$, (3.1) holds for any $\phi \in W^{1,2}(\Omega, \mathbb{C})$ and any $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3)$. Since $\nabla_{q\boldsymbol{n}_0}\psi_0 = \nabla_{q\boldsymbol{e}}(ce^{iq\varphi}) = 0$ and $|\psi_0| = 1$, we have, from (2.9),

$$\begin{aligned} \mathcal{B}(\psi_0, \boldsymbol{n}_0; \phi, \boldsymbol{v}) = \int_\Omega &\{|\nabla_{q\boldsymbol{e}}\phi - iq\boldsymbol{n}_1\psi_0|^2 + 2\kappa^2(\Re(\overline{\phi}\psi_0))^2 \\ &+ K_1((\operatorname{div}\boldsymbol{n}_1)^2 + 2(\operatorname{div}\boldsymbol{e})(\operatorname{div}\boldsymbol{n}_2)) + K_2(\boldsymbol{e}\cdot\operatorname{curl}\boldsymbol{n}_1)^2 \\ &+ K_3|\boldsymbol{e}\times\operatorname{curl}\boldsymbol{n}_1|^2\}dx. \end{aligned}$$

For any $\phi \in W^{1,2}(\Omega, \mathbb{C})$, we can write $\phi = icqe^{iq\varphi}u$, $u \in W^{1,2}(\Omega, \mathbb{C})$. Therefore,

$$\nabla_{q\boldsymbol{e}}\phi - iq\boldsymbol{n}_1\psi_0 = icqe^{iq\varphi}(\nabla u - \boldsymbol{n}_1)$$

and $\Re(\overline{\phi}\psi_0) = |c|^2\Re(iqu) = -q\Im(u)$. Here since $\boldsymbol{n}_2 \in W_0^{1,2}(\Omega, \mathbb{R}^3)$,

$$2\int_\Omega (\operatorname{div} \boldsymbol{e})(\operatorname{div} \boldsymbol{n}_2)dx = -2\int_\Omega \nabla(\operatorname{div} \boldsymbol{e}) \cdot \boldsymbol{n}_2 dx.$$

By the formula (2.10) and the hypothesis (H.1), we have $\nabla(\operatorname{div} \boldsymbol{e}) = \Delta \boldsymbol{e} = -|\nabla \boldsymbol{e}|^2 \boldsymbol{e}$. Moreover, we have $2\boldsymbol{e} \cdot \boldsymbol{n}_2 = (\boldsymbol{v} \cdot \boldsymbol{e})^2 - |\boldsymbol{v}|^2 = -|\boldsymbol{n}_1|^2$. If we write $\boldsymbol{n}_1 = \boldsymbol{w}$, then $\boldsymbol{w} \in W_0^{1,2}(\Omega, \mathbb{R}^3)$ and $\boldsymbol{w}(x) \cdot \boldsymbol{e}(x) = 0$ a.e. in Ω. Hence we can rewrite

$$\begin{aligned}\mathcal{B}(\psi_0, \boldsymbol{n}_0; \phi, \boldsymbol{v}) = \int_\Omega \{&q^2|\nabla u - \boldsymbol{w}|^2 + 2\kappa^2 q^2(\Im(u))^2 - K_1|\nabla \boldsymbol{e}|^2|\boldsymbol{w}|^2 \\ &+ K_1(\operatorname{div} \boldsymbol{w})^2 + K_2(\boldsymbol{e} \cdot \operatorname{curl} \boldsymbol{w})^2 + K_3|\boldsymbol{e} \times \operatorname{curl} \boldsymbol{w}|^2\}dx.\end{aligned}$$

If $(\phi, \boldsymbol{v})$ minimizes $\mathcal{B}(\psi_0, \boldsymbol{n}_0; \phi, \boldsymbol{v})/\|\boldsymbol{v} \cdot \boldsymbol{h}\|^2_{L^2(\Omega)}$, then u is real valued. Thus we may assume that $u = -\frac{i}{cq}e^{-iq\varphi}\phi$ is a real valued function. We write $\mathcal{B}(\psi_0, \boldsymbol{n}_0; \phi, \boldsymbol{v})$ by $B(u, \boldsymbol{w})$. That is to say,

$$B(u, \boldsymbol{w}) = \int_\Omega \{q^2|\nabla u - \boldsymbol{w}|^2 dx + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1\int_\Omega |\nabla \boldsymbol{e}|^2|\boldsymbol{w}|^2\}dx \tag{3.2}$$

where

$$\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] = \int_\Omega \{K_1(\operatorname{div} \boldsymbol{w})^2 + K_2(\boldsymbol{e} \cdot \operatorname{curl} \boldsymbol{w})^2 + K_3|\boldsymbol{e} \times \operatorname{curl} \boldsymbol{w}|^2\}dx.$$

Here we note that under the hypothesis (H.2), we can show the following.

Lemma 3.1. *Assume that* (H.2) and (H.3) hold. Then there exists a constant $c > 0$ such that

$$\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1\int_\Omega |\nabla \boldsymbol{e}|^2|\boldsymbol{w}|^2 dx \geq c\|\boldsymbol{w}\|^2_{W^{1,2}(\Omega, \mathbb{R}^3)} \tag{3.3}$$

for all $\boldsymbol{w} \in W_0^{1,2}(\Omega, \mathbb{R}^3)$.

Proof. Since $(\boldsymbol{e} \cdot \operatorname{curl} \boldsymbol{w})^2 + |\boldsymbol{e} \times \operatorname{curl} \boldsymbol{w}|^2 = |\operatorname{curl} \boldsymbol{w}|^2$ and $\boldsymbol{w} \in W_0^{1,2}(\Omega, \mathbb{R}^3)$, we have

$$\begin{aligned}&\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1\int_\Omega |\nabla \boldsymbol{e}|^2|\boldsymbol{w}|^2 dx \\ &\geq K_1\int_\Omega \{|\operatorname{div} \boldsymbol{w}|^2 + |\operatorname{curl} \boldsymbol{w}|^2\}dx \\ &\quad - K_1 \max_{x\in\overline{\Omega}} |\nabla \boldsymbol{e}|^2 \int_\Omega |\boldsymbol{w}|^2 dx \\ &\geq K_1\int_\Omega |\nabla \boldsymbol{w}|^2 dx - K_1 \max_{x\in\overline{\Omega}} |\nabla \boldsymbol{e}|^2 c(\Omega)\int_\Omega |\nabla \boldsymbol{w}|^2 dx \\ &\geq K_1(1 - c(\Omega)\max_{x\in\overline{\Omega}} |\nabla \boldsymbol{e}|^2)\int_\Omega |\nabla \boldsymbol{w}|^2 dx \\ &\geq c_1 K_1(1 - c(\Omega)\max_{x\in\overline{\Omega}} |\nabla \boldsymbol{e}|^2)\|\boldsymbol{w}\|^2_{W_0^{1,2}(\Omega, \mathbb{R}^3)}\end{aligned}$$

for some positive constant c_1. Thus (3.3) holds with

$$c = c_1 K_1 (1 - c(\Omega) \max_{x \in \overline{\Omega}} |\nabla \boldsymbol{e}|^2). \ \square$$

Definition 3.2. *For $q \geq 0, \kappa > 0, K_1 > 0, K_2 > 0$ and $K_3 > 0$, define $H_{sh} = H_{sh}(q, \kappa, K_1, K_2, K_3, \Omega, \boldsymbol{h}, \boldsymbol{e})$ by*

$$H_{sh}^2 = \frac{1}{\chi_a} \inf \Big\{ \frac{B(u, \boldsymbol{w})}{\|\boldsymbol{h} \cdot \boldsymbol{w}\|_{L^2(\Omega)}^2}; (u, \boldsymbol{w}) \in W^{1,2}(\Omega) \times W_0^{1,2}(\Omega, \mathbb{R}^3), \\ \boldsymbol{w}(x) \cdot \boldsymbol{e}(x) = 0 \ \text{a.e. in } \Omega, \boldsymbol{h} \cdot \boldsymbol{w}(x) \not\equiv 0 \ \text{in } \Omega \Big\}.$$

From the above arguments, we have the following lemma.

Lemma 3.3. *If $\sigma < H_{sh}$, then we see that the pure smectic state is weakly stable and if $\sigma > H_{sh}$, then the pure smectic state is not weakly stable.*

Proof. If $\sigma < H_{sh}$, then

$$\sigma^2 \chi_a \|\boldsymbol{h} \cdot \boldsymbol{w}\|_{L^2(\Omega)}^2 < B(u, \boldsymbol{w}) \tag{3.4}$$

for all $(u, \boldsymbol{w}) \in W^{1,2}(\Omega) \times W_0^{1,2}(\Omega, \mathbb{R}^3)$ with $\boldsymbol{w}(x) \cdot \boldsymbol{e}(x) = 0$ a.e. in Ω and $\boldsymbol{h} \cdot \boldsymbol{w}(x) \not\equiv 0$. For any $\phi \in W^{1,2}(\Omega, \mathbb{C})$ and any $\boldsymbol{v} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$, we define ψ_t and $\boldsymbol{n}_t$ by (2.4). We show that

$$\mathcal{E}[\psi_0, \boldsymbol{n}_0] \leq \mathcal{E}[\psi_t, \boldsymbol{n}_t] \tag{3.5}$$

for small $|t| > 0$.

In the case where $\boldsymbol{v} = 0$, we see that $\boldsymbol{n}_t = \boldsymbol{e}$. Thus we have

$$\begin{aligned} \mathcal{E}[\psi_t, \boldsymbol{n}_t] &= \mathcal{G}[\psi_t, \boldsymbol{n}_t] + \mathcal{F}[\boldsymbol{n}_t] - \chi_a \sigma^2 \int_\Omega (\boldsymbol{h} \cdot \boldsymbol{n}_t)^2 dx \\ &= \mathcal{G}[\psi_t, \boldsymbol{e}] + \mathcal{F}[\boldsymbol{e}] \\ &\geq K_1 \int_\Omega |\operatorname{div} \boldsymbol{e}|^2 dx = \mathcal{E}[\psi_0, \boldsymbol{n}_0]. \end{aligned} \tag{3.6}$$

Thus we see that (3.5) holds for small t.

In the case where $\boldsymbol{v} \neq 0$, we may assume that $|\boldsymbol{v}| = 1$. When $\boldsymbol{v} = \pm \boldsymbol{e}$, then $\boldsymbol{w} = \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{e})\boldsymbol{e} = 0$ and $\boldsymbol{n}_t = \boldsymbol{e}$. Thus since (3.6) holds, we see that (3.5) holds. When $\boldsymbol{v} \neq \pm \boldsymbol{e}$, $\boldsymbol{w} = \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{e})\boldsymbol{e} \neq 0$. If $\boldsymbol{h} \cdot \boldsymbol{w}(x) \equiv 0$, putting $u = -i \frac{1}{cq} e^{-iq\varphi} \phi$, it follows from (3.3) and the Poincaré inequality that

$$B(u, \boldsymbol{w}) \geq c \int_\Omega |\nabla \boldsymbol{w}|^2 dx \geq c_1 \int_\Omega |\boldsymbol{w}|^2 dx > 0 = \int_\Omega (\boldsymbol{h} \cdot \boldsymbol{w})^2 dx.$$

Thus it follows from (2.7) that (3.5) holds. If $\boldsymbol{h}(x) \cdot \boldsymbol{w}(x) \neq 0$, using (3.4) we can see that (3.5) also holds.

If $\sigma > H_{sh}$, there exists $(u, \boldsymbol{w}) \in W^{1,2}(\Omega) \times W_0^{1,2}(\Omega, \mathbb{R}^3)$ with $\boldsymbol{w}(x) \cdot \boldsymbol{h}(x) \not\equiv 0$ and $\boldsymbol{w}(x) \cdot \boldsymbol{e}(x) = 0$ in Ω such that

$$B(u, \boldsymbol{w}) < \chi_a \sigma^2 \|\boldsymbol{h} \cdot \boldsymbol{w}\|_{L^2(\Omega)}^2.$$

It follows from Lemma 2.5 (ii) that $(\psi_0, \boldsymbol{n}_0)$ is not weakly stable. □

For a further simple expression of (3.2), let $(u, \boldsymbol{w})$ be a minimizer of H_{sh}. Then u satisfies the equation

$$\begin{cases} \Delta u = \operatorname{div} \boldsymbol{w} & \text{in } \Omega, \\ \frac{\partial u}{\partial \boldsymbol{\nu}} = \boldsymbol{w} \cdot \boldsymbol{\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.7}$$

If we impose $\int_\Omega u dx = 0$, the solution of (3.7) is unique. We write $u = \xi_{\boldsymbol{w}}$. Then it is well known that $\xi_{\boldsymbol{w}}$ is a minimizer of

$$\omega(\boldsymbol{w}) = \inf_{\xi \in W^{1,2}(\Omega)} \frac{1}{|\Omega|} \int_\Omega |\nabla \xi - \boldsymbol{w}|^2 dx.$$

Hence $\xi_{\boldsymbol{w}}$ satisfies

$$\int_\Omega |\nabla \xi_{\boldsymbol{w}} - \boldsymbol{w}|^2 dx = \omega(\boldsymbol{w})|\Omega|.$$

