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CAUCHY PROBLEM ON NON-SOLVABLE SYSTEMS OF FIRST ORDER PARTIAL DIFFERENTIAL EQUATIONS WITH APPLICATIONS*

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Abstract. Let $L_1, L_2, \cdots, L_m$ be m partial differential operators of first order and $h_1, h_2, \cdots, h_m$ continuously differentiable functions. Then *is the partial differential equation system* $L_i[u] = h_i,\ 1 \leq i \leq m$ *solvable for a differential mapping* $u : \mathbb{R}^n \to \mathbb{R}^n$ *or not*? Similarly, let φ be a continuous function $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$. Then *is the Cauchy problem* $L_i[u] = h_i,\ 1 \leq i \leq m$ *with* $u(x_1, x_2, \cdots, x_{n-1}, t_0) = \varphi(x_1, x_2, \cdots, x_{n-1})$ *solvable or not*? If not, *how can we characterize the behavior of such a function* u? All these questions are ignored in classical mathematics only by saying *not solvable*! In fact, non-solvable equation systems are nothing but Smarandache systems, i.e., contradictory systems themselves, in which a ruler behaves in at least two different ways within the same system, i.e., validated and invalided, or only invalided but in multiple distinct ways. They are widely existing in the natural world and our daily life. In this paper, we discuss non-solvable partial differential equation systems of first order by a combinatorial approach, classify these systems by underlying graphs, particularly, these non-solvable linear systems, characterize their behaviors, such as those of global stability, energy integral and their geometry, which enables one to find a differentiable manifold with preset m vector fields. Applications of such non-solvable systems to interaction fields and flows in network are also included in this paper.

Key words. Non-solvable partial-differential equation, vertex-edge labeled graph, global stability, energy integral, combinatorial manifold.

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1. Introduction.

A partial differential equation

$$F(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) = 0 \qquad (PDE)$$

on functions $u(x_1, \cdots, x_n)$ is *non-solvable* if there are no function $u(x_1, \cdots, x_n)$ on a domain $D \subset \mathbb{R}^n$ with (PDE) holds. For example, the equation $e^{u_{x_1}+u_{x_2}} = 0$ is such a PDE. Similarly, a system of partial differential equations

$$\left\{\begin{array}{l} F_1(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) = 0 \end{array}\right. \qquad (PDES)$$

is *non-solvable* if there are no function $u(x_1, \cdots, x_n)$ on a domain $D \subset \mathbb{R}^n$ with $(PDES)$ holds.

Such non-solvable systems of partial differential equations are indeed existing. For example, H.Lewy [5] proved that there exists a function $F(x_1, x_2, x_3) \in C^\infty(\mathbb{R}^3)$ such that the partial differential equation

$$-u_{x_1} - iu_{x_2} + 2i\,(x_1 + ix_2)\,u_{x_3} = 0$$

is non-solvable. R.Rubinsten [14] proved that

$$\begin{array}{ll} u_{tt} + t^n u_{xx} + (i - t^m)u_x = 0, & n > 4m + 2,\ m \equiv 1(mod2) \\ u_t - t^n u_{xx} + it^m u_x = 0, & n > 2m + 1,\ n \equiv 0(mod2) \end{array}$$

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are non-solvable locally at the origin. It should be noted that these non-solvable linear algebraic or ordinary differential equation systems have been characterized recently by the author in the references [12]-[13].

The objective of this paper is to characterize those non-solvable partial differential equation systems of first order on one function $u(x_1, x_2, \cdots, x_n)$ by a combinatorial approach, classify these systems and characterize their behaviors with some applications. For such an objective, we should know its counterpart, i.e., solvable conditions on partial differential equations (PDE). The following result is well-known from standard textbooks, such as those of [4] or [15].

THEOREM 1.1. *Let*

$$\begin{cases} x_i = x_i(t, s_1, s_2, \cdots, s_{n-1}) \\ u = u(t, s_1, s_2, \cdots, s_{n-1}) \\ p_i = p_i(t, s_1, s_2, \cdots, s_{n-1}), \quad i = 1, 2, \cdots, n \end{cases} \qquad (SDE)$$

be a solution of system

$$\frac{dx_1}{F_{p_1}} = \frac{dx_2}{F_{p_2}} = \cdots = \frac{dx_n}{F_{p_n}} = \frac{du}{\sum\limits_{i=1}^{n} p_i F_{p_i}}$$

$$= -\frac{dp_1}{F_{x_1} + p_1 F_u} = \cdots = -\frac{dp_n}{F_{x_n} + p_n F_u} = dt$$

with initial values

$$\begin{cases} x_{i_0} = x_{i_0}(s_1, s_2, \cdots, s_{n-1}) \\ u_0 = u_0(s_1, s_2, \cdots, s_{n-1}) \\ p_{i_0} = p_{i_0}(s_1, s_2, \cdots, s_{n-1}), \quad i = 1, 2, \cdots, n \end{cases} \qquad (IDE)$$

such that

$$\begin{cases} F(x_{1_0}, x_{2_0}, \cdots, x_{n_0}, u, p_{1_0}, p_{2_0}, \cdots, p_{n_0}) = 0 \\ \dfrac{\partial u_0}{\partial s_j} - \sum\limits_{i=0}^{n} p_{i_0} \dfrac{\partial x_{i_0}}{\partial s_j} = 0, \quad j = 1, 2, \cdots, n-1. \end{cases}$$

Then (SDE) is the solution of partial differential equation

$$F(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \qquad (PDE)$$

of first order with initial values (IDE), where $p_i = \dfrac{\partial u}{\partial x_i}$ *and* $F_{p_i} = \dfrac{\partial F}{\partial p_i}$ *for integers* $1 \leq i \leq n$.

Particularly, if such a partial differential equation (PDE) of first order is linear or quasilinear, let

$$L = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$$

be a partial differential operator of first order with continuously differentiable functions a_i, $1 \leq i \leq n$. Then such a linear or quasilinear partial differential equation (PDE) of first order can be denoted by

$$L[u] \equiv \sum_{i=1}^{n} a_i \frac{\partial u}{\partial x_i} = c, \qquad (LPDE)$$

where c is a continuously differentiable function. Let $L_1, L_2, \cdots, L_m$ be m partial differential operators of first order (linear or non-linear) and h_i, $1 \leq i \leq m$ continuously differentiable functions on $\mathbb{R}^n$. Then *is the partial differential equation system*

$$L_i[u(x_1, x_2, \cdots, x_n)] = h_i, \qquad 1 \leq i \leq m \qquad (PDES_m)$$

solvable or not for a differential mapping $u : \mathbb{R}^n \to \mathbb{R}^n$? Similarly, let ϖ_i $1 \leq i \leq m$ be continuous functions on $\mathbb{R}^n$. Then *is the Cauchy problem*

$$\begin{cases} L_i[u] = h_i \\ u(x_1, x_2, \cdots, x_{n-1}, x_n^0) = \varpi_i, \quad 1 \leq i \leq m \end{cases} \qquad (PDES_m^C)$$

solvable or not? Denoted by S_i^0 the solution of ith equation in system $(PDES_m)$ or $(DEPS_m^C)$. Then the partial differential equation system $(PDES_m)$ or $(DEPS_m^C)$ is solvable only if $\bigcap_{i=1}^{m} S_i^0 \neq \emptyset$. Notice that $u : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable. Thus the systems $(PDES_m)$ or $(DEPS_m^C)$ is solvable only if $\bigcap_{i=1}^{m} S_i^0$ is a non-empty functional set on a domain $D \subset \mathbb{R}^n$. Otherwise, non-solvable, i.e., $\bigcap_{i=1}^{m} S_i^0 = \emptyset$ for any domain $D \subset \mathbb{R}^n$.

An equation or a system of equations is said *reducible* if it can be reduced from another(s) with the same solutions. Now let $(PDES_m)$ be a system of partial differential equations with

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, u_{x_1}, \cdots, u_{x_n}, u_{x_1x_2}, \cdots, u_{x_1x_n}, \cdots) = 0 \end{cases}$$

on a function $u(x_1, \cdots, x_n, t)$. Then its *symbol* is determined by

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n, p_1p_2, \cdots, p_1p_n, \cdots) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n, p_1p_2, \cdots, p_1p_n, \cdots) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, p_1, \cdots, p_n, p_1p_2, \cdots, p_1p_n, \cdots) = 0, \end{cases}$$

i.e., substitute $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ into $(PDES_m)$ for the term $u_{x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}}$, where $\alpha_i \geq 0$ for integers $1 \leq i \leq n$.

DEFINITION 1.2. *A non-solvable $(PDES_m)$ is algebraically contradictory if its symbol is non-solvable. Otherwise, differentially contradictory.*

For example, the system of partial differential equations following

$$\begin{cases} u_x + 2u_y + 3u_z = 2 + y^2 + z^2 \\ yzu_x + xzu_y + xyu_z = x^2 - y^2 - z^2 \\ (yz+1)u_x + (xz+2)u_y + (xy+3)u_z = x^2 + 1 \end{cases}$$

is algebraically contradictory because its symbol

$$\begin{cases} p_1 + 2p_2 + 3p_3 = 2 + y^2 + z^2 \\ yzp_1 + xzp_2 + xyp_3 = x^2 - y^2 - z^2 \\ (yz+1)p_1 + (xz+2)p_2 + (xy+3)p_3 = x^2 + 1 \end{cases}$$

is non-solvable.

All terminologies and notations in this paper are standard. For those not mentioned here, we follow the [4] and [15] for partial differential equation. [8]-[10], [16] for algebra, topology and Smarandache systems, and [1]-[2] for mechanics.

2. Non-solvable systems of partial differential equations. First, we get the non-solvability of Cauchy problem of partial differential equations of first order following.

THEOREM 2.1. *A Cauchy problem on systems*

$$\begin{cases} F_1(x_1,x_2,\cdots,x_n,u,p_1,p_2,\cdots,p_n)=0 \\ F_2(x_1,x_2,\cdots,x_n,u,p_1,p_2,\cdots,p_n)=0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1,x_2,\cdots,x_n,u,p_1,p_2,\cdots,p_n)=0 \end{cases}$$

of partial differential equations of first order is non-solvable with initial values

$$\begin{cases} x_i|_{x_n=x_n^0}=x_i^0(s_1,s_2,\cdots,s_{n-1}) \\ u|_{x_n=x_n^0}=u_0(s_1,s_2,\cdots,s_{n-1}) \\ p_i|_{x_n=x_n^0}=p_i^0(s_1,s_2,\cdots,s_{n-1}), \quad i=1,2,\cdots,n \end{cases}$$

if and only if the system

$$F_k(x_1,x_2,\cdots,x_n,u,p_1,p_2,\cdots,p_n)=0,\ 1\le k\le m$$

is algebraically contradictory, in this case, there must be an integer k_0, $1\le k_0\le m$ such that

$$F_{k_0}(x_1^0,x_2^0,\cdots,x_{n-1}^0,x_n^0,u_0,p_1^0,p_2^0,\cdots,p_n^0)\neq 0$$

or it is differentially contradictory itself, i.e., there is an integer j_0, $1\le j_0\le n-1$ such that

$$\frac{\partial u_0}{\partial s_{j_0}}-\sum_{i=0}^{n-1}p_i^0\frac{\partial x_i^0}{\partial s_{j_0}}\neq 0.$$

Proof. If the Cauchy problem

$$\left.\begin{array}{l} F_k(x_1,x_2,\cdots,x_n,u,p_1,p_2,\cdots,p_n)=0 \\ x_i|_{x_n=x_n^0}=x_i^0,\ u|_{x_n=x_n^0}=u_0,\ p_i|_{x_n=x_n^0}=p_i^0,\ i=1,2,\cdots,n \end{array}\right\},\quad 1\le k\le m$$

of partial differential equations of first order is solvable, it is clear that the symbol of system of partial differential equations

$$\begin{cases} F_1(x_1,x_2,\cdots,x_n,u,p_1,p_2,\cdots,p_n)=0 \\ F_2(x_1,x_2,\cdots,x_n,u,p_1,p_2,\cdots,p_n)=0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1,x_2,\cdots,x_n,u,p_1,p_2,\cdots,p_n)=0 \end{cases}$$

can not be contradictory, i.e., compatible. Furthermore, if it is algebraically contradictory, then there must be an integer k_0 such that

$$F_{k_0}(x_1^0,x_2^0,\cdots,x_{n-1}^0,x_n^0,u_0,p_1^0,p_2^0,\cdots,p_n^0)\neq 0.$$

Otherwise, the $x_i^0, u_0, p_i^0,\ 1 \leq i \leq n$ is a solution of the system, a contradiction.

Notice that $u_0 = u(x_1^0, \cdots, x_n^0) = u(x_1^0(s_1, \cdots, s_{n-1}), \cdots, x_n^0(s_1, \cdots, s_{n-1}))$. There must be

$$\frac{\partial u_0}{\partial s_j} - \sum_{i=0}^{n-1} p_i^0 \frac{\partial x_i^0}{\partial s_j} = 0$$

for any integer $j,\ 1 \leq j \leq n-1$.

Now if the system of partial differential equations

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \end{cases}$$

is not algebraically contradictory, we can find its non-trivial solutions x_i^0, u_0, p_i^0, $1 \leq i \leq n$. Furthermore, if

$$\frac{\partial u_0}{\partial s_j} - \sum_{i=0}^{n-1} p_i^0 \frac{\partial x_i^0}{\partial s_j} = 0$$

for any integer $j,\ 1 \leq j \leq n-1$, then the Cauchy problem

$$\begin{cases} F_k(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ x_i|_{x_n = x_n^0} = x_i^0,\ u|_{x_n = x_n^0} = u_0,\ p_i|_{x_n = x_n^0} = p_i^0,\ i = 1, 2, \cdots, n \end{cases}$$

is solvable by Theorem 1.1. Let

$$\begin{cases} x_i^{[k]} = x_i^{[k]}(t, s_1, s_2, \cdots, s_{n-1}) \\ u^{[k]} = u^{[k]}(t, s_1, s_2, \cdots, s_{n-1}) \\ p_i^{[k]} = p_i^{[k]}(t, s_1, s_2, \cdots, s_{n-1}), \quad i = 1, 2, \cdots, n \end{cases}$$

be the solution of Cauchy problem

$$\begin{cases} F_k(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ x_i|_{x_n = x_n^0} = x_i^0,\ u|_{x_n = x_n^0} = u_0,\ p_i|_{x_n = x_n^0} = p_i^0,\ i = 1, 2, \cdots, n \end{cases}$$

of partial differential equation of first order for an integer $1 \leq k \leq m$, i.e., the solution of its characteristic system

$$\frac{dx_1}{F_{kp_1}} = \frac{dx_2}{F_{kp_2}} = \cdots = \frac{dx_n}{F_{kp_n}} = \frac{du}{\sum\limits_{i=1}^{n} p_i F_{kp_i}}$$
$$= -\frac{dp_1}{F_{kx_1} + p_1 F_{ku}} = \cdots = -\frac{dp_n}{F_{kx_n} + p_n F_{ku}} = dt$$

with initial values $x_i^0, u_0, p_i^0,\ 1 \leq i \leq n$.

Without loss of generality, denoted by $S^{[k]}$ all of its solutions $x_i^{[k]}, u^{[k]}, p_i^{[k]}, 1 \leq i \leq n$. Then

$$\bigcap_{k=1}^{m} S^{[k]} \supseteq \{x_i^0, u_0, p_i^0,\ 1 \leq i \leq n\} \neq \emptyset.$$

Thus the Cauchy problem on partial differential equations

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \end{cases}$$

of first order with initial values

$$\begin{cases} x_i^0 = x_i^0(s_1, s_2, \cdots, s_{n-1}) \\ u_0 = u_0(s_1, s_2, \cdots, s_{n-1}) \\ p_i^0 = p_i^0(s_1, s_2, \cdots, s_{n-1}), \quad i = 1, 2, \cdots, n \end{cases}$$

is solvable. This completes the proof. □

COROLLARY 2.2. *Let*

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \end{cases}$$

be an algebraically contradictory system of partial differential equations of first order. Then there are no values $x_i^0, u_0, p_i^0,\ 1 \leq i \leq n$ *such that*

$$\begin{cases} F_1(x_1^0, x_2^0, \cdots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \cdots, p_n^0) = 0, \\ F_2(x_1^0, x_2^0, \cdots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \cdots, p_n^0) = 0. \end{cases}$$

COROLLARY 2.3. *A Cauchy problem*

$$\begin{cases} F_1(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ F_2(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0 \end{cases}$$

of first order with

$$F_k(x_1^0, x_2^0, \cdots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \cdots, p_n^0) = 0, \quad 1 \leq k \leq m$$

for values $x_i^0, u_0, p_i^0,\ 1 \leq i \leq n$ *is non-solvable if and only if there is an integer* $j_0,\ 1 \leq j_0 \leq n-1$ *such that*

$$\frac{\partial u_0}{\partial s_{j_0}} - \sum_{i=0}^{n-1} p_i^0 \frac{\partial x_i^0}{\partial s_{j_0}} \neq 0.$$

If the system $(PDES_m)$ is linear or quasilinear, i.e.,

$$\sum_{i=1}^{n} a_i^{[k]} \frac{\partial u}{\partial x_i} = c^{[k]},\ 1 \leq k \leq m, \qquad (LPDES_m)$$

then

$$F_k = \sum_{i=1}^{n} a_i^{[k]} \frac{\partial u}{\partial x_i} - c^{[k]} = \sum_{i=1}^{n} a_i^{[k]} p_i + b^{[k]} p_t - c^{[k]}$$

for integers $1 \le k \le m$.

Calculation shows that $F_{kp_i} = a_i^{[k]}, \ 1 \le k \le m$ and

$$\sum_{i=1}^{n} p_i F_{kp_i} = \sum_{i=1}^{n} a_i^{[k]} p_i = c^{[k]},$$

$$F_{kx_l} = \sum_{i=1}^{n} a_{ix_l}^{[k]} p_i - c_{x_i}^{[k]}, \quad F_{ku} = \sum_{i=1}^{n} a_{iu}^{[k]} p_i - c_u^{[k]}$$

and

$$\frac{\partial}{\partial x_l}\left(\sum_{i=1}^{n} a_i^{[k]} p_i - c^{[k]}\right) = \sum_{i=1}^{n}\left(a_{ix_l}^{[k]} p_i + a_{iu}^{[k]} p_l p_i + a_i^{[k]} p_{ix_l}\right) - \left(c_{x_l}^{[k]} + c_u^{[k]} p_l\right) = 0.$$

We know that

$$F_{kx_l} + p_l F_u = \sum_{i=1}^{n} a_i^{[k]} p_{ix_l} = \sum_{i=1}^{n} a_i^{[k]} p_{lx_i}.$$

Notice that on a solution surface $u(x_1, \cdots, x_n)$,

$$\frac{dp_l}{dx_l} = \sum_{i=1}^{n} p_{lx_i} \frac{dx_i}{dx_l} = \sum_{i=1}^{n} p_{lx_i} \frac{a_i^{[k]}}{a_l^{[k]}},$$

which implies that

$$\frac{dx_i}{a_i^{[k]}} = \frac{dp_l}{F_{kx_l} + p_l F_u} = \frac{dp_l}{\sum\limits_{i=1}^{n} a_i^{[k]} p_{lx_i}}$$

is an identity. Thus, if the system $(PDES_m)$ is linear or quasilinear system $(LPDES_m)$, we only need to consider the characteristic system

$$\frac{dx_1}{F_{kp_1}} = \frac{dx_2}{F_{kp_2}} = \cdots = \frac{dx_n}{F_{kp_n}} = \frac{du}{\sum\limits_{i=1}^{n} p_i F_{kp_i}}$$

for finding solutions $u(x_1, \cdots, x_n)$. Furthermore, we only need to prescribe the initial data by $u|_{x_n = x_n^0}$, then the condition

$$\frac{\partial u_0}{\partial s_{j_0}} - \sum_{i=0}^{n} p_i^0 \frac{\partial x_i^0}{\partial s_{j_0}} = 0$$

is naturally hold by $p_{i_0} = \left.\dfrac{\partial u}{\partial x_i}\right|_{x_n = x_n^0}$ in this case. Consequently, we can get simpler conditions for linear or quasilinear non-solvable $(LPDES_m)$ than that of Theorem 2.1.

Corollary 2.4. *A Cauchy problem* $(LPDES_m^C)$ *of quasilinear partial differential equations with initial values* $u|_{x_n = x_n^0} = u_0$ *is non-solvable if and only if the system* $(LPDES_m)$ *of partial differential equations is algebraically contradictory.*

Particularly, if the Cauchy problem $(LPDES_m^C)$ of partial differential equations is linear with $c^{[i]}=0,\ 1\leq i\leq m$, we know the following conclusion.

Corollary 2.5. *A Cauchy problem* $(LPDES_m^C)$ *of linear partial differential equations with* $c^{[i]}=0,\ 1\leq i\leq m$ *and initial values* $u|_{x_n=x_n^0}=u_0$ *is non-solvable if and only if the system* $(LPDES_m)$ *of partial differential equations is algebraically contradictory.*

If a $(PDES_m)$ is not algebraic contradictory, we can find initial values x_i^0, u_0, p_i^0, $1\leq i\leq n$ by solving the algebraic system

$$\begin{cases} F_1(x_1,x_2,\cdots,x_{n-1},x_n^0,u,p_1,p_2,\cdots,p_n)=0\\ F_2(x_1,x_2,\cdots,x_{n-1},x_n^0,u,p_1,p_2,\cdots,p_n)=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ F_m(x_1,x_2,\cdots,x_{n-1},x_n^0,u,p_1,p_2,\cdots,p_n)=0.\end{cases}$$

Generally, all these datum $x_i^0, u_0, p_i^0,\ 1\leq i\leq n$ appeared in Theorem 2.1, particularly, these datum $x_i^0,\ 1\leq i\leq n$ in Corollaries 2.4 and 2.5 consist a domain, i.e., a manifold for u, such as those shown in the following example.

Example 2.6. Let us consider the partial differential equation system

$$\begin{cases} (z-y)u_x+(x-z)u_y+(y-x)u_z=0\\ xzu_x+yzu_y+zu_z=0\\ u|_{z=0}=e^{x+y}\end{cases}$$

The characteristic system of its first equation is

$$\frac{dx}{z-y}=\frac{dy}{x-z}=\frac{dz}{y-x}$$

and we are easily to find two independent initial integrals

$$\varphi_1=x+y+z,\quad \varphi_2=x^2+y^2+z^2.$$

Consequently,

$$x+y=\varphi_1|_{z=0}=\overline{\varphi}_1,\ \ x^2+y^2=\varphi_2|_{z=0}=\overline{\varphi}_2.$$

Solving this algebraic system, we know that

$$\begin{cases} x=\dfrac{\overline{\varphi}_1\pm\sqrt{\overline{\varphi}_2-\overline{\varphi}_1^2}}{2},\\ y=\dfrac{\overline{\varphi}_1\mp\sqrt{\overline{\varphi}_2-\overline{\varphi}_1^2}}{2}.\end{cases}$$

Hence, the solution of

$$\begin{cases} (z-y)u_x+(x-z)u_y+(y-x)u_z=0\\ u|_{z=0}=e^{x+y}\end{cases}$$

is $u=e^{x+y}=e^{\overline{\varphi}_1}=e^{x+y+z}$.

Similarly, the characteristic system of its second equation is

$$\frac{dx}{xz}=\frac{dy}{yz}=\frac{dz}{z}$$

with two independent initial integrals

$$\psi_1 = \frac{x}{e^z}, \quad \psi_2 = \frac{y}{e^z},$$

which implies that $x = \overline{\psi}_1, \ y = \overline{\psi}_2$. Whence, the solution of

$$\begin{cases} xzu_x + yzu_y + zu_z = 0 \\ u|_{z=0} = e^{x+y} \end{cases}$$

is $u = e^{x+y} = e^{\frac{x+y}{e^z}}$. Consequently, $u = e^{x+y+z} = e^{\frac{x+y}{e^z}}$. Calculation shows that

$$x + y + \frac{e^z}{e^e - 1} z = 0,$$

which is the domain of solutions of the partial differential equation system.

For the non-solvability of shifted partial differential equations of first order, we know the following result.