Write $B(\boldsymbol{w}) = B(\xi_{\boldsymbol{w}}, \boldsymbol{w})$. It is clear that for any $b > 0$, $\xi_{b\boldsymbol{w}} = b\xi_{\boldsymbol{w}}$, and so $B(b\boldsymbol{w}) = b^2 B(\boldsymbol{w})$. Therefore we can write

$$H_{sh}^2 = \frac{1}{\chi_a} \inf\{B(\boldsymbol{w}); \boldsymbol{w} \in W_0^{1,2}(\Omega, \mathbb{R}^3), \boldsymbol{w}(x) \cdot \boldsymbol{e}(x) = 0 \text{ a.e. in } \Omega, \\ \|\boldsymbol{h} \cdot \boldsymbol{w}\|_{L^2(\Omega)} = 1\}.$$

Here we note that the pure smectic state involves a complex number c, but $B(u, \boldsymbol{w})$ and $B(\boldsymbol{w})$ are independent of c. From now we write H_{sh} by $H_{sh}(q)$. Then we see that

$$H_{sh}^2(0) = \frac{1}{\chi_a} \inf\{\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{w}|^2 dx; \boldsymbol{w} \in W_0^{1,2}(\Omega, \mathbb{R}^3), \\ \boldsymbol{w}(x) \cdot \boldsymbol{e}(x) = 0 \text{ a.e. in } \Omega, \|\boldsymbol{h} \cdot \boldsymbol{w}\|_{L^2(\Omega)} = 1\}.$$

PROPOSITION 3.4. *Assume that* Ω *is simply connected domain with smooth boundary, and* (H.1), (H.2), (H.3) *and* (H.4) *hold. Then* $H_{sh}(q) > 0$ *is achieved. For fixed* κ, K_1, K_2, K_3, Ω, $\boldsymbol{h}$ *and* $\boldsymbol{e}$, *we have*

$$\lim_{q \to +\infty} H_{sh}(q) = +\infty.$$

Proof. Step 1. Let $\boldsymbol{w}_j \in W_0^{1,2}(\Omega, \mathbb{R}^3)$, $\xi_j = \xi_{\boldsymbol{w}_j}$ satisfy $\boldsymbol{w}_j(x) \cdot \boldsymbol{e}(x) = 0$ a.e. in Ω, $\|\boldsymbol{h} \cdot \boldsymbol{w}_j\|_{L^2(\Omega)} = 1$ and $B(\boldsymbol{w}_j) \to \chi H_{sh}^2(q)$ as $j \to \infty$. Since $\boldsymbol{w}_j \in W_0^{1,2}(\Omega, \mathbb{R}^3)$, it follows from (3.3) that there exist constants $c, C > 0$ such that $\|\boldsymbol{w}_j\|_{W_0^{1,2}(\Omega, \mathbb{R}^3)}^2 \le cB(\boldsymbol{w}_j) \le C$. Thus passing to a subsequence, we may assume that $\boldsymbol{w}_j \to \boldsymbol{w}_0$ weakly in $W_0^{1,2}(\Omega, \mathbb{R}^3)$, strongly in $L^2(\Omega, \mathbb{R}^3)$ and a.e. in Ω. Hence $\boldsymbol{w}_0(x) \cdot \boldsymbol{e}(x) = 0$ a.e. in Ω. Since

$$\int_\Omega |\nabla \xi_j - \boldsymbol{w}_j|^2 dx \le \frac{1}{q^2} B(\boldsymbol{w}_j) \le C,$$

we see that $\|\nabla \xi_j\|_{L^2(\Omega, \mathbb{R}^3)}$ is bounded. Since $\int_\Omega \xi_j dx = 0$, again applying the Poincaré inequality, we see that $\{\xi_j\}$ is bounded in $W^{1,2}(\Omega, \mathbb{R})$. Moreover, since

$\|\operatorname{div} \boldsymbol{w}_j\|_{L^2(\Omega)} \le C\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} \le C$, $\boldsymbol{w}_j = 0$ on $\partial\Omega$ and $\xi_j = \xi_{\boldsymbol{w}_j}$ is a unique solution of (3.7), it follows from the elliptic estimate that $\{\xi_j\}$ is bounded in $W^{2,2}(\Omega)$. After passing to a subsequence, we may assume that $\xi_j \to \xi_0$ weakly in $W^{2,2}(\Omega)$ and strongly in $W^{1,2}(\Omega)$. Then ξ_0 satisfies (3.7) for $\boldsymbol{w} = \boldsymbol{w}_0$, i.e., $\xi_0 = \xi_{\boldsymbol{w}_0}$. Therefore,

$$B(\boldsymbol{w}_0) = B(\xi_{\boldsymbol{w}_0}, \boldsymbol{w}_0) \le \liminf_{j\to\infty} B(\xi_j, \boldsymbol{w}_j) = \chi_a H_{sh}^2(q).$$

Since $\|\boldsymbol{w}_0 \cdot \boldsymbol{h}\|_{L^2(\Omega)} = \lim_{j\to\infty} \|\boldsymbol{w}_j \cdot \boldsymbol{h}\|_{L^2(\Omega)} = 1$, we see that $B(\boldsymbol{w}_0) \ge \chi H_{sh}^2(q)$. Thus $\boldsymbol{w}_0$ is a minimizer of $B(\boldsymbol{w})$, so $(\xi_{\boldsymbol{w}_0}, \boldsymbol{w}_0)$ achieves $H_{sh}(q)$.

We show that $H_{sh}(q) > 0$. If $H_{sh}(q) = 0$, then $B(\boldsymbol{w}_0) = 0$. So using Lemma 3.1 we see that $\boldsymbol{w}_0 = 0$. This contradicts the fact that $\|\boldsymbol{h} \cdot \boldsymbol{w}_0\|_{L^2(\Omega)} = 1$.

Step 2. Suppose that $H_{sh}(q) \le c$ for all $q \ge 0$. Choose $q_j \to \infty$ and choose $u_j \in W^{1,2}(\Omega)$ and $\boldsymbol{w}_j \in W_0^{1,2}(\Omega, \mathbb{R}^3)$ such that

$$\int_\Omega u_j dx = 0, \boldsymbol{e}(x) \cdot \boldsymbol{w}_j(x) = 0 \text{ a.e. in } \Omega, \|\boldsymbol{h} \cdot \boldsymbol{w}_j\|_{L^2(\Omega)} = 1,$$

and $(u_j, \boldsymbol{w}_j)$ achieves $H_{sh}(q_j)$. Then

$$\int_\Omega q_j^2 |\nabla u_j - \boldsymbol{w}_j|^2 dx + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] - K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{w}_j|^2 dx$$
$$\le \chi_a c^2 \int_\Omega (\boldsymbol{h} \cdot \boldsymbol{w}_j)^2 dx = \chi_a c^2.$$

Thus from (3.3) , $\{\boldsymbol{w}_j\}$ is bounded in $W_0^{1,2}(\Omega, \mathbb{R}^3)$. Passing to a subsequence, we may assume that $\boldsymbol{w}_j \to \widehat{\boldsymbol{w}}$ weakly in $W_0^{1,2}(\Omega, \mathbb{R}^3)$, strongly in $L^4(\Omega, \mathbb{R}^3)$ and a.e. in Ω. Hence this implies that $\|\boldsymbol{h} \cdot \widehat{\boldsymbol{w}}\|_{L^2(\Omega)} = 1$ and $\widehat{\boldsymbol{w}}(x) \cdot \boldsymbol{e}(x) = 0$ a.e. in Ω. Since $\|\nabla u_j - \boldsymbol{w}_j\|_{L^2(\Omega)} = O(q_j^{-1})$ and $\boldsymbol{w}_j \to \widehat{\boldsymbol{w}}$ strongly in $L^2(\Omega, \mathbb{R}^3)$, $\{\nabla u_j\}$ is bounded in $L^2(\Omega, \mathbb{R}^3)$. Since $\int_\Omega u_j dx = 0$, it follows from the Poincaré inequality that $\{u_j\}$ is bounded in $W^{1,2}(\Omega)$. Passing to a subsequence, we may assume that $u_j \to \widehat{u}$ weakly in $W^{1,2}(\Omega)$ and strongly in $L^2(\Omega)$. Since $\nabla u_j \to \nabla \widehat{u}$ weakly in $L^2(\Omega, \mathbb{R}^3)$ and $\nabla u_j \to \widehat{\boldsymbol{w}}$ strongly in $L^2(\Omega, \mathbb{R}^3)$, we have $\nabla \widehat{u} = \widehat{\boldsymbol{w}}$ and $\nabla u_j \to \nabla \widehat{u}$ strongly in $L^2(\Omega, \mathbb{R}^3)$. Thus we see that $u_j \to \widehat{u}$ strongly in $W^{1,2}(\Omega)$. Moreover, $\nabla \widehat{u} = \widehat{\boldsymbol{w}} = 0$ on $\partial\Omega$ and $\nabla \widehat{u} \cdot \boldsymbol{e} = 0$ in Ω. By the hypothesis (H.4), we see that $\nabla \widehat{u} = 0$ in Ω. In fact, assume that $\nabla \widehat{u}(p) \neq 0$ for some point $p \in \Omega$. Let $\boldsymbol{x} = \boldsymbol{x}(t)$ be the integral curve of $\boldsymbol{e}$ through p. Then since

$$\frac{d}{dt} \nabla \widehat{u}(\boldsymbol{x}(t)) = \nabla(\nabla \widehat{u}(\boldsymbol{x}(t)) \cdot \boldsymbol{e}(\boldsymbol{x}(t)) = 0,$$

$\nabla \widehat{u}(\boldsymbol{x}(t))$ is independent of t. By the hypothesis (H.4), $\boldsymbol{x}(t)$ intersects with $\partial\Omega$. This contradicts the fact that $\nabla \widehat{u} = 0$ on $\partial\Omega$. Thus $\nabla \widehat{u} = 0$ in Ω and so $\widehat{\boldsymbol{w}} = \nabla \widehat{u} = 0$ in Ω. This contradicts $\|\boldsymbol{h} \cdot \widehat{\boldsymbol{w}}\|_{L^2(\Omega)} = 1$. □

We shall derive the Euler-Lagrange equation for the minimizer of $H_{sh}(q)$. Let $(\xi_{\boldsymbol{w}}, \boldsymbol{w})$ be a minimizer of $H_{sh}(q)$. Then we see that $\boldsymbol{w}$ satisfies

$$H_{sh}^2(q) = \frac{1}{\chi_a} \frac{q^2 \|\nabla \xi_{\boldsymbol{w}} - \boldsymbol{w}\|_{L^2(\Omega,\mathbb{R}^3)}^2 + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{w}|^2 dx}{\|\boldsymbol{h} \cdot \boldsymbol{w}\|_{L^2(\Omega)}^2}.$$

For $\boldsymbol{v} \in W^{1,2}(\Omega, \mathbb{R}^3)$ with $\boldsymbol{v}(x) \cdot \boldsymbol{e}(x) = 0$, since $\xi_{\boldsymbol{w}+t\boldsymbol{v}} = \xi_{\boldsymbol{w}} + t\xi_{\boldsymbol{v}}$ and

$$\int_\Omega (\boldsymbol{h} \cdot (\boldsymbol{w} + t\boldsymbol{v}))^2 dx = \int_\Omega (\boldsymbol{h} \cdot \boldsymbol{w})^2 dx + 2t \int_\Omega (\boldsymbol{h} \cdot \boldsymbol{w})(\boldsymbol{h} \cdot \boldsymbol{v}) dx + O(t^2),$$

we have

$$\begin{aligned}
&\|\nabla\xi_{\boldsymbol{w}+t\boldsymbol{v}}-(\boldsymbol{w}+t\boldsymbol{v})\|^2_{L^2(\Omega,\mathbb{R}^3)}=\|\nabla\xi_{\boldsymbol{w}}-\boldsymbol{w}\|^2_{L^2(\Omega,\mathbb{R}^3)}\\
&\quad+2t\int_\Omega(\nabla\xi_{\boldsymbol{w}}-\boldsymbol{w})\cdot(\nabla\xi_{\boldsymbol{v}}-\boldsymbol{v})dx+O(t^2),\\
&\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}+t\boldsymbol{v}]-K_1\int_\Omega|\nabla\boldsymbol{e}|^2|\boldsymbol{w}+t\boldsymbol{v}|^2dx\\
&\quad=\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}]-K_1\int_\Omega|\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2dx\\
&\quad+2t\int_\Omega\{K_1(\operatorname{div}\boldsymbol{w})(\operatorname{div}\boldsymbol{v})+K_2(\boldsymbol{e}\cdot\operatorname{curl}\boldsymbol{w})(\boldsymbol{e}\cdot\operatorname{curl}\boldsymbol{v})\\
&\quad+K_3(\boldsymbol{e}\times\operatorname{curl}\boldsymbol{w})\cdot(\boldsymbol{e}\times\operatorname{curl}\boldsymbol{v})-K_1|\nabla\boldsymbol{e}|^2(\boldsymbol{w}\cdot\boldsymbol{v})\}dx+O(t^2).
\end{aligned}$$

Here we note that

$$(\boldsymbol{e}\times\operatorname{curl}\boldsymbol{w})\cdot(\boldsymbol{e}\times\operatorname{curl}\boldsymbol{v})=\operatorname{curl}\boldsymbol{w}\cdot\operatorname{curl}\boldsymbol{v}-(\operatorname{curl}\boldsymbol{w}\cdot\boldsymbol{e})(\operatorname{curl}\boldsymbol{v}\cdot\boldsymbol{e}).$$