THEOREM 2.7. *A Cauchy problem on systems*

$$\begin{cases} F_1(x_1, \cdots, x_n, x_{n+1}^{[1]}, \cdots, x_{n_1}^{[1]}, u, p_1, \cdots, p_n, p_{n+1}^{[1]}, \cdots, p_{n_1}^{[1]}) = 0 \\ F_2(x_1, \cdots, x_n, x_{n+1}^{[2]}, \cdots, x_{n_2}^{[2]}, u, p_1, \cdots, p_n, p_{n+1}^{[2]}, \cdots, p_{n_2}^{[2]}) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, \cdots, x_n, x_{n+1}^{[m]}, \cdots, x_{n_m}^{[m]}, u, p_1, \cdots, p_n, p_{n+1}^{[m]}, \cdots, p_{n_m}^{[m]}) = 0 \end{cases}$$

of partial differential equations of first order is non-solvable with initial values

$$\begin{cases} u_0 = u_0(s_1, s_2, \cdots, s_{n_m-1}) \\ x_i^0 = x_i^0(s_1, s_2, \cdots, s_{n_m-1}), \ 1 \le i \le n \ \text{and} \ x_i^0 = 0, \ i \ge n+1 \\ p_i^0 = p_i^0(s_1, s_2, \cdots, s_{n_m-1}), \ 1 \le i \le n \ \text{and} \ p_i^0 = 0, \ i \ge n+1, \end{cases}$$

where $x_1, \cdots, x_n, x_{n+1}^{[1]}, \cdots, x_{n_1}^{[1]}, x_{n+1}^{[2]}, \cdots, x_{n_2}^{[2]}, \cdots, x_{n+1}^{[m]}, \cdots, x_{n_m}^{[m]}$ *are independent,* $p_k^{[i]} = \partial u / \partial x_k^{[i]}$ *and* $n \le n_1 \le n_2 \le \cdots \le n_m$ *if and only if there are integers* $\{n_{i_1}, n_{i_2}, \cdots, n_{i_l}\} \subset \{n_1, n_2, \cdots, n_m\}$, $n_{i_1} = n_{i_2} = \cdots = n_{i_l} = n$ *such that the system*

$$\begin{cases} F_{i_1}(x_1, \cdots, x_n, u, p_1, \cdots, p_n) = 0 \\ F_{i_2}(x_1, \cdots, x_n, u, p_1, \cdots, p_n) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_{i_l}(x_1, \cdots, x_n, u, p_1, \cdots, p_n) = 0 \end{cases}$$

is algebraically contradictory, in this case, there must be an integer $k_0, \ 1 \le k_0 \le l$ *such that*

$$F_{i_{k_0}}(x_1^0, x_2^0, \cdots, x_{n-1}^0, x_n^0, u_0, p_1^0, p_2^0, \cdots, p_n^0) \ne 0$$

or it is differentially contradictory itself, i.e., there is an integer $j_0, \ 1 \le j_0 \le n_m - 1$ *such that*

$$\frac{\partial u_0}{\partial s_{j_0}} - \sum_{i=0}^{n} p_i^0 \frac{\partial x_i^0}{\partial s_{j_0}} \ne 0.$$

Proof. Notice that $x_1,\cdots,x_n,x_{n+1}^{[1]},\cdots,x_{n_1}^{[1]},x_{n+1}^{[2]},\cdots,x_{n_2}^{[2]},\cdots,x_{n+1}^{[m]},\cdots,x_{n_m}^{[m]}$ are independent. Whence, if $n_i\neq n_j$, the system

$$\begin{cases} F_i(x_1,\cdots,x_n,x_{n+1}^{[1]},\cdots,x_{n_1}^{[1]},u,p_1,\cdots,p_n,p_{n+1}^{[1]},\cdots,p_{n_1}^{[1]})=0\\ F_j(x_1,\cdots,x_n,x_{n+1}^{[2]},\cdots,x_{n_2}^{[2]},u,p_1,\cdots,p_n,p_{n+1}^{[2]},\cdots,p_{n_2}^{[2]})=0\end{cases}$$

must be algebraically compatible. Furthermore, for any integer $1\leq k\leq m$ we know the Cauchy problem

$$\begin{cases} F_k(x_1,\cdots,x_n,x_{n+1}^{[k]},\cdots,x_{n_1}^{[k]},u,p_1,\cdots,p_n,p_{n+1}^{[k]},\cdots,p_{n_1}^{[k]})=0,\\ x_i|_{x_n=x_n^0}=x_i^0(s_1,s_2,\cdots,s_{n_m-1}),\ 1\leq i\leq n;\ x_i^{[k]}|_{x_n=x_n^0}=0,\ i\geq n+1,\\ u|_{x_n=x_n^0}=u_0(s_1,s_2,\cdots,s_{n_m-1}),\\ p_i|_{x_n=x_n^0}=p_i^0(s_1,s_2,\cdots,s_{n_m-1}),\ 1\leq i\leq n;\ p_i^{[k]}|_{x_n=x_n^0}=0,\ i\geq n+1\end{cases}$$

is solvable by Theorem 1.1 if

$$\frac{\partial u_0}{\partial s_j}-\sum_{i=0}^{n}p_i^0\frac{\partial x_i^0}{\partial s_j}=0$$

for any integer $1\leq j\leq n_m-1$. Whence, the conclusion follows by a similar way to that of Theorem 2.1. □

3. Combinatorial classification of partial differential equations. According to Theorem 2.1 and Corollary 2.2, if the system of partial differential equations

$$\begin{cases} F_1(x_1,x_2,\cdots,x_n,u,p_1,p_2,\cdots,p_n)=0\\ F_2(x_1,x_2,\cdots,x_n,u,p_1,p_2,\cdots,p_n)=0\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ F_m(x_1,x_2,\cdots,x_n,u,p_1,p_2,\cdots,p_n)=0\end{cases}$$

is algebraically contradictory, there are no initial values $x_i^0,u_0,p_i^0,\ 1\leq i\leq n$ such that the Cauchy problem

$$\begin{cases} F_k(x_1,\cdots,x_n,u,p_1,\cdots,p_n,)=0,\\ x_i|_{x_n=x_n^0}=x_i^0,\ u|_{x_n=x_n^0}=u_0,\ p_i|_{x_n=x_n^0}=p_i^0,\quad i=1,2,\cdots,n\end{cases}$$

is all solvable for any integer $1\leq k\leq m$. Whence, we need to prescribe different initial values $x_i^{[k^0]},u_0^{[k]},p_i^{[k^0]},\ 1\leq i\leq n$ for integers $1\leq k\leq m$. In fact, the following two steps enable one to find these initial values with minimum numbers:

Step 1: Decompose $(PDES_m)$ into minimal compatible families $\mathscr{F}_1,\mathscr{F}_2,\cdots,\mathscr{F}_s$ such that:

(1) All equations in $\mathscr{F}_i$ is maximal algebraically compatible for any integer $1\leq i\leq s$;

(2) $|\mathscr{F}_1|+|\mathscr{F}_2|+\cdots+|\mathscr{F}_s|=m$.

Step 2: Solve family $\mathscr{F}_i$ and prescribe initial values $x_j^{[i^0]},u_0^{[i]},p_j^{[i^0]},\ 1\leq j\leq n$ in the algebraic solution of $\mathscr{F}_i$ for integers $1\leq i\leq s$.

Furthermore, we assume these initial values $x_i^{[k^0]},u_0^{[k]},p_i^{[k^0]},\ 1\leq i\leq n$ hold with

$$\frac{\partial u_0^{[k^0]}}{\partial s_j}-\sum_{i=0}^{n}p_i^{[k^0]}\frac{\partial x_i^{[k^0]}}{\partial s_j}=0$$

for integers $1 \leq j \leq n-1$, $1 \leq k \leq s$ and denote the solution space of Cauchy problem

$$\begin{cases} F_k(x_1, \cdots, x_n, u, p_1, \cdots, p_n,) = 0, \\ x_i|_{x_n=x_n^0} = x_i^{[k^0]},\ u|_{x_n=x_n^0} = u_0^{[k]},\ p_i|_{x_n=x_n^0} = p_i^{[k^0]} \end{cases}$$

by $S^{[k]}$. Then we can define a vertex-edge labeled graph $G[PDES_m^C]$ as follows:

$$V(G[PDES_m^C]) = \{S^{[i]} | 1 \leq i \leq m\},$$

$$E(G[PDES_m^C]) = \{(S^{[i]}, S^{[j]}) | S^i \bigcap S^j \neq \emptyset,\ 1 \leq i, j \leq m\}$$

with labels $l(S^{[i]}) = S^{[i]}$, $l(S^{[i]}, S^{[j]}) = S^i \bigcap S^j$ for integers $1 \leq i, j \leq m$. Its underlying graph of $G[PDES_m^C]$, i.e., without labels is denoted by $\widehat{G}[PDES_m^C]$. Particularly, by replacing each label $S^{[i]}$ with $S_0^{[i]} = \{u_0^{[i]}\}$ and $S^{[i]} \bigcap S^{[j]}$ by $S_0^{[i]} \bigcap S_0^{[j]}$ for integers $1 \leq i, j \leq m$, we get a new vertex-edge labeled graph, denoted by $G_0[PDES_m^C]$. Clearly, $\widehat{G}[PDES_m^C] \simeq \widehat{G}_0[PDES_m^C]$.

Then the following results on $\widehat{G}[PDES_m^C]$ are easily know by definition.

THEOREM 3.1. *If $\widehat{G}[PDES_m^C] \not\simeq K_m$, or $\widehat{G}[PDES_m^C] \simeq K_m$ but there are integers $1 \leq i, j, k \leq m$ such that $S^{[i]} \bigcap S^{[j]} \bigcap S^{[k]} = \emptyset$, where m is the number of equations in $(PDES_m^C)$, then $(PDES_m^C)$ is non-solvable.*

Proof. Clearly, if the system $(PDES_m^C)$ is solvable, then any subsystem of equations in $(PDES_m^C)$ is solvable. This fact implies that $\widehat{G}[PDES_m^C]$ is a complete graph and for three integers $1 \leq i, j, k \leq m$, $S^{[i]} \bigcap S^{[j]} \bigcap S^{[k]} \neq \emptyset$. Thus, if $\widehat{G}[PDES_m^C] \not\simeq K_m$, or $S^{[i]} \bigcap S^{[j]} \bigcap S^{[k]} = \emptyset$ for three integers $1 \leq i, j, k \leq m$, then the Cauchy problem $(PDES_m^C)$ is non-solvable. □

The following result enables one to introduce the conception of *G-solution* of partial differential equations of first order.

THEOREM 3.2. *For any system $(PDES_m^C)$ of partial differential equations of first order, $\widehat{G}[PDES_m^C]$ is simple. Conversely, for any simple graph G, there is a system $(PDES_m^C)$ of partial differential equations of first order such that $\widehat{G}[PDES_m^C] \simeq G$.*

Proof. By definition, it is clear that the graph $\widehat{G}[PDES_m^C]$ is simple for any system $(PDES_m^C)$ of partial differential equations of first order. Notice that for any partial differential equation

$$F(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) = 0,$$

there are infinitely partial differential equations algebraically contradictory with it, for example, the equation

$$F(x_1, x_2, \cdots, x_n, u, p_1, p_2, \cdots, p_n) + s = 0,$$

and there are also infinitely partial differential equations not algebraically contradictory with it, for example, the equation

$$F(x_1, x_2, \cdots, x_n, u, p_1 + s, p_2 + s, \cdots, p_n + s) = 0$$

for a real number $s \neq 0$. All of these facts enables one to construct a system $(PDES_m^C)$ of partial differential equations such that $G[PDES_m^C] \simeq G$.