Thus if we define $\boldsymbol{k}(x)=\boldsymbol{h}(x)\times\boldsymbol{e}(x)$ and an orthogonal projection P_x onto the space $[\boldsymbol{k}(x),\boldsymbol{h}(x)]$ spanned by $\boldsymbol{k}(x)$ and $\boldsymbol{h}(x)$, we get the Euler-Lagrange equation

$$\begin{cases}
P_x\big[-K_1\nabla(\operatorname{div}\boldsymbol{w})-K_2\boldsymbol{e}\times\nabla(\boldsymbol{e}\cdot\operatorname{curl}\boldsymbol{w})\\
+K_3\big(\operatorname{curl}^2\boldsymbol{w}+\boldsymbol{e}\times\nabla(\boldsymbol{e}\cdot\operatorname{curl}\boldsymbol{w})\big)\\
+q^2(\boldsymbol{w}-\nabla\xi_{\boldsymbol{w}})-K_1|\nabla\boldsymbol{e}|^2\boldsymbol{w}\big]\\
=\chi H^2_{sh}(q)(\boldsymbol{h}\cdot\boldsymbol{w})\boldsymbol{h} & \text{in } \Omega,\\
\boldsymbol{w}=0 & \text{on } \partial\Omega.
\end{cases}$$

In particular case where $K_1=K_2=K_3=K$, since $\operatorname{curl}^2\boldsymbol{w}=-\Delta\boldsymbol{w}+\nabla\operatorname{div}\boldsymbol{w}$, we have

$$\begin{cases}
P_x[-K\Delta\boldsymbol{w}+q^2(\boldsymbol{w}-\nabla\xi_{\boldsymbol{w}})-K_1|\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2]=\chi_a H^2_{sh}(\boldsymbol{h}\cdot\boldsymbol{w})\boldsymbol{h} & \text{in } \Omega,\\
\boldsymbol{w}=0 \quad \text{on } \partial\Omega & .
\end{cases}$$

4. Loss of global minimality of pure smectic states. In this section, we examine loss of global minimality of pure smectic states. In order to do so, let $(\psi_0,\boldsymbol{n}_0)$ be a pure smectic state. That is to say, $\psi_0=ce^{iq\varphi},\boldsymbol{n}_0=\boldsymbol{e}$ with $c\in\mathbb{C},|c|=1$ and φ is the function as in (1.5).

If a global minimizer $(\psi,\boldsymbol{n})\in W^{1,2}(\Omega,\mathbb{C})\times W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$ is not a pure smectic state, then we claim that

$$\boldsymbol{h}\cdot\boldsymbol{n}\not\equiv 0 \quad \text{in } \Omega. \tag{4.1}$$

In fact, if $\boldsymbol{h}\cdot\boldsymbol{n}\equiv 0$ in Ω,

$$\mathcal{E}[\psi,\boldsymbol{n}]=\mathcal{G}[\psi,\boldsymbol{n}]+\mathcal{F}[\boldsymbol{n}]\le\mathcal{E}[\psi_0,\boldsymbol{n}_0]=\int_\Omega K_1|\operatorname{div}\boldsymbol{e}|^2dx.$$

Hence since $\mathcal{F}[\boldsymbol{n}]\le K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}=\mathcal{F}[\boldsymbol{e}]$, we have $\boldsymbol{n}=\boldsymbol{e}$ from (H.1). Moreover, we have

$$0=\mathcal{G}[\psi,\boldsymbol{n}]=\int_\Omega\Big\{|\nabla\psi-iq\boldsymbol{n}\psi|^2+\frac{\kappa^2}{2}(1-|\psi|^2)^2\Big\}dx.$$

Therefore, since $|\psi| = 1$, we can write $\psi = ce^{iq\widetilde{\varphi}(x)}$ with $|c| = 1$ locally for some function $\widetilde{\varphi}$. Therefore, $0 = \nabla\psi - iq\boldsymbol{e}\psi = icq(\nabla\widetilde{\varphi} - \boldsymbol{e})e^{iq\widetilde{\varphi}}$. Thus $\nabla\widetilde{\varphi} = \boldsymbol{e}$, so we can write $\psi = ce^{iq\varphi}$ with $|c| = 1$ locally where φ is the function as in (1.5). Since Ω is connected, $\psi = ce^{iq\varphi}$ in Ω. Then $(\psi, \boldsymbol{n}) = (\psi_0, \boldsymbol{n}_0)$ is a pure smectic state. Hence (4.1) holds.

Thus if $(\psi, \boldsymbol{n})$ is a global minimizer of $\mathcal{E}$ which is not a pure smectic state, we have

$$\mathcal{G}[\psi, \boldsymbol{n}] + \mathcal{F}[\boldsymbol{n}] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)} \le \chi_a\sigma^2\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}.$$

Definition 4.1. *For given $q \ge 0, \kappa > 0, K_1, K_2, K_3 > 0$ and $\boldsymbol{h}, \boldsymbol{e}_0$, define $H_s = H_s(q, \kappa, K_1, K_2, K_3, \Omega, \boldsymbol{h}, \boldsymbol{e}_0)$ by*

$$(4.2)\quad H_s^2 = \frac{1}{\chi_a}\inf\left\{\frac{\mathcal{G}[\psi, \boldsymbol{n}] + \mathcal{F}[\boldsymbol{n}] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}}{\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}};\right.$$
$$\left.(\psi, \boldsymbol{n}) \in W^{1,2}(\Omega, \mathbb{C}) \times W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0), \boldsymbol{h}(x)\cdot\boldsymbol{n}(x) \not\equiv 0 \text{ in } \Omega\right\}$$

Note that since $\mathcal{F}[\boldsymbol{n}] \ge \mathcal{F}[\boldsymbol{e}] = K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}$, the definition of H_s^2 has a meaning.

Then we have the following lemma.

Lemma 4.2. *Under the assumptions* (H.1), (H.2), (H.3) *and* (H.4), *we have following.*

(i) *If there exists a global minimizer $(\psi, \boldsymbol{n})$ of $\mathcal{E}$ which is not a pure smectic state, then $\sigma \ge H_s$.*

(ii) *If $\sigma > H_s$, then the global minimizers of $\mathcal{E}$ are not pure smectic states.*

(iii) *If $0 \le \sigma < H_s$, then the only global minimizers of $\mathcal{E}$ are pure smectic states.*

Proof. (i) If $(\psi, \boldsymbol{n})$ is a global minimizer of $\mathcal{E}$ which is not a pure smectic state, then from (4.1) $\boldsymbol{h}\cdot\boldsymbol{n} \not\equiv 0$ in Ω and $\mathcal{E}[\psi, \boldsymbol{n}] \le \mathcal{E}[\psi_0, \boldsymbol{n}_0] = K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}$. Therefore, we have

$$\mathcal{G}[\psi, \boldsymbol{n}] + \mathcal{F}[\boldsymbol{n}] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)} \le \chi_a\sigma^2\int_\Omega(\boldsymbol{h}\cdot\boldsymbol{n})^2dx.$$

This implies that $H_s \le \sigma$.

(ii) If $\sigma > H_s$, there exists $(\psi, \boldsymbol{n}) \in W^{1,2}(\Omega, \mathbb{C}) \times W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ with $\boldsymbol{h}\cdot\boldsymbol{n} \not\equiv 0$ in Ω and

$$\frac{1}{\chi_a}\frac{\mathcal{G}[\psi, \boldsymbol{n}] + \mathcal{F}[\boldsymbol{n}] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}}{\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}} < \sigma^2.$$

Thus we have

$$\mathcal{E}[\psi, \boldsymbol{n}] = \mathcal{G}[\psi, \boldsymbol{n}] + \mathcal{F}[\boldsymbol{n}] - \chi_a\sigma^2\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)} < \mathcal{E}[\psi_0, \boldsymbol{n}_0] = K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}.$$

This implies that global minimizers of $\mathcal{E}$ are not pure smectic states.

(iii) If $0 \le \sigma < H_s$ and there exists a global minimizer which is not a pure smectic state, it follows from (i) that $\sigma \ge H_s$. ☐

In the following we write H_s by $H_s(\kappa,q)$. Since pure smectic states lose the global minimality at $H_s(\kappa,q)$ and lose local minimality at $H_{sh}(q)$, we see that

$$H_s(\kappa,q) \le H_{sh}(q). \tag{4.3}$$

We define a number H_n which is closely related to H_s.

Definition 4.3. *$H_n = H_n(q) = H_n(q,\kappa,K_1,K_2,K_3,\Omega,\boldsymbol{h},\boldsymbol{e}_0)$ is defined by*

$$H_n^2(q) = \frac{1}{\chi_a}\inf\Big\{\frac{q^2\|\nabla u-\boldsymbol{n}\|^2_{L^2(\Omega,\mathbb{R}^3)}+\mathcal{F}[\boldsymbol{n}]-K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}}{\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}};$$
$$(u,\boldsymbol{n})\in W^{1,2}(\Omega)\times W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0), \boldsymbol{h}\cdot\boldsymbol{n}\not\equiv 0 \text{ in } \Omega\Big\}$$

We note that

$$H_n^2(0) = \frac{1}{\chi_a}\inf\Big\{\frac{\mathcal{F}[\boldsymbol{n}]-K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}}{\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}};\ \boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0), \boldsymbol{h}\cdot\boldsymbol{n}\not\equiv 0 \text{ in } \Omega\Big\}.$$

Lemma 4.4. *For any $\kappa>0$, we have $H_s(\kappa,0)=H_n(0)$, and for any $\kappa>0$ and $q\ge 0$, we have $H_s(\kappa,q)\ge H_n(0)$.*

Proof. We choose a test field $\psi=1$ and any $\boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$ with $\boldsymbol{h}\cdot\boldsymbol{n}\not\equiv 0$ in Ω, we see that $\mathcal{G}[\psi,\boldsymbol{n}]\big|_{q=0}=0$. Thus $H_s^2(\kappa,0)\le H_n^2(0)$. On the other hand, for any $\kappa>0$ and $q\ge 0$, it is easily seen that

$$\begin{aligned} H_s^2(\kappa,q) &\ge \frac{1}{\chi_a}\inf\Big\{\frac{\mathcal{F}[\boldsymbol{n}]-K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}}{\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}};\boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0),\ \boldsymbol{h}\cdot\boldsymbol{n}\not\equiv 0\Big\}\\ &= H_n^2(0). \end{aligned}$$

$\square$

We state the relations of H_s, H_n and H_{sh} and that if they are achieved.

Theorem 4.5. *Let Ω be a simply connected bounded domain in $\mathbb{R}^3$ with smooth boundary and assume that* (H.1), (H.2), (H.3) *and* (H.4) *hold. Then we can get the following.*

(i) *For any $\kappa>0$ and any $q\ge 0$,*

$$0<H_s(\kappa,q)\le H_n(q)\le H_{sh}(q).$$

(ii) *For any $\kappa>0$ and any $q\ge 0$, if $H_s(\kappa,q)<H_{sh}(q)$, then $H_s(\kappa,q)$ is achieved.*

(iii) *For any $q\ge 0$, if $H_n(q)<H_{sh}(q)$, then $H_n(q)$ is achieved. In this case, we have $H_s(\kappa,q)<H_{sh}(q)$, so $H_s(\kappa,q)$ is achieved.*

Proof. From now, we denote various constants by c, C, C_1 which may vary from line to line.

(i) Step 1. We show that $H_s(\kappa,q)\le H_n(q)$.

For any $\phi\in W^{1,2}(\Omega)$ and any $\boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$ with $\boldsymbol{n}\cdot\boldsymbol{h}\not\equiv 0$ in Ω, we take $(e^{iq\phi},\boldsymbol{n})$ as a test function of $H_s(\kappa,q)$. Then we have

$$H_s^2(\kappa,q)\le\frac{1}{\chi_a}\frac{q^2\|\nabla\phi-\boldsymbol{n}\|^2_{L^2(\Omega,\mathbb{R}^3)}+\mathcal{F}[\boldsymbol{n}]-K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}}{\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}}.$$

This implies that $H_s^2(\kappa, q) \le H_n^2(q)$.

Step 2. We show that $H_n(q) \le H_{sh}(q)$.