For $\forall v_1 \in V(G)$, label it with $S^{[v_1]}$, where $S^{[v_1]}$ is the solution space of Cauchy problem

$$\begin{cases} F_{v_1}(x_1, \cdots, x_n, u, p_1, \cdots, p_n,) = 0, \\ x_i|_{x_n=x_n^0} = x_{i0}^{v_1},\ u|_{x_n=x_n^0} = u_0^{v_1},\ p_i|_{x_n=x_n^0} = p_{i0}^{v_1}. \end{cases}$$

If vertices $v_1, v_2, \cdots, v_k$ have been labeled and $V(G) \setminus \{v_1, v_2, \cdots, v_k\} \neq \emptyset$, let $v_{k+1} \in V(G) \setminus \{v_1, v_2, \cdots, v_k\}$. Not loss of generality, assume $\{v_1, v_2, \cdots, v_k\} = \{v_{i_1}, v_{i_2}, \cdots, v_{i_l}\} \bigcup \{v_{j_1}, v_{j_2}, \cdots, v_{j_{k-l}}\}$ such that $v_{k+1}v_{i_s} \in E(G),\ 1 \leq s \leq k$ and $v_{k+1}v_{j_t} \notin E(G),\ 1 \leq t \leq k-l$. Label the vertex v_{k+1} by $S^{[v_{k+1}]}$, where $S^{[v_{k+1}]}$ is the solution space of such a Cauchy problem

$$\begin{cases} F_{v_{k+1}}(x_1, \cdots, x_n, u, p_1, \cdots, p_n,) = 0, \\ x_i|_{x_n=x_n^0} = x_{i0}^{v_{k+1}},\ u|_{x_n=x_n^0} = u_0^{v_{k+1}},\ p_i|_{x_n=x_n^0} = p_{i0}^{v_{k+1}} \end{cases}$$

that

$$\begin{cases} F_{v_{k+1}}(x_1, \cdots, x_n, u, p_1, \cdots, p_n,) = 0, \\ F_{v_{i_s}}(x_1, \cdots, x_n, u, p_1, \cdots, p_n,) = 0 \end{cases}$$

is algebraically compatible for integers $1 \leq s \leq l$ but the system

$$\begin{cases} F_{v_{k+1}}(x_1, \cdots, x_n, u, p_1, \cdots, p_n,) = 0, \\ F_{v_{j_t}}(x_1, \cdots, x_n, u, p_1, \cdots, p_n,) = 0 \end{cases}$$

is algebraically contradictory for integers $1 \leq t \leq k-l$. As we discussed previous, such a partial differential equation

$$F_{v_{k+1}}(x_1, \cdots, x_n, u, p_1, \cdots, p_n,) = 0$$

can be always chosen.

Continuing this process, all vertices in G are labeled by the induction and we get a system $(PDES_m^C)$ of partial differential equations

$$\begin{cases} F_v(x_1, \cdots, x_n, u, p_1, \cdots, p_n,) = 0, \quad v \in V(G), \\ x_i|_{x_n=x_n^0} = x_{i0}^{v},\ u|_{x_n=x_n^0} = u_0^{v},\ p_i|_{x_n=x_n^0} = p_{i0}^{v}. \end{cases}$$

Clearly, such a system $(PDES^C)$ with $\widehat{G}[PDES_m^C] \simeq G$ by construction. In fact, the bijection $\varphi:\ S^{[v]} \in V(G[PDES_m^C]) \to v \in V(G)$ is a graph isomorphism from $\widehat{G}[PDES_m^C]$ to G. This completes the proof. □

Notice that the symbol of a linear partial differential equation

$$F(x_1, \cdots, x_n, u, p_1, \cdots, p_n,) = 0$$

of first order is a superplane in $\mathbb{R}^{2n+1}$. Thus for an algebraically contradictory linear system

$$\begin{cases} F_i(x_1, \cdots, x_n, u, p_1, \cdots, p_n,) = 0 \\ F_j(x_1, \cdots, x_n, u, p_1, \cdots, p_n,) = 0, \end{cases}$$

if

$$F_k(x_1, \cdots, x_n, u, p_1, \cdots, p_n,) = 0$$

is contradictory to one of there two partial differential equations, then it must be contradictory to another. This fact enables one to classify equations in $(LPDES_m)$ by contradictory property and determine its $\widehat{G}[LPDES_m^C]$ following.

THEOREM 3.3. *Let $(LPDES_m)$ be a system of linear partial differential equations of first order with maximal contradictory classes $\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_s$ on equations in $(LPDES)$. Then $\widehat{G}[LPDES_m^C] \simeq K(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_s)$, i.e., an s-partite complete graph.*

Proof. By definition, these equations in a contradictory class $\mathscr{C}_i$, $1 \leq i \leq s$ are contradictory. Thus there are no edges between them. Similarly, these equations in two different contradictory classes $\mathscr{C}_i, \mathscr{C}_j$, $1 \leq i \neq j \leq s$ can not be contradictory. Thus there are edges between them. Whence, $\widehat{G}[LPDES_m^C]$ is nothing else but the s-partite complete graph $K(\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_s)$. □

EXAMPLE 3.4. Let us consider the following Cauchy problems

$$\begin{cases} u_t + au_x = 0 \\ u_t + xu_x = 0 \\ u_t + au_x + e^t = 0 \\ u|_{t=0} = \phi(x). \end{cases} \qquad (3-1)$$

Clearly, it is algebraically contradictory because $e^t \neq 0$ for any value t but

$$\begin{cases} u_t + au_x = 0 \\ u_t + xu_x = 0 \\ u|_{t=0} = \phi(x) \end{cases} \quad \text{and} \quad \begin{cases} tu_t + u_x = 0 \\ u_t + au_x + e^t = 0 \\ u|_{t=0} = \phi(x) \end{cases}$$

are not algebraically contradictory. The vertex-edge labeled graph $G[(3-1)]$ of Cauchy problem $(3-1)$ is shown in Fig.1,

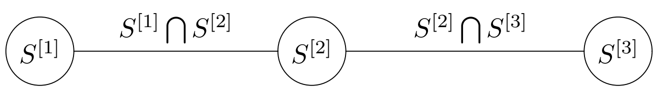

FIG. 1

where $S^{[1]}, S^{[2]}$ and $S^{[3]}$ are determined by solving these Cauchy problems

$$(1)\begin{cases} u_t + au_x = 0 \\ u|_{t=0} = \phi(x) \end{cases}, \quad (2)\begin{cases} u_t + xu_x = 0 \\ u|_{t=0} = \phi(x) \end{cases} \quad \text{and} \quad (3)\begin{cases} u_t + au_x + e^t = 0 \\ u|_{t=0} = \phi(x), \end{cases}$$

respectively. Calculation shows that

$$S^{[1]} = \{\phi(x - at)\}, \quad S^{[2]} = \{\phi\left(\frac{x}{e^t}\right)\}, \quad S^{[3]} = \{\phi(x - at) - e^t + 1\}$$

and

$$S^{[1]} \bigcap S^{[2]} = \{\phi(x - at) = \phi\left(\frac{x}{e^t}\right)\}, \quad S^{[2]} \bigcap S^{[3]} = \{\phi\left(\frac{x}{e^t}\right) = \phi(x - at) - e^t + 1\}.$$

DEFINITION 3.5. *Let $(PDES_m^C)$ be the Cauchy problem of a partial differential equation system of first order. Then the vertex-edge labeled graph $G[PDES_m^C]$ is called* *its topological graph solution and $G_0[PDES_m^C]$ the initial topological graph solution, abbreviated to G-solution, initial G-solution, respectively.*

Combining this definition with that of Theorems 3.2 and 3.3, the following conclusion is holden immediately.

THEOREM 3.6. *A Cauchy problem on system ($PDES_m$) of partial differential equations of first order with initial values $x_i^{[k^0]}, u_0^{[k]}, p_i^{[k^0]}$, $1 \leq i \leq n$ for the kth equation in ($PDES_m$), $1 \leq k \leq m$ such that*

$$\frac{\partial u_0^{[k]}}{\partial s_j} - \sum_{i=0}^{n} p_i^{[k^0]} \frac{\partial x_i^{[k^0]}}{\partial s_j} = 0$$

is uniquely G-solvable, i.e., $G[PDES]$ is uniquely determined.

Applying the combinatorial structures of G-solutions of partial differential equations, we classify them following.

DEFINITION 3.7. *Let $(PDES)_1$ and $(PDES)_2$ be two reduced systems of partial differential equations of first order in $\mathbb{R}^n$ with vertex-edge labeled graphs $G_1[PDES]$, $G_2[PDES]$. The two systems $(PDES)_1$ and $(PDES)_2$ are called to be isometric if $\widehat{G}_1[PDES] \overset{\theta}{\simeq} \widehat{G}_2[PDES]$ with $h(l(v)) = l(\theta(v))$ for $\forall v \in \widehat{G}_1[PDES]$, where h is an isometry on $\mathbb{R}^{n+1}$, denoted by $(PDES)_1 \overset{\theta}{\sim} (PDES)_2$. Particularly, if h =identity, i.e., $l(v) = l(\theta(v))$ for $\forall v \in \widehat{G}_1[PDES]$, $(PDES)_1$ and $(PDES)_2$ are called to be isotopy, denoted by $(PDES)_1 \overset{\theta}{=} (PDES)_2$.*

Let h be an isometry on $\mathbb{R}^{n+1}$. Denoted by $(PDES)^h$ such a system replaced $x_1, x_2, \cdots, x_n$ by $h(x_1), h(x_2), \cdots, h(x_n)$ and p_i by $\partial u/\partial h(x_i)$ for each equation in $(PDES)$. Then we know the following result on isometric equations.

THEOREM 3.8. *$(PDES)_1 \overset{\theta}{\sim} (PDES)_2$ if and only if there is an isometry h on $\mathbb{R}^{n+1}$ such that $(PDES)_1^h \overset{\theta}{=} (PDES)_2$. Particularly, $(PDES)_1 \overset{\theta}{=} (PDES)_2$ if and only if $G_1[PDES] \overset{\theta}{\simeq} G_2[PDES]$, i.e., reduced partial differential equations in $(PDES)_1$ are the same as those of reduced equations in $(PDES)_2$.*

Proof. Notice that $G_1[PDES] \overset{\theta}{\simeq} G_2[PDES]$ in $\mathbb{R}^{n+1}$ if and only if the G-solutions of $(PDES)_1$ and $(PDES)_2$ are coincident. By definition, if $(PDES)_1 \overset{\theta}{\sim} (PDES)_2$, then there is an isometry h such that $\widehat{G}_1[PDES] \simeq \widehat{G}_2[PDES]$ with $h(l(v)) = l(\theta(v))$ for $\forall v \in \widehat{G}_1[PDES]$, i.e., h is an isometry between the G-solutions of $(PDES)_1$ and $(PDES)_2$. Without loss of generality, let h map the G_1-solution to G_2-solution. Then it implies that $G[(PDES)_1^h] \overset{\theta}{\simeq} G_2[PDES]$. Thus $(PDES)_1^h \overset{\theta}{=} (PDES)_2$.

Particularly, if $(PDES)_1 \overset{\theta}{=} (PDES)_2$, there must be $\widehat{G}_1[PDES] \overset{\theta}{\simeq} \widehat{G}_2[PDES]$ and $l(v) = l(\theta(v))$ for $\forall v \in \widehat{G}_1[PDES]$. Thus $G_1[PDES] \overset{\theta}{\simeq} G_2[PDES]$, i.e., the G_1-solutions of $(PDES)_1$ are coincident with that of $(PDES)_2$. This fact implies that all reduced partial differential equations in $(PDES)_1$ are the same as those of reduced equations in $(PDES)_2$. □

COROLLARY 3.9. *Let $(PDES)$ be a system of partial differential equations of first order in $\mathbb{R}^n$, $[A]_{n\times n}$ an orthogonal matrix and $h = [A]_{n\times n}(x_1, x_2, \cdots, x_n)^T$. Then $(PDES)^h \overset{\theta}{\sim} (PDES)$.*

For example, let h be a linear transformation on $\mathbb{R}^2$ determined by

$$\begin{cases} x_1 = ax + by \\ y_1 = -bx + ay \end{cases}$$

with $a^2 + b^2 = 1,\ a, b \in \mathbb{R}$. Then

$$\begin{cases} \dfrac{\partial u}{\partial x} = a\dfrac{\partial u}{\partial x_1} - b\dfrac{\partial u}{\partial y} \\ \dfrac{\partial u}{\partial y} = b\dfrac{\partial u}{\partial x_1} + a\dfrac{\partial u}{\partial y_1}. \end{cases}$$

Thus, the equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

is isometric to

$$(a+b)\frac{\partial u}{\partial x} + (a-b)\frac{\partial u}{\partial y} = 0$$

since G-solution of them is K_2 with labels transformed by h each other.

4. Characterizing G-solutions.

4.1. Global stability of G-solutions. Denoted a solution $u(x_1, x_2, \cdots, x_n)$ by $u(x_1, x_2, \cdots, x_{n-1}, t)$ and G-solution, G_0-solution by $G[t]$-solution, $G[0]$-solution in this section. We discuss the global stability of $G(t)$-solutions of partial differential equation systems of first order, i.e., *sum-stability* and *prod-stability* following.