Let $u \in W^{1,2}(\Omega)$, $\boldsymbol{w} \in W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$ with $\boldsymbol{e} \cdot \boldsymbol{w} \equiv 0$ in Ω and $\boldsymbol{h} \cdot \boldsymbol{w} \not\equiv 0$ in Ω and put

$$\phi_t = \varphi + tu, \quad \boldsymbol{n}_t = \frac{\boldsymbol{e} + t\boldsymbol{w}}{|\boldsymbol{e} + t\boldsymbol{w}|} = \boldsymbol{e} + t\boldsymbol{w} + O(t^2)$$

where φ is as in (1.5). Then we have

$$q^2 \|\nabla \phi_t - \boldsymbol{n}_t\|^2_{L^2(\Omega, \mathbb{R}^3)} = t^2 q^2 \|\nabla u - \boldsymbol{w}\|^2_{L^2(\Omega, \mathbb{R}^3)} + O(t^3),$$

$$\mathcal{F}[\boldsymbol{n}_t] = \mathcal{F}[\boldsymbol{e}] + t^2 \left\{ \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] + 2K_1 \int_\Omega (\operatorname{div} \boldsymbol{e})(\operatorname{div} \boldsymbol{n}_2) dx \right\} + O(t^3),$$

$$\int_\Omega (\boldsymbol{h} \cdot \boldsymbol{n}_t)^2 dx = t^2 \int_\Omega (\boldsymbol{h} \cdot \boldsymbol{w})^2 dx + O(t^3).$$

Since $\boldsymbol{n}_2 = -\frac{1}{2}|\boldsymbol{w}|^2 \boldsymbol{e}$, it follows from the hypothesis (H.1) that

$$2K_1 \int_\Omega (\operatorname{div} \boldsymbol{e})(\operatorname{div} \boldsymbol{n}_2) dx = K_1 \int_\Omega \nabla(\operatorname{div} \boldsymbol{e}) \cdot |\boldsymbol{w}|^2 \boldsymbol{e} dx = -K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{w}|^2 dx.$$

Therefore,

$$\begin{aligned}
&\frac{q^2 \|\nabla \phi_t - \boldsymbol{n}_t\|^2_{L^2(\Omega, \mathbb{R}^3)} + \mathcal{F}[\boldsymbol{n}_t] - \mathcal{F}[\boldsymbol{e}]}{\|\boldsymbol{h} \cdot \boldsymbol{n}_t\|^2_{L^2(\Omega)}} \\
&\quad = \frac{q^2 \|\nabla u - \boldsymbol{w}\|^2_{L^2(\Omega, \mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] + 2K_1 \int_\Omega (\operatorname{div} \boldsymbol{e})(\operatorname{div} \boldsymbol{n}_2) dx}{\|\boldsymbol{h} \cdot \boldsymbol{w}\|^2_{L^2(\Omega)}} + O(t) \\
&\qquad = \frac{q^2 \|\nabla u - \boldsymbol{w}\|^2_{L^2(\Omega, \mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{w}|^2 dx}{\|\boldsymbol{h} \cdot \boldsymbol{w}\|^2_{L^2(\Omega)}} + O(t).
\end{aligned}$$

Hence, we have

$$\begin{aligned}
\chi_a H_n^2(q) &\le \frac{q^2 \|\nabla \phi_t - \boldsymbol{n}_t\|^2_{L^2(\Omega, \mathbb{R}^3)} + \mathcal{F}[\boldsymbol{n}_t] - \mathcal{F}[\boldsymbol{e}]}{\|\boldsymbol{h} \cdot \boldsymbol{n}_t\|^2_{L^2(\Omega)}} \\
&= \frac{q^2 \|\nabla u - \boldsymbol{w}\|^2_{L^2(\Omega, \mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{w}|^2 dx}{\|\boldsymbol{h} \cdot \boldsymbol{w}\|^2_{L^2(\Omega)}} + O(t).
\end{aligned}$$

Letting $t \to 0$, we have

$$\chi_a H_n^2(q) \le \frac{q^2 \|\nabla u - \boldsymbol{w}\|^2_{L^2(\Omega, \mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}] - K_1 \int_\Omega |\nabla \boldsymbol{e}|^2 |\boldsymbol{w}|^2 dx}{\|\boldsymbol{h} \cdot \boldsymbol{w}\|^2_{L^2(\Omega)}}.$$

Since $W_0^{1,2}(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$ is dense in $W_0^{1,2}(\Omega, \mathbb{R}^3)$, we get $H_n(q) \le H_{sh}(q)$.

Step 3. We show that $H_s(\kappa, q) > 0$.

Since $H_{sh}(q) > 0$ from Proposition 3.4, if $H_s(\kappa, q) = H_{sh}(q)$, the result is trivial. So we assume that $H_s(\kappa, q) < H_{sh}(q)$. We borrow the result of (ii) which is proved independently of (i). Since $H_s(\kappa, q)$ is achieved, let $(\psi, \boldsymbol{n})$ be a minimizer of $H_s(\kappa, q)$. Assume that $H_s(\kappa, q) = 0$. Then we have

$$0 \le \mathcal{G}[\psi, \boldsymbol{n}] + \mathcal{F}[\boldsymbol{n}] - K_1 \|\operatorname{div} \boldsymbol{e}\|^2_{L^2(\Omega)} = \chi_a H_s^2(\kappa, q) \|\boldsymbol{h} \cdot \boldsymbol{n}\|^2_{L^2(\Omega)} = 0,$$

and $\boldsymbol{h} \cdot \boldsymbol{n} \not\equiv 0$ in Ω. This implies that $\mathcal{F}[\boldsymbol{n}] = K_1\|\operatorname{div} \boldsymbol{e}\|^2_{L^2(\Omega)}$. By the hypothesis (H.1), we see that $\boldsymbol{n} = \boldsymbol{e}$ in Ω. This contradicts the fact that $\boldsymbol{h} \cdot \boldsymbol{n} \not\equiv 0$. Thus we see that (i) holds if (ii) is proved independently.

Proof of (ii). We assume that $H_s(\kappa, q) < H_{sh}(q)$.

Step 4. Let $\{(\psi_j, \boldsymbol{n}_j)\}$ be a minimizing sequence of $H_s(\kappa, q)$. Then

$$\mathcal{G}[\psi_j, \boldsymbol{n}_j] + \mathcal{F}[\boldsymbol{n}_j] - K_1\|\operatorname{div} \boldsymbol{e}\|^2_{L^2(\Omega)} = (\chi_a H_s(\kappa, q) + o(1))\|\boldsymbol{h} \cdot \boldsymbol{n}_j\|^2_{L^2(\Omega)} \tag{4.4}$$

Since $|\boldsymbol{h}\cdot\boldsymbol{n}_j| \le 1$, the right hand side of (4.4) is bounded. Thus $\{\operatorname{div} \boldsymbol{n}_j\}$ is bounded in $L^2(\Omega)$, $\{\operatorname{curl} \boldsymbol{n}_j\}$ is bounded in $L^2(\Omega, \mathbb{R}^3)$ and $\boldsymbol{n}_j = \boldsymbol{e}_0$ on $\partial\Omega$. It follows from Dautray and Lions [10, Proposition 6] (or Girault and Raviart [14, Corollary 3.7], Temam [27, Appendix I Proposition 1.4]) that $\{\boldsymbol{n}_j\}$ is bounded in $W^{1,2}(\Omega, \mathbb{R}^3)$. Passing to a subsequence, we may assume that $\boldsymbol{n}_j \to \widehat{\boldsymbol{n}}$ weakly in $W^{1,2}(\Omega, \mathbb{R}^3)$, strongly in $L^2(\Omega, \mathbb{R}^3)$ and a.e. in Ω. Thus we have $|\widehat{\boldsymbol{n}}| = 1$ a.e. in Ω and $\widehat{\boldsymbol{n}} = \boldsymbol{e}_0$ on $\partial\Omega$, so $\widehat{\boldsymbol{n}} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$. On the other hand, we see from (4.4) that $\{\nabla_{q\boldsymbol{n}_j}\psi_j\}$ is bounded in $W^{1,2}(\Omega, \mathbb{C}^3)$ and $\{\psi_j\}$ is bounded in $L^4(\Omega, \mathbb{C})$. Since

$$\|\nabla\psi_j\|_{L^2(\Omega,\mathbb{C}^3)} \le \|\nabla_{q\boldsymbol{n}_j}\psi_j\|_{L^2(\Omega,\mathbb{C}^3)} + \|q\boldsymbol{n}_j\psi_j\|_{L^2(\Omega,\mathbb{C}^3)},$$

we see that $\{\psi_j\}$ is bounded in $W^{1,2}(\Omega, \mathbb{C})$. After passing to a subsequence, we may assume that $\psi_j \to \widehat{\psi}$ weakly in $W^{1,2}(\Omega, \mathbb{C})$ and strongly in $L^4(\Omega, \mathbb{C})$. Thus we have

$$\begin{aligned} &\mathcal{G}[\widehat{\psi}, \widehat{\boldsymbol{n}}] + \mathcal{F}[\widehat{\boldsymbol{n}}] - K_1\|\operatorname{div} \boldsymbol{e}\|^2_{L^2(\Omega)} \\ &\le \liminf_{j\to\infty}\{\mathcal{G}[\psi_j, \boldsymbol{n}_j] + \mathcal{F}[\boldsymbol{n}_j] - K_1\|\operatorname{div} \boldsymbol{e}\|^2_{L^2(\Omega)}\} \\ &= \chi_a H_s^2(\kappa, q)\|\boldsymbol{h} \cdot \widehat{\boldsymbol{n}}\|^2_{L^2(\Omega)}. \end{aligned} \tag{4.5}$$

If we can show that $\boldsymbol{h} \cdot \widehat{\boldsymbol{n}} \not\equiv 0$ in Ω, we see that $H_s(\kappa, q)$ is achieved.

Step 5. Assume that $\boldsymbol{h}\cdot\widehat{\boldsymbol{n}} \equiv 0$ in Ω. Then it follows from (4.5) and the hypothesis (H.1) that $\widehat{\boldsymbol{n}} = \boldsymbol{e}$ in Ω. Moreover, $\nabla\widehat{\psi} - iq\boldsymbol{e}\widehat{\psi} = 0$, $|\widehat{\psi}| = 1$. Then we can write $\widehat{\psi} = ce^{iq\varphi}$ for some $c \in \mathbb{C}$ with $|c| = 1$ where φ is as in (1.5).

Since $\boldsymbol{h} \cdot \boldsymbol{n}_j \not\equiv 0$, we have $\boldsymbol{n}_j \not\equiv \boldsymbol{e}$. We write

$$\boldsymbol{n}_j = \boldsymbol{e} + \varepsilon_j\boldsymbol{w}_j, \quad \psi_j = e^{iq\varphi}(1 + iq\varepsilon_j g_j) \tag{4.6}$$

such that $\varepsilon_j = \|\boldsymbol{n}_j - \boldsymbol{e}\|_{W^{1,2}(\Omega,\mathbb{R}^3)} > 0$, $\boldsymbol{w}_j \in W_0^{1,2}(\Omega, \mathbb{R}^3)$ and $\boldsymbol{w}_j$ satisfies that $\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} = 1$. Using the Poincaré inequality and the formula $|\boldsymbol{e}\cdot\operatorname{curl}\boldsymbol{w}|^2 + |\boldsymbol{e}\times\operatorname{curl}\boldsymbol{w}|^2 = |\operatorname{curl}\boldsymbol{w}|^2$, we have

$$\begin{aligned} \mathcal{F}[\boldsymbol{n}_j] - \mathcal{F}[\boldsymbol{e}] &= \varepsilon_j^2\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] \\ &= \varepsilon_j^2\int_\Omega\{K_1|\operatorname{div}\boldsymbol{w}_j|^2 + K_2|\boldsymbol{e}\cdot\operatorname{curl}\boldsymbol{w}_j|^2 + K_3|\boldsymbol{e}\times\operatorname{curl}\boldsymbol{w}_j|^2\}dx \\ &\ge \varepsilon_j^2K_1\int_\Omega\{|\operatorname{div}\boldsymbol{w}_j|^2 + |\operatorname{curl}\boldsymbol{w}_j|^2\}dx \\ &= C\varepsilon_j^2K_1\int_\Omega|\nabla\boldsymbol{w}_j|^2dx \\ &\ge c\varepsilon_j^2K_1. \end{aligned} \tag{4.7}$$

Thus we have

$$\varepsilon_j^2 \le \frac{1}{cK_1}(\mathcal{F}[\boldsymbol{n}_j] - \mathcal{F}[\boldsymbol{e}]) = o(1)$$

as $j \to \infty$. Since $\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} = 1$, after passing to a subsequence, we may assume that $\boldsymbol{w}_j \to \widehat{\boldsymbol{w}}$ weakly in $W^{1,2}(\Omega,\mathbb{R}^3)$ and strongly in $L^4(\Omega,\mathbb{R}^3)$. Since

$$1 = |\boldsymbol{n}_j|^2 = |\boldsymbol{e} + \varepsilon_j \boldsymbol{w}_j|^2 = 1 + 2\varepsilon_j \boldsymbol{e}\cdot\boldsymbol{w}_j + \varepsilon_j^2|\boldsymbol{w}_j|^2,$$

we have $\boldsymbol{e}\cdot\boldsymbol{w}_j = -\frac{\varepsilon_j}{2}|\boldsymbol{w}_j|^2 \to 0$ strongly in $L^2(\Omega)$, so $\boldsymbol{e}\cdot\widehat{\boldsymbol{w}} = 0$ a.e. in Ω. Since

$$\begin{aligned} |\nabla_{q\boldsymbol{n}_j}\psi_j|^2 &= q^2\varepsilon_j^2|\nabla g_j - (1+iq\varepsilon_j g_j)\boldsymbol{w}_j|^2,\\ |\psi_j|^2 &= 1 + q\varepsilon_j(-2\Im(g_j) + q\varepsilon_j|g_j|^2),\end{aligned}$$

using (4.4), we have

$$\begin{aligned}(4.8)\qquad &(\chi_a H_s^2(\kappa,q) + o(1))\|\boldsymbol{h}\cdot\boldsymbol{w}_j\|^2_{L^2(\Omega)}\\ &= \frac{1}{\varepsilon_j^2}(\chi_a H_s^2(\kappa,q)+o(1))\|\boldsymbol{h}\cdot\boldsymbol{n}_j\|^2_{L^2(\Omega)}\\ &= \frac{1}{\varepsilon_j^2}\mathcal{G}[\psi_j,\boldsymbol{n}_j] + \frac{1}{\varepsilon_j^2}(\mathcal{F}[\boldsymbol{n}_j] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)})\\ &= \int_\Omega\left\{q^2|\nabla g_j - (1+iq\varepsilon_j g_j)\boldsymbol{w}_j|^2 + \frac{\kappa^2q^2}{2}(-2\Im(g_j)+q\varepsilon_j|g_j|^2)^2\right\}dx\\ &\quad + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j].\end{aligned}$$

Thus we have

$$\int_\Omega |\nabla g_j - \psi_j e^{-iq\varphi}\boldsymbol{w}_j|^2dx = \int_\Omega|\nabla g_j - (1+iq\varepsilon_j g_j)\boldsymbol{w}_j|^2dx \le C_1.$$

Therefore,

$$\begin{aligned}\|\nabla g_j\|_{L^2(\Omega,\mathbb{C}^3)} &\le \|\nabla g_j - \psi_j e^{-iq\varphi}\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{C}^3)} + \|\psi_j\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{C}^3)}\\ &\le C + \|\psi_j\|_{W^{1,2}(\Omega,\mathbb{C})}\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} \le C_1.\end{aligned}$$

Put $\widetilde{g}_j = g_j - b_j$ where $b_j = \frac{1}{|\Omega|}\int_\Omega g_j dx$. Since $\int_\Omega \widetilde{g}_j dx = 0$, it follows from the Poincaré inequality that $\|\widetilde{g}_j\|_{L^2(\Omega,\mathbb{C})} \le c(\Omega)\|\nabla g_j\|_{L^2(\Omega,\mathbb{C}^3)} \le C$, so $\|\widetilde{g}_j\|_{W^{1,2}(\Omega,\mathbb{C})} \le C$. By the Sobolev embedding theorem,

$$\|\widetilde{g}_j\|_{L^4(\Omega,\mathbb{C})} \le C\|\widetilde{g}_j\|_{W^{1,2}(\Omega,\mathbb{C})} \le C_1.$$

Hence,

$$\|\widetilde{g}_j\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{C}^3)} \le \|\widetilde{g}_j\|_{L^4(\Omega,\mathbb{C})}\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} \le C.$$

Now we estimate b_j.

Since

$$\psi_j = e^{iq\varphi}(1+iq\varepsilon_j g_j) = \widehat{\psi} + iq\varepsilon_j g_j e^{iq\varphi}$$

and $\psi_j \to \widehat{\psi}$ in $L^4(\Omega,\mathbb{C})$, we have $\varepsilon_j g_j \to 0$ in $L^4(\Omega,\mathbb{C})$. Thus $\varepsilon_j b_j = \varepsilon_j g_j - \varepsilon_j\widetilde{g}_j = o(1)$. On the other hand, we have

$$\begin{aligned} C &\ge \|\nabla g_j - (1+iq\varepsilon_j g_j)\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{C}^3)}\\ &= \|\nabla\widetilde{g}_j - (1+iq\varepsilon_j b_j)\boldsymbol{w}_j - iq\varepsilon_j\widetilde{g}_j\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{C}^3)}\\ &\ge \|\nabla\widetilde{g}_j - (1+iq\varepsilon_j b_j)\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{C}^3)} - O(\varepsilon_j\|\widetilde{g}_j\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{C}^3)})\\ &= \|\nabla\widetilde{g}_j - (1+iq\varepsilon_j b_j)\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{C}^3)} - O(\varepsilon_j).\end{aligned}$$

Put $u_j = \widetilde{g}_j/(1+iq\varepsilon_j b_j)$, then

$$\|\nabla\widetilde{g}_j - (1+iq\varepsilon_j b_j)\boldsymbol{w}_j\|^2_{L^2(\Omega,\mathbb{C}^3)} = |1+iq\varepsilon_j b_j|^2\|\nabla u_j - \boldsymbol{w}_j\|^2_{L^2(\Omega,\mathbb{R}^3)}.$$

Therefore, we have $\|\nabla u_j - \boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{C}^3)} \le C$, so $\|\nabla u_j\|_{L^2(\Omega,\mathbb{C}^3)} \le C_1$. Since $\int_\Omega u_j dx = 0$, it follows from the Poincaré inequality that $\{u_j\}$ is bounded in $W^{1,2}(\Omega,\mathbb{C})$. Passing to a subsequence, we may assume that $u_j \to \widehat{u}$ weakly in $W^{1,2}(\Omega,\mathbb{C})$ and strongly in $L^2(\Omega,\mathbb{C})$. From (4.8), it follows that

$$\begin{aligned}&(\chi_a H_s^2(\kappa,q)+o(1))\|\boldsymbol{h}\cdot\boldsymbol{w}_j\|^2_{L^2(\Omega)}\\ &= \int_\Omega\left\{q^2(1+o(1))|\nabla u_j - \boldsymbol{w}_j|^2 + \frac{\kappa^2 q^2}{2}(-2\Im(g_j)+q\varepsilon_j|g_j|^2)^2\right\}dx + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j].\end{aligned}$$

This implies that

$$q^2\|\nabla u_j - \boldsymbol{w}_j\|^2_{L^2(\Omega,\mathbb{C}^3)} + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] \le (\chi_a H_s^2(\kappa,q)+o(1))\|\boldsymbol{h}\cdot\boldsymbol{w}_j\|^2_{L^2(\Omega)}. \tag{4.9}$$

Letting $j\to\infty$, we get

$$q^2\|\nabla\widehat{u} - \widehat{\boldsymbol{w}}\|^2_{L^2(\Omega,\mathbb{C}^3)} + \mathcal{F}(\boldsymbol{e})[\widehat{\boldsymbol{w}}] \le \chi_a H_s^2(\kappa,q)\|\boldsymbol{h}\cdot\widehat{\boldsymbol{w}}\|^2_{L^2(\Omega)}. \tag{4.10}$$

We note that we may take $\widehat{u}$ to be a real valued function.

Step 6. We show that $\boldsymbol{h}\cdot\widehat{\boldsymbol{w}} \not\equiv 0$ in Ω.

Assume that $\boldsymbol{h}\cdot\widehat{\boldsymbol{w}} \equiv 0$ in Ω. Since $\boldsymbol{w}_j \to \widehat{\boldsymbol{w}}$ strongly in $L^2(\Omega,\mathbb{R}^3)$, we have $\|\boldsymbol{h}\cdot\boldsymbol{w}_j\|_{L^2(\Omega)} \to 0$. Since $\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j]\to 0$ from (4.8), we see that $\operatorname{div}\boldsymbol{w}_j \to 0$ in $L^2(\Omega)$ and $\operatorname{curl}\boldsymbol{w}_j \to 0$ in $L^2(\Omega,\mathbb{R}^3)$. Since $\operatorname{div}\widehat{\boldsymbol{w}} = 0$, $\operatorname{curl}\widehat{\boldsymbol{w}} = 0$, $\nabla\widehat{u} = \widehat{\boldsymbol{w}}$ and $\widehat{\boldsymbol{w}} = 0$ on $\partial\Omega$ from (4.10), we see that $\Delta\widehat{u} = 0$ in Ω and $\nabla\widehat{u} = 0$ on $\partial\Omega$. Applying the maximum principal, we see that $\widehat{u}$ is a constant in Ω, so $\widehat{\boldsymbol{w}} = 0$. Thus $\boldsymbol{w}_j \to 0$ strongly in $L^2(\Omega,\mathbb{R}^3)$. Therefore, it follows from [10] that

$$\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} \le C(\|\operatorname{div}\boldsymbol{w}_j\|_{L^2(\Omega)} + \|\operatorname{curl}\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{R}^3)} + \|\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{R}^3)}) \to 0$$

as $j\to\infty$. This contradicts the fact that $\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} = 1$.

Thus from (4.10),

$$\chi_a H_s^2(\kappa,q) \ge \frac{q^2\|\nabla\widehat{u} - \widehat{\boldsymbol{w}}\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\widehat{\boldsymbol{w}}]}{\|\boldsymbol{h}\cdot\widehat{\boldsymbol{w}}\|^2_{L^2(\Omega)}} \ge \chi_a H_{sh}^2(q).$$

Hence we get $H_s(\kappa,q) = H_{sh}(q)$. This contradicts our hypothesis. Thus we get $\boldsymbol{h}\cdot\widehat{\boldsymbol{n}} \not\equiv 0$ in Ω. By Step 4, we see that $H_s(\kappa,q)$ is achieved. Therefore (ii) holds, so (i) also holds.

Proof of (iii). Assume that $H_n(q) < H_{sh}(q)$.

Step 7. Let $\{(u_j,\boldsymbol{n}_j)\} \subset W^{1,2}(\Omega)\times W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e})$ with $\boldsymbol{h}\cdot\boldsymbol{n}_j \not\equiv 0$ in Ω be a minimizing sequence of $H_n(q)$. Then we have

$$\begin{aligned}&q^2\|\nabla u_j - \boldsymbol{n}_j\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}[\boldsymbol{n}_j] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}\\ &\qquad\qquad = (\chi_a H_n^2(q)+o(1))\|\boldsymbol{h}\cdot\boldsymbol{n}_j\|^2_{L^2(\Omega)}.\end{aligned} \tag{4.11}$$

Since $|\boldsymbol{h}\cdot\boldsymbol{n}_j| \le 1$, the right hand side of (4.11) is bounded. Thus we see that $\{\operatorname{div}\boldsymbol{n}_j\}$ is bounded in $L^2(\Omega)$, $\{\operatorname{curl}\boldsymbol{n}_j\}$ is bounded in $L^2(\Omega,\mathbb{R}^3)$, $|\boldsymbol{n}_j| = 1$ a.e. in Ω and $\boldsymbol{n}_j = \boldsymbol{e}$ on $\partial\Omega$. Therefore, it follows from [10] that $\{\boldsymbol{n}_j\}$ is bounded in $W^{1,2}(\Omega,\mathbb{R}^3)$.

After passing to a subsequence, we may assume that $\boldsymbol{n}_j \to \widehat{\boldsymbol{n}}$ weakly in $W^{1,2}(\Omega, \mathbb{R}^3)$, strongly in $L^2(\Omega, \mathbb{R}^3)$ and a.e. in Ω. As in (ii), we get $\widehat{\boldsymbol{n}} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$. When $q > 0$, it follows from (4.11) that $\{\nabla u_j\}$ is bounded in $L^2(\Omega, \mathbb{R}^3)$. Put $\widehat{u}_j = u_j - d_j$ where $d_j = \frac{1}{|\Omega|}\int_\Omega u_j dx$. Applying the Poincaré inequality, we see that $\{\widehat{u}_j\}$ is bounded in $W^{1,2}(\Omega)$. Passing to a subsequence, we may assume that $\widehat{u}_j \to \widehat{u}$ weakly in $W^{1,2}(\Omega)$ and strongly in $L^4(\Omega)$. Letting $j \to \infty$ in (4.11), we have

$$\begin{aligned}
(4.12)\qquad & q^2\|\nabla\widehat{u} - \widehat{\boldsymbol{n}}\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}[\widehat{\boldsymbol{n}}] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)} \\
&\le \liminf_{j\to\infty}\{q^2\|\nabla\widehat{u}_j - \boldsymbol{n}_j\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}[\boldsymbol{n}_j] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}\} \\
&= \chi_a H_n^2(q)\|\boldsymbol{h}\cdot\widehat{\boldsymbol{n}}\|^2_{L^2(\Omega)}.
\end{aligned}$$

If we show that $\boldsymbol{h}\cdot\widehat{\boldsymbol{n}} \not\equiv 0$ in Ω, we see that $H_n(q)$ is achieved. When $q = 0$, if we show that $\boldsymbol{h}\cdot\widehat{\boldsymbol{n}} \not\equiv 0$, by the definition of $H_n(0)$, we also see that $H_n(0)$ is achieved.

Step 8. Assume that $\boldsymbol{h}\cdot\widehat{\boldsymbol{n}} \equiv 0$ in Ω.

Then from (4.12), $\mathcal{F}[\widehat{\boldsymbol{n}}] \le K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}$. Thus we have $\widehat{\boldsymbol{n}} = \boldsymbol{e}$ and $\nabla\widehat{u} = \widehat{\boldsymbol{n}} = \boldsymbol{e}$ if $q > 0$. Hence if we write $\boldsymbol{n}_j = \boldsymbol{e} + \varepsilon_j\boldsymbol{w}_j$, $\widehat{u}_j = \widehat{u} + \varepsilon_j g_j$, then $\varepsilon_j = \|\boldsymbol{n}_j - \boldsymbol{e}\|_{W^{1,2}(\Omega,\mathbb{R}^3)} > 0$, $\boldsymbol{w}_j \in W_0^{1,2}(\Omega, \mathbb{R}^3)$ and $\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} = 1$. According to Lemma 3.1, we have

$$\begin{aligned}
\mathcal{F}[\boldsymbol{n}_j] - \mathcal{F}[\boldsymbol{e}] &= \varepsilon_j^2\left\{\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{w}_j|^2 dx\right\} \\
&\ge c\varepsilon_j^2\|\boldsymbol{w}_j\|^2_{W^{1,2}(\Omega,\mathbb{R}^3)}.
\end{aligned}$$