Definition 4.1. *Let $(PDES_m^C)$ be a Cauchy problem on a system of partial differential equations of first order in $\mathbb{R}^n$, and $u^{[v]}$ the solution of the vth equation with initial value $u_0^{[v]}$. Then*

(1) *The system $(PDES_m^C)$ is sum-stable if for any number $\varepsilon > 0$ there exists $\delta_v > 0,\ v \in V(\widehat{G}[0])$ such that each $G(t)$-solution with*

$$\left| u'^{[v]}_0 - u^{[v]}_0 \right| < \delta_v, \qquad \forall v \in V(\widehat{G}[0])$$

exists for all $t \geq 0$ and with the inequality

$$\left| \sum_{v \in V(\widehat{G}[t])} u'^{[v]} - \sum_{v \in V(\widehat{G}[t])} u^{[v]} \right| < \varepsilon$$

holds, denoted by $G[t] \overset{\Sigma}{\sim} G[0]$. Furthermore, if there exists a number $\beta_v > 0,\ v \in V(\widehat{G}[0])$ such that every $G'[t]$-solution with

$$\left| u'^{[v]}_0 - u^{[v]}_0 \right| < \beta_v, \qquad \forall v \in V(\widehat{G}[0])$$

satisfies

$$\lim_{t \to \infty} \left| \sum_{v \in V(\widehat{G}[t])} u'^{[v]} - \sum_{v \in V(\widehat{G}[t])} u^{[v]} \right| = 0,$$

then the $G[t]$-solution is called asymptotically stable, denoted by $G[t] \overset{\Sigma}{\to} G[0]$.

(2) *The system ($PDES_m^C$) is prod-stable if for any number $\varepsilon > 0$ there exists $\delta_v > 0,\ v \in V(\widehat{G}[0])$ such that each $G(t)$-solution with*

$$\left|u'^{[v]}_0 - u^{[v]}_0\right| < \delta_v, \qquad \forall v \in V(\widehat{G}[0])$$

exists for all $t \geq 0$ and with the inequality

$$\left|\prod_{v \in V(\widehat{G}[t])} u'^{[v]} - \prod_{v \in V(\widehat{G}[t])} u^{[v]}\right| < \varepsilon$$

holds, denoted by $G[t] \overset{\Pi}{\sim} G[0]$. Furthermore, if there exists a number $v \in V(\widehat{G}[0])$ such that every $G'[t]$-solution with

$$\left|u'^{[v]}_0 - u^{[v]}_0\right| < \beta_v, \qquad \forall v \in V(\widehat{G}[0])$$

satisfies

$$\lim_{t \to \infty} \left|\prod_{v \in V(\widehat{G}[t])} u'^{[v]} - \prod_{v \in V(\widehat{G}[t])} u^{[v]}\right| = 0,$$

Then the $G[t]$-solution is called asymptotically prod-stable, denoted by $G[t] \overset{\Pi}{\to} G[0]$.

Denote by $\ln G[t]$ such a $G[t]$-solution replaced $u^{[v]}$ by $\ln u^{[v]}$ for $\forall v \in V(G[t])$. The following result follows immediately from the definition of sum and prod-stability of $G[t]$-solution.

THEOREM 4.2. *Let ($PDES_m^C$) be a Cauchy problem of partial differential equations of first order in $\mathbb{R}^n$. Then*

(1) *$G[t] \overset{\Pi}{\sim} G[0]$ if and only if $\ln G[t] \overset{\Sigma}{\sim} \ln G[0]$, and $G[t] \overset{\Pi}{\to} G[0]$ if and only if $\ln G[t] \overset{\Sigma}{\to} \ln G[0]$.*

(2) *If there is a permutation π action on $V(G[t])$ such that*

$$\left|v'^{[v]}_0 - u^{[v]}_0\right| < \delta_v, \qquad \forall v \in V(\widehat{G}[0])$$

exists with the inequality

$$\left|u'^{[v]} - u^{[v^\pi]}\right| < \varepsilon$$

holds for $\forall v \in V(G[t])$, then $G[t] \overset{\Sigma}{\sim} G[0]$. Furthermore, if there exists a number $\beta_v > 0,\ v \in V(\widehat{G}[0])$ such that every $G'[t]$-solution with

$$\left|u'^{[v]}_0 - u^{[v]}_0\right| < \beta_v, \qquad \forall v \in V(\widehat{G}[0])$$

satisfies

$$\lim_{t \to \infty} \left|u'^{[v]} - u^{[v^\pi]}\right| = 0,$$

then $G[t] \xrightarrow{\Sigma} G[0]$. *Particularly, if* $u^{[v]}$ *is stable or asymptotically stable for* $\forall v \in V(G[t])$, *then* $G[t] \overset{\Sigma}{\sim} G[0]$ *or* $G[t] \xrightarrow{\Sigma} G[0]$.

Proof. Notice that

$$\ln\left|\prod_{v\in V(\widehat{G}[0])} u^{[v]}\right| = \sum_{v\in V(\widehat{G}[0])} \ln\left|u^{[v]}\right|$$

and if a $G[t]$-solution is prod-stable or asymptotically prod-stable, its $G'[t]$-solution replacing some $u^{[v]}$ by $-u^{[v]}$ is also prod-stable or asymptotically prod-stable, we get the conclusion (1).

For any permutation π on $V(G[t])$, it is clear that

$$\sum_{v\in V(G[t])} u^{[v^\pi]} = \sum_{v\in V(G[t])} u^{[v]},$$

which implies the conclusion (2) by definition. □

Notice that the characteristic system of the ith equation in $(PDES_m)$ is

$$\frac{dx_1}{F_{kp_1}} = \frac{dx_2}{F_{kp_2}} = \cdots = \frac{dx_n}{F_{kp_n}} = \frac{du}{\sum\limits_{i=1}^{n} p_i F_{kp_i}}$$
$$= -\frac{dp_1}{F_{kx_1} + p_1 F_{ku}} = \cdots = -\frac{dp_n}{F_{kx_n} + p_n F_{ku}} = dt.$$

Whence, the sum and prod-stability of Cauchy problem $(PDES_m^C)$ are equivalent to that of the ordinary differential equations consisting of all characteristic systems of partial differential equations in $(PDES_m^C)$ with the same initial values. Particularly, let the system $(PDES_m^C)$ be

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial t} = H_i(t, x_1, \cdots, x_{n-1}, p_1, \cdots, p_{n-1}) \\ u|_{t=t_0} = u_0^{[i]}(x_1, x_2, \cdots, x_{n-1}) \end{array}\right\} \ 1 \le i \le m \qquad (APDES_m^C)$$

A point $X_0^{[i]} = (t_0, x_{10}^{[i]}, \cdots, x_{(n-1)0}^{[i]})$ with $H_i(t_0, x_{10}^{[i]}, \cdots, x_{(n-1)0}^{[i]}) = 0$ for an integer $1 \le i \le m$ is called an *equilibrium point* of the ith equation in $(APDES_m)$. Then a result on the global stability of $(APDES_m)$ is found in the following.

THEOREM 4.3. *Let* $X_0^{[i]}$ *be an equilibrium point of the* i*th equation in* $(APDES_m^C)$,

$$X_0^\Sigma = \sum_{i=1}^{m} X_0^{[i]}, \qquad X^\Sigma(G[t]) = \sum_{v\in V(\widehat{G}[0])} X_v(t),$$
$$X_0^\Pi = \prod_{i=1}^{m} X_0^{[i]}, \qquad X^\Pi(G[t]) = \prod_{v\in V(\widehat{G}[0])} X_v(t)$$

and $L : \mathscr{O} \subset \mathbb{R}^n \to \mathbb{R}$ *a differentiable function on an open set* $\mathscr{O} \subset \mathbb{R}^n$ *containing* X_0^Σ *and* X_0^Π. *If*

$$L\left(X^\Sigma(G[t])\right) > 0 \ \ and \ \ \dot{L}\left(X^\Sigma(G[t])\right) \le 0$$

for $X \in \mathscr{O} - X_0^{\Sigma}$, *the system* ($APDES_m^C$) *is sum-stability, i.e.,* $G[t] \overset{\Sigma}{\sim} G[0]$. *Furthermore, if*

$$\dot{L}\left(X^{\Sigma}(G[t])\right) < 0$$

for $X \in \mathscr{O} - X_0^{\Sigma}$, *then* $G[t] \overset{\Sigma}{\to} G[0]$.

Similarly, if

$$L\left(X^{\Pi}(G[t])\right) > 0 \quad and \quad \dot{L}\left(X^{\Pi}(G[t])\right) \leq 0$$

for $X \in \mathscr{O} - X_0^{\Pi}$, *the system* ($APDES_m^C$) *is prod-stability, i.e.,* $G[t] \overset{\Pi}{\sim} G[0]$. *Furthermore, if*

$$\dot{L}\left(X^{\Pi}(G[t])\right) < 0$$

for $X \in \mathscr{O} - X_0^{\Pi}$, *then* $G[t] \overset{\Pi}{\to} G[0]$.

Proof. Let $\epsilon > 0$ be a so small number that the closed ball $B_\epsilon(X_0^{\Sigma})$ centered at X_0^{Σ} with radius ϵ entirely lies in $\mathscr{O}$ and let Λ_0 be the minimum value of $L\left(X^{\Sigma}(G[t])\right)$ on the boundary of $B_\epsilon(X_0^{\Sigma})$, i.e., the sphere $S_\epsilon(X_0^{\Sigma})$. Clearly, $\Lambda_0 > 0$ by assumption. Define $U = \{X \in B_\epsilon(X_0^{\Sigma}) | L(X) < \Lambda_0\}$. Notice that $X_0^{\Sigma} \in U$ and L is non-increasing on $\left(X^{\Sigma}(G[t])\right)$ by definition in $\mathscr{O} - X_0^{\Sigma}$. There are no solutions $X_v(t),\ v \in V(\widehat{G}[0])$ starting in U such that $L\left(X^{\Sigma}(G[t])\right)$ meet the sphere $S_\epsilon(X_0^{\Sigma})$ because of the decrease of $L\left(X^{\Sigma}(G[t])\right)$. Thus all solutions $X_v(t),\ v \in V(\widehat{G}[0])$ starting in U enable $L\left(X^{\Sigma}(G[t])\right)$ included in ball $B_\epsilon(X_0^{\Sigma})$. Consequently, $G[t] \overset{\Sigma}{\sim} G[0]$ by definition.

Now assume that $\dot{L}\left(X^{\Sigma}(G[t])\right) < 0$ for $X^{\Sigma}(G[t]) \neq X_0^{\Sigma}$. Thus L is strictly decreasing on $X^{\Sigma}(G[t])$ in $\mathscr{O} - X_0^{\Sigma}$. If $X_v(t_n),\ v \in V(\widehat{G}[0])$ are all solutions of ($APDES_m^C$) starting in $U - X_0^{\Sigma}$ such that $X^{\Sigma}(G[t_n]) \to Y_0$ for $n \to \infty$ with $Y_0 \in B_\epsilon(X_0^{\Sigma})$, then it must be $Y_0 = X_0^{\Sigma}$. Otherwise, since $L\left(X^{\Sigma}(G[t_n])\right) > L(Y_0)$ by the assumption $\dot{L}\left(X^{\Sigma}(G[t])\right) < 0$ for $X^{\Sigma}(G[t]) \in \mathscr{O} - X_0^{\Sigma}$ and $L\left(X^{\Sigma}(G[t])\right) \to L(Y_0)$ for $n \to \infty$ by the continuity of L, if $Y_0 \neq X_0^{\Sigma}$, let $Y_v(t), v \in V(\widehat{G}[0])$ be the solutions starting at Y_0. Then for any $\eta > 0$,

$$L\left(\sum_{v \in V(\widehat{G}[0])} Y_v(\eta)\right) < L(Y_0).$$

But then a contradiction

$$L\left(\sum_{v \in V(\widehat{G}[0])} X_v(t_n + \eta)\right) < L(Y_0)$$

yields by letting $Y_0 = X^{\Sigma}(G[t_n])$ for sufficiently large n. So there must be $Y_0 = X_0^{\Sigma}$. Thus $G[t] \overset{\Sigma}{\to} G[0]$.