As the proof of (ii) we get $\varepsilon_j^2 = o(1)$ as $j \to \infty$. Since $\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} = 1$, after passing to a subsequence, we may assume that $\boldsymbol{w}_j \to \widehat{\boldsymbol{w}}$ weakly in $W_0^{1,2}(\Omega, \mathbb{R}^3)$ and strongly in $L^4(\Omega, \mathbb{R}^3)$. Since $\boldsymbol{e}\cdot\boldsymbol{w}_j = -\frac{\varepsilon_j}{2}|\boldsymbol{w}_j|^2 \to 0$ strongly in $L^2(\Omega)$, we see that $\boldsymbol{e}\cdot\widehat{\boldsymbol{w}} = 0$ a.e. in Ω. Since $\nabla\widehat{u}_j = \nabla\widehat{u} + \varepsilon_j\nabla g_j = \boldsymbol{e} + \varepsilon_j\nabla g_j$, we have

$$\begin{aligned}
(4.13)\qquad & (\chi_a H_n^2(q) + o(1))\|\boldsymbol{h}\cdot\boldsymbol{w}_j\|^2_{L^2(\Omega)} \\
&= \frac{1}{\varepsilon_j^2}(\chi_a H_n^2(q) + o(1))\|\boldsymbol{h}\cdot\boldsymbol{n}_j\|^2_{L^2(\Omega)} \\
&= \frac{1}{\varepsilon_j^2}\{q^2\|\nabla\widehat{u}_j - \boldsymbol{n}_j\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}[\boldsymbol{n}_j] - K_1\|\operatorname{div}\boldsymbol{e}\|^2_{L^2(\Omega)}\} \\
&= \frac{1}{\varepsilon_j^2}q^2\|\nabla\widehat{u}_j - \boldsymbol{n}_j\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{w}_j|^2 dx \\
&= q^2\|\nabla g_j - \boldsymbol{w}_j\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{w}_j|^2 dx
\end{aligned}$$

Since $\int_\Omega g_j dx = 0$, $\|g_j\|_{L^2(\Omega)} \le C\|\nabla g_j\|_{L^2(\Omega,\mathbb{R}^3)} \le C_1$. After passing to a subsequence, we may assume that $g_j \to \widehat{g}$ weakly in $W^{1,2}(\Omega)$ and strongly in $L^4(\Omega)$. By (4.12),

$$\begin{aligned}
(4.14)\quad & q^2\|\nabla\widehat{g} - \widehat{\boldsymbol{w}}\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\widehat{\boldsymbol{w}}] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\widehat{\boldsymbol{w}}|^2 dx \\
&\le \liminf_{j\to\infty}\{q^2\|\nabla g_j - \boldsymbol{w}_j\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{w}_j|^2 dx\} \\
&\qquad = \chi_a H_n^2(q)\|\boldsymbol{h}\cdot\widehat{\boldsymbol{w}}\|^2_{L^2(\Omega)}.
\end{aligned}$$

Step 9. We shall show that $\boldsymbol{h}\cdot\widehat{\boldsymbol{w}} \not\equiv 0$ in Ω.

Assume that $\boldsymbol{h}\cdot\widehat{\boldsymbol{w}} \equiv 0$ in Ω. Since $\boldsymbol{w}_j \to \widehat{\boldsymbol{w}}$ strongly in $L^2(\Omega,\mathbb{R}^3)$, we have $\|\boldsymbol{h}\cdot\boldsymbol{w}_j\|_{L^2(\Omega)} \to 0$. Since $\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}_j] \to 0$ from (4.13), we see that $\operatorname{div}\boldsymbol{w}_j \to 0$ in $L^2(\Omega)$ and $\operatorname{curl}\boldsymbol{w}_j \to 0$ in $L^2(\Omega,\mathbb{R}^3)$. Thus $\operatorname{div}\widehat{\boldsymbol{w}} = 0$, $\operatorname{curl}\widehat{\boldsymbol{w}} = 0$, $\nabla\widehat{g} = \widehat{\boldsymbol{w}}$ in Ω and $\widehat{\boldsymbol{w}} = 0$ on $\partial\Omega$. Therefore, $\Delta\widehat{g} = 0$ in Ω and $\nabla\widehat{g} = 0$ on $\partial\Omega$. By the maximum principle, we see that $\widehat{g}$ is a constant and so $\widehat{\boldsymbol{w}} = 0$ in Ω. Thus $\boldsymbol{w}_j \to 0$ strongly in $L^2(\Omega,\mathbb{R}^3)$. According to [10], $\|\boldsymbol{w}_j\|_{W^{1,2}(\Omega,\mathbb{R}^3)} \to 0$. This is a contradiction. Hence we have

$$\chi_a H_{sh}^2(q) \le \frac{q^2\|\nabla\widehat{g} - \widehat{\boldsymbol{w}}\|^2_{L^2(\Omega,\mathbb{R}^3)} + \mathcal{F}(\boldsymbol{e})[\widehat{\boldsymbol{w}}] - K_1\int_\Omega |\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2dx}{\|\boldsymbol{h}\cdot\widehat{\boldsymbol{w}}\|^2_{L^2(\Omega)}} \le \chi_a H_n^2(q).$$

This completes the proof. $\square$

5. Instabilities in pure nematic states. In this section we examine the local minimality as well as global minimality of the pure nematic states. Let $\psi = 0$ and $\boldsymbol{n} = \boldsymbol{n}_\sigma$ where $\boldsymbol{n}_\sigma$ is a global minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$:

$$\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_\sigma] = \inf_{\boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)} \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}]$$

where $\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] = \mathcal{F}[\boldsymbol{n}] - \chi_a\sigma^2\|\boldsymbol{h}\cdot\boldsymbol{n}\|^2_{L^2(\Omega)}$. We note that $(0,\boldsymbol{n})$ is a critical point of $\mathcal{E}$ if and only if $\boldsymbol{n}$ is a critical point of $\mathcal{F}_{\sigma\boldsymbol{h}}$. Define

$$C(\sigma) = C(\sigma,\kappa,K_1,K_2,K_3,\boldsymbol{h},\boldsymbol{e}_0) = \inf_{\boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)} \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}]$$

and

$$\mathcal{M}(\sigma) = \mathcal{M}(\sigma,\kappa,K_1,K_2,K_3,\boldsymbol{h},\boldsymbol{e}_0) = \{\boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0); \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] = C(\sigma)\}.$$

If $\boldsymbol{n}\in\mathcal{M}(\sigma)$, then $(0,\boldsymbol{n})$ is a critical point of $\mathcal{E}$. When $\boldsymbol{n}$ is a minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$, we look for the Euler-Lagrange equation for $\boldsymbol{n}$. For any $\boldsymbol{v}\in W_0^{1,2}(\Omega,\mathbb{R}^3)$, we compute

$$\frac{d}{dt}\bigg|_{t=0}\left\{\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}+t\boldsymbol{v}] - \int_\Omega \lambda(|\boldsymbol{n}+t\boldsymbol{v}|^2-1)dx\right\} = 0$$

where λ is the Lagrange multiplier which depends on x. By the standard arguments, we get the Euler-Lagrange equation for $\boldsymbol{n}$:

$$(5.1)\quad \begin{cases} -K_1\nabla(\operatorname{div}\boldsymbol{n}) + K_2\{(\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})\operatorname{curl}\boldsymbol{n} + \operatorname{curl}\big((\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})\boldsymbol{n}\big)\} & \\ \quad +K_3\{|\operatorname{curl}\boldsymbol{n}|^2\boldsymbol{n} - (\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})\operatorname{curl}\boldsymbol{n} + \operatorname{curl}^2\boldsymbol{n} & \\ \quad -\operatorname{curl}\big((\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})\boldsymbol{n}\big)\} & \\ \quad -\chi_a\sigma^2(\boldsymbol{h}\cdot\boldsymbol{n})\boldsymbol{h} - \lambda\boldsymbol{n} = 0 & \text{in } \Omega, \\ \boldsymbol{n} = \boldsymbol{e}_0 & \text{on } \partial\Omega. \end{cases}$$

We can compute the Lagrange multiplier λ:

$$\begin{aligned} \lambda = \lambda(x) = \boldsymbol{n}\cdot\big[&-K_1\nabla(\operatorname{div}\boldsymbol{n}) + K_2\{(\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})\operatorname{curl}\boldsymbol{n} \\ &+ \operatorname{curl}\big((\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})\boldsymbol{n}\big)\} + K_3\{|\operatorname{curl}\boldsymbol{n}|^2\boldsymbol{n} - (\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})\operatorname{curl}\boldsymbol{n} \\ &+ \operatorname{curl}^2\boldsymbol{n} - \operatorname{curl}\big((\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})\boldsymbol{n}\big)\} - \chi_a\sigma^2(\boldsymbol{h}\cdot\boldsymbol{n})\boldsymbol{h}\big]. \end{aligned}$$

In the particular case where $K_1 = K_2 = K_3 = K$, we use the formulas: $\operatorname{curl}^2\boldsymbol{n} = -\Delta\boldsymbol{n} + \nabla\operatorname{div}\boldsymbol{n}$ and $-\Delta\boldsymbol{n}\cdot\boldsymbol{n} = |\nabla\boldsymbol{n}|^2$ which follows from $\boldsymbol{n}\cdot\boldsymbol{n} = 1$. In this case, we have

$$(5.2)\quad \begin{cases} -K\Delta\boldsymbol{n} = K|\nabla\boldsymbol{n}|^2\boldsymbol{n} + \chi_a\sigma^2\big((\boldsymbol{h}\cdot\boldsymbol{n})\boldsymbol{h} - (\boldsymbol{h}\cdot\boldsymbol{n})^2\boldsymbol{n}\big) & \text{in } \Omega, \\ \boldsymbol{n} = \boldsymbol{e}_0 & \text{on } \partial\Omega. \end{cases}$$

Since $\boldsymbol{h}\cdot\boldsymbol{e}=0$, $\boldsymbol{e}$ is a critical point of $\mathcal{F}_{\sigma\boldsymbol{h}}$ for any σ. Recall that

$$H_{sh}^2(0)=\frac{1}{\chi_a}\inf\Big\{\mathcal{F}(\boldsymbol{e})[\boldsymbol{w}]-K_1\int_\Omega|\nabla\boldsymbol{e}|^2|\boldsymbol{w}|^2dx;\boldsymbol{w}\in W_0^{1,2}(\Omega,\mathbb{R}^3),\\ \boldsymbol{w}(x)\cdot\boldsymbol{e}(x)=0\text{ in }\Omega,\|\boldsymbol{h}\cdot\boldsymbol{w}\|_{L^2(\Omega)}=1\Big\},$$

$$H_n^2(0)=\frac{1}{\chi_a}\inf\left\{\frac{\mathcal{F}[\boldsymbol{n}]-K_1\|\operatorname{div}\boldsymbol{e}\|_{L^2(\Omega)}^2}{\|\boldsymbol{h}\cdot\boldsymbol{n}\|_{L^2(\Omega)}^2};\ \boldsymbol{n}\in W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0),\ \boldsymbol{h}\cdot\boldsymbol{n}\not\equiv0\text{ in }\Omega\right\}$$

and $0<H_n(0)\le H_{sh}(0)$ from Theorem 4.5 (i).

We give a simple criterion for $\boldsymbol{n}=\boldsymbol{e}$ to be a global minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$.

LEMMA 5.1. (i) *If $0\le\sigma<H_n(0)$, then $\boldsymbol{n}=\boldsymbol{e}$ is the only global minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$ in $W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$.*

(ii) *If $H_n(0)<H_{sh}(0)$ and $H_n(0)<\sigma<H_{sh}(0)$, then $\boldsymbol{n}=\boldsymbol{e}$ is not a global minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$ in $W^{1,2}(\Omega,\mathbb{S}^2,\boldsymbol{e}_0)$, but it is weakly stable (a local minimizer).*

(iii) *If $\sigma>H_{sh}(0)$, $\boldsymbol{n}=\boldsymbol{e}$ is not weakly stable.*

For the proof, see [3] and [19].

Next, we consider a question: When $\boldsymbol{n}_\sigma\in\mathcal{M}(\sigma,\kappa,K_1,K_2,K_3,\boldsymbol{h},\boldsymbol{e}_0)$ is a global minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$, and is $(0,\boldsymbol{n}_\sigma)$ a global minimizer of $\mathcal{E}$?

Let $\mu=\mu(q\boldsymbol{n})$ be the lowest eigenvalue of the magnetic Neumann problem

$$\begin{cases}-\nabla_{q\boldsymbol{n}}^2\phi=\mu\phi & \text{in }\ \Omega,\\ \nabla_{q\boldsymbol{n}}\phi\cdot\boldsymbol{\nu}=0 & \text{on }\ \partial\Omega.\end{cases}\tag{5.3}$$

That is to say,

$$\mu(q\boldsymbol{n})=\inf_{0\ne\phi\in W^{1,2}(\Omega,\mathbb{C})}\frac{\|\nabla_{q\boldsymbol{n}}\phi\|_{L^2(\Omega,\mathbb{C}^3)}^2}{\|\phi\|_{L^2(\Omega,\mathbb{C})}^2}.$$

Define

$$\mu_*(q,\sigma)=\mu_*(q,\sigma,K_1,K_2,K_3,\boldsymbol{h},\boldsymbol{e}_0)=\inf_{\boldsymbol{n}\in\mathcal{M}(\sigma,\kappa,K_1,K_2,K_3,\boldsymbol{h},\boldsymbol{e}_0)}\mu(q\boldsymbol{n}).$$

LEMMA 5.2. (i) *If $(\psi,\boldsymbol{n})$ is a global minimizer of $\mathcal{E}$ which is not a pure nematic state, then $\mu(q\boldsymbol{n})<\kappa^2$.*

(ii) *If $\mu_*(q,\sigma)<\kappa^2$, then pure nematic states are not global minimizers of $\mathcal{E}$.*

For the proof, see [3] or [19].