It should be noted that replacing $X_0^{\Sigma}, X^{\Sigma}(G[t])$ by $X_0^{\Pi}, X^{\Pi}(G[t])$ and $B_\epsilon(X_0^{\Sigma})$ by $B_\epsilon(X_0^{\Pi})$ in the previous discussion, the conclusion is also hold, which enables one to know that $G[t] \overset{\Pi}{\sim} G[0]$ or $G[t] \overset{\Pi}{\to} G[0]$. This completes the proof. □

According to Theorem 4.3, if we find a differential function $L : \mathscr{O} \subset \mathbb{R}^n \to \mathbb{R}$, then we are easily known the sum or prod-stability of $(APDES_m^C)$. Calculation shows that the characteristic system of the ith equation in $(APDES_m)$ is

$$dt = \frac{dx_1}{\frac{\partial H_i}{\partial p_1}} = \cdots = \frac{dx_{n-1}}{\frac{\partial H_i}{\partial p_{n-1}}} = -\frac{dp_1}{\frac{\partial H_i}{\partial x_1}} = \cdots = -\frac{dp_{n-1}}{\frac{\partial H_i}{\partial x_{n-1}}} = \frac{du}{\sum\limits_{l=0}^{n-1} p_l \frac{\partial H_i}{\partial p_l} + \frac{\partial u}{\partial t}}$$

and

$$\frac{dx_l}{dt} = \frac{\partial H_i}{\partial p_l}, \qquad \frac{dp_l}{dt} = -\frac{\partial H_i}{\partial x_l},$$

for integers $1 \leq i \leq m,\ 1 \leq l \leq n-1$. Whence,

$$\begin{aligned}\frac{dH_i}{dt} &= \frac{\partial H_i}{\partial t} + \sum_{l=1}^{n-1} \frac{\partial H_i}{\partial x_l}\frac{dx_l}{dt} + \sum_{l=1}^{n-1} \frac{\partial H_i}{\partial p_l}\frac{dp_l}{dt} \\ &= \frac{\partial H_i}{\partial t} + \sum_{l=1}^{n-1} \frac{\partial H_i}{\partial x_l}\frac{\partial H_i}{\partial p_l} - \sum_{l=1}^{n-1} \frac{\partial H_i}{\partial p_l}\frac{\partial H_i}{\partial x_l} \equiv \frac{\partial H_i}{\partial t}\end{aligned}$$

for integers $1 \leq i \leq m$. This fact enables us to find conditions for the global stability of partial differential systems $(APDES_m^C)$.

Theorem 4.4. *Let $X_0^{[i]}$ be an equilibrium point of the ith equation in $(APDES_m)$ for each integer $1 \leq i \leq m$. If*

$$\sum_{i=1}^{m} H_i(X) > 0 \quad and \quad \sum_{i=1}^{m} \frac{\partial H_i}{\partial t} \leq 0$$

for $X \neq \sum\limits_{i=1}^{m} X_0^{[i]}$, then the system $(APDES_m)$ is sum-stability, i.e., $G[t] \overset{\Sigma}{\sim} G[0]$. Furthermore, if

$$\sum_{i=1}^{m} \frac{\partial H_i}{\partial t} < 0$$

for $X \neq \sum\limits_{i=1}^{m} X_0^{[i]}$, then $G[t] \overset{\Sigma}{\to} G[0]$.

Similarly, if

$$\prod_{i=1}^{m} H_i(X) > 0 \quad and \quad \sum_{i=1}^{m} \frac{1}{H_i(X)}\frac{\partial H_i}{\partial t} \leq 0$$

for $X \neq \prod\limits_{i=1}^{m} X_0^{[i]}$, then $G[t] \overset{\Pi}{\sim} G[0]$. Furthermore, if

$$\sum_{i=1}^{m} \frac{1}{H_i(X)}\frac{\partial H_i}{\partial t} < 0$$

for $X \neq \prod\limits_{i=1}^{m} X_0^{[i]}$, then $G[t] \overset{\Pi}{\to} G[0]$.

Proof. Define $L(X)=\sum\limits_{i=1}^{m}H_i(X)$. Then $\dot{L}(X)=\sum\limits_{i=1}^{m}\dot{H}_i(X)$. By assumption, if

$$\sum_{i=1}^{m}H_i(X)>0,\quad \sum_{i=1}^{m}\frac{\partial H_i}{\partial t}\leq 0 \text{ or } \sum_{i=1}^{m}\frac{\partial H_i}{\partial t}<0,$$

we know that

$$L(X)>0,\quad \dot{L}(X)\leq 0 \text{ or } \dot{L}(X)<0$$

for $X\neq\sum\limits_{i=1}^{m}X_0^{[i]}$. Applying Theorem 4.3, we get that $G[t]\overset{\Sigma}{\sim}G[0]$, or furthermore, $G[t]\overset{\Sigma}{\to}G[0]$ if $\sum\limits_{i=1}^{m}\frac{\partial H_i}{\partial t}<0$. Thus we get the sum-stability of $G[t]$-solution of $(APDES_m^C)$.

For the prod-stability of $G[t]$-solution of $(APDES_m^C)$, let $L(X)=\prod\limits_{i=1}^{m}H_i(X)$. Then

$$\dot{L}(X)=\sum_{j=1}^{m}\frac{\dot{H}_j(X)\prod\limits_{i=1}^{m}H_i(X)}{H_j(X)}=\prod_{i=1}^{m}H_i(X)\left(\sum_{j=1}^{m}\frac{\dot{H}_j(X)}{H_j(X)}\right).$$

Whence, if

$$\prod_{i=1}^{m}H_i(X)>0,\quad \sum_{i=1}^{m}\frac{1}{H_i(X)}\frac{\partial H_i}{\partial t}\leq 0 \quad\text{or}\quad \sum_{i=1}^{m}\frac{1}{H_i(X)}\frac{\partial H_i}{\partial t}<0$$

for integers $1\leq i\leq m$, then

$$L(X)>0,\quad \dot{L}(X)\leq 0\quad\text{or}\quad \dot{L}(X)<0$$

for $X\neq\prod\limits_{i=1}^{m}X_0^{[i]}$. Applying Theorem 4.3, we know that $G[t]\overset{\Pi}{\sim}G[0]$, or furthermore, $G[t]\overset{\Pi}{\to}G[0]$ if

$$\sum_{i=1}^{m}\frac{1}{H_i(X)}\frac{\partial H_i}{\partial t}<0.$$

for $X\neq\prod\limits_{i=1}^{m}X_0^{[i]}$. □

Corollary 4.5. *An equilibrium point X^* of the Cauchy problem*

$$\begin{cases}\dfrac{\partial u}{\partial t}=H(t,x_1,\cdots,x_{n-1},p_1,\cdots,p_{n-1})\\ u|_{t=t_0}=u_0(x_1,x_2,\cdots,x_{n-1})\end{cases}$$

is stable if $H(X)>0$, $\frac{\partial H}{\partial t}\leq 0$, and is asymptotically stable if $\frac{\partial H}{\partial t}\leq 0$ for $X\neq X^*$.

Let us see a simple example in the following.

EXAMPLE 4.6. Let $(APDES_m^C)$ be

$$\begin{cases} \frac{\partial u}{\partial t} = H_1(t,x) = x^2 e^{-t+x} \\ u|_{t=0} = \varphi(x), \end{cases} \qquad \begin{cases} \frac{\partial u}{\partial t} = H_2(t,x) = x^4 e^{-5t+x^2} \\ u|_{t=0} = \zeta(x). \end{cases}$$

Clearly, $(t,0)$ is its an equilibrium point. Calculation shows that

$$\begin{aligned} H_1(t,x) + H_2(t,x) &= x^2 e^{-t+x} + x^4 e^{-5t+x^2} > 0, \\ \dot{H}_1(t,x) + \dot{H}_2(t,x) &= -x^2 e^{-t+x} - 5x^4 e^{-5t+x^2} < 0 \end{aligned}$$

and

$$\begin{aligned} H_1(t,x)H_2(t,x) &= x^6 e^{-6t+x+x^2} > 0, \\ H_1 \dot{H}_2(t,x) &= -6x^6 e^{-6t+x+x^2} < 0 \end{aligned}$$

if $x \neq 0$. Thus the equilibrium points $(t,0)$ of $(APDES_m^C)$ are both sum and prod-stable by Theorem 4.4.

4.2. Energy integral of G-solution.

DEFINITION 4.7. *Let $G[t]$ be the G-solution of Cauchy problem $(APDES_m^C)$. The v-energy $E(v[t])$ and G-energy $E(G[t])$ are defined respectively by*

$$E(v[t]) = \int_{\mathscr{O}_v} \left(\frac{\partial u^{[v]}}{\partial t} \right)^2 dx_1 dx_2 \cdots dx_{n-1},$$

where $\mathscr{O}_v \subset \mathbb{R}^n$ is determined by the vth equation

$$\begin{cases} \frac{\partial u}{\partial t} = H_v(t, x_1, \cdots, x_{n-1}, p_1, \cdots, p_{n-1}) \\ u|_{t=t_0} = u_0^{[v]}(x_1, x_2, \cdots, x_{n-1}) \end{cases}$$

and

$$E(G[t]) = \sum_{G \leq \widehat{G}[0]} (-1)^{|G|+1} \int_{\mathscr{O}_G} \left(\frac{\partial u^G}{\partial t} \right)^2 dx_1 dx_2 \cdots dx_{n-1},$$

where u^G is the $\mathbb{C}^2$ solution of system

$$\left. \begin{aligned} \frac{\partial u}{\partial t} &= H_v(t, x_1, \cdots, x_{n-1}, p_1, \cdots, p_{n-1}) \\ u|_{t=t_0} &= u_0^{[v]}(x_1, x_2, \cdots, x_{n-1}) \end{aligned} \right\} v \in V(G)$$

and $\mathscr{O}_G = \bigcap_{v \in V(G)} \mathscr{O}_v$. Particularly, if $\widehat{G}[0] \simeq \overline{K}_n$, i.e., all equations in $(APDES_m^C)$ is non-solvable two by two, then

$$E(G[t]) = \sum_{v \in \widehat{G}[0]} \int_{\mathscr{O}_v} \left(\frac{\partial u^v}{\partial t} \right)^2 dx_1 dx_2 \cdots dx_{n-1} = \sum_{v \in \widehat{G}[0]} E(v[t]).$$

We determine the non-empty domain $\mathscr{O}_G \subset \mathbb{R}^n$ in the following.

THEOREM 4.8. *Let the Cauchy problem be ($APDES_m^C$), $G \subset \widehat{G}[0]$ with $\mathscr{O}_G \neq \emptyset$. Then*

$$\bigcap_{v\in V(G)} \mathscr{O}_v = \{X \in \mathbb{R}^n | H_u(X) = H_v(X),\ \forall u, v \in V(G)\}.$$

if $|G| \geq 2$.

Proof. Noticing that if $\mathscr{O}_G \neq \emptyset$, there is a solution u^G of the system

$$\left.\begin{array}{l} \dfrac{\partial u^G}{\partial t} = H_v(t, x_1, \cdots, x_{n-1}, p_1, \cdots, p_{n-1}) \\ u^G|_{t=t_0} = u_0^{[v]}(x_1, x_2, \cdots, x_{n-1}) \end{array}\right\} v \in V(G).$$

Whence, $H_v = u_t^G$ for $\forall v \in V(G)$ in $\mathscr{O}_G$, which implies that

$$\bigcap_{v\in V(G)} \mathscr{O}_v \subset \{X \in \mathbb{R}^n | H_v(X) = H_u(X),\ \forall u, v \in V(G)\}.$$

Conversely, for $\forall X \in \{X \in \mathbb{R}^n | H_v(X) = H_u(X),\ \forall u, v \in V(G)\}$, there are $H_v(X) = H_u(X) = H(X)$ for $\forall u, v \in V(G)$. Thus the system

$$\frac{\partial u^G}{\partial t} = H_v(t, x_1, \cdots, x_{n-1}, p_1, \cdots, p_{n-1}), \quad v \in V(G)$$

is equivalent to the partial differential equation

$$\frac{\partial u}{\partial t} = H(t, x_1, \cdots, x_{n-1}, p_1, \cdots, p_{n-1}).$$

Now by Theorem 1.1, this equation is always solvable with suitable initial values, which means that

$$\{X \in \mathbb{R}^n | H_v(X) = H_u(X),\ \forall u, v \in V(G)\} \subset \bigcap_{v\in V(G)} \mathscr{O}_v. \qquad \square$$

Theorem 4.8 enables one to introduce the conception of *energy-index* for the system ($APDES_m^C$) following.