PROPOSITION 5.3. *If $0\le\sigma\le H_n(0)$ and $\kappa>0$, or $\sigma>H_n(0)$ and $\mu_*(q,\sigma)<\kappa^2$, then the pure nematic states are not global minimizer of $\mathcal{E}$.*

Proof. When $0\le\sigma<H_n(0)$, it follows from Lemma 5.1 (i) that $\mathcal{M}(\sigma)=\{\boldsymbol{e}\}$. When $\sigma=H_n(0)$,

$$\sigma^2\le\frac{1}{\chi_a}\frac{\mathcal{F}[\boldsymbol{n}]-K_1\|\operatorname{div}\boldsymbol{e}\|_{L^2(\Omega)}^2}{\|\boldsymbol{h}\cdot\boldsymbol{n}\|_{L^2(\Omega)}^2}$$

for any $\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$ with $\boldsymbol{h} \cdot \boldsymbol{n} \not\equiv 0$ in Ω. Therefore,

$$\mathcal{F}[\boldsymbol{n}] - K_1 \|\operatorname{div} \boldsymbol{e}\|^2_{L^2(\Omega)} - \chi \sigma^2 \|\boldsymbol{h} \cdot \boldsymbol{n}\|^2_{L^2(\Omega)} \geq 0. \tag{5.4}$$

This inequality holds even in the case $\boldsymbol{h} \cdot \boldsymbol{n} = 0$. Therefore, (5.4) hold for any $\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$. This implies that $C(\sigma) \geq K_1 \|\operatorname{div} \boldsymbol{e}\|^2_{L^2(\Omega)}$.

On the other hand, $C(\sigma) \leq \mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{e}] = K_1 \|\operatorname{div} \boldsymbol{e}\|^2_{L^2(\Omega)}$. Therefore, we have $C(\sigma) = K_1 \|\operatorname{div} \boldsymbol{e}\|^2_{L^2(\Omega)} = \mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{e}]$, so $\boldsymbol{e} \in \mathcal{M}(\sigma)$. If we put $\psi = e^{iq\varphi}$ where φ is the function defined in (1.5), then $\nabla_{q\boldsymbol{e}} \psi = 0$. So $\mu(q\boldsymbol{e}) = 0$. Thus we have

$$\mu_*(q, \sigma) \leq \mu(q\boldsymbol{e}) = 0 < \kappa^2$$

for any $\kappa > 0$. Therefore, from Lemma 5.2 (ii), we see that pure nematic states are not global minimizers of $\mathcal{E}$. If $\sigma > H_n(0)$ and $\mu_*(q, \sigma) < \kappa^2$, it suffices to apply Lemma 5.2 (ii). □

Now define

$$\begin{aligned}
\sigma_*(\kappa, q) &= \inf\{\sigma > 0; \mu_*(q, \sigma) \geq \kappa^2\}, \\
Q_*(\kappa, q) &= \inf\{q > 0; \mu_*(q, \sigma) \geq \kappa^2\}.
\end{aligned}$$

Let $\sigma > H_n(0)$. Summing up the above, the pure nematic states are not global minimizers in the following cases.

(1) $0 < \sigma < \sigma_*(\kappa, q)$.
(2) $\mu_*(q, \sigma) < \kappa^2$.
(3) $0 \leq q < Q_*(q, \sigma)$.

The following theorem indicates the difference between liquid crystals and superconductors under strong external field.

THEOREM 5.4. *Let $q, \kappa, K_1, K_2, K_3, \boldsymbol{h}, \boldsymbol{e}_0$ with $K_1 = K_2 = K_3 = K$ be given. Assume that* (H.1), (H.2), (H.3) *and* (H.4) *hold.*

(i) *In the case where* $\operatorname{curl} \boldsymbol{h} = 0$ *in Ω, then if σ is sufficiently large, the pure nematic states are not global minimizers.*

(ii) *There exists a constant $c > 0$ such that if $|\operatorname{curl} \boldsymbol{h}(x)| \leq c$ for all $x \in B_R$ where B_R is an open ball with center* 0 *and radius $R > 0$ containing $\overline{\Omega}$ and $\boldsymbol{h}(x)$ denotes an C^2-extension of $\boldsymbol{h}(x)$ $(x \in \Omega)$ to $\overline{B}_R$ (We used the same notation), then the conclusion of* (i) *also holds.*

REMARK 5.5. *For our example* (1.4), *we have for large $|a_1|$,*

$$|\operatorname{curl} \boldsymbol{h}(x)| = \frac{1}{\sqrt{(x_1 - a_1)^2 + x_2^2}} \leq C(a_1) \text{ for } x \in \overline{\Omega}.$$

We note that $C(a_1) \to 0$ as $|a_1| \to \infty$. Therefore the hypothesis of Theorem 5.4 (ii) *holds in this case.*

In order to prove Theorem 5.4, we need a lemma.

LEMMA 5.6. (i) *For large σ, $C(\sigma) \leq -\chi_a \sigma^2 |\Omega| + C_1 \sigma$ where $C_1 > 0$ depends only on $K_1, K_2, K_3, \boldsymbol{h}, \boldsymbol{e}_0$ and Ω.*

(ii) *Let $\boldsymbol{n}_\sigma$ be a global minimizer of $\mathcal{F}_{\sigma \boldsymbol{h}}$. Then $|\boldsymbol{h} \cdot \boldsymbol{n}_\sigma| \to 1$ in $L^2(\Omega)$ as $\sigma \to +\infty$.*

(iii) *Assume that $K_1 = K_2 = K_3$. Then $\boldsymbol{h} \cdot \boldsymbol{n}_\sigma \to 1$ or -1 in $L^2(\Omega)$ as $\sigma \to +\infty$.*

Proof. Define $\boldsymbol{k}(x) = \boldsymbol{h}(x) \times \boldsymbol{e}(x)$. Then $(\boldsymbol{e}(x), \boldsymbol{k}(x), \boldsymbol{h}(x))$ is a orthonormal basis in $\mathbb{R}^3$. For $\boldsymbol{n} \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$, we can write

$$\boldsymbol{n} = n_{\boldsymbol{e}}\boldsymbol{e} + n_{\boldsymbol{k}}\boldsymbol{k} + n_{\boldsymbol{h}}\boldsymbol{h}, \quad n_{\boldsymbol{e}}^2 + n_{\boldsymbol{k}}^2 + n_{\boldsymbol{h}}^2 = 1 \text{ a.e. in } \Omega.$$

We see that

$$\begin{aligned}
\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] &= \mathcal{F}[\boldsymbol{n}] - \chi_a\sigma^2 \int_\Omega (\boldsymbol{h}\cdot\boldsymbol{n})^2 dx \\
&= \mathcal{F}[\boldsymbol{n}] - \chi_a\sigma^2 \int_\Omega n_{\boldsymbol{h}}^2 dx \\
&= \mathcal{F}[\boldsymbol{n}] - \chi_a\sigma^2 \int_\Omega (1 - n_{\boldsymbol{e}}^2 - n_{\boldsymbol{k}}^2) dx \\
&= \mathcal{T}_\sigma[\boldsymbol{n}] - \chi_a\sigma^2|\Omega|
\end{aligned}$$

where

$$\mathcal{T}_\sigma[\boldsymbol{n}] = \mathcal{F}[\boldsymbol{n}] + \chi_a\sigma^2 \int_\Omega (n_{\boldsymbol{e}}^2 + n_{\boldsymbol{k}}^2) dx.$$

Proof of (i). Choose a test field

$$\begin{aligned}
\boldsymbol{n} &= (\cos\phi)\boldsymbol{e} + (\sin\phi)\boldsymbol{h} \\
&= \sum_{i=1}^3 \{(\cos\phi)e_i(x) + (\sin\phi)h_i(x)\}\boldsymbol{e}_i
\end{aligned}$$

where $\boldsymbol{e}(x) = \sum_{i=1}^3 e_i(x)\boldsymbol{e}_i$ and $\boldsymbol{h} = \sum_{i=1}^3 h_i(x)\boldsymbol{e}_i$ and $\boldsymbol{e}_1 = (1,0,0), \boldsymbol{e}_2 = (0,1,0), \boldsymbol{e}_3 = (0,0,1)$. Then since

$$\operatorname{div}\boldsymbol{n} = \sum_{i=1}^3 \{-(\sin\phi)(\partial_i\phi)e_i + (\cos\phi)(\partial_i\phi)h_i\} + \sum_{i=1}^3 \{(\cos\phi)(\partial_i e_i) + (\sin\phi)(\partial_i h_i)\}$$

and $\boldsymbol{e}, \boldsymbol{h} \in C^2(\overline{\Omega}, \mathbb{R}^3)$, we see that $|\operatorname{div}\boldsymbol{n}|^2 \le C(|\nabla\phi|^2 + 1)$. Similarly we have

$$|\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n}|^2 + |\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n}|^2 \le C_1(|\nabla\phi|^2 + 1).$$

Thus if we write $\mathcal{T}_\sigma[\boldsymbol{n}] = \int_\Omega f_{\sigma,\boldsymbol{h}}(\phi)dx$, we have

$$|f_{\sigma,\boldsymbol{h}}(\phi)| \le C\max\{K_2, K_3\}(|\nabla\phi|^2 + 1) + \chi_a\sigma^2\cos^2\phi.$$

For any $\varepsilon > 0$, define $\Omega_\varepsilon = \{x \in \Omega; d(x, \partial\Omega) < \varepsilon\}$ and $\Omega^\varepsilon = \{x \in \Omega; d(x.\partial\Omega) \ge \varepsilon\}$, and decompose $\mathcal{T}_\sigma[\boldsymbol{n}]$ as follows: $\mathcal{T}_\sigma[\boldsymbol{n}] = \mathcal{T}_{\sigma,1}[\boldsymbol{n}] + \mathcal{T}_{\sigma,2}[\boldsymbol{n}]$ where

$$\mathcal{T}_{\sigma,1}[\boldsymbol{n}] = \int_{\Omega_\varepsilon} f_{\sigma,\boldsymbol{h}}(\phi)dx, \quad \mathcal{T}_{\sigma,2}[\boldsymbol{n}] = \int_{\Omega^\varepsilon} f_{\sigma,\boldsymbol{h}}(\phi)dx.$$

Choose ϕ such that

$$\phi = \begin{cases} \frac{\pi}{2} & \text{in } \Omega^\varepsilon, \\ 0 & \text{on } \partial\Omega, \\ |\nabla\phi| \le \frac{C_2}{\varepsilon} & \text{in } \Omega. \end{cases}$$

Then $\mathcal{T}_{\sigma,2}[\boldsymbol{n}] \le C\max\{K_2, K_3\}$ and

$$\begin{aligned}\mathcal{T}_{\sigma,1}[\boldsymbol{n}] &\le \int_{\Omega_\varepsilon}\{C\max\{K_2,K_3\}(|\nabla\phi|^2+1)+\chi_a\sigma^2(\cos\phi)^2\}dx\\ &\le \big[C\max\{K_2,K_3\}\big(\frac{C_2^2}{\varepsilon^2}+1\big)+\chi_a\sigma^2\big]|\Omega_\varepsilon|.\end{aligned}$$

Since $\partial\Omega$ is smooth, there exists $C_0 > 0$ depending only on $\partial\Omega$ such that $|\Omega_\varepsilon| \le C_0\varepsilon$ for any small $\varepsilon > 0$. For large σ, choose $\varepsilon > 0$ so that

$$\varepsilon = \min\Big\{\frac{\sqrt{C}C_2}{\sigma}\sqrt{\frac{\max\{K_2,K_3\}}{\chi_a}}, 1\Big\}.$$

Then we have $\mathcal{T}_{\sigma,1}[\boldsymbol{n}] \le C_3 + C_4\sigma$. Therefore, we have

$$C(\sigma) \le \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] \le -\chi_a\sigma^2|\Omega| + C_5\sigma + C_6.$$

Thus (i) holds.

Proof of (ii). From (i), we see that

$$\mathcal{T}_\sigma[\boldsymbol{n}_\sigma] = \mathcal{F}[\boldsymbol{n}_\sigma] + \chi_a\sigma^2\int_\Omega(n_{\boldsymbol{e}}^2+n_{\boldsymbol{k}}^2)dx \le C_1\sigma.$$

This implies that

$$\int_\Omega(n_{\boldsymbol{e}}^2+n_{\boldsymbol{k}}^2)dx \le \frac{C_1}{\chi_a\sigma} \to 0$$

as $\sigma \to +\infty$. Thus $\int_\Omega(1-|n_{\sigma,\boldsymbol{h}}|^2)dx \to 0$ as $\sigma \to +\infty$. Since $|n_{\sigma,\boldsymbol{h}}| \le 1$, for any $1 \le p < +\infty$,

$$\int_\Omega(1-|n_{\sigma,\boldsymbol{h}}|)^p dx \le 2^{p-1}\int_\Omega(1-|n_{\sigma,\boldsymbol{h}}|)dx \to 0.$$

Since $\boldsymbol{h}\cdot\boldsymbol{n}_\sigma = n_{\sigma,\boldsymbol{h}}$, we see that $|\boldsymbol{h}\cdot\boldsymbol{n}_\sigma| \to 1$ in $L^p(\Omega)$ as $\sigma \to +\infty$.

Proof of (iii). When $K_1 = K_2 = K_3 = K$, we shall show that $\boldsymbol{n}_\sigma$ has the following property: $n_{\sigma,\boldsymbol{h}} > 0$ in Ω or $n_{\sigma,\boldsymbol{h}} < 0$ in Ω or $n_{\sigma,\boldsymbol{h}} \equiv 0$ in Ω.

In fact, $\boldsymbol{n}_\sigma$ satisfies the Euler-Lagrange equation (5.2). That is to say, if we write $\boldsymbol{n}_\sigma = \boldsymbol{n}$ for brevity,

$$\left\{\begin{array}{ll} -\Delta\boldsymbol{n} = |\nabla\boldsymbol{n}|^2\boldsymbol{n} + b^2\sigma^2[n_{\boldsymbol{h}}\boldsymbol{h} - n_{\boldsymbol{h}}^2\boldsymbol{n}] & \text{in } \Omega,\\ \boldsymbol{n} = \boldsymbol{e}_0 & \text{on } \partial\Omega\end{array}\right. \tag{5.5}$$

where $b^2 = \chi_a/K$. Define $\boldsymbol{u}_\sigma = H(n_{\sigma,\boldsymbol{h}})\boldsymbol{n}_\sigma$ where $H(t) = 1$ if $t \ge 0$ and $H(t) = -1$ if $t < 0$ and write

$$\boldsymbol{u}_\sigma = u_{\sigma,\boldsymbol{e}}\boldsymbol{e}(x) + u_{\sigma,\boldsymbol{k}}\boldsymbol{k}(x) + u_{\sigma,\boldsymbol{h}}\boldsymbol{h}(x).$$

Since $n_{\sigma,\boldsymbol{h}} = 0$ on $\partial\Omega$, we see that $\boldsymbol{u}_\sigma = \boldsymbol{n}_\sigma = \boldsymbol{e}_0$ on $\partial\Omega$ and $\nabla\boldsymbol{u}_\sigma = \boldsymbol{n}_\sigma\nabla H(n_{\sigma,\boldsymbol{h}}) + H(n_{\sigma,\boldsymbol{h}})\nabla\boldsymbol{n}_\sigma$. It is well known that

$$\nabla H(n_{\sigma,\boldsymbol{h}}) = 2\nabla n_{\sigma,\boldsymbol{h}}\delta_{\{n_{\sigma,\boldsymbol{h}}=0\}} = 0.$$

Thus $|\nabla \boldsymbol{u}_\sigma| = |\nabla \boldsymbol{n}_\sigma|$ and $\boldsymbol{u}_\sigma \in W^{1,2}(\Omega, \mathbb{S}^2, \boldsymbol{e}_0)$. Therefore $\boldsymbol{u}_\sigma$ is also a minimizer of $\mathcal{F}_{\sigma \boldsymbol{h}}$ and so $\boldsymbol{u}_\sigma$ satisfies the Euler-Lagrange equation

$$\left\{ \begin{array}{ll} -\Delta \boldsymbol{u}_\sigma = |\nabla \boldsymbol{u}_\sigma|^2 \boldsymbol{u}_\sigma - b^2\sigma^2\big((\boldsymbol{h}\cdot\boldsymbol{u}_\sigma)\boldsymbol{h} - (\boldsymbol{h}\cdot\boldsymbol{u}_\sigma)^2\boldsymbol{u}_\sigma\big) & \text{in } \Omega, \\ \boldsymbol{u}_\sigma = \boldsymbol{e}_0 & \text{on } \partial\Omega. \end{array} \right. \tag{5.6}$$

From the first equation of (5.6), we have

$$-(\Delta \boldsymbol{u}_\sigma)\cdot \boldsymbol{h} = |\nabla \boldsymbol{u}_\sigma|^2 u_{\sigma,\boldsymbol{h}} + b^2\sigma^2(u_{\sigma,\boldsymbol{h}} - u_{\sigma,\boldsymbol{h}}^3). \tag{5.7}$$

Using the Leibniz formula, we can see that

$$\Delta u_{\sigma,\boldsymbol{h}} = (\Delta \boldsymbol{u}_\sigma)\cdot \boldsymbol{h} = 2\mathrm{Tr}[\nabla \boldsymbol{u}_\sigma (\nabla \boldsymbol{h})^t] - \boldsymbol{u}_\sigma \cdot \Delta \boldsymbol{h} \tag{5.8}$$

where A^t denotes the transposed matrix for any matrix A. Since $\boldsymbol{e}\cdot\boldsymbol{h} = 0, \boldsymbol{k}\cdot\boldsymbol{h} = 0$ and $\boldsymbol{h}\cdot\boldsymbol{h} = 1$, we have

$$\begin{aligned} \Delta \boldsymbol{e}\cdot\boldsymbol{h} + 2\mathrm{Tr}[\nabla \boldsymbol{e}(\nabla \boldsymbol{h})^t] + \boldsymbol{e}\cdot\Delta\boldsymbol{h} &= 0, \\ \Delta \boldsymbol{k}\cdot\boldsymbol{h} + 2\mathrm{Tr}[\nabla \boldsymbol{k}(\nabla \boldsymbol{h})^t] + \boldsymbol{k}\cdot\Delta\boldsymbol{h} &= 0, \\ \Delta \boldsymbol{h}\cdot\boldsymbol{h} + 2\mathrm{Tr}[\nabla \boldsymbol{h}(\nabla \boldsymbol{h})^t] + \boldsymbol{h}\cdot\Delta\boldsymbol{h} &= 0. \end{aligned} \tag{5.9}$$

From (5.7), (5.8) and (5.9), we can get the equation

$$\begin{aligned} &\Delta u_{\sigma,\boldsymbol{h}} - 2\nabla \boldsymbol{h}(\boldsymbol{h})\cdot\nabla u_{\sigma,\boldsymbol{h}} + (\Delta\boldsymbol{h}\cdot\boldsymbol{h})u_{\sigma,\boldsymbol{h}} \\ &\qquad = -|\nabla \boldsymbol{u}_\sigma|^2 u_{\sigma,\boldsymbol{h}} - b^2\sigma^2 u_{\sigma,\boldsymbol{h}}(1 - u_{\sigma,\boldsymbol{h}}^2) + g + \sum_{i=1}^3 \partial_i f^i \end{aligned}$$

where

$$\begin{aligned} g &= -u_{\sigma,\boldsymbol{e}}\{(\Delta\boldsymbol{e}\cdot\boldsymbol{h}) - 2\mathrm{div}\,(\nabla\boldsymbol{h}(\boldsymbol{e}))\} - u_{\sigma,\boldsymbol{k}}\{(\Delta\boldsymbol{k}\cdot\boldsymbol{h}) - 2\mathrm{div}\,(\nabla\boldsymbol{h}(\boldsymbol{k}))\}, \\ f^i &= 2\{(\nabla\boldsymbol{h}(\boldsymbol{e}))_i u_{\sigma,\boldsymbol{e}} + (\nabla\boldsymbol{h}(\boldsymbol{k}))_i u_{\sigma,\boldsymbol{k}}\}. \end{aligned}$$

Clearly we see that $g \in L^{q/2}(\Omega)$ and $f^i \in L^q(\Omega)$ for any $q > 3$. By the weak Harnack inequality for non-negative superharmonic function (cf. Gilbarg and Trudinger [12, Theorem 8.18]), for any $B_R(y) \subset \Omega$,

$$\left(\frac{1}{|B_R(y)|}\int_{B_R(y)} u_{\sigma,\boldsymbol{h}}^p dx\right)^{1/p} \le C\left\{\operatorname*{ess\,inf}_{B_{\theta R}(y)} u_{\sigma,\boldsymbol{h}} + k_\sigma(R)\right\}$$

for some $p \in [1, 3)$ where

$$k_\sigma(R) = R^\delta \sum_{i=1}^3 \|f^i\|_{L^q(\Omega)} + R^{2\delta}\|g\|_{L^{q/2}(\Omega)},$$

$\delta = 1 - 3/q$ and $0 < \theta < 1$. Here we note that the constant C is independent of σ. Since $u_{\sigma,\boldsymbol{e}}, u_{\sigma,\boldsymbol{k}} \to 0$ in $L^q(\Omega)$ as $\sigma \to \infty$ and $u_{\sigma,\boldsymbol{e}}^2 + u_{\sigma,\boldsymbol{k}}^2 \le 1$, we see that $k_\sigma(R) \to 0$ as $\sigma \to \infty$. On the other hand, it follows from (ii) that

$$\left\{\frac{1}{|B_R(y)|}\int_{B_R(y)} u_{\sigma,\boldsymbol{h}}^p dx\right\}^{1/p} \to 1 \quad \text{as } \sigma \to \infty.$$

This implies that if $u_{\sigma,\boldsymbol{h}} \not\equiv 0$, then $u_{\sigma,\boldsymbol{h}} > 0$ in Ω. By (ii), $n_{\sigma,\boldsymbol{h}} \not\equiv 0$ in Ω for large σ. Therefore, $n_{\sigma,\boldsymbol{h}} > 0$ in Ω or $n_{\sigma,\boldsymbol{h}} < 0$ in Ω. Assume that $n_{\sigma,\boldsymbol{h}} > 0$ in Ω. By (ii), $n_{\sigma,\boldsymbol{h}} \to 1$ in $L^2(\Omega)$ as $j \to \infty$. Hence $(n_{\boldsymbol{e}}^2 + n_{\boldsymbol{k}}^2)^{1/2} \to 0$ in $L^2(\Omega, \mathbb{R}^3)$. Thus we have $\boldsymbol{n}_\sigma \to \boldsymbol{h}$ in $L^2(\Omega, \mathbb{R}^3)$, that is to say, $\boldsymbol{h} \cdot \boldsymbol{n}_\sigma \to 1$ in $L^2(\Omega)$. □

Proof of Theorem 5.4. Let $\boldsymbol{n}_\sigma$ be a global minimizer of $\mathcal{F}_{\sigma\boldsymbol{h}}$. We shall estimate $\mu(q\boldsymbol{n}_\sigma)$ for large σ. By Lemma 5.6, we may assume that $\boldsymbol{n}_\sigma \to \boldsymbol{h}$ strongly in $L^2(\Omega, \mathbb{R}^3)$.

Proof of (i). Since $\operatorname{curl} \boldsymbol{h} = 0$ in Ω, there exists a smooth function $\chi(x)$ such that $\boldsymbol{h} = \nabla\chi$. If we define $\phi(x) = e^{iq\chi(x)}$, then $\nabla_{q\boldsymbol{n}_\sigma}\phi = iq(\boldsymbol{h} - \boldsymbol{n}_\sigma)e^{iq\chi(x)}$. Therefore, we have

$$\int_\Omega |\nabla_{q\boldsymbol{n}_\sigma}\phi|^2 dx = q^2 \int_\Omega |\boldsymbol{h} - \boldsymbol{n}_\sigma|^2 dx \to 0$$

as $\sigma \to +\infty$. Hence

$$\begin{aligned}\mu(q\boldsymbol{n}_\sigma) &\le \frac{\|\nabla_{q\boldsymbol{n}_\sigma}\phi\|^2_{L^2(\Omega,\mathbb{C}^3)}}{\|\phi\|^2_{L^2(\Omega,\mathbb{C})}} \\ &= \frac{q^2\|\boldsymbol{h} - \boldsymbol{n}_\sigma\|^2_{L^2(\Omega,\mathbb{R}^3)}}{|\Omega|} \to 0\end{aligned}$$

as $\sigma \to +\infty$. Thus for any $\kappa > 0$, $\mu(q\boldsymbol{n}_\sigma) < \kappa^2$ for large σ. Thus from Lemma 5.2 (ii), we see that pure nematic states are not global minimizers of $\mathcal{E}$.

Proof of (ii). Let $\sup_{x\in\overline{B}_R} |\operatorname{curl} \boldsymbol{h}(x)| \le c$ for all $x \in \overline{B}_R$ where c is determined later. If we define a function

$$\chi(x) = \int_0^1 \boldsymbol{h}(tx) \cdot x dt,$$

we have

$$\partial_i\chi(x) = h_i(x) + \sum_{j\neq i} \int_0^1 \{(\partial_i h_j)(tx) - (\partial_j h_i)(tx)\} t x_j dt.$$

Thus we have $|\nabla\chi(x) - \boldsymbol{h}(x)| \le \alpha c|x| \le \alpha c R$ where α is a universal constant (cf. Raymond [24, Lemma 2.2]). Define $\phi(x) = e^{iq\chi(x)}$ for $x \in \Omega$. Then we have $\nabla_{q\boldsymbol{n}_\sigma}\phi = iq(\nabla\chi - \boldsymbol{n}_\sigma)e^{iq\chi(x)}$. Hence

$$\begin{aligned}\int_\Omega |\nabla_{q\boldsymbol{n}_\sigma}\phi|^2 dx &= q^2 \int_\Omega |\nabla\chi - \boldsymbol{n}_\sigma|^2 dx \\ &\le 2q^2 \int_\Omega \{|\nabla\chi - \boldsymbol{h}|^2 + |\boldsymbol{h} - \boldsymbol{n}_\sigma|^2\} dx \\ &\le 2q^2c^2\alpha^2R^2|\Omega| + 2q^2 \int_\Omega |\boldsymbol{h} - \boldsymbol{n}_\sigma|^2 dx.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\mu(q\boldsymbol{n}_\sigma) &\le \frac{\|\nabla_{q\boldsymbol{n}_\sigma}\phi\|^2_{L^2(\Omega,\mathbb{R}^3)}}{\|\phi\|^2_{L^2(\Omega)}} \\ &\le 2q^2c^2\alpha^2R^2 + 2q^2\frac{1}{|\Omega|}\int_\Omega |\boldsymbol{h} - \boldsymbol{n}_\sigma|^2 dx.\end{aligned}$$

We choose $c > 0$ so that $2q^2c^2\alpha^2R^2 < \kappa^2$. Since $\boldsymbol{n}_\sigma \to \boldsymbol{h}$ in $L^2(\Omega, \mathbb{R}^3)$ as $\sigma \to \infty$, we have $\mu(q\boldsymbol{n}_\sigma) < \kappa^2$ for large σ. Then it suffices to apply Lemma 5.2 (ii).

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