DEFINITION 4.9. *Let the Cauchy problem be ($APDES_m^C$) with each H_i in $\mathbb{C}^2$ for integers $1 \leq i \leq m$. Its energy-index $ind^E(G)$ is defined by*

$$ind^E(G) = \sum_{G \leq \widehat{G}[0]} (-1)^{|G|+1} \int\limits_{\bigcap_{v\in V(G)} \mathscr{O}_v} H_G \dot{H}_G dx_1 dx_2 \cdots dx_{n-1},$$

where $H_G = H_v$ for $\forall v \in V(G)$ with $\mathscr{O}_G \neq \emptyset$.

Denoted by

$$\overline{ind}_G(v) = \int\limits_{\bigcap_{v\in V(G)} \mathscr{O}_v} H_v \frac{\partial H_v}{\partial t} dx_1 dx_2 \cdots dx_{n-1}$$

for $G \leq \widehat{G}[0]$. We know a result on the energy-index following.

THEOREM 4.10. *Let $(APDES_m^C)$ be a Cauchy problem with G-solution $G[t]$ and all H_i in $\mathbb{C}^2$ for integers $1 \leq i \leq m$. Then*

$$ind^E(G) = \sum_{i=1}^{m} \frac{(-1)^{i+1}}{i} \sum_{v \in V(K_i),\ K_i \leq \widehat{G}[0]} \overline{ind}_{K_i}(v).$$

Proof. Clearly, $\overline{ind}_G(v) = ind^E(v)$ if $G = \langle v \rangle$ and $\overline{ind}_G(v) = 0$ if $G \not\simeq K_s$ for some integer $1 \leq s \leq m$. By definition, we know that

$$\begin{aligned}
ind^E(G) &= \sum_{G \leq \widehat{G}[0]} (-1)^{|G|+1} \int\limits_{\bigcap\limits_{v \in V(G)} \mathscr{O}_v} H_G \dot{H}_G dx_1 dx_2 \cdots dx_{n-1} \\
&= \sum_{G \leq \widehat{G}[0]} (-1)^{|G|+1} \int\limits_{\bigcap\limits_{v \in V(G)} \mathscr{O}_v} H_G \frac{\partial H_G}{\partial t} dx_1 dx_2 \cdots dx_{n-1} \\
&= \sum_{i=1}^{m} \sum_{K_i \leq \widehat{G}[0]} (-1)^{i+1} \frac{1}{i} \sum_{v \in V(K_i)} \overline{ind}_{K_i}(v) \\
&= \sum_{i=1}^{m} \frac{(-1)^{i+1}}{i} \sum_{v \in V(K_i),\ K_i \leq \widehat{G}[0]} \overline{ind}_{K_i}(v).
\end{aligned}$$

□

Particularly, if $\widehat{G}[0]$ is K_3-free, i.e., there are no induced subgraphs isomorphic to K_3 in $\widehat{G}[0]$, then $\bigcap\limits_{v \in V(K_i)} \mathscr{O}_v = \emptyset$ for integers $i \geq 3$. We get the following conclusion.

COROLLARY 4.11. *For a Cauchy problem $(APDES_m^C)$ with G-solution $G[t]$, if $\widehat{G}[0]$ is K_3-free, then*

$$ind^E(G) = \sum_{v \in V(\widehat{G}[0])} ind^E(v) - \frac{1}{2} \sum_{e \in E(\widehat{G}[0])} \int\limits_{\mathscr{O}_u \bigcap \mathscr{O}_v} H_e \frac{\partial H_e}{\partial t} dx_1 dx_2 \cdots dx_{n-1},$$

where $H_e = H_u = H_v$ for $e = (u, v) \in E(\widehat{G}[0])$.

Applying the energy-index $ind^E(G)$, we know a G-energy inequality following.

THEOREM 4.12. *Let $G[t]$ be the G-solution of Cauchy problem $(APDES_m^C)$. If $in^E(G) > 0$, then $E(G[t_1]) > E(G[t_0])$ for $t_1 > t_0$.*

Proof. By definition we know that

$$\begin{aligned}
E(G[t_1]) - E(G[t_0]) &= \sum_{G\leq \widehat{G}[0]} (-1)^{|G|+1} \int\limits_{\mathscr{H}_G} \left(\frac{\partial u^G}{\partial t}\right)^2 dx_1\cdots dx_{n-1}\Bigg|_{t=t_1} \\
&\quad - \sum_{G\leq \widehat{G}[0]} (-1)^{|G|+1} \int\limits_{\mathscr{H}_G} \left(\frac{\partial u^G}{\partial t}\right)^2 dx_1\cdots dx_{n-1}\Bigg|_{t=t_0} \\
&= \sum_{G\leq \widehat{G}[0]} (-1)^{|G|+1} \int\limits_{\mathscr{H}_G} \int\limits_{t_0}^{t_1} \frac{dH_G^2}{dt} dt dx_1\cdots dx_{n-1} \\
&= \sum_{G\leq \widehat{G}[0]} (-1)^{|G|+1} \int\limits_{\mathscr{H}_G} \int\limits_{t_0}^{t_1} H_G \dot{H}_G dt dx_1\cdots dx_{n-1} \\
&= \int\limits_{t_0}^{t_1} \left(\sum_{G\leq \widehat{G}[0]} (-1)^{|G|+1} \int\limits_{\mathscr{H}_G} H_G \dot{H}_G dx_1\cdots dx_{n-1} \right) dt \\
&= \int\limits_{t_0}^{t_1} in^G(G) dt \ > \ 0
\end{aligned}$$

if $t_1 > t_0$, where the interchangeable of integral orders is holden by the $\mathbb{C}^2$ property. Therefore, $E(G[t_1]) > E(G[t_0])$. □

Particularly, let $G = \langle v\rangle$, we get a v-energy inequality following.

COROLLARY 4.13. *Let $G[t]$ be the G-solution of Cauchy problem ($APDES_m^C$), $v \in V(\widehat{G}[0])$ with $H_v\dot{H}_v > 0$. Then $E(v[t_1]) > E(v[t_0])$ if $t_1 > t_0$.*

4.3. Geometry of G-solution. Let $u:\ \mathbb{R}^n \to \mathbb{R}^n$ be differentiable. We define its n-dimensional graph $\Gamma[u]$ by the set of ordered pairs

$$\Gamma[u] = \{((x_1,\cdots,x_n), u(x_1,\cdots,x_n)))\,|(x_1,\cdots,x_n)\in\mathbb{R}^n\}.$$

Similarly, for a system ($PDES_m^C$) of partial differential equations of first order (solvable or non-solvable), its n-geometrical graph is defined by

$$\Gamma[PDES_m^C] = \{((x_1,\cdots,x_n), u_v(x_1,\cdots,x_n)))\,|(x_1,\cdots,x_n)\in\mathbb{R}^n,\ v\in V(\widehat{G}[0])\}.$$

Then, a conclusion on $\Gamma[PDES_m^C]$ can be determined in the following.

THEOREM 4.14. *Let the Cauchy problem be ($PDES_m^C$). Then every connected component of $\Gamma[PDES_m^C]$ is a differentiable n-manifold with atlas $\mathscr{A} = \{(U_v,\phi_v)|v\in V(\widehat{G}[0])\}$ underlying graph $\widehat{G}[0]$, where U_v is the n-dimensional graph $G[u^{[v]}] \simeq \mathbb{R}^n$ and ϕ_v the projection*

$$\phi_v : ((x_1,x_2,\cdots,x_n), u(x_1,x_2,\cdots,x_n))) \to (x_1,x_2,\cdots,x_n)$$

for $\forall\,(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n$.

Proof. Clearly, U_v is open and

$$\phi_v^{-1}:\ (x_1,x_2,\cdots,x_n) \to ((x_1,x_2,\cdots,x_n), u(x_1,x_2,\cdots,x_n)))$$

for $\forall v\in V(\widehat{G}[0])$. Notice that u is differentiable in $\mathbb{R}^n$ and $\phi_v\phi_v^{-1} = \mathbf{1}_{U_u\cap U_v}$ and $\phi_v\phi_u^{-1} = \mathbf{1}_{U_u\cap U_v}$ on $U_u\bigcap U_v$ are also differentiable by definition of $U_u\bigcap U_v$ for $u,v\in$

$V(\widehat{G}[0])$. Thus the connected n-dimensional component of $\Gamma[PDES_m^C]$ is a differential manifold. □

Notice that it is shown in [11] that manifolds can be classified by n-dimensional graphs and listed by graphs. However, Theorem 4.14 enables one to get such n-dimensional graphs for differentiable manifolds by systems $(PDES_m^C)$ of partial differential equations. We know that the standard basis of a vector field $T(M)$ on a differentiable n-manifold M is

$$\left\{\frac{\partial}{\partial x_i},\ 1 \leq i \leq n\right\}$$

and a vector field X can be viewed as a first order partial differential operator

$$X = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i},$$

where a_i is $\mathbb{C}^\infty$-differentiable for integers $1 \leq i \leq n$. Combining Theorems 3.6 and 4.14 enables one to get the following result on vector fields.

THEOREM 4.15. *For any integer $m \geq 1$, let $U_i, 1 \leq i \leq m$ be open sets in $\mathbb{R}^n$ underlying a connected graph defined by*

$$V(G) = \{U_i | 1 \leq i \leq m\}, \qquad E(G) = \{(U_i, U_j) | U_i \bigcap U_j \neq \emptyset, 1 \leq i, j \leq m\}.$$

If X_i is a vector field on U_i for integers $1 \leq i \leq m$, then there always exists a differentiable manifold $M \subset \mathbb{R}^n$ with atlas $\mathscr{A} = \{(U_i, \phi_i) | 1 \leq i \leq m\}$ underlying graph G and a function $u_G \in \Omega^0(M)$ such that

$$X_i(u_G) = 0, \quad 1 \leq i \leq m.$$

Proof. For any integer $1 \leq k \leq m$, let

$$X_k = \sum_{i=1}^{n} a_i^{[k]} \frac{\partial}{\partial x_i}.$$

Notice that the system $(PDES_m^C)$ of partial differential equations

$$\left.\begin{array}{l} a_1^{[v]} \dfrac{\partial u}{\partial x_1} + a_2^{[v]} \dfrac{\partial u}{\partial x_2} + \cdots + a_1^{[v]} \dfrac{\partial u}{\partial x_n} = 0 \\ u\big|_{x_n = x_n^{[0]}} = u_v^{[0]} \end{array}\right\} v \in V(G)$$

has a G-solution by Theorem 3.6. According to Theorem 4.14, its n-dimensional graph $\Gamma[PDES_m^C]$ is an n-dimensional manifold M. We construct a differentiable function u_G on M. In fact, let u_v be a solution of the vth equation of system $(PDES_m^C)$ and $\{h_v, v \in V(G)\}$ a partition of unity on open sets $\{U_v, v \in V(G)\}$. Define

$$u_G = \sum_{v \in V(G)} h_v u_v.$$

Then, it is clear that

$$X_k(u_G) = \sum_{i=1}^{n} a_i^{[k]} \frac{\partial u}{\partial x_i} = 0$$

for any integers $1 \leq k \leq m$. □

Generally, we can also characterize these systems of shifted partial differential equations introduced in Theorem 2.7 by that of a generalization of manifold, i.e. *differentiable combinatorial manifold* defined following.

DEFINITION 4.16 ([6], [10]). *Let $n_\nu, \nu \in \Lambda$ be positive integers. A differentiable combinatorial manifold $\widetilde{M}(n_\nu, \nu \in \Lambda)$ is a second countable Hausdorff space with a maximal atlas $\mathscr{A} = \{(U_\nu, \phi_\nu)|\nu \in \Lambda\}$ for a countable set Λ such that $\phi_\nu : U_\nu \to \mathbb{R}^{n_\nu}$, $\phi_\nu\phi_\mu^{-1} : \phi_\mu(U_\mu \cap U_\nu) \to \phi_\nu(U_\mu \cap U_\nu)$ are differentiable for $\forall \mu, \nu \in \Lambda$.*

Clearly, a combinatorial manifold underlies a connected graph $G[\widetilde{M}(n_\nu, \nu \in \Lambda)]$ defined by $V(G[\widetilde{M}(n_\nu, \nu \in \Lambda)]) = \{U_\nu, \ \nu \in \Lambda\}$ and $E(G[\widetilde{M}(n_\nu, \nu \in \Lambda)]) = \{(U_\mu, U_\nu)|U_\mu \bigcap U_\nu \neq \emptyset, \ \mu, \nu \in \Lambda\}$. Particularly, if Λ is finite, then $G[\widetilde{M}(n_\nu, \nu \in \Lambda)]$ is nothing but an finite connected graph., i.e., a compact $\widetilde{M}(n_\nu, \nu \in \Lambda)$. The following results are a generalization of Theorems 4.14 and 4.15, which can be similarly obtained.

THEOREM 4.17. *Let the Cauchy problem be*

$$\begin{cases} F_1(x_1, \cdots, x_n, x_{n+1}^{[1]}, \cdots, x_{n_1}^{[1]}, u, p_1, \cdots, p_n, p_{n+1}^{[1]}, \cdots, p_{n_1}^{[1]}) = 0 \\ F_2(x_1, \cdots, x_n, x_{n+1}^{[2]}, \cdots, x_{n_2}^{[2]}, u, p_1, \cdots, p_n, p_{n+1}^{[2]}, \cdots, p_{n_2}^{[2]}) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, \cdots, x_n, x_{n+1}^{[m]}, \cdots, x_{n_m}^{[m]}, u, p_1, \cdots, p_n, p_{n+1}^{[m]}, \cdots, p_{n_m}^{[m]}) = 0 \end{cases}$$

of partial differential equations of first order with initial values $u_0, x_i^0, p_i^0, \ 1 \leq i \leq n$ and $x_i^0 = 0, p_i^0 = 0$ for integers $i \geq n+1$, where $x_1, \cdots, x_n, x_{n+1}^{[1]}, \cdots, x_{n_1}^{[1]}, x_{n+1}^{[2]}, \cdots, x_{n_2}^{[2]}, \cdots, x_{n+1}^{[m]}, \cdots, x_{n_m}^{[m]}$ are independent, $p_k^{[i]} = \partial u/\partial x_k^{[i]}$ and $n \leq n_1 \leq n_2 \leq \cdots \leq n_m$. Then every connected component of $\Gamma[PDES_m^C]$ is a differentiable combinatorial manifold $\widetilde{M}(n_i, \ 1 \leq i \leq m)$ with atlas $\mathscr{A} = \{(U_v, \phi_v)|v \in V(\widehat{G}[0])\}$ underlying graph $\widehat{G}[0]$, where U_v is the n_v-dimensional graph $G[u^{[v]}] \simeq \mathbb{R}^{n_v}$ and ϕ_v is a projection determined by

$$\phi_v : ((x_1, x_2, \cdots, x_{n_v}), u(x_1, x_2, \cdots, x_{n_v}))) \to (x_1, x_2, \cdots, x_{n_v})$$

for $\forall (x_1, x_2, \cdots, x_{n_v}) \in \mathbb{R}^{n_v}$.

THEOREM 4.18. *For any integer $m \geq 1$, let $U_i, 1 \leq i \leq m$ be open sets in $\mathbb{R}^{n_i}$ underlying a connected graph defined by*

$$V(G) = \{U_i|1 \leq i \leq m\}, \qquad E(G) = \{(U_i, U_j)|U_i \bigcap U_j \neq \emptyset, 1 \leq i, j \leq m\}.$$

If X_i is a vector field on U_i for integers $1 \leq i \leq m$, then there always exists a differentiable combinatorial manifold $\widetilde{M} \subset \mathbb{R}^{\widehat{m} + \sum\limits_{i=1}^{m}(n_i - \widehat{m})}$ with atlas $\mathscr{A} = \{(U_i, \phi_i)|1 \leq i \leq m\}$ underlying graph G, where

$$\widehat{m} = \dim\left(\bigcap_{i=1}^{m} \mathbb{R}^{n_i}\right)$$

and a function $u_G \in \Omega^0(\widetilde{M})$ such that

$$X_i(u_G) = 0, \quad 1 \leq i \leq m.$$

Theorems 4.14, 4.15 and 4.17, 4.18 show the differentiable geometry on combinatorial manifolds discussed in [6] and [10] is more valuable for knowing the global behavior of a thing in the world.

5. Applications.

5.1. Interaction fields. Let $\mathscr{F}_1, \mathscr{F}_2, \cdots, \mathscr{F}_m$ be m interaction fields with respective Hamiltonians $H^{[1]}, H^{[2]}, \cdots, H^{[m]}$, i.e., a combinatorial field $\widetilde{\mathscr{F}}$ introduced in [7], where

$$H^{[k]}: (q_1, \cdots, q_n, p_2, \cdots, p_n, t) \to H^{[k]}(q_1, \cdots, q_n, p_1, \cdots, p_n, t)$$

for integers $1 \le k \le m$. Thus

$$\left.\begin{array}{l} \dfrac{\partial H^{[k]}}{\partial p_i} = \dfrac{dq_i}{dt} \\ \dfrac{\partial H^{[k]}}{\partial q_i} = -\dfrac{dp_i}{dt}, \quad 1 \le i \le n \end{array}\right\} \quad 1 \le k \le m.$$

Such an interaction system naturally underlies a graph G with

$V(G) = \{H^{[i]} | 1 \le i \le m\}$,

$E(G) = \{(H^{[i]}, H^{[j]}) | H^{[i]}$ interacts with $H^{[j]}$ for integers $1 \le i, j \le m\}$.

For example, let $m = 4$. Then such an interaction system are shown in Fig.2. Such a system is equivalent to the system $(APDES_m^C)$ of non-solvable partial differential equations

$$\left.\begin{array}{l} \dfrac{\partial u}{\partial t} = H_i(t, x_1, \cdots, x_{n-1}, p_1, \cdots, p_{n-1}) \\ u|_{t=t_0} = u_0^{[k]}(x_1, x_2, \cdots, x_{n-1}) \end{array}\right\} \quad 1 \le k \le m.$$

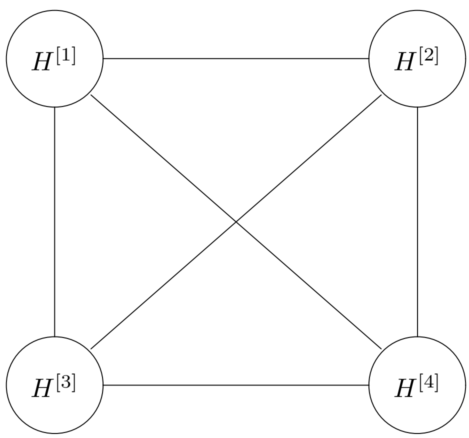

FIG. 2

Whence, if $X_0^{[i]}$ be an equilibrium point of the ith equation in this system,

$$\sum_{k=1}^{m} H^{[k]}(X) > 0 \quad \text{and} \quad \sum_{k=1}^{m} \frac{\partial H^{[k]}}{\partial t} \le 0$$

for $X \ne \sum\limits_{k=1}^{m} X_0^{[k]}$, then $\widetilde{\mathscr{F}}$ is sum-stable and furthermore, if

$$\sum_{k=1}^{m} \frac{\partial H^{[k]}}{\partial t} < 0$$

for $X \neq \sum\limits_{k=1}^{m} X_0^{[k]}$, then it is also asymptotically sum-stable by Theorem 4.4.

Similarly, if

$$\prod_{k=1}^{m} H^{[k]}(X) > 0 \quad \text{and} \quad \sum_{k=1}^{m} \frac{1}{H^{[k]}(X)} \frac{\partial H^{[k]}}{\partial t} \leq 0$$

for $X \neq \prod\limits_{k=1}^{m} X_0^{[k]}$, then $\widetilde{\mathscr{F}}$ is prod-stable and furthermore, if

$$\sum_{k=1}^{m} \frac{1}{H^{[k]}(X)} \frac{\partial H^{[k]}}{\partial t} < 0$$

for $X \neq \prod\limits_{k=1}^{m} X_0^{[k]}$, then it is also asymptotically prod-stable by Theorem 4.4. Such combinatorial fields are extensively existed in theoretical physics (See references [7], [9]-[10] for details).

5.2. Flows in network. Let N be a network and let $q(x,t)$, $\rho(x,t)$, $u(x,t)$ be the respective rate, density and velocity of 1-dimensional flow on an arc (x,y) of N at time t. Then the continuity equation of 1-dimension enables one knowing that

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0 \quad \text{and} \quad q = \rho u.$$

Particularly, if $u(x,t)$ depends on $\rho(x,t)$, the density, let $u(x,t) = u(\rho(x,t))$, then $q(x,t) = \rho(x,t)u(\rho(x,t))$ and

$$\frac{\partial q}{\partial x} = \left(u + \rho \frac{\partial u}{\partial x}\right) \frac{\partial \rho}{\partial x} = \phi(\rho) \frac{\partial \rho}{\partial x},$$

where, $\phi(\rho) = u + \rho \dfrac{\partial u}{\partial x}$. Consequently,

$$\frac{\partial \rho}{\partial t} + \phi(\rho) \frac{\partial \rho}{\partial x} = 0.$$

Now let O be a node in N incident with m in-flows and 1 out-flow. Such as those shown in Fig.3.

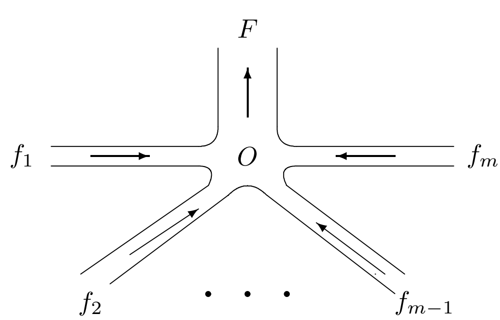

FIG.3

Then *how can we characterize the behavior of flow F*? Denote the rate, density of flow f_i by $\rho^{[i]}$ for integers $1 \leq i \leq m$ and that of F by $\rho^{[F]}$, respectively. Then we know that

$$\frac{\partial \rho^{[i]}}{\partial t} + \phi_i(\rho^{[i]})\frac{\partial \rho^{[i]}}{\partial x} = 0, \ 1 \leq i \leq m.$$

Assume these flows $f^{[i]}$, $1 \leq i \leq m$ to be conservation at the node O. Then we know that $\rho^{[F]} = \sum\limits_{i=1}^{m} \rho^{[i]}$. Whence,

$$\frac{\partial \rho^{[F]}}{\partial t} = \sum_{i=1}^{m} \frac{\partial \rho^{[i]}}{\partial t} = -\sum_{i=1}^{m} \phi_i(\rho^{[i]})\frac{\partial \rho^{[i]}}{\partial x}.$$

Thus

$$\frac{\partial \rho^{[F]}}{\partial t} + \sum_{i=1}^{m} \phi_i(\rho^{[i]})\frac{\partial \rho^{[i]}}{\partial x} = 0$$

by the continuity equation of 1-dimension. Generally, it is difficult to determine the behavior of flow F by this equation.

We prescribe the initial value of $\rho^{[i]}$ by $\rho^{[i]}(x, t_0)$ at time t_0. Replacing each $\rho^{[i]}$ by ρ in these flow equations of f_i, $1 \leq i \leq m$, we then get a non-solvable system $(PDES_m^C)$ of partial differential equations following.

$$\left.\begin{array}{l} \dfrac{\partial \rho}{\partial t} + \phi_i(\rho)\dfrac{\partial \rho}{\partial x} = 0 \\ \rho\,|_{t=t_0} = \rho^{[i]}(x, t_0) \end{array}\right\} \ 1 \leq i \leq m.$$

Let $\rho_0^{[i]}$ be an equilibrium point of the ith equation, i.e., $\phi_i(\rho_0^{[i]})\dfrac{\partial \rho_0^{[i]}}{\partial x} = 0$. Applying Theorem 4.4, if

$$\sum_{i=1}^{m} \phi_i(\rho) < 0 \ \text{ and } \ \sum_{i=1}^{m} \phi_(\rho)\left[\frac{\partial^2 \rho}{\partial t \partial x} - \phi'(\rho)\left(\frac{\partial \rho}{\partial x}\right)^2\right] \geq 0$$

for $X \neq \sum\limits_{k=1}^{m} \rho_0^{[i]}$, then we know that the flow F is stable and furthermore, if

$$\sum_{i=1}^{m} \phi_(\rho)\left[\frac{\partial^2 \rho}{\partial t \partial x} - \phi'(\rho)\left(\frac{\partial \rho}{\partial x}\right)^2\right] < 0$$

for $X \neq \sum\limits_{k=1}^{m} \rho_0^{[i]}$, then it is also asymptotically stable.

